Math 10A
August 30, 2016
Prof. Ken Ribet

Announcements. Chapter 1 of the textbook is available to you via bCourses. The Cal Student Store has ordered more copies.

Wiley: "I know that the Follett store across the street from the Cal store will be receiving an additional order of the Schreiber texts tomorrow (the standard version at the custom price). I don't have tracking for the Cal store yet, but it's likely that they will arrive tomorrow too."

Only a few students from last semester's 10B course have offered their books for sale. The demand from this class vastly outstrips the supply, but I hope to coax out a few more copies from the sophomores.

Richard Bamler has sent me some "handouts" that illustrate interesting functions, especially in the context of biology. I'll add them to bCourses as well.

Lecturing technique. On Tuesday, August 30, I plan to lecture using the chalkboards. There won't be any slides or "documents" (for the document camera). After the lecture, let me know what you think of that mode of communication. I'm writing these notes with some incidental remarks that might have made their way to the slides for the lecture—if there were any.

Sequences. The book says that a sequence is a real-valued function defined on the set of *natural numbers*. You should be a tiny bit careful about the phrase "natural numbers." In North America, it is most common to refer to the set of positive integers $\{1, 2, 3, \ldots\}$ as the set of natural numbers. Outside of North America, most authors include 0 as a natural number. But, more importantly, a sequence (a_n) can well have a 0th term even people are skittish as including 0 as a natural number. Also, some sequences are defined only for a finite number of values of n. For example, we could let a_n be the elevation (in meters) above sea level of Floor n of Evans Hall. In this example, there'd be an a_0 (for the ground floor) and an a_{-1} for the basement. There'd be an a_n for $n = 1, \ldots, 10$ but no a_{11} (or anything beyond).

Visit `https://oeis.org` for an online oracle that identifes a sequence for you if you type in the first few terms. In spring, 2013, I was lecturing in Math 55 with an open laptop and told my students about the On-Line Encyclopedia of Integer Sequences. I decided to trick OEIS by typing in the street numbers for stops on a certain New York City subway line. To my amazement, the oracle wasn't tricked at all—it identified the subway line!

On page 87, the textbook mentions "difference equations"; these are situations where a sequence is defined recursively. For example, the famous Fibonacci sequence is defined by decreeing that the first two values of the sequence are 0 and 1 and that further values are computed from previous values by the rule

$$a_{n+2} = a_{n+1} + a_n.$$

There is a fairly long section on difference equations in Math 10B.

In the textbook, the authors consider only difference equations in which each value of the sequence is computed from the previous value (i.e., from only one previous value) by some algorithm. This can be written as the formula

$$a_{n+1} = f(a_n),$$

where f is some function.

Here's an interesting example: we start with $a_1 = 1$ and define

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$$

for $n \geq 1$. The sequence begins 1, 1.5, 1.41666666666667, 1.41421568627451, 1.41421356237469, 1.41421356237309 and 1.41421356237309 . Note that $\sqrt{2} \approx$ 1.41421356237310. It looks as if the sequence is *converging to* the square root of 2. (Spoiler: it is!)

As the book explains (p. 93), an equilibrium for the difference equation $a_{n+1} = f(a_n)$ is a number a such that $f(a) = a$. If you start with $a_1 = a$, then the sequence (a_n) is constant: all of its values are equal to a. In the example that we just considered, $\sqrt{2}$ is an equilibrium for the equation because $2/\sqrt{2} = \sqrt{2}$: the average of a_1 and $2/a_1$ will be equal to a_1 if $a_1 = \sqrt{2}$.

You can find equilibrium points for $a_{n+1} = f(a_n)$ by looking for numbers a that satisfy $a = f(a)$.

Sums. Very often, sequences are defined as "partial sums"; for example, we might have

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

or

$$b_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}$$

for $n \geq 1$. The word "partial" is appropriate because of the allusion to the corresponding *infinite series*

$$\sum_{j=1}^{\infty} \frac{1}{j}, \qquad \sum_{j=1}^{\infty} \frac{1}{2^j}.$$

These series are given meaning by defining them to be the limits (if they exist) of the sequences a_n and b_n defined just above.

Limits, did we say "limits"? In discussing a sequence before, did we using the phrase "converging to"?

Limits. Faced with a sequence (a_n), we might want to figure out what happens for n very big. What is the limit of a_n as n tends to infinity? A simple principle is that $\lim_{n\to\infty} \frac{1}{n} = 0$. You can get a lot of mileage out of this fact. For example, you'll discover that

$$\lim_{n\to\infty} \frac{3n^4 + 5n2 + 17n + 9}{96n^4 + 39} = \frac{3}{96}$$

by dividing numerator and denominator of $\frac{3n^4+5n2+17n+9}{96n^4+39}$ by n^4 and looking at the behavior of all the individual terms. You'll want to use the fact that a limit of a sum is the sum of the limits and that similar statements are true for products and quotients. Most students have a pretty good intuitive understanding of what's going on; the important piece of advice I'll give you is to keep calm and have confidence that you can figure out the correct answer.

On the other hand, you'll need more experience before figuring out limits like

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n, \qquad \lim_{n\to\infty} \sqrt[n]{n}.$$

In the first limit, the base $\left(1 + \frac{1}{n}\right)$ is approaching 1, making you think that the limit will be 1; on the other hand, the exponent is getting big, making you think that the limit will be large. In fact, the limit is a number around 2.718, which we've defined to be e, so it's not 1 and not infinity. In the second limit, the fact that we're taking the nth root of a "number" makes you think that the limit will be close to 1; recall that the nth root of a fixed number like 49 approaches 1 as $n \to \infty$. On the other hand, we're taking the nth root of a bigger and bigger number, so it's not really clear what's going on. As we'll see later, l'Hôpital's Rule helps us clarify limits in which forces are competing to raise and lower the limit.

The next thing to look at is a limit like

$$\lim_{x \to a} f(x),$$

where f is some reasonable function like x^2. To say the limit is L is to say that $f(x)$ is getting close to L as x approaches a. In 99% of the situations when there is an $f(a)$, the function $f(x)$ gets close to $f(a)$, and that's the answer: $L = f(a)$. In the remaining 1% of the situations, either there is no $f(a)$ or there is an $f(a)$ that has been defined malevolently.

In some sense, differential calculus is about that 1%. However, it's about the 1% where there is no $f(a)$ but there ought to be an $f(a)$, and you can figure out what it is. A quintessential example is given by

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}.$$

We will discuss this type of example because it occurs all over the place when we compute derivatives.

Derivatives. The derivative of a function f at a point a is the slope of the line tangent to the graph of $y = f(x)$ at the point $(a, f(a))$. We think of the tangent line as the "limiting case" of the secant line that connects $(a, f(a))$ to $(b, f(b))$ where b is a number close to a. As $b \to a$, the secant line approaches the tangent line, and the slope of the secant line approaches the slope of the tangent line.

Thus a derivative is a limit.