Inequalities (Discussion)

Worksheet 6: Applications of (Weighted) Jensen's Inequality[1]

Date: 12/3/2020

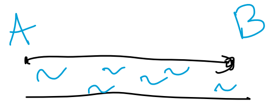
MATH 74: Transition to Upper-Division Mathematics
with Professor Zvezdelina Stankova, UC Berkeley

Read: *Session 9: Introduction to Inequalities. Part I* (vol. II)

- Theorem 3 on Continuity and Midpoints (pp. 220)

Write: clearly. Supply your reasoning in words and/or symbols. Show calculations and relevant pictures.

1. **(Applying Jensen's Inequality to leftover Power Mean Inequalities)**
   (a) Sketch the graph $f(x) = -\log_3 x$ for $x > 0$. Geometrically show it is convex.
   (b) Prove $P_1 \geq P_0$ for any $x_1, x_2, \ldots, x_n > 0$. (*Hint:* Apply JI to $f(x) = -\log_3 x$ and simplify.)
   (c) Prove that $P_{\frac{1}{3}} \geq P_0$. (*Hint:* Substitute something for each x_i in AM-GM.)
   (d) Prove that $f(x) = -\log_3 x$ is convex by applying the shortcut Theorem 3 and Baby AM-GM.
2. **(Weighted Means and Linear Combinations)** Your teacher would like to give the two exams scores x_1 and x_2 in her class relative weights in the ratio of $3:7$, or more generally, $m:n$.
   (a) Express the final score as a linear combination of the two exam scores. What do the two coefficients of your formula add up to? If $x_1 = 65$ and $x_2 = 95$, what will your final score be?
   (b) Let $ABCD$ be a right trapezoid, $AD||BC$ and $AB \perp AD$, X on side AB such that $AX : XB = 3:7$, and Y on side CD such that $XY||AD$. Express the length of XY as a linear combination of the lengths of the two bases AD and BC. Explain. (*Hint:* If $AD \leq BC$, construct a segment $AF||DC$ with F on side BC and $AF \cap XY - \{E\}$. Use similar $\triangle$s and a few parallelograms. See Fig. 5b on p. 224.)
3. **(Ultimate Applications of JI and WJI)**
   (a) Do Exercise 12 on p. 221;
   (b) Do Exercise 13 on p. 222;
   (c) Do Exercise 15 on p. 223;
   (d) Do Exercise 16 on p. 223.
4. **(Motion Shake&-Bake)** A motorboat must travel from port A to port B and back in no more than 9 hours. What should the motorboat's speed (in calm waters) be, if the speed of the current of the river is 7 km/h and the distance between A and B is 84 km?
5. **(Log Shake-&-Bake)** Solve the inequality $\log^2_{\frac{1}{2}} x - |\log_2 x| - 2 < 0$.



Speed $= v$
w/ current $= v+7$
against $= v-7$

6. **(Extra Background Practice)** L78: Test 2 on "Rational Inequalities."

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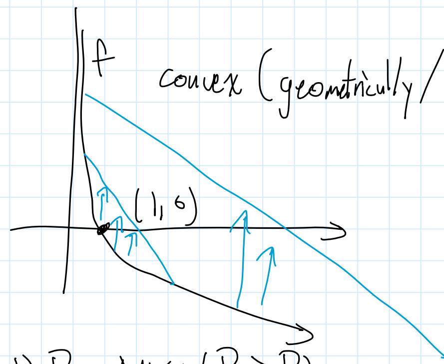
1) $f(x) = -\log_3(x)$



convex (geometrically / graphically)

Recall: $\log a + \log b = \log(ab)$

b) Prove AM-GM ($P_1 \geq P_0$)

Jensen's Inequality

If f is convex func, then $f\left(\frac{x_1+x_2+\cdots+x_n}{n}\right) \leq \frac{f(x_1)+\cdots+f(x_n)}{n}$

Note: when $n=2$, this is just definition of convexity w/ $\lambda = \frac{1}{2}$

Apply JI to $-\log_3(x)$

$$-\log_3\left(\frac{x_1+\cdots+x_n}{n}\right) \leq \frac{-\log_3(x_1)-\log_3(x_2)-\cdots-\log_3(x_n)}{n} = \frac{-\log_3(x_1x_2\cdots x_n)}{n}$$

$$3^{\log_3\left(\frac{x_1+\cdots+x_n}{n}\right)} \geq 3^{\frac{1}{n}\log_3(x_1\cdots x_n)}$$

$$\Rightarrow \frac{x_1+x_2+\cdots+x_n}{n} \geq (x_1x_2\cdots x_n)^{1/n} = \sqrt[n]{x_1x_2\cdots x_n} = GM.$$

c) Prove $P_{1/3} \geq P_0$ (this is a corollary AM-GM)

$$P_{1/3} = \left(\frac{x_1^{1/3}+x_2^{1/3}+\cdots+x_n^{1/3}}{n}\right)^{1/(1/3)} = \left(\frac{\sqrt[3]{x_1}+\cdots+\sqrt[3]{x_n}}{n}\right)^3$$

Apply AMGM to $\sqrt[3]{x_1}, \ldots, \sqrt[3]{x_n}$

$$\left(\frac{\sqrt[3]{x_1}+\cdots+\sqrt[3]{x_n}}{n}\right)^3 \geq \sqrt[n]{\sqrt[3]{x_1}\cdots\sqrt[3]{x_n}} = \left((x_1\cdots x_n)^{1/3}\right)^{1/n} = \left(\sqrt[3n]{x_1\cdots x_n}\right)^3$$

$$P_{1/3} \geq x_1\cdots x_n = P_0 \checkmark$$

Remark: We can use this technique to prove $P_r \geq P_0$ for any $r > 0$.

d) Algebraically prove $-\log_3 x$ is convex.

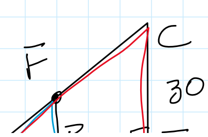
Sufficient to show for $\lambda = \frac{1}{2}$: $-\log_3\left(\frac{x+y}{2}\right) \leq \frac{-\log_3(x)-\log_3(y)}{2}$

Hint: Use baby AM-GM $\frac{x+y}{2} \geq (xy)^{1/2}$

$-\log_3(x)$ is a decreasing function (For $r \leq s$, $-\log_3(r) \geq -\log_3(s)$) because $\frac{d}{dx} -\log_3(x) = \frac{-1}{x \ln 3} < 0$

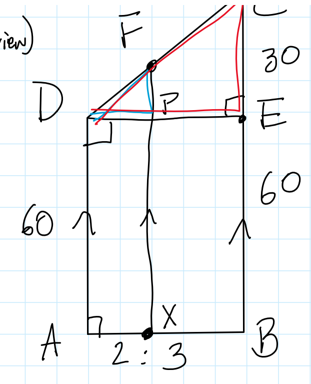
So $-\log_3\left(\frac{x+y}{2}\right) \leq -\log_3(xy)^{1/2} = -\frac{1}{2}\log_3(xy) = \frac{-\log_3(x)-\log_3(y)}{2}$ ✓

2) (Geo Review)



$$\frac{AX}{XB} = \frac{2}{3}$$

2) (Geo Review)



$\frac{AX}{XB} = \frac{2}{3}$

What is $FX = FP + PX = 12 + 60 = 72 = \frac{3}{5} \cdot 60 + \frac{2}{5} \cdot 90$

$PX = 60$

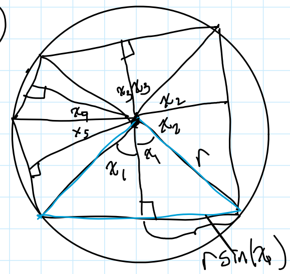
$\triangle DPF \sim \triangle DEC$ by AA similarity. $\Rightarrow \frac{FP}{CE} = \frac{DP}{DE} = \frac{DP}{DP+PE} = \frac{2}{2+3} = \frac{2}{5}$

$\angle PDF = \angle EDC$ (both just $\angle D$)

$\frac{FP}{30} = \frac{2}{5} \Rightarrow FP = \frac{2}{5} \cdot 30 = 12$

$\angle DPF = \angle DEC = 90°$

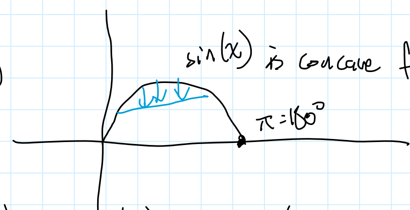
3) b)



Prove that the largest pentagon perimeter is achieved when the pentagon is regular.

$2x_1 + 2x_2 + \cdots + 2x_5 = 360° = 2\pi \Rightarrow x_1 + x_2 + \cdots + x_5 = 180°$

Perimeter $= 2r\sin(x_1) + 2r\sin(x_2) + \cdots + 2r\sin(x_5) = 2r(\sin(x_1) + \sin(x_2) + \cdots + \sin(x_5))$



$\sin(x)$ is concave for $x \in (0, 180°)$

Apply Concave JI: $\frac{\sin(x_1) + \cdots + \sin(x_5)}{5} \leq \sin\left(\frac{x_1 + x_2 + \cdots + x_5}{5}\right) = \sin\left(\frac{180}{5}\right) = \sin(36°)$

Perimeter $= 2r(\sin(x_1) + \cdots + \sin(x_5)) \leq 2r \cdot 5\sin(36°)$

Remark: This perimeter is achievable when $x_1 = x_2 = \cdots = x_5 = 36°$ (regular pentagon)