Discussion 2 12/1/20

Tuesday, December 1, 2020 2:05 PM

Inequalities (Discussion)

Worksheet 4: Limits, Sandwiching, Combinatorics, and Convexity[1]

Date: 12/1/2020

MATH 74: Transition to Upper-Division Mathematics

with Professor Zvezdelina Stankova, UC Berkeley

Write: clearly. Supply your reasoning in words and/or symbols. Show calculations and relevant pictures.

1. **(Sequence Starters)** Consider $\sqrt{3}$, $\sqrt[3]{3}$, $\sqrt[4]{3}$, $\sqrt[5]{3}$, $\sqrt[10]{3}$, $\sqrt[100]{3}$, and $\sqrt[1000]{3}$.
   (a) Calculate the answers correct to 4 decimal places. What are these numbers approaching? 1
   (b) Explain why $\lim_{r\to\infty} \sqrt[r]{3} = 1$. Can you prove it? How about $\lim_{r\to-\infty} \sqrt[r]{3} = 1$?
2. **(Comparing Power Means)** You received the scores $x_1 = 75$, $x_2 = 60$, and $x_3 = 90$ on your exams. Find the following power means of these numbers, rounding to 4 decimal places. (Write also the formulas for these power means that you used!)
   (a) $P_{-\infty}$, P_{-2}, P_{-1}, $P_{-\frac{1}{2}}$, P_0, $P_{\frac{1}{2}}$, P_1, P_2, P_3, P_∞. How do they compare with each other? Explain.
   (b) $P_{-0.1}$, $P_{-0.01}$, $P_{-0.001}$; $P_{0.1}$, $P_{0.01}$, $P_{0.001}$. Do they approach something? Explain.
   (c) P_{10}, P_{100}, P_{500}; P_{-10}, P_{-100}, P_{-500}. Do they approach something? Explain.
3. **(Sandwiching Power Means)** For the three exam scores above, prove that
   (a) $\frac{90}{\sqrt[5]{3}} \le P_5 \le 90$ and, in general, $\frac{90}{\sqrt[r]{3}} \le P_r \le 90$ for any $r > 0$.
   (b) Show that, as $r \to \infty$, the power means P_r approach 90. (*Hint:* Use #1.)
   How does this explain the definition $P_\infty = \max\{75, 60, 90\}$?
   (c) Prove that $\lim_{r\to-\infty} P_r = 60$. Explain why $P_{-\infty} = \min\{75, 60, 90\}$. Idea $a^{-r} = \frac{1}{a^r}$
4. **(Combinatorics Recall)** In Problem 2, imagine that you have multiplied out all the factors in the LHS and then in the RHS, and written the results as long summations.
   (a) When $n = 4$, how many summands of the form $a_i a_j$ for $i \ne j$ are there? Of the form $a_i a_j a_k$?
   (b) Repeat part (a) for any natural number n.
   (c) What is $(1+g)^4$ equal to? $(1+g)^n$ for any $n \in \mathbb{N}$? Where can we find all coefficients in RHS?
   (d) Do Problem 2 for $n = 4$ numbers. (*Hint:* Use (a), (c), combinations, + a bunch of AM-GM's.)
   (e) Do Problem 2 for any n positive numbers $a_1, a_2, \dots, a_n$.
5. **(Adding Convex Functions)** Draw the graphs of $f(x)$ and $g(x)$ below for all $x \in \mathbb{R}$. Explain why they are convex by using the *geometric* definition of convexity. Then draw the graph of their sum $f(x) + g(x)$ and again explain why it is also convex.
   (a) $f(x) = x^2$, $g(x) = 2x$; (b) $f(x) = x^2$, $g(x) = -2x + 5$.
   (c) Prove that the sum of two convex functions is a convex function.
6. **(Trapezoidal Reasoning)** Do Exercise 19 on p. 224 for $\lambda = \frac{1}{2}$, $\lambda = \frac{1}{3}$, and then for general $\lambda \in (0, 1)$. How does this exercise explain why the geometric and algebraic definitions of convexity are equivalent?

1

1) $\lim_{r\to\infty} \sqrt[r]{3} = \lim_{r\to\infty} 3^{1/r} = 3^0 = 1$

$P_r = \sqrt[r]{\frac{x_1^r + x_2^r + \dots + x_n^r}{n}} = \left(\frac{x_1^r + x_2^r + \dots + x_n^r}{n}\right)^{1/r}$

$P_1 = \left(\frac{x_1^1 + x_2^1 + \dots + x_n^1}{n}\right)^{1/1} = \frac{x_1 + x_2 + \dots + x_n}{n} = AM.$

$P_\infty = \max\{x_1, \dots, x_n\}$, $P_{-\infty} = \min\{x_1, \dots, x_n\}$.

$P_0 = \sqrt[n]{x_1 x_2 \cdots x_n} = GM.$

Expect

$P_{-\infty} \le \ \le P_{-1} \le P_{-1/2} \le P_0 \le P_{1/2} \le P_1 \le P_{10} \le P_{100} \le P_\infty$

3. Prove that $\lim_{r\to\infty} P_r = P_\infty$

$x_1 = 75 \quad x_2 = 60 \quad x_3 = 90$

$P_r = \left(\frac{75^r + 60^r + 90^r}{3}\right)^{1/r} \ge \left(\frac{90^r}{3}\right)^{1/r} = \frac{(90^r)^{1/r}}{3^{1/r}} = \frac{90}{3^{1/r}}$

Want to take $r \to \infty$, so assume $r > 0$ Then $75^r \le 90^r$, $60^r \le 90^r$ so $\left(\frac{75^r + 60^r + 90^r}{3}\right)^{1/r} \le \left(\frac{90^r + 90^r + 90^r}{3}\right)^{1/r} = (90^r)^{1/r} = 90$

$\lim_{r\to\infty} \frac{90}{3^{1/r}} \le \lim_{r\to\infty} P_r \le \lim_{r\to\infty} 90 = 90 \Rightarrow 90 \le \lim_{r\to\infty} P_r \le 90 \Rightarrow \lim_{r\to\infty} P_r = 90 = P_\infty.$

$\| \quad \frac{90}{1} = 90$

$g = \sqrt[4]{a_1 a_2 a_3 a_4}$, $a_i \ge 0$

4. $(1+a_1)(1+a_2)(1+a_3)(1+a_4) \ge (1+g)^4 = \binom{4}{0} + \binom{4}{1} \cdot g + \binom{4}{2} g^2 + \binom{4}{3} g^3 + \binom{4}{4} g^4$

	# of a's	LHS	# terms
S_0	0	1	$1 = \binom{4}{0}$
S_1	1	$a_1 + a_2 + a_3 + a_4$	$4 = \binom{4}{1}$
S_2	2	$a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4$	$6 = \binom{4}{2}$
S_3	3	$a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4$	$4 = \binom{4}{3}$
S_4	4	$a_1a_2a_3a_4$	$1 = \binom{4}{4}$

Idea:

$S_0 \ge \binom{4}{0}$ ✓

$S_1 \ge \binom{4}{1} g$ ✓

$S_2 \ge \binom{4}{2} g^2$ ✓

$S_3 \ge \binom{4}{3} g^3$ ✓

$S_4 \ge \binom{4}{4} g^4$ ✓

$\Rightarrow S_0 + \dots + S_4 = (1+a_1) \cdots (1+a_4) \ge (1+g)^4.$ $\square$

$S_0 = 1 \ge \binom{4}{0} = 1$ ✓

AM-GM ⇒ $\frac{a_1+a_2+a_3+a_4}{4} \geq \sqrt[4]{a_1a_2a_3a_4} \Rightarrow S_1 \geq 4g$ ✓

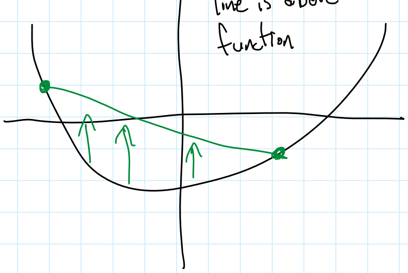
AM-GM ⇒ $\frac{a_1a_2+a_1a_3+\cdots+a_3a_4}{\binom{4}{2}} \geq \sqrt[6]{a_1^3a_2^3a_3^3a_4^3} = \left(\sqrt[4]{a_1a_2a_3a_4}\right)^2 = g^2 \Rightarrow S_2 \geq \binom{4}{2} g^2$ ✓

AM-GM ⇒ $\frac{a_1a_2a_3+\cdots+a_2a_3a_4}{4} \geq \sqrt[4]{a_1^3a_2^3a_3^3a_4^3} = g^3 \Rightarrow S_3 \geq 4g^3$ ✓

$S_4 = a_1a_2a_3a_4 = g^4$ ✓

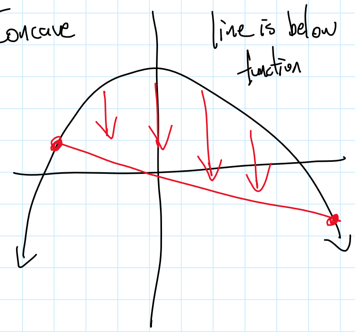
5. Convex

line is above function

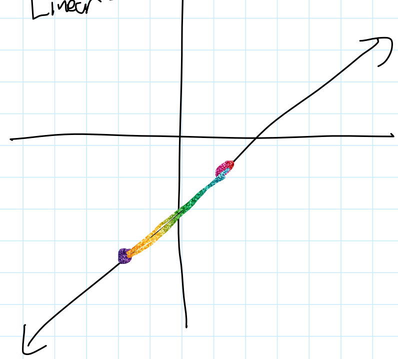


Concave

line is below function



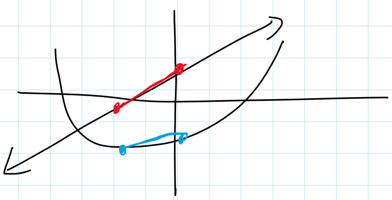
Linear = Convex + concave



Algebraically: For $0 \leq \lambda \leq 1$ and x_1, x_2

$$\lambda f(x_1) + (1-\lambda) f(x_2) \geq f(\lambda x_1 + (1-\lambda) x_2).$$

Convex + Convex



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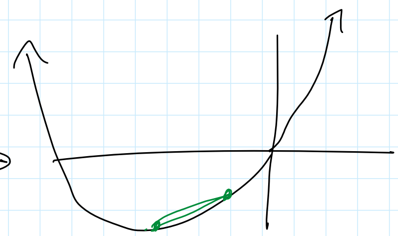

Idea: Green line = Red + Blue ≥ orig line + orig parabola

= new parabola