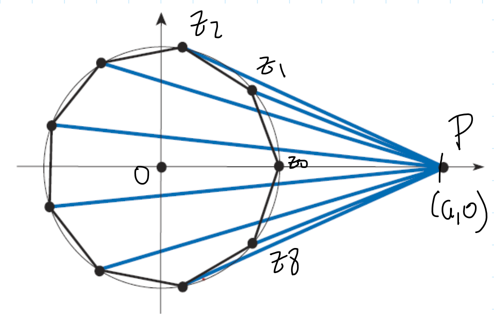
Office Hours 12/10/20

Thursday, December 10, 2020 12:06 PM



$\prod$ blw lines $= |OP|^9 - 1$, $|OP| = a$

$x^9 - 1 = (x - z_0)(x - z_1) \cdots (x - z_8)$

$|a^9 - 1| = |a - z_0||a - z_1| \cdots |a - z_8|$

$$\underbrace{\frac{1}{1\cdot 3} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 5} + \frac{1}{4\cdot 6} \cdots + \frac{1}{n(n+2)}}\ = ?$$

$$\left[\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}\right] n(n+2) \quad \text{This is true for all } n$$

$1 = A(n+2) + Bn$

$n=1 \quad 1 = 3A + B = 2A$

$n=2 \quad 1 = 4A + 2B$

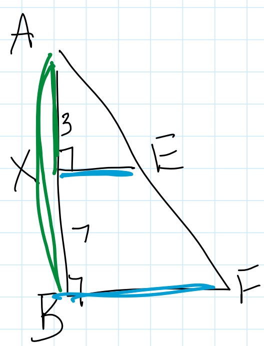
$\Rightarrow A + B = 0 \Rightarrow B = -A \quad A = \frac{1}{2},\ B = -\frac{1}{2}$

$$\frac{1}{n(n+2)} = \frac{1}{2}\cdot\frac{1}{n} - \frac{1}{2}\cdot\frac{1}{n+2} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)$$

$$\frac{1}{2}\left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \frac{1}{2}\left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \frac{1}{2}\left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \cdots + \frac{1}{2}\left(\cancel{\frac{1}{n-1}} - \frac{1}{n+1}\right) + \frac{1}{2}\left(\cancel{\frac{1}{n}} - \frac{1}{n+2}\right)$$

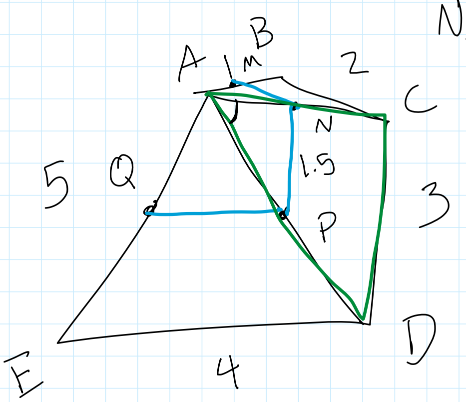
$$= \frac{1}{2}\left[\frac{1}{1} + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right] = \frac{1}{2}\ \frac{2(n+1)(n+2) + (n+1)(n+2) - 2(n+2) - 2(n+1)}{2(n+1)(n+2)} = \frac{3n^2 + 9n + 6 - 2n - 4 - 2n - 2}{4(n+1)(n+2)}$$

$$= \boxed{\frac{3n^2 + 5n}{4(n+1)(n+2)}}$$



$\triangle AXE \sim \triangle ABF$

$$\frac{XE}{BF} = \frac{AX}{AB} = \frac{3}{3+7}$$



N, P midpoints of AC, AD

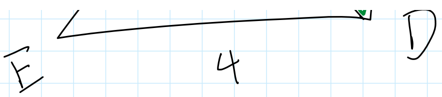
$AM + MN + NP + PQ + QA$

NP ✓

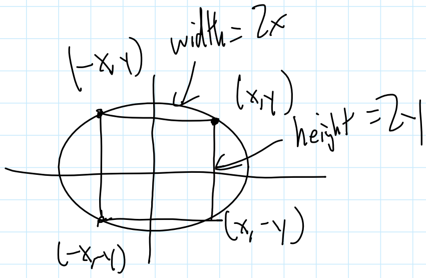
$\triangle ANP \sim \triangle ACD$

RAR $\angle NAP = \angle CAD$ ✓

$$\frac{NP}{CD} = \frac{AN}{AC} = \frac{AP}{AD} = \frac{1}{2} \checkmark$$



$$\frac{NP}{CD} = \frac{AN}{AC} = \frac{AP}{PD} = \frac{1}{2} \checkmark$$



Area $= (2x)(2y)$

Volume $= (2x)(2y)(2z) = \underline{\underline{8}}xyz$

In class, we proved for 16-puzzle

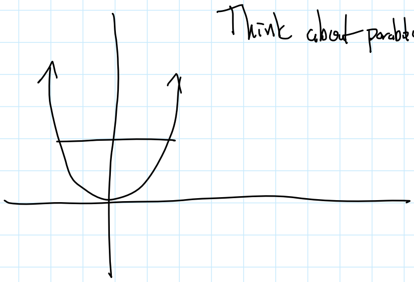
9	8	5	6
7	10	11	1
2	3	4	

is this solvable?

We can label the missing tile as 12, so every move will be a transposition w/ 12 and another number.
12 has to start and end in the bottom right corner ⇒ forms closed loop.

In class we proved a closed loop has even length. If a board is solvable ⇒ perm. is even.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 9 & 8 & 5 & 6 & 7 & 10 & 11 & 12 & 3 & 4 \end{pmatrix} \Rightarrow \underbrace{(1\ 9\ 2\ 8)}_{4\text{ cycle} = 3 \text{ trans}}\underbrace{(3\ 5\ 7\ 11\ 4\ 6\ 10)}_{7\text{cycle} = 6\text{ trans}} = \pi\ 3+6 = 9 \text{ trans} = \text{odd} \Rightarrow \text{unsolvable}$$



Think about parabola = convex

$$f\left(\frac{x+y}{2}\right) \underset{\uparrow \text{ Convex}}{\leq} \frac{f(x)+f(y)}{2}$$

$$f\left(\frac{x+y}{2}\right) \underset{\uparrow \text{ Concave}}{\geq} \frac{f(x)+f(y)}{2}$$

$$\left(\frac{x+y}{2}\right)^3 \leq \frac{x^3+y^3}{2}$$

$$\Rightarrow \frac{x+y}{2} \leq \sqrt[3]{\frac{x^3+y^3}{2}}$$

$$AM = P_1 \leq P_3$$

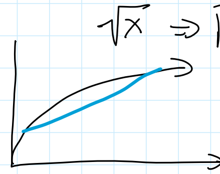
$$P_{-\infty} \leq P_{-100} \leq P_{-1} \leq P_0 \leq P_1 \leq P_{100} \leq P_{\infty}$$

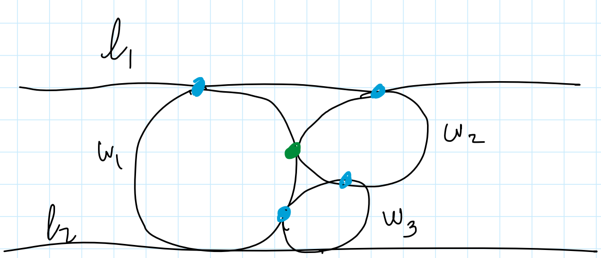
We can use convexity of $f(x) = x^3$ to prove $P_3 \geq P_1$

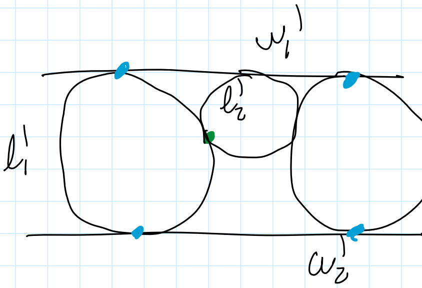
If x^r is convex ⟺ then $P_r \geq P_1$

If x^r is concave ⟺ then $P_r \leq P_1$

$\sqrt{x} \Rightarrow P_{\frac{1}{2}} \leq P_1$







For Inversion, start at the center then move out

lines through center ⟺ themself

lines not through center ⟺ Circles touching center

$w_1, w_2 \Rightarrow$ parallel lines

$l_1 \Rightarrow$ circle touch w_1', w_3'

$w_3 \Rightarrow$ circle touch w_1', w_3'

$l_2 \Rightarrow$ circle touch w_3, w_1

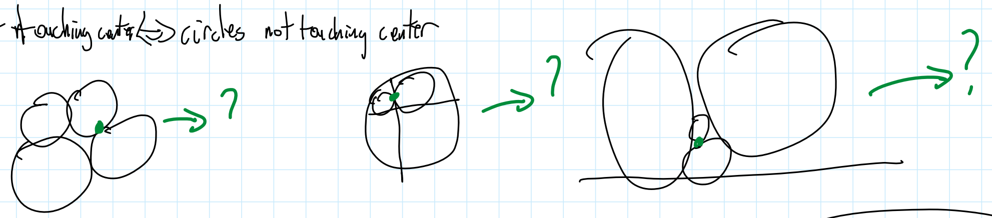
4 pts = rectangle are cyclic

⇒ 4 pts are concyclic

$w_2 \Rightarrow$ circle touch w_3, w_1

lines not through center $\Leftrightarrow$ circles touching center

circles not touching center $\Leftrightarrow$ circles not touching center



replace $(a,b) \to (\gcd(a,b), \operatorname{lcm}(a,b))$

Show largest # written is a mono.

1	5	10	30	15	60
	a			b	
a_1	a_2	a_3	a_4	a_5	a_6

$$\gcd(a,b) \le \min(a,b) \le \max(a,b) \le \operatorname{lcm}(a,b)$$

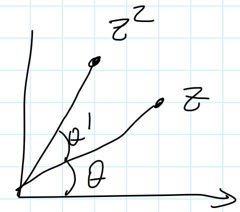
Upper bound $\operatorname{lcm}(a_1, \dots, a_n)$ Largest # = integer + upper bound

lower bound $\gcd(a_1, \dots, a_n)$

Complex #s Polar $\longleftrightarrow$ Coordinate form

Viete's formulas $x^n + a_{n-1}x^{n-1} + \cdots + a_0$ $\sum$ roots $= -a_{n-1}$

n^{th} Roots of #s $\prod$ roots $= (-1)^n a_0$



Prove the double angle formula: $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

$$z = \underset{\text{Polar}}{(1, \theta)} = 1 \cdot (\cos\theta + i\sin\theta)$$

x-coor y-coor

$$z^2 = (1 \cdot 1, \theta + \theta) = (\cos\theta + i\sin\theta)^2 = \cos^2\theta + 2i\cos\theta\sin\theta + (i\sin\theta)^2 = \underbrace{\cos^2\theta - \sin^2\theta}_{} + i\underbrace{(2\cos\theta\sin\theta)}_{}$$

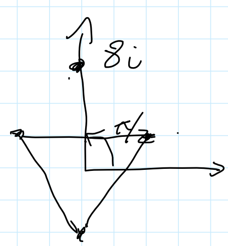
$$\| \quad (1, 2\theta) = 1 \cdot \cos 2\theta + i\sin 2\theta \qquad = \underbrace{\cos 2\theta}_{} + i\underbrace{\sin(2\theta)}_{}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \quad \sin 2\theta = 2\cos\theta \cdot \sin\theta$$

What is $\sqrt[3]{8i} = \sqrt[3]{(8, \frac{\pi}{2})} = \sqrt[3]{8}\left(\cos\frac{\frac{\pi}{2}+2\pi k}{3} + i\sin\frac{\frac{\pi}{2}+2\pi k}{3}\right)$

Root Formula $\sqrt[n]{z} = \sqrt[n]{|z|}\left(\cos\frac{\theta+2\pi k}{n} + i\sin\frac{\theta+2\pi k}{n}\right)$

In general, n^{th} roots of any # for a regular n-gon



Inv + Mono

Chameleons why did we need # red $\equiv$ # green (mod 3)?

Mono. $\Rightarrow$ What conditions force game to end

Pointless Machine what pairs are achievable from a given pair

Stomp = what colorings did we use to prove a board wasn't solvable

Escape of Clones $\Rightarrow$ What was "green land" and how did we use it to prove certain certain prisons unescapable

Conway Checkers $\Rightarrow$ Why does an inv. not work

Comb.

NT

Induction

Comb.

Combinatorial proofs

PHP

Counting, Dogs + Biscuits

NT

FLT $a^{p-1} \equiv 1 \mod p$ if $(a,p)=1$

EFT $a^{\phi(d)} \equiv 1 \mod d$ if $(a,d)=1$

If $(a,d) \neq 1$, write first few values to find a pattern

When is $x^n \equiv x^m \pmod d$

$x^n \equiv y^n \pmod d$

Induction

MMI, MMI^s

Remember multiple base cases

Telescoping series

Geo + Inversion

Similar triangles

Parallel lines, Parallel axiom (Prove axioms are equivalent)

Be comfortable inverting a picture (practice w/ textbook images)

Distance Formulas

Real Analysis

RT, FT, IVT

Proving consequences e.g. MVT $\Rightarrow$ FT

Prove at least n roots IVT

at most n roots RT

Group Theory

Understand permutations, cycles

16 puzzle

rotations + symmetry grps.

Ineq.

Power Means

Convexity/Concavity

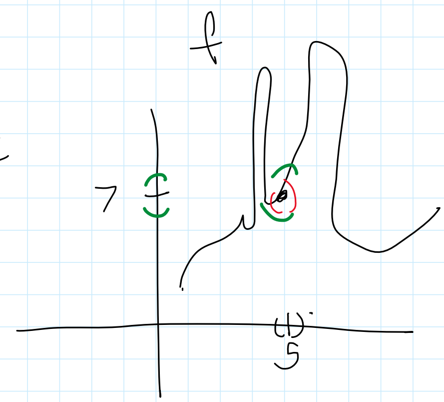
JI

ε, δ = small values, N = large value

$\lim_{x \to 7} f(x) = 5$ $\quad \forall \varepsilon > 0, \exists \delta > 0$ s.t. $|x-7| < \delta \Rightarrow |f(x) - 5| < \varepsilon$

$\lim_{x \to 7} f(x) = \infty$ $\quad \forall N > 0\ \exists \delta > 0$ s.t. $|x-7| < \delta \Rightarrow f(x) > N$

$\lim_{x \to \infty} f(x) = 5$ $\quad \forall \varepsilon > 0, \exists N > 0$ s.t. $x > N \Rightarrow |f(x) - 5| < \varepsilon$



Prove $\lim_{x \to 3} x^2 = 9$ $\quad \forall \varepsilon > 0, \exists \delta > 0$ s.t. $\underline{|x-3| < \delta} \Rightarrow \underline{|x^2 - 9| < \varepsilon}$ $\quad |x^2-9| = |x-3|\,|x+3| = \varepsilon$

$\delta \Rightarrow \delta = \frac{\varepsilon}{\cancel{|x+3|}} = \frac{\varepsilon}{7}$

$|x+3| < 7$

Goal is to explicitly find δ (in terms of ε)

Since x is close to 3, $|x+3| < 7$

Set $\delta = \varepsilon/7$

We are given $\underline{|x-3| < \delta = \varepsilon/7}$

Want to prove $\underline{|x^2-9|} < \varepsilon$, $\quad \underline{|x^2-9|} = \underline{|x-3|\,|x+3|} < \boxed{\frac{\varepsilon}{7} \cdot |x+3|} < \varepsilon$ ✓ $\quad < 1$

Strategy $\Rightarrow$ look at $|x^2-9| < \varepsilon$, set $|x-3| = \delta$, $|x^2-9| = \varepsilon$, solve for δ.

$\lim_{x\to 9} \sqrt{x} = 3$ $\quad |x-9|<\delta \Rightarrow |\sqrt{x}-3|<\varepsilon$

$x-9=\delta \qquad \sqrt{x}-3=\varepsilon$

$x=\varepsilon^2+6\varepsilon+9$

$\delta = x-9 = \underline{\varepsilon^2+6\varepsilon}$

Given $|x-9|<\varepsilon^2+6\varepsilon$, want to prove $|\sqrt{x}-3|<\varepsilon$.

$\Downarrow \qquad \Downarrow$

$x-9<\varepsilon^2+6\varepsilon \Rightarrow x\le \varepsilon^2+6\varepsilon+9 = (\varepsilon+3)^2 \qquad \sqrt{x}-3<\sqrt{\varepsilon^2+6\varepsilon+9}-3=\varepsilon+3-3=\varepsilon$

$\sup S$ = least upper bound of S.

Suppose $a_1, a_2, \dots, a_n$ is a $\underbrace{\text{monotonically incr. seq. w/ upperbound}}$.

Prove that $\lim a_i = \sup\{a_i\} = S \quad$ Want $\forall \varepsilon>0\ \exists N>0$ s.t. $i>N \Rightarrow |a_i - S|<\varepsilon$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \parallel$

Mathematical $\quad \forall\varepsilon>0,\ \exists a_n$ s.t. $a_n > \sup S - \varepsilon$. $\qquad S-a_i$ because $S>a_i$

$\forall \varepsilon>0\ \exists a_n$ s.t. $a_n < \inf S + \varepsilon$. $\qquad$ Want. $S-a_i<\varepsilon \Rightarrow S-\varepsilon<a_i$.

$-1, -2, -3, \dots$

sup

From defn of sup, $\forall\varepsilon>0$, $\exists a_n$ s.t. $a_n > S-\varepsilon$

Set $N=n$, if $i>N$ then $a_i \ge a_n > S-\varepsilon$

$\Rightarrow S-a_i<\varepsilon$

In general $z^n + a_{n-1}z^{n-1} + \cdots + a_0 = \prod (z-\text{roots})$

$z^5-1 = \prod(z-\{\text{roots of } z^5-1\}) = (z-z_0)(z-z_1)(z-z_2)(z-z_3)(z-z_4)$.

Find when $z^5-1=0$

$z^5 = 1 \Rightarrow z=\sqrt[5]{1}\left(\cos\frac{0+2\pi k}{5} + i\sin\frac{0+2\pi k}{5}\right)$

$(1,0) \qquad = \cos\frac{2\pi k}{5} + i\sin\frac{2\pi k}{5} \qquad k=0,1,2,3,4$

$z_0 = \cos 0 + i\sin 0$

$z_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

$z_4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$.