Discussion 2 11/5/20

Thursday, November 5, 2020 2:05 PM

(calculus) (MAT 104)

Real Analysis (Discussion)

Worksheet 1: Optimization in Practice and in Theory[1]

Date: 11/5/2020

MATH 74: Transition to Upper-Division Mathematics
with Professor Zvezdelina Stankova, UC Berkeley

Read: *Session 12: Geometric Re-Constructions III* (vol. II)
- §1.1. The initial set-up (p. 288); §1.3. A Calculus Drill (pp. 290-292) • Appendix.

Write: clearly. Supply your reasoning in words and/or symbols. Show calculations and relevant pictures.

1. **(Farmer and Cow)** Using Calculus, solve the following problems:
   (a) (Problem 2, p. 4) During a hot summer day, a farmer and a cow find themselves on the same side of a river. The farmer is 2 km from the river and the cow is 6 km from the river. If each of them walks straight to the river, they will be 4 km from each other. But the cow has a broken leg! The farmer must get to the river, dip his bucket there, and take the water to the cow. To which point on the riverbank should the farmer walk so that his total path is as short as possible?
   (*Hint*: Work with the function $F(x)=f(x,4-x)=\sqrt{2^2+x^2}+\sqrt{6^2+(4-x)^2}$ on the interval $[0,4]$.)
   (b) (Exercise 5, p.292) Using the sign of $F'(x)$, show that $F(x)$ from (a) decreases for all $x<0$ and increases for all $x>4$. Make a conclusion about $F(x)$ on all of $\mathbb{R}$.
   (c) (Exercise 6, p. 292) Show that $F(x)=f(x,4-x)=\sqrt{2^2+x^2}+\sqrt{6^2+(4-x)^2}$ has two *slant* asymptotes: $y=2x-4$ when $x\to\infty$ and $y=4-2x$ when $x\to-\infty$. Why does this imply $F(x)\approx|x|+|4-x|$ when $|x|$ is very large? Use all of the above to sketch the graph of $F(x)$ on $\mathbb{R}$.
   (d) Generalize and solve the problem for distances a, b, and c (instead of 2 km, 6 km, and 4 km).

w/ a,b,c

2. **(Logical Tidbits)**
   (a) What does it mean to *apply* a theorem in a problem? Which theorem(s) did you apply in the optimization problem above? How exactly did you do it?
   (b) How are *definitions* useful in solving problems? Did you use any definitions above? Why?
   (c) State the hypotheses and conclusions of EVT, FT, and CIM. (See the Key Takeaways below.)
   (d) Why do we need to assume that $f(x)$ is *continuous* in EVT and CIM, but not in FT? How do you convince me that a condition is *necessary*?
   (e) Why do we need a *closed finite* interval in EVT and CIM, but not in FT?
   (f) Which theorem in this lesson is *existential* and which shows us *how to find* the extremes of f?
   (g) Formulate the *converses* and the *contrapositives* of EVT, FT, and CIM. Which are true and why?
   (h) Why does a continuous function on $\mathbb{R}$ with *no* critical points have *no* local or global extrema?
   (i) How do you prove that something *can* happen or a feature *can* occur?

FT

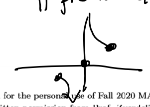
$f(x)=e^x$ on $\mathbb{R}$

effective

3. **(Concrete Applications)** Prove the following statements over **R**. Justify everything rigorously.
   (a) $f(x)=2020+(x-2021)^7$ has a critical number but no extremum there.
   (b) $g(x)=\frac{x^{201}}{201}-\frac{x^{101}}{101}+x$ has no local or global extrema.
   (c) $f(x)=ax^3+bx^2+cx+d$ $(a\neq 0)$ *can* have 0, 1, or 2 critical numbers and 0 or 2 local extrema.

To show condition is necessary, provide ex. w/ condition not fulfilled nor conclusion

$f(x)=1/x$ on $[-1,1]$



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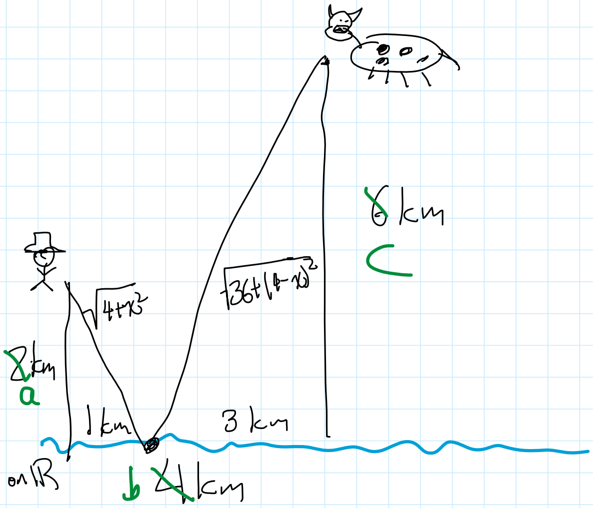


Goal: Minimize $f(x)=\sqrt{4+x^2}+\sqrt{36+(4-x)^2}$ for $x\in[0,4]$

Use CIM: f is cont. ✓ ($\sqrt{\ }$ is cont.) closed int. ✓ ($[0,4]$)

1) Find where $f'(c)=0$ or $\nexists$. $\frac{d}{dx}\sqrt{x}=\frac{1}{2}\cdot\frac{1}{\sqrt{x}}$

$$f'(x)=\cancel{\frac{1}{2}}\cdot\frac{\cancel{2}x}{\sqrt{4+x^2}}+\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}(4-x)(-1)}{\sqrt{36+(4-x)^2}}=0$$

$$\frac{x}{\sqrt{4+x^2}}=\frac{(4-x)}{\sqrt{36+(4-x)^2}}\longrightarrow\frac{x^2}{4+x^2}=\frac{(4-x)^2}{36+(4-x)^2}$$

$$9\,\cancel{36}x^2+\cancel{x^2(4-x)^2}=\cancel{4}(4-x)^2+\cancel{x^2(4-x)^2}$$

$$9x^2=x^2-8x+16\Rightarrow 8x^2+8x-16=0$$

$$\Rightarrow x^2+x-2=(x+2)(x-1)=0$$

$x=-2\notin[0,4]$ $\quad x=1$

global min.

Also check $x=0$ $x=4$

$f(1)=\sqrt{2^2+1^2}+\sqrt{36+3^2}=\sqrt{5}+\sqrt{45}=4\sqrt{5}\approx 9$

$f(0)=\sqrt{4+0^2}+\sqrt{36+4^2}=2+\sqrt{52}\approx 9.1$

$f(4)=\sqrt{4+4^2}+\sqrt{36+0^2}=6+\sqrt{20}\approx 10.5$

b) $$f'(x)=\cancel{\frac{1}{2}}\cdot\frac{\cancel{2}x}{\sqrt{4+x^2}}+\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}(4-x)(-1)}{\sqrt{36+(4-x)^2}}=$$

$x<0$ $\quad=\frac{x}{\sqrt{4+x^2}}\ {}^{<0}_{>0}+\frac{x-4}{\sqrt{36+(4-x)^2}}\ {}^{<0}_{>0}\quad <0$

$x>4$ $\quad >0$

f is decreasing when $x<0$, incr. $x>4$ } $x=1$ is a global min. on all of $\mathbb{R}$.

$x=1$ is a minimum of f on $[0,4]$

Appendix: Key Takeaways and Review

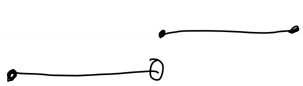
- **(Strategy)** Real-life problem $\xrightarrow{\text{translate}}$ Calculus problem $\xrightarrow{\text{solve}}$ Calculus answer $\xrightarrow{\text{translate back}}$ Real-life answer
- **(Applications)** To apply a theorem, we must check that its hypothesis is satisfied in the particular problem and then state its conclusion without further proof.
- **(Implications)** Consider the statement "$A\Rightarrow B$" (if A then B).
  1. Its **converse** is "$B\Rightarrow A$" (if B then A).
  2. Its **contrapositive** is "not $B\Rightarrow$ not A" (if not B then not A).
  3. Its **inverse** is "not $A\Rightarrow$ not B" (if not A then not B).
- **(Definitions)** If $f: D\subset\mathbb{R}\to\mathbb{R}$, then f has a:
  1. **global maximum** $f(c)$ if $f(c)\geq f(x)$ $\forall x\in D$.
  2. **local maximum** $f(c)$ if $f(c)\geq f(x)$ for all x in an *open* interval containing c: $x\in(c-\delta,c+\delta)$.
  3. **critical number** $c\in D$ if $f'(c)=0$ or $f'(c)$ $\nexists$.
  4. **slant asymptote** $y=ax+b$ for $x\to\infty$ if $\lim_{x\to\infty}[f(x)-(ax+b)]=0$.
- **(Continuity)** f **is continuous** at c if $\lim_{x\to c}f(x)=f(c)$. This means that three things happen simultaneously:
  1. $\lim_{x\to c}=L$ exists and is finite (nearby behavior);
  2. $f(c)$ is well-defined (behavior exactly at c);
  3. $L=f(c)$ (the two behaviors match).
- **(Fermat's Theorem, FT)** If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c)=0$.
- **(Extreme Value Theorem, EVT)** If f is continuous on a closed finite interval $[a,b]$, then f attains a global maximum $f(c)$ and a global minimum value $f(d)$ at some numbers $c,d\in[a,b]$.
- **(Closed Interval Method, CIM)** To find the *global* maximum and minimum values of a continuous function f on a closed interval $[a,b]$:
  1. Find the values of f at its critical numbers in (a,b).
  2. Find the values of f at the endpoints of $[a,b]$.
  3. The largest of the values from Steps 1 and 2 is the global maximum value; the smallest of these values is the global minimum value.
- **(CIM on an open interval)** If f is defined and continuous on an open interval and has no critical points, then it has *no* local and *no* global extrema.
- **(Monotonicity)** If $f'(x)>0$ on an interval I, then $f(x)$ is increasing (not necessarily strictly) on I.
- **(1st Derivative Test)** If $f'(x)$ changes its sign at c:
  1. from $+$ to $-$, then $f(c)$ is a local maximum;
  2. from $-$ to $+$, then $f(c)$ is a local minimum.
- **(2nd Derivative Test)** If $f'(c)=0$ and
  1. $f''(c)>0$, then $f(c)$ is a local minimum;
  2. $f''(c)<0$, then $f(c)$ is a local maximum.

$A\Rightarrow B$

Converse of EVT (Swap if, then) $B\Rightarrow A$

X If f attains global extrema, then f is cont. on a closed interval.

false

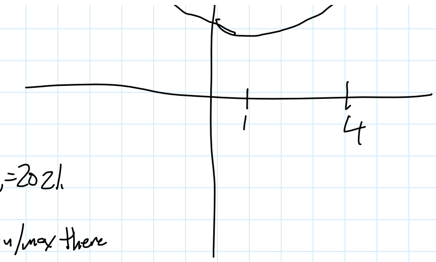


Contrapositive $\neg B\Rightarrow\neg A$

If f doesn't attain a global extremum, then f is discont. or f is not on a closed interval. True

Contrapositive is always true.

3a) $f(x)=2020+(x-2021)^7$



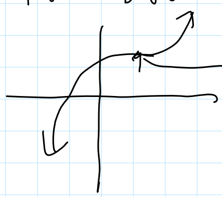
3a) $f(x) = 2020 + (x-2021)^7$

Critical pt. $f'(x) = 0$ or $\nexists$

$f'(x) = 7(x-2021)^6 = 0 \Rightarrow x - 2021 = 0 \Rightarrow x = 2021$

If $f'(x)$ doesn't change signs, then x doesn't have local min/max there

$x < 2021$, $f'(x) = 7(x-2021)^6 > 0$
$x > 2021$



no local extrema.