Discussion 1 11/3/20

Tuesday, November 3, 2020 12:19 PM

Inversion in the Plane (Discussion)

Worksheet 7: Ptolemy's Inequality and Circle vs. Circle[1]

Date: 11/3/2020

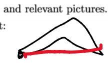
MATH 74: Transition to Upper-Division Mathematics
with Professor Zvezdelina Stankova, UC Berkeley

Read: *Session 1: Inversion in the Plane. Part I* (vol. I)

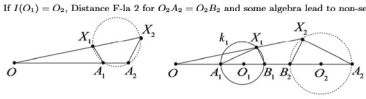
- §8. Problem 7. Ptolemy's Inequality (p. 18)

Write: clearly. Supply your reasoning in words and/or symbols. Show calculations and relevant pictures.

1. **(Ptolemy's Inequality)** Let $ABCD$ be *any* convex quadrilateral. Prove that: $AB \cdot CD + AD \cdot BC \geq AC \cdot BD$, with equality if and only if $ABCD$ is *cyclic*. (*Hint:* Mimic the Proof of PtT, but in the end apply the Triangle Inequality.)



2. **(Circle-to-Circle Proof)** Let $k(O,r)$ be the circle of inversion. Let $k_1(O_1)$ be a circle **not** through O. In this problem, you will show that k_1 goes to a circle k_2, also **not** through O.
   (a) Let A_1 and X_1 be points such that O, A_1, and X_1 do not lie on a line. Let $I(A_1) = A_2$ and $I(X_1) = X_2$. Prove that $\angle OX_1A_1 = \angle OA_2X_2$ and conclude that A_1, A_2, X_2, and X_1 are concyclic. (*Hint:* Distance F-la 1, similar $\triangle$s, and cyclic quadrilaterals.)
   (b) A line through O and the center O_1 of k_1 intersects k_1 in points A_1 and B_1 (A_1 is closer to O). Let $I(A_1) = A_2$ and $I(B_1) = B_2$. Let X_1 be any other point on k_1 and $I(X_1) = X_2$. Prove that $\angle B_2X_2A_2 = 90°$. (*Hint:* The famous $\angle A_1X_1B_1$ is the difference of two other angles with vertices at X_1. Using part (a) for A_1, X_1, and then for B_1, X_1, make conclusions about the angles of $\triangle X_2A_2B_2$.)
   (c) On what figure does X_2 lie? How does part (b) show that k_1 goes to a circle k_2 under I?
   (d) Prove that the center O_1 of k_1 does **not** go to the center O_2 of k_2 under I.
   (*Hint:* If $I(O_1) = O_2$, Distance F-la 2 for $O_2A_2 = O_2B_2$ and some algebra lead to non-sense!)

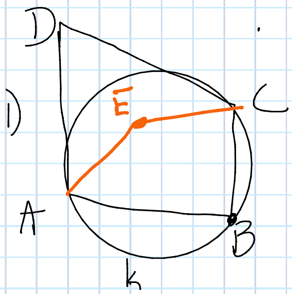


3. **(Circle-⊥-Circle Proof)** Let $k(O,r)$ be the circle of inversion. Let $k_1(O_1)$ be a circle such that $k_1 \perp k$. In this problem, you will show that k_1 goes to *itself* under inversion.
   (a) Let k_1 and k intersect in points A and B. The condition $k_1 \perp k$ forces two angles on the picture to be special. Which are these two angles and what are they equal to?
   (b) Let X be any point on k_1 (other than A and B). Let ray $\overrightarrow{OX}$ intersect k_1 for a second time in point X_1. Prove that $I(X) = X_1$. How does this show that k_1 goes to itself under I?
   (*Hint:* Similar $\triangle$s or Power of a Point and Distance Formula 1!)
4. **(Algebra Shake-&-Bake)** Find the roots of the equation $x^2 + x + 1 = \dfrac{156}{x + x^2}$.
   (*Hint:* Brute-force-degree-4 equation or slick-substitution-quadratic equation?)

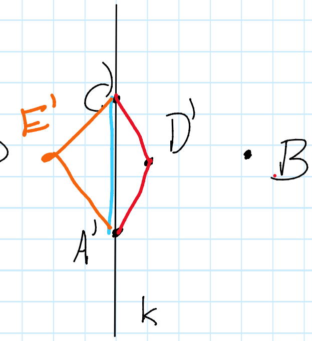
Extra Background and Practice: *Metric Relations between Segments in a Circle:* L148

6. **(Fundamentals)** LC148: #1, 2, 3, 4, 5*, 6*.
   (*Hint:* In #6, extend the line connecting the center and the point until the line intersects the circle again.)

1)



$I(B,r)$

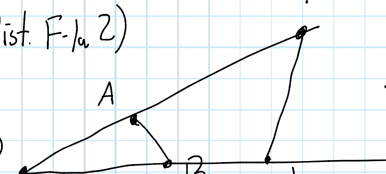


$A'D' + C'D' \geq A'C'$

$A' = I(O,r)$

Recall (Dist. F-la 2)

Dist. F-la



$\dfrac{AB \cdot r^2}{OA \cdot OB} = A'B'$

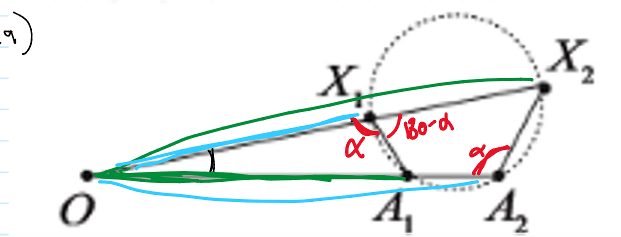
$OA \cdot OA' = r^2 = OB \cdot OB'$

$$\left[\frac{AD \cdot r^2}{BA \cdot BD} + \frac{CD \cdot r^2}{BC \cdot BD} \geq \frac{AC \cdot r^2}{BA \cdot BC}\right] \cdot \frac{\cancel{BA} \cdot BC \cdot BD}{r^2}$$

$AD\,BC + AB \cdot CD \geq AC \cdot BD$

Equality holds iff D' is on the line $\to$ D is on the circle $\Rightarrow ABCD$ is cyclic.

2a)



Know:

$I(A_1) = A_2$

$I(X_1) = X_2$

Want to prove $\triangle OA_2X_2 \sim \triangle OX_1A_1$

We know: share $\angle O$.

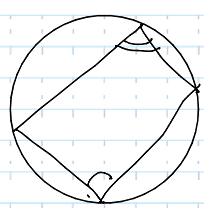
Can use: ~~AA~~, RAR $\quad \dfrac{OA_1}{OX_2} \overset{?}{=} \dfrac{OX_1}{OA_2} \xrightarrow{\text{Cross-Multiply}} OA_1\,OA_2 \overset{?}{=} OX_1\,OX_2 = r^2$

and $\angle O = \angle O$

Proof Dist. F-la gives $OA_1 OA_2 = r^2 = OX_1 \cdot OX_2$ so $\dfrac{OA_1}{OX_2} = \dfrac{OX_1}{OA_2}$. Then by RAR sim., $\triangle OA_2X_2 \sim \triangle OX_1A_1$ so

$\angle OX_1A_1 = \angle OA_2X_2$

Thm $ABCD$ is a cyclic iff $\angle A + \angle C = 180°$.

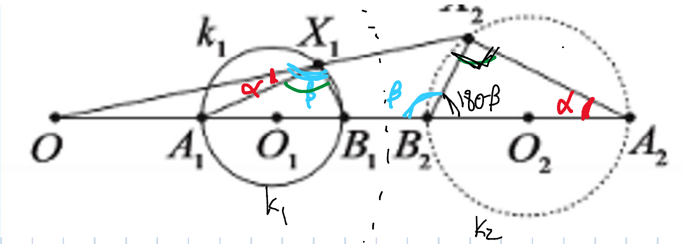


$OA \cdot OA' = r^2 \Rightarrow OA' = \dfrac{r^2}{OA}$

2b)

A_1B_1 is a diameter of circle O_1.

What does 2a tell us? for AX and BX.



A_1B_1 is a diameter of circle O_1.

What does 2a tell us? for AX and BX,

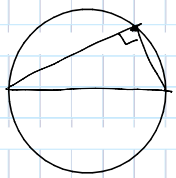
$\angle OX_1B_1 = \angle OB_2X_2$

$\angle OX_1A_1 = O A_2X_2$

$\angle B_2X_2A_2 = \beta - \alpha = \angle A_1XB_1 = 90°$

Therefore X_2 is on circle w/diameter A_2B_2

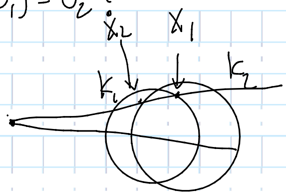
Thm Any angle inscircle overlooking a diameter is a right angle



2c) X_1 is any pt on circle w/diameter A_1B_1, it is always sent to X_2 on circle w/ diameter A_2B_2. This proves $I(k_1) \subseteq k_2$

Now we need to prove $I(k_1) \supset k_2$.

2d) Is $I(O_1) = O_2$?



Answer: No

Suppose for contradiction $I(O_1) = O_2$

$$\frac{\cancel{O_1A_1}\cdot \cancel{r^2}}{OA_1 \cdot \cancel{OO_1}} = O_2A_2 = O_2B_2 = \frac{\cancel{O_1B_1}\cdot \cancel{r^2}}{\cancel{OO_1}\cdot OB_1}$$

$\Rightarrow OA_1 = OB_1 \Rightarrow A_1 = B_1$. Therefore $I(O_1) \neq O_2$.