Inequalities (Discussion)

Worksheet 2: Applications of AM-GM[1]

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MATH 74: Transition to Upper-Division Mathematics
with Professor Zvezdelina Stankova, UC Berkeley

Read: *Session 9: Introduction to Inequalities. Part I* (vol. II)

- §2.4. More variables, more challenge for baby AM-GM (p. 214)
- §2.5. Need more strength (p. 215)

1. **(Applications of Baby AM-GM)** From the BMC book (pp. 214):
   (a) Create a version of Exercise 3 with 4 numbers and solve it.
   (b) State what Exercise 4 says for $n = \cancel{2018}$ 2020, and then prove it. (*Hint:* How much is the sum $1 + 2 + 3 + \cdots + 2018$?) $2020! < \left(\frac{2020+1}{2}\right)^{2020}$
   (c) Create a version of Exercise 3 with n numbers. Solve it.
   (d) Do Exercise 4 using *only* Baby AM-GM. (Try with baby and then with general AM-GM!)
2. **(Algebra Helpers)**
   (a) Find all roots of the equation $2z^3 - 3z^2 + 1 = 0$.
   (b) Find a formula to factor $a^5 - b^5$ and prove that it is correct.
   (c) Generalize your formula in (b) to any natural n and prove that it is correct.
3. **(Applications of General AM-GM)** From the BMC book (pp. 215):
   (a) Do Exercise 6 for $n = 6$ and then for any natural n.
   (b) State and solve a 2d-version of Exercise 7 for the ellipse $\frac{x^2}{5^2} + \frac{y^2}{7^2} = 1$.
   (c) Solve the full 3d-version of Exercise 7, as stated in the book. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
4. **(Systems Shake-&-Bake)** Solve the system. Plot the solutions in the coordinate plane.
   (a) $\begin{cases} x^2 = y; \\ y^2 = x. \end{cases}$ (b) $\begin{cases} x^2 - xy + x - y = 0; \\ 2x^2 - y^2 + 3x - 2y - 2 = 0. \end{cases}$
5. **(Algebra Shake-&-Bake)** How many *real* solutions does the equation below have?
   (a) $\sqrt{x+1} = \sqrt{x} + 1$; (b) $\sqrt[3]{x+1} = \sqrt[3]{x} + 1$; (c) $\sqrt[4]{x+1} = \sqrt[4]{x} + 1$; (d) $\sqrt[n]{x+1} = \sqrt[n]{x} + 1$, $n \in \mathbb{N}$?
6. **(Geo Shake-&-Bake)** In a rectangle with side $\sqrt{2}$, the perpendiculars from two opposite vertices to the diagonal that does not pass through them divide that diagonal into three equal parts. What is the area of the rectangle? Is the answer unique? How do know we haven't missed a case? (*Hint:* In $\triangle ABC$ with a right $\angle C$ right and an altitude CH, $AH \cdot AB = AC^2$ and $BH \cdot BA = BC^2$. Why?)

7. **(Extra Background and Practice)** Test 1 on "Rational Inequalities" (L78).

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1

1a) $a, b, c > 0$, Prove:

$$(a+b)(b+c)(c+a) \geq 8abc$$

Baby AM-GM $x, y > 0$

$$\frac{x+y}{2} \geq \sqrt{xy}$$

AM-GM $\Rightarrow$
$$\left.\begin{aligned} \frac{a+b}{2} &\geq \sqrt{ab} \\ \frac{b+c}{2} &\geq \sqrt{bc} \\ \frac{c+a}{2} &\geq \sqrt{ca} \end{aligned}\right\} \overset{\times}{\Longrightarrow} \frac{(a+b)(b+c)(c+a)}{8} \geq \sqrt{abbcca} = abc \checkmark$$

a) Generalize w/ 5 #s $a, b, c, d, e > 0$

HW ☆ $(a+b)(b+c)(c+d)(d+e)(e+a) \geq 32 abcde$ $\quad 32 = 2^5$

$$(a+b)(a+c)(a+d)(a+e)(b+c)(b+d)(b+e)(c+d)(c+e)(d+e) \geq 2^{10}(abcde)^2$$

b) $n! < \left(\frac{n+1}{2}\right)^n$

$n=6$ $\quad 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \overset{?}{<} \left(\frac{6+1}{2}\right)^6$

AMGM w/
$$\left.\begin{aligned} (1,6) &\quad \sqrt{1 \cdot 6} < \frac{1+6}{2} \\ (2,5) &\quad \sqrt{2 \cdot 5} < \frac{2+5}{2} \\ (3,4) &\quad \sqrt{3 \cdot 4} < \frac{3+4}{2} \end{aligned}\right\} \overset{\times}{\Longrightarrow} \sqrt{6!} < \left(\frac{7}{2}\right)^3$$

$\Downarrow$ square

$$6! < \left(\frac{7}{2}\right)^6$$

$n=5$ $\quad 5! < \left(\frac{5+1}{2}\right)^5 = 3^5$

$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$

$$\left.\begin{aligned} \sqrt{1 \cdot 5} &< \frac{1+5}{2} \\ \sqrt{2 \cdot 4} &< \frac{2+4}{2} \\ \sqrt{3} &= \sqrt{3} \end{aligned}\right\} \sqrt{5!} < 3 \cdot 3 \cdot \sqrt{3} \Rightarrow 5! < 3^5$$

OR
$$\begin{matrix} 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \\ 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \end{matrix}$$

$$\left.\begin{aligned} \sqrt{1 \cdot 5} &< \frac{1+5}{2} \\ \sqrt{2 \cdot 4} &< \frac{2+4}{2} \\ \sqrt{3 \cdot 3} &= \frac{3+3}{2} \\ \sqrt{4 \cdot 2} &< \frac{4+2}{2} \\ \sqrt{5 \cdot 1} &< \frac{5+1}{2} \end{aligned}\right\} \overset{\times}{\Longrightarrow} 5! < 3^5$$

HW ☆ $2020! < \left(\frac{2020+1}{2}\right)^{2020}$

3a) $a \geq b \geq 0$

$$a^5 - b^5 \geq 5(a-b)(ab)^2$$

Can't use AMGM on $a^5 + (-b^5)$ because $(-b^5) < 0$ $\quad$ AM-GM

Can't use AM-GM on $a^5+(-b^5)$ because $(-b^5)<0$

AM-GM

Try factoring first. $a^5-b^5=(a-b)\underbrace{\boxed{a^4+a^3b+a^2b^2+ab^3+b^4}}_{\text{Usefull AM-GM}} \geq (a-b)5(ab)^2$

Full AM-GM: $x_1, \dots, x_n \geq 0$

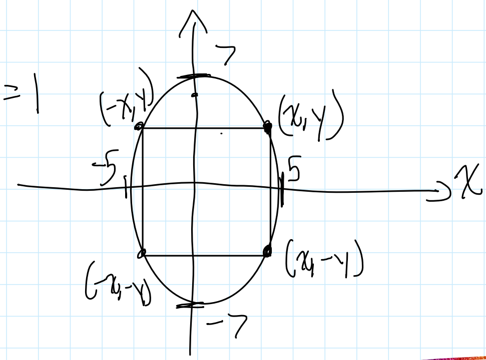
$$\frac{x_1+x_2+\cdots+x_n}{n} \geq \sqrt[n]{x_1 \cdot x_2 \cdots x_n}$$

Applied to $a^4, a^3b, a^2b^2, ab^3, b^4$:

$$\frac{\boxed{a^4+a^3b+a^2b^2+ab^3+b^4}}{5} \geq \sqrt[5]{a^4 \cdot a^3b \cdot a^2b^2 \cdot ab^3 \cdot b^4} = \sqrt[5]{a^{10}\, b^{10}} = (ab)^2$$

When do we get equality? We need $\geq$ to be equality: $a^4=a^3b=a^2b^2=ab^3=b^4 \Rightarrow a=b$

3b) $\frac{x^2}{5^2}+\frac{y^2}{7^2}=1$



Optimize Area: $(2x)(2y)$ subject to $\boxed{\frac{x^2}{5^2}+\frac{y^2}{7^2}=1}$

Use AM-GM! Using on $\left(\frac{x^2}{5^2}, \frac{y^2}{7^2}\right) \Rightarrow \frac{1}{2} = \frac{\left(\frac{x^2}{5^2}+\frac{y^2}{7^2}\right)}{2} \geq \sqrt{\frac{x^2y^2}{5^2 \cdot 7^2}} = \frac{xy}{35}$

$$xy \leq \frac{35}{2} \Rightarrow \underbrace{(2x)(2y)}_{\text{Area}} \leq \frac{4 \cdot 35}{2} = 70$$

Need to show we can get an area of 70. AM-GM tells us equality iff $\boxed{\frac{x^2}{5^2}=\frac{y^2}{7^2}=\frac{1}{2}} \Rightarrow \boxed{x=5/\sqrt{2},\ y=7/\sqrt{2}}$