# Discussion 1 11/19/20

Thursday, November 19, 2020 12:26 PM

## Action Groups (Discussion)

**Worksheet 4: Writing and Inverting Permutations, Abelian Groups**[1]

Date: 11/19/2020

MATH 74: Transition to Upper-Division Mathematics
with Professor Zvezdelina Stankova, UC Berkeley

**Read:** *Session 5: Introduction to Group Theory.* (vol. II)

- §4. General Groups. (pp. 112)
- §5. Some More Examples of Groups. (pp. 116)
- §6. Permutation (or Symmetric) Groups. (pp. 119-120)

**Write:** clearly. Supply your reasoning in words and/or symbols. Show calculations and relevant pictures.

1. **(Famous Groups)** Show that the sets below are groups under the given operation. ($S^* = S - \{0\}$)
   (a) $(\mathbb{Q}, +)$; (b) $(\mathbb{C}, +)$; (c) $(\mathbb{Q}^*, \cdot)$;
   Why aren't they *cyclic*? (*Hint*: By contradiction.)
2. **(Unitary Group)** Let's find an infinite subgroup of $(\mathbb{C}^*, \cdot)$ that is not the whole group! Let $\mathcal{C}$ be the unit circle in the complex plane, centered at (0,0). Let $z_1$ and $z_2$ be any two complex numbers on $\mathcal{C}$.
   (a) Prove that $z_1 z_2$ is also in $\mathcal{C}$. (*Hint*: Use geo-interpretation of $\mathbb{C}$-multiplication or $|z|$.)
   (b) Prove that $\frac{1}{z_1}$ is also in $\mathcal{C}$. (*Hint*: Same as (a).)
   (c) Prove that $\mathcal{C}$ is a group under complex multiplication. (*Hint*: Verify the definition of a group. Don't forget about identity and associativity!)
   (d) What is the order of $\mathcal{C}$?
3. **(Cycle Notation)** Calculate $(1356)^2, (1356)^3$, and $(1356)^4$. What is $o((1356))$?
4. **(2-row to Notation)** Write the permutation $\phi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 2 & 5 & 6 & 7 & 8 & 1 & 4 \end{pmatrix}$ as a product of cycles.
5. **(Order)** Prove that an $r$-cycle is of order $r$.
6. **(Product of Transpositions)** Represent each of the following permutations (written in cyclic notation) as a product of *transpositions*, and determine which permutations are even and which are odd.

   HW

   (a) $(1, 3, 6, 7, 4, 10)$;
   (b) $(1, 3, 6, 7, 4, 8, 10)$;
   (c) $r$-cycle $(a_1, a_2, \ldots, a_r)$.
   (d) cycle of length 2020? of length 2021?
7. **(Symmetries)** Recall the symmetry group $S(F)$ of a figure $F$. The set of permutations $S_n$ can be thought as the symmetry group of which set?
8. **(Commute)** Two elements $g_1$ and $g_2$ *commute* in a group $G$ if $g_1 g_2 = g_2 g_1$. A group $G$ in which any two elements commute is called *abelian*. Which:
   (a) cycles commute?
   (b) transpositions commute?
   (c) groups on this page are abelian?
   (d) action/symmetry groups we saw are abelian?
9. **(Product$^{-1}$)** Let $G$ be a group and let $a, b \in G$.
   (a) Prove that $(ab)^{-1} = b^{-1}a^{-1}$.
   (b) Why did we switch the order of the inverses on the RHS. Why $(ab)^{-1} \neq a^{-1}b^{-1}$ in general? Give examples and counterexamples. (*Hint*: To show that two elements are inverses of each other in $G$, what do you have to show?
   (c) What kind of a group is $G$ if for all $a, b \in G$, we have $(ab)^{-1} = a^{-1}b^{-1}$?
   (d) Show that for any $a_1, a_2, \ldots, a_r \in G$:
   $(a_1 a_2 \cdots a_r)^{-1} = a_r^{-1} \cdots a_2^{-1} a_1^{-1}$.
   (e) Find the inverses of (25), (25)(63), (25)(256), $\boxed{(153429), (1342)(123),}$ $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 2 & 5 & 6 & 7 & 8 & 1 & 4 \end{pmatrix}$.
10. **(Geometry Shake-&-Bake)** In isosceles $\triangle ABC$, we have leg $AC = 28$ cm. Through midpoint $M$ of altitude $CC_1$ to base $AB$ we have constructed a line $l$ parallel to leg $BC$. What is the length of the part of $l$ that is inside the triangle? (*Hint*: The three midpoints of $\triangle ABC$'s sides are the vertices of the *medial* $\triangle A_1B_1C_1$, one side of which passes through $M$ and another side is parallel to $l$.)



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3) $(1356)(1356) = (15)(\underbrace{36}_{\text{transposition}})$

$5 \leftarrow 3 \leftarrow 1$

$(1356)^4 = ((1356)^2)^2 = (15)(36)(15)(36) = (1)(3)(5)(6) = e$

4 cycle = odd cycle

$1 \leftarrow 5 \leftarrow 5 \leftarrow 1 \leftarrow 1$

$3 \leftarrow 3 \leftarrow 6 \leftarrow 6 \leftarrow 3$

$o((1356)) = 4$

4) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 2 & 5 & 6 & 7 & 8 & 1 & 4 \end{pmatrix} = (1\ 3\ 5\ 7)(2)(468)$
$= (1357)(468).$

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 6 & 8 & 1 & 3 & 5 & 7 \end{pmatrix} = (124875)(36)$
$= (36)(124875)$

6.a) $\underbrace{(1\ 3\ 6\ 7\ 4\ 10)}_{5 \text{ transpositions} \to \text{ odd permutation.}} = (1\ 3)(36)(6\ 7)(7\ 4)(4\ 10)$

$1 \leftarrow 3 \leftarrow 6 \leftarrow 7 \leftarrow 4 \leftarrow 10$

c) $(a_1, a_2, \ldots, a_r) = (a_1, a_2)(a_2, a_3) \cdots (a_{r-1}, a_r)$

✓ $a_2 \leftarrow a_1 \leftarrow a_1 \leftarrow a_1$

✓ $a_3 \leftarrow a_3 \leftarrow a_2 \leftarrow a_2$

✓ $a_1 \leftarrow a_2 \leftarrow a_3 \leftarrow a_{r-1} \leftarrow a_r$

Is an $r$-cycle even or odd? = $\begin{cases} \text{even} & \text{if } r \text{ is odd} \\ \text{odd} & \text{if } r \text{ is even} \end{cases}$

$e = (12)(12)$

8) Two elements commute if $g_1 g_2 = g_2 g_1$

G is abelian if all pairs of elements commute.

Is $S_n$ abelian? $\quad S_2 = \{e, (12)\}$ abelian ✓

$S_n$ ← permutation of n elements.

$S_3 = \{e, (12), (13), (23), (123), (132)\}$ $\quad S_3$ is not abelian.

$(21)(13) \qquad (31)(12)$

$(12)(13) = (132) \neq (13)(12) = (123)$

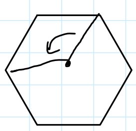
$3 \leftarrow 3 \leftarrow 1 \qquad 2 \leftarrow 2 \leftarrow 1$

$2 \leftarrow 1 \leftarrow 3 \qquad 3 \leftarrow 1 \leftarrow 2$

$1 \leftarrow 2 \leftarrow 2 \qquad 1 \leftarrow 3 \leftarrow 3$

$(12)(34) \overset{\checkmark}{=} (34)(12)$

$K$ = rotations of a hexagon. $= \{e, r_1, r_2, r_3, r_4, r_5\}$ is abelian because $r_i r_j = r_{i+j} = r_j r_i$



Let $r_i$ be rotation by $60 \cdot i°$ cw

Is $D_4$ = symmetries of a square abelian? No, $r \cdot \underset{\text{reflection}}{s_1} \neq s_1 \cdot r$ (From Tuesday)

T and S also both abelian (verify on your own)

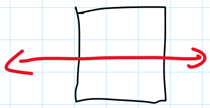
9) Think in $D_4$, $r$ = rotate clockwise by 90° $\quad r^{-1}$ = rotate ccw by 90°

$s$ = reflect $\quad s^{-1} = s$ (because $s^2 = e$)

rotate back, then reflect back

a) Think in $D_4$, $r$ = rotate clockwise by $90^\circ$ $\quad$ $r^{-1}$ = rotate ccw by $90^\circ$

$s$ = reflect ⟷ $\quad$ $s^{-1} = s$ (because $s^2 = e$)



rotate back then reflect back

$$(\underbrace{rs}_{\text{first reflect then rotate}})^{-1} = s^{-1}r^{-1}$$

Should have $(rs)(rs)^{-1} = e$

$$(r\cancel{s})(\cancel{s^{-1}})(r^{-1}) = \cancel{r r^{-1}} = e$$

c) What if $a^{-1}b^{-1} = (ab)^{-1} = b^{-1}a^{-1} \Rightarrow a^{-1}$ and $b^{-1}$ commute.

If $(ab)^{-1} = a^{-1}b^{-1}$ for all pairs $a, b$, $\quad a^{-1}, b^{-1}$ commute for all pairs $\Rightarrow$ group is abelian.

$$a\,[\,a^{-1}b^{-1} = b^{-1}a^{-1}\,]$$

$$(MN)^{-1} = N^{-1} \cdot M^{-1}$$

$$[\,\cancel{a \cdot a^{-1}} \cdot b^{-1} = a \cdot b^{-1} \cdot a^{-1}\,]\,a$$

$$b\,[\,b^{-1}a = a \cdot b^{-1} \cdot \cancel{a^{-1}a}\,]\,b$$

$$\cancel{bb^{-1}}ab = ba\cancel{bb^{-1}}$$

$$ab = ba$$