Discussion 2 11/17/20

Tuesday, November 17, 2020 2:04 PM

Action Groups (Discussion)

Worksheet 2: Symmetries of Famous Polygons[1]

Date: 11/17/2020

MATH 74: Transition to Upper-Division Mathematics
with Professor Zvezdelina Stankova, UC Berkeley

Read: *Session 5: Introduction to Group Theory* (vol. II, pp. 109-110, 112)
- §3.5. A Group within a group
- §3.6. Twin groups
- §5.1. Total "recall"

Write: clearly. Supply your reasoning in words and/or symbols. Show calculations and relevant pictures.

1. **(Quadrilateral)** Write the multiplication table for the symmetries of the figures:
   (a) a rectangle that is NOT a square;
   (b) a square;
   (c) a rhombus that is NOT a square;
   (d) a parallelogram that is none of the above.
   Are any of these groups "the same"?

1ad

2. **(Pentagon)** A regular pentagon is labelled $ABCDE$ counterclockwise. Describe in words and draw pictures for the following symmetries.
   (a) Let r be the rotation about the center of the pentagon that takes vertex B to D. Where do the other four vertices go under this rotation? What is the angle of this rotation?
   (b) What is the symmetry of the pentagon that is inverse to r? Describe r^{-1} and say where it takes each vertex of the pentagon.
   (c) Which symmetries of the pentagon are represented by powers of r: r^2, r^3, r^4, etc.? Do you get all rotations of the pentagon this way?
   (d) Let s_1 and s_2 be the reflections of the pentagon that switch points B and D, and and points B and C, respectively. What are the axes of reflection for s_1 and s_2? Where do the other vertices go under s_1? under s_2?
   (e) Which symmetry of the pentagon is the product $s_1 s_2$? Is it a reflection or a rotation?
   (f) Repeat part (e) for rs_1 and for $s_1 r$.
   (g) Are $s_1 r$ and rs_1 the same symmetry? Explain.
3. **(Square)** Describe all of the subgroups of D_4.
   (a) For each subgroup, list its elements, order, and operation table. Did you miss any subgroups?
   (b) For each subgroup of order 4, is it isomorphic to the Turning Soldier or to One Sock group?

T:

$\cdot$	s	r	b	l
s	s	r	b	l
r	r	b	l	s
b	b	l	s	r
l	l	s	r	b

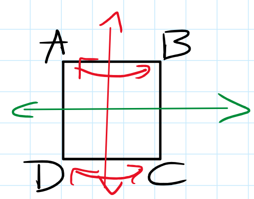
S:

$\cdot$	n	c	i	t
n	n	c	i	t
c	c	n	l	s
i	i	l	n	r
t	t	s	r	n

4. **(Triangle)** Let R be the subgroup of D_3 of rotational symmetries of an equilateral triangle.
   (a) Construct R's table, with appropriate labeling of R's elements. What is $o(R)$? Is R cyclic?
   (b) Now let G be *any* group of order 3. Prove that $G \cong R$. (*Hint*: Investigate *all* possibilities for G's table.... There is only *one* such possibility!)
   (c) Give examples of two "non-action" groups of order 3 (with different elements and operation than R). Are these groups isomorphic to R?
5. **(No Reflections?)** Find and describe a figure in the plane with exactly 3 symmetries. Give the multiplication table for the group of symmetries. (*Hint*: An equilateral triangle has too many symmetries: eliminate the reflections.)

Key Takeaways:
- **(Isomorphic groups)** If we can relabel the elements of one group G_1 to the elements of the other groups G_2 so that the table of G_1 becomes the table of G_2. We write $G_1 \cong G_2$.
- **(Subgroup)** A subset G_1 of a group G that is a group in itself under the multiplication in G.
- **(Cyclic subgroup)** A group all of whose elements are powers of a single element, called a generator.
- **(Permutation)** Every symmetry of a polygon induces a permutation of its vertices.

1



Symmetries

reflection axis 1 s_1

$\leftrightarrow$ s_2

s_3

s_4

Rotation by $0°$ cw e

$90°$ cw r_1

$180°$ cw r_2

$270°$ cw r_3

	e	r_1	r_2	r_3	s_1	s_2	s_3	s_4
e	e	r_1					s_3	s_4
r_1	r_1							
r_2	r_2				s_2			
r_3								
s_1					e			
s_2						e		
s_3	s_3						e	
s_4	s_4							e

$$\begin{matrix}A & B\\ C & D\end{matrix} \xrightarrow{r_3} \begin{matrix}B & C\\ A & D\end{matrix}$$

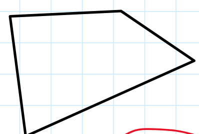
$s_2 = r_2 \cdot s_1$ $\quad f \circ g = f(g(x))$

$$\begin{matrix}A & B\\ D & C\end{matrix} \xrightarrow{s_1} \begin{matrix}B & A\\ C & D\end{matrix} \xrightarrow{r_2} \begin{matrix}D & C\\ A & B\end{matrix}$$

$s_2 \cdot s_2 = e$

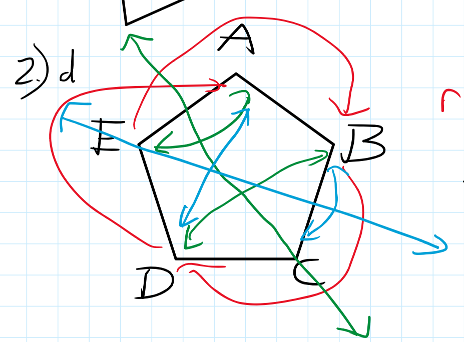
$s_1 \cdot s_2 = r_2$

$$\begin{matrix}A & B\\ D & C\end{matrix} \xrightarrow{s_2} \begin{matrix}D & C\\ A & B\end{matrix} \xrightarrow{s_1} \begin{matrix}C & D\\ B & A\end{matrix}$$



only has "do nothing" symmetry.

2) d



r: rotation that sends $B \to D$ = rotation by $\frac{2}{5} \cdot 360° = 144°$

s_1: reflection that swaps B and D s_1

s_2: reflection that swaps B and C. s_2

$s_1 \cdot s_2$

$$\begin{matrix} & A & \\ E & & B\\ D & & C\end{matrix} \xrightarrow{s_2} \begin{matrix} & D & \\ E & & C\\ A & & B\end{matrix} \xrightarrow{s_1} \begin{matrix} & E & \\ D & & A\\ C & & B\end{matrix}$$ = rotation clockwise by $72°$.

$r \cdot s_1$

$$\begin{matrix} & A & \\ E & & B\\ D & & C\end{matrix} \xrightarrow{s_1} \begin{matrix} & E & \\ A & & D\\ B & & C\end{matrix} \xrightarrow{r} \begin{matrix} & B & \\ C & & A\\ D & & E\end{matrix}$$

$r \cdot s_1$ = reflection through line through D, midpoint AB

$s_1 \cdot r$

$$\begin{matrix} & A & \\ E & & B\\ D & & C\end{matrix} \xrightarrow{r} \begin{matrix} & D & \\ C & & E\\ B & & A\end{matrix} \xrightarrow{s_1} \begin{matrix} & C & \\ D & & B\\ E & & A\end{matrix}$$

$s_1 \circ r$ = reflection through B, midpoint DE

Keypoint! $r \cdot s_1 \neq s_1 \cdot r$

3a

	e	r_1	r_2	r_3	s_1	s_2	s_3	s_4
e	e	r_1					s_3	s_4
r_1	r_1							
r_2	r_2				s_2			
r_3								
s_1					e	r_2		
s_2						e		
s_3	s_3						e	
s_4	s_4							e

$T:$

$\cdot$	s	r	b	l
s	s	r	b	l
r	r	b	l	s
b	b	l	s	r
l	l	s	r	b

$S:$

$\cdot$	n	c	i	t
n	n	c	i	t
c	c	n	l	s
i	i	l	n	r
t	t	s	r	n

What are the subgroups of D_4 = sym. of a square.

Subgroup: Subset s.t. multiplication keeps you inside.

T $\{e, r_1, r_2, r_3\}$ ✓ (cyclic, gen. r_1, $r_2 = r_1^2, r_3 = r_1^3$)

order 4

$\{e, s_1, s_2, s_3, s_4\}$ ✗ $s_1 \cdot s_2 = r_2$

$\{e\}$ ✓

$\{e, r_1, r_2, r_3, s_1, s_2, s_3, s_4\}$ ✓

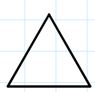
$\{e, s_1\}$ ✓

S $\{\underline{e}, \underline{s_1}, \underline{s_2}, \underline{r_2}\}$ ✓ (every element is its own inverse)

$s, r, b = r^2, l = r^3$

T is cyclic (generated by r)

5). Construct a shape w/ exactly 3 symmetries.



Symmetries

e

r = rotate by $120°$

r^2 = rotate by $240°$

s_1 = reflection

s_2

s_3

Try to alter triangle to get rid of reflection symmetries but preserve rotation symmetries