Discussion 1 11/12/20

Thursday, November 12, 2020 11:59 AM

Real Analysis (Discussion)

Worksheet 5: Monotone, Bounded, and Convergent[1]

Date: 11/12/2020

MATH 74: Transition to Upper-Division Mathematics
with Professor Zvezdelina Stankova, UC Berkeley

Read: *Session 12: Geometric Re-Constructions III* (vol. II)

- §3. Infinitely Many Angles and Infinite Series (pp. 296-300);
- Appendices for definitions/theorems.

Write: clearly. Supply your reasoning in words and/or symbols. Show calculations and relevant pictures.

1. **(Logic Tidbits)**
   (a) **(Negations)** What does it mean that $\{x_n\}$:
   - is *not* monotone?
   - is *not* bounded?
   - does *not* converge to 6: $\lim_{n\to\infty} x_n \neq 6$?

   (b) **(Infimum)** If $S \subset \mathbb{R}$ that is bounded below, show that $\inf S$ is exists and is unique.

   (c) **(No "holes")** How does the Completeness Axiom guarantee that $\mathbb{R}$ has no "holes"?

2. **(Proofs)** Prove the following theorems:
   (a) **(Convergent ⇒ Bounded, CBT)** ∀ convergent sequence is bounded.

   (b) **(Monotone Bounded Theorem, MBT)** ∀ bounded monotone sequence is convergent.

   (c) **(Convergent-Monotone, CMT)** ∀ convergent sequence has a monotone subsequence.

3. **(Concrete Applications)** Prove the following statements over R. Justify everything rigorously.
   (a) Show that the equation $2x - 1 - \sin x = 0$ has exactly one real root. (*Hint*: Use IVT for *at least* and RT for *at most* one real root.)

   (b) Prove that if $f'(x) \neq 1$ for all $x \in \mathbb{R}$, then f has at most one *fixed point*; i.e., some $c \in \mathbb{R}$ with $f(c) = c$. If in addition $f : [0,1] \to [0,1]$, conclude that f has *exactly* one fixed point. (*Hint*: What if there were *two* fixed points?)

   (c) Find the limit of the sequence. (*Hint*: Show that it is bounded and either monotone or splits into a decreasing and increasing subsequences.)
   - $a_{n+1} - \frac{1}{3 - a_n}$, $a_1 - 2$.
   - $a_{n+1} = 1 + \frac{1}{1+a_n}$, $a_1 = 1$. How does this yield the *continued fraction* expansion $\sqrt{2} = 1 + \frac{1}{2+\frac{1}{2+\cdots}}$?

   ← draw the graph + find limit. (no need to prove monotonicity)

4. **(Challenge Resolved)** Complete the steps below (two ways?) and solve the $\aleph_0$–Squares Problem:
   (a) Show that $\sum_{n=1}^{\infty} \arctan \frac{1}{n} \geq \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{3n^3}\right)$.

   (b) Prove that $\sum_{n=1}^{\infty} \frac{1}{n^3} \leq 2$ and $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

5. **(Combinatorics Shake-&-Bake: Erdős-Szekeres Theorem)** In any sequence of $k \cdot m + 1$ distinct numbers there is either an increasing sequence of length $k+1$ or a decreasing sequence of length $m+1$.
   (a) **(Monotone Pigeons)** There are 10 people of different heights $a_1, \ldots, a_{10}$ standing in a line. Prove that no matter how they are lined up, there are some 4 people standing in increasing order or some 4 people standing in decreasing order (not necessarily consecutive). (*Hint*: If no increasing subsequence of length 4 and show there is a decreasing subsequence $a_{j_1} > a_{j_2} > a_{j_3} > a_{j_4}$. For every a_i, the maximum increasing subsequence of heights *starting with* a_i has length $m_i = 1, 2$, or 3. The m_i's are the pigeons and their values 1, 2, or 3 are the holes.) Try your proof on the sequence $\{16, 24, 6, 8, 3, 14, 2, 10, 7, 5\}$.

   (b) **(Can the statement be strengthened?)** Show that 10 is the smallest number with the above property. (*Hint*: Find a sequence of 9 (different) numbers that contains neither an increasing subsequence of length 4 nor a decreasing subsequence of length 4. Is this a counterexample? To what?)

Appendix 0: Types of Limits. Notation for Laws. Indeterminates

I. LIMITS OF SEQUENCES: $\lim_{n\to\square_1} x_n = \square_2$; e.g., $\lim_{n\to+\infty} x_n = 7$, $\lim_{n\to-\infty} x_n = +\infty$, etc.

Mix and match goals and answers below:

Limit $\square_2$	Goal for x_n	Want for x_n:
L	ϵ-goal around L	$L - \epsilon < x_n < L + \epsilon$
$+\infty$	M-goal, $M > 0$	$x_n > M$
$-\infty$	M-goal, $M < 0$	$x_n < M$

$n \to \square_1$	Answer for n	Expect for n:
$n \to +\infty$	N-answer, $N > 0$	$n > N$
$n \to -\infty$	N-answer, $N < 0$	$n < N$

II. LIMITS OF FUNCTIONS: $\lim_{x\to\square_1} f(x) = \square_2$; e.g., $\lim_{x\to5} f(x) = 7$, $\lim_{x\to-\infty} f(x) = 7$, $\lim_{x\to+\infty} f(x) = -\infty$, etc.

Mix and match goals and answers below:

Limit $\square_2$	Goal for $f(x)$	Want for $f(x)$:
L	ϵ-goal around L	$L - \epsilon < f(x) < L + \epsilon$
$+\infty$	M-goal, $M > 0$	$f(x) > M$
$-\infty$	M-goal, $M < 0$	$f(x) < M$

$x \to \square_1$	Answer for x	Expect for x:
$x \to a$	δ-interval around a	$a - \delta < x < a + \delta$
$x \to +\infty$	N-answer, $N > 0$	$x > N$
$x \to -\infty$	N-answer, $N < 0$	$x < N$

III. FOR INDETERMINATES: $\lim_{x\to\square}(f(x)+g(x))$, $\lim_{x\to\square} cf(x)$, $\lim_{x\to\square} f(g(x))$, $\lim_{x\to\square} f(x)^{g(x)}$, etc.

Why are some expressions *in*determinates? What are the *de*terminates equal to?

Operation	LL	CL	DL	Indeterminates	Determinates
sum	LL+	CL+	DL+	$\infty + (-\infty)$	$\infty + c$
difference	LL−	CL−	DL−	$\infty - \infty$	$\infty - c$
× constant	LLc	CLc	DLc	$\infty \cdot 0$	$\infty \cdot c$ for $c \neq 0$
product	LL∗	CL∗	DL∗ = PR	$\infty \cdot 0$	$\infty \cdot \infty$
division	LL÷	CL÷	DL÷ = QR	$\frac{0}{0}, \frac{\pm\infty}{\pm\infty}$	$\frac{d}{\infty}, \frac{\infty}{0^+}, \frac{\infty}{0^-}, \frac{\infty}{c}$ $(c \neq 0)$
composition	LL∘	CL∘	DL∘ = CR		
exponentiation	LLx^a, LLa^x	CLx^a, CLa^x	DLx^a, DLa^x	0^0, ∞^0, 1^∞	1^1, 1^0, 0^1, 0^∞, ∞^1, ∞^∞

Appendix 1: Key Takeaways and Review with Extensions

- **(PST 1)** Reduce a more general theorem to a special case of it by creating an object (e.g., function) that satisfies the conditions of the special case.
- **(PST 2)** To find the limit L of a recurrence sequence, first show that it is bounded and monotone (or split it into an increasing and a decreasing subsequences). Conclude that it is convergent and hit the recurrence equation with $\lim_{n\to\infty}$ on both sides to solve for L.
- **(Strategy)** Real-life problem $\xrightarrow{\text{translate}}$ Calculus problem $\xrightarrow{\text{solve}}$ Calculus answer $\xrightarrow{\text{translate back}}$ Real-life answer
- **(Applications)** To apply a theorem, we must check that its hypothesis is satisfied in the particular problem and then state its conclusion without further proof.
- **(Implications)** Consider "$A \Rightarrow B$" (if A then B).
  1. Its **converse** is "$B \Rightarrow A$".
  2. Its **contrapositive** is "not $B \Rightarrow$ not A".
  3. Its **inverse** is "not $A \Rightarrow$ not B".
- **(Convergent-Monotone Theorem, CMT)** ∀ convergent sequence has a monotone subsequence.
- **(Bolzano-Weirstrass Theorem, BWT)** ∀ infinite sequence has a monotone subsequence.
- **(Monotonicity)** If $f'(x) > 0$ on an interval I, then $f(x)$ is increasing (not necessarily strictly) on I.
- **(1st Derivative Test)** If $f'(x)$ changes its sign at c:
  1. from + to −, then $f(c)$ is a local maximum;
  2. from − to +, then $f(c)$ is a local minimum.
- **(2nd Derivative Test)** If $f'(c) = 0$ and $f''(c) > 0$ (or $f''(c) < 0$), then $f(c)$ is a local min (or max).
- **(Rolle's Theorem, RT)** If f is cont. on $[a,b]$ & diff. on (a,b) w/ $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a,b)$.
- **(Mean Value Theorem, MVT)** If f is continuous on $[a,b]$ and differentiable on (a,b), then some tangent slope is equal to the total secant slope: $f'(c) = \frac{f(b)-f(a)}{b-a}$ for some $c \in (a,b)$.

4)

$\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots$

diverges

$a_n = \arctan 1/n$

$$\sum_{n=1}^{\infty} \alpha_n = \sum_{n=1}^{\infty} \arctan\frac{1}{n} \geq \sum_{n=1}^{\infty}\left(\frac{1}{n} - \left(\frac{1}{n}\right)^3/3\right) = \sum_{n=1}^{\infty}\frac{1}{n} - \sum_{n=1}^{\infty}\frac{1}{3n^3}$$

$$\geq \infty - 2 = \infty$$

$\arctan x \geq x - x^3/3$ for $x \in [0,1]$

$\frac{1}{n} \in [0,1]$

b) $$\sum_{n=1}^{\infty}\frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots \geq \frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

($\frac{1}{3} + \frac{1}{4} \geq \frac{1}{4} + \frac{1}{4}$, $\frac{1}{5} + \cdots + \frac{1}{8} \geq \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$)

$= \infty$.

p-series test $\sum \frac{1}{n^p} = \begin{cases} \text{diverges } p \leq 1 \\ \text{converges } p > 1 \end{cases}$

$$\sum_{n=1}^{\infty}\frac{1}{n^3} \leq \sum_{n=1}^{\infty}\frac{1}{n^2} \overset{\text{Euler}}{=} \frac{\pi^2}{6} \leq 2 < \infty$$

$\sum \frac{1}{n} \geq \sum \frac{1}{2^n} = 1$ $\exists L$ s.t. $-L \leq a_i \leq L$

2b) Prove that all bounded monotone sequences are convergent

$a_1, a_2, a_3, \ldots$ $a_1 \leq a_2 \leq \cdots$ or $a_1 \geq a_2 \geq a_3 \geq \cdots$

$\lim_{n\to\infty} a_n$ exists and is finite.

WLOG suppose $a_1 \leq a_2 \leq a_3 \leq \cdots$

Guess that $\lim_{n\to\infty} a_n = \sup a_n = S$ ← exists because sequence is bounded

What does it mean for S to be "least upper bound"?

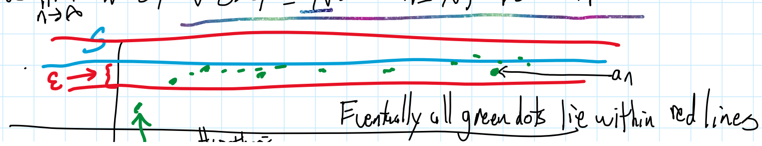
For every $\varepsilon > 0$ $S - \varepsilon$ isn't an upper bound: There exists some N s.t. $a_N > S - \varepsilon$.

Ex $\sup\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\} = 1$ ($1 - \frac{1}{n}$) $.99 < \frac{100}{101}$ $.\overline{999} = 1$

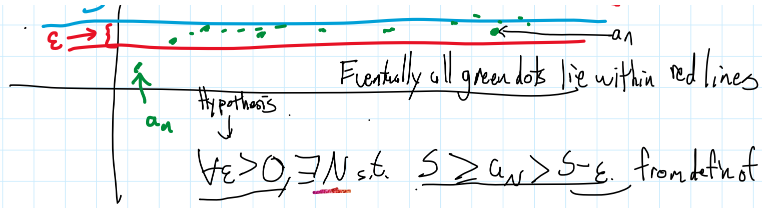
$\sup\{1, 2, 3, 4, \ldots\} = \text{DNE} = \infty$

Prove that $S = L$

To prove $\lim_{n\to\infty} a_n = S$, If $\forall \varepsilon > 0$, Then $\exists N$ s.t. $\forall n \geq N$, $|S - a_n| < \varepsilon$.



- **(Applications)** To apply a theorem, we must check that its hypothesis is satisfied in the particular problem and then state its conclusion without further proof.
- **(Implications)** Consider "$A \Rightarrow B$" (if A then B).
  1. Its **converse** is "$B \Rightarrow A$".
  2. Its **contrapositive** is "not $B \Rightarrow$ not A".
  3. Its **inverse** is "not $A \Rightarrow$ not B".
- **(Sequence Limit)** A sequence $\{x_n\}$ is **convergent** with limit L if for every $\varepsilon > 0$ there is N such that $x_n \overset{\varepsilon}{\approx} L$ for any $n \geq N$; i.e., $|L - x_n| < \varepsilon$ for $n \geq N$. In such a case, we write $\lim_{n\to\infty} x_n = L$.
- **(Supremum)** For a nonempty set S of real numbers that is bounded above, the *least upper bound* for S in $\mathbb{R}$ is called the *supremum* of S and is denoted $\sup S$.
- **(Infimum)** For a nonempty set S of real numbers that is bounded below, the *greatest lower bound* for S in $\mathbb{R}$ is called the *infimum* of S and is denoted $\inf S$.
- **(Completeness Axiom)** Every nonempty set $S \subset \mathbb{R}$ that is bounded above has a supremum $\sup S$.
- **(Definitions)** If $f : D \subset \mathbb{R} \to \mathbb{R}$, then f has a:
  1. **global maximum** $f(c)$ if $f(c) \geq f(x)\ \forall x \in D$.
  2. **local maximum** $f(c)$ if $f(c) \geq f(x)$ for all x in an *open* interval containing c: $x \in (c-\delta, c+\delta)$.
  3. **critical number** $c \in D$ if $f'(c) = 0$ or $f'(c)\ \nexists$.
  4. **slant asymptote** $y = ax + b$ for $x \to \infty$ if $\lim_{x\to\infty}[f(x) - (ax+b)] = 0$.
- **(Continuity)** f is **continuous** at c if $\lim_{x\to c} f(x) = f(c)$.
- **(Derivative)** f is **differentiable** at a if the limit L exists and is finite: $L = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}\ (= f'(a))$.
  Alternatively, $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.
- **(Squeeze Theorem, ST)** If $f(x) \leq g(x) \leq h(x)$ for all x in an open interval that contains a (except possibly at a) and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.
- **(Convergent $\Rightarrow$ Bounded Theorem, CBT)** $\forall$ convergent sequence is bounded.
- **(Monotone Bounded Theorem, MBT)** $\forall$ bounded monotone sequence is convergent.
- **(Bounded-Convergent Theorem, BCT)** $\forall$ bounded sequence has a convergent subsequence.
- **(Rolle's Theorem, RT)** If f is cont. on $[a,b]$ & diff. on (a,b) w/ $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a,b)$.
- **(Mean Value Theorem, MVT)** If f is continuous on $[a,b]$ and differentiable on (a,b), then some tangent slope is equal to the total secant slope: $f'(c) = \frac{f(b)-f(a)}{b-a}$ for some $c \in (a,b)$.
- **(Cauchy's Mean Value Theorem, CMVT)** If f and g are continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) \neq 0$ for all $x \in (a,b)$, then for some $c \in (a,b)$ we have $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$.
- **(Constant Function Theorem, CFT)** If $f'(x) = 0$ on $[a,b]$, then f is constant on (a,b).
- **(Antiderivatives Theorem, AT)** If $f'(x) = g'(x)$ on $[a,b]$, then $f(x) = g(x) + c$ for some constant c.
- **(Fermat's Theorem, FT)** If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.
- **(Intermediate Value Theorem, IVT)** If f is continuous on $[a,b]$ and N is an intermediate value for f (i.e., N is betwee $f(a)$ and $f(b)$), then f attains the value N somewhere on $[a,b]$: $f(c) = N$ for some $c \in [a,b]$.
- **(Extreme Value Theorem, EVT)** If f is continuous on a closed finite interval $[a,b]$, then f attains a global maximum $f(c)$ and a global minimum value $f(d)$ at some numbers $c, d \in [a,b]$.
- **(Closed Interval Method, CIM)** To find the *global* maximum and minimum values of a continuous function f on a closed interval $[a,b]$:
  1. Find the values of f at its critical numbers in (a,b).
  2. Find the values of f at the endpoints of $[a,b]$.
  3. The largest of the values from Steps 1 and 2 is the global max and the smallest is the global min.
- **(CIM on an open interval)** If f is defined and continuous on an open interval and has no critical points, then it has *no* local and *no* global extrema.
- **(L'Hopital's Rule, LH)** Let f and g be diff., with $g'(x) \neq 0$ on an open interval $\ni a$ (except possibly at a). If $\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$, as long as the RHS limit exists (or is ∞ or $-\infty$).



hypothesis

$\forall \varepsilon > 0, \exists N$ s.t. $S \geq a_N > S - \varepsilon$ from defn of sup.

and $\forall n \geq N$, $S \geq a_n \underset{\text{monotonicity}}{\geq} a_N > S - \varepsilon$ so $|S - a_n| < \varepsilon$ ✓

So $L = S$ and $\lim a_n$ exists, a_n are convergent.

$S \geq a_n > S - \varepsilon$

$0 \geq a_n - S > -\varepsilon \Rightarrow 0 \leq |S - a_n| < \varepsilon$ ($\times -1$)

$\varepsilon = 0.01$

$a_n = 1 - \frac{1}{n}$, want to prove $\lim a_n = S = 1$.

We know $S - \varepsilon$ ($= 0.99$) is not an upper bound because $1 - \frac{1}{101} > 0.99$

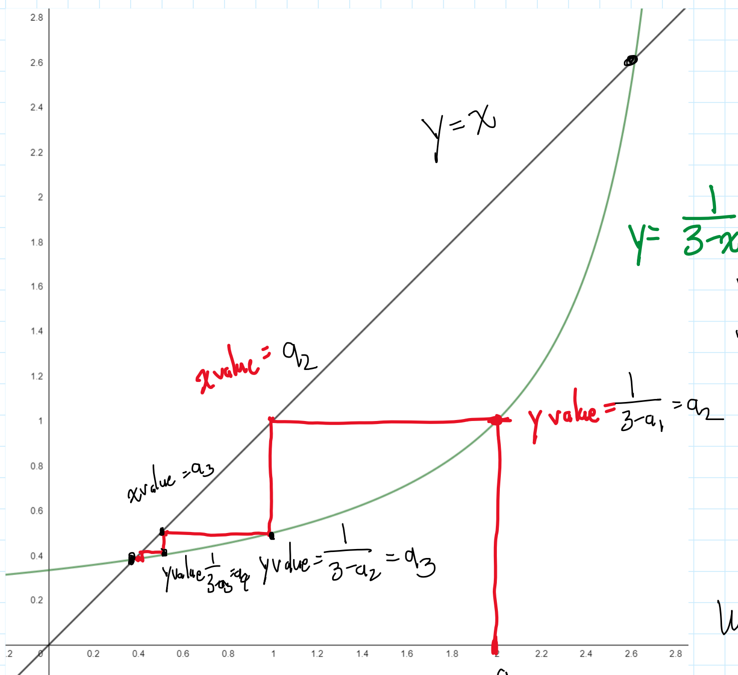
$\forall n \geq 101$, $1 \geq 1 - \frac{1}{n} \geq 1 - \frac{1}{101} > 0.99$ and so

$\forall n \geq 101$, $|S - a_n| = |1 - (1 - \frac{1}{n})| = \frac{1}{n} < \varepsilon = 0.01$.

$\frac{1}{n}$ $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \to 0$

Guess $\lim a_n = \inf a_n$

3c) $a_{n+1} = \frac{1}{3 - a_n}$ $a_1 = 2$ $a_2 = \frac{1}{3-2} = 1$ $a_3 = \frac{1}{3-1} = \frac{1}{2}$ $a_4 = \frac{1}{3 - \frac{1}{2}} = \frac{2}{5}, \ldots$



$\lim a_n =$ (smaller) intersection of $y = x$, $y = 1/(3-x)$

$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{3 - a_n}$ If $\lim_{n\to\infty} a_n = L$

$L = \frac{1}{3 - L}$

$3L - L^2 = 1 \to L^2 - 3L + 1 = 0$

$L = \frac{3 \pm \sqrt{5}}{2} \to \boxed{\frac{3 - \sqrt{5}}{2}}$ (use smaller)

Why does $\lim a_n$ exist? $\to$ a_n is a bounded monotone decreasing sequence. $\Rightarrow$ convergent

Bounded: If $\frac{3-\sqrt{5}}{2} < a_N < 3$, then $\frac{3-\sqrt{5}}{2} < \frac{1}{3 - a_N} < 3$

Monotone: $\frac{1}{3 - a_N} < a_N \to a_N^2 - 3a_N + 1 < 0$ true for $\frac{3-\sqrt{5}}{2} < a_N < 3$

FT $\Rightarrow$ RT

Assume FT + hypothesis of RT $\Rightarrow$ Conc. of RT.

Assume FT + hypothesis of RT $\Rightarrow$ Conc. of RT.