Discussion 2 11/10/20

Tuesday, November 10, 2020 2:03 PM

Real Analysis (Discussion)

Worksheet 4: Differentiable $\Rightarrow$ Continuous, Fermat's $\Rightarrow$ Rolle's [1]

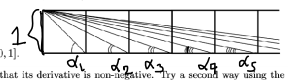
Date: 11/10/2020

MATH 74: Transition to Upper-Division Mathematics
with Professor Zvezdelina Stankova, UC Berkeley

Read: *Session 12: Geometric Re-Constructions III* (vol. II)
• §3. Infinitely Many Angles and Infinite Series (pp. 296); • Appendices for Indeteminants and Theorems

Write: clearly. Supply your reasoning in words and/or symbols. Show calculations and relevant pictures.

1. **(Logical Relations and Proofs)** See Appendix 1 for a list of relevant definitions and theorems.
   (a) **(Diff. $\to$ Cont.)** Prove that a differentiable function at a is also continuous at a.
   (b) **(FT)** Prove Fermat's Theorem. (*Hint*: Apply EVT and use one-sided def'n $f'(c) = \lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h}$ to show that $f'(c) \le 0$ and $f'(c) \ge 0$.)
   (c) **(FT $\Rightarrow$ RT)** Show that FT implies RT.
   (d) **(AT)** Prove the Antiderivative Theorem. (*Hint*: Apply CF to $h(x) = f(x) \quad g(x)$.)
   (e) **(Baby LH)** Prove LH when $f(a) = g(a) = 0$, f' and g' are continuous, and $g'(a) \neq 0$.
   (f) **(∞-LH)** Prove LH when $a = \infty$. (*Hint*: Set $t = 1/x$ and use LH for a finite a.)
2. **(Challenge Tackle)** In the $\aleph_0$–Squares Problem 4, we conjecture that $\alpha_1 + \alpha_2 + \cdots + \alpha_n + \cdots = \infty$.
   (a) Show that $\sum_{n=1}^{\infty} \alpha_n = \sum_{n=1}^{\infty} \arctan \frac{1}{n}$.
   (b) Prove that $\arctan x \ge x - \frac{x^3}{3}$ for $x \in [0,1]$.



   (*Hint*: Pull everything to the LHS and show that its derivative is non-negative. Try a second way using the Taylor polynomial of $\arctan x$. What do you need to know about this Taylor polynomial in order to use it here?)
3. **(Concrete Applications)** Prove the following statements over **R**. Justify everything rigorously.
   (a) Show that a deg. n polynomial of has at most n real roots, $n \ge 1$. (*Hint*: RT and induction?)
   (b) If f and g are cont. on $[a,b]$ and diff. on (a,b), $f(a) = g(a)$ and $f'(x) < g'(x)$ for all $x \in (a,b)$, prove that $f(b) < g(b)$. (*Hint*: MVT to $f - g$.)
   (c) Let $f(x) = 2 - |2x - 1|$. Show that there is no value c such that $f(3) - 3f'(c) + f(0)$. Why does this not contradict MVT?
   (d) Is there a function f with $f(0) = -1$, $f(2) = 4$, and $f'(x) < 2$ for all x?
   (e) If f is odd and is differentiable on $\mathbb{R}$, prove that $\forall b > 0\ \exists c \in (-b,b)$ with $f'(c) = f(b)/b$.
   (f) If $f(x) = \frac{1}{x}$ and $g(x) = \begin{cases} 1/x & \text{if } x > 0 \\ 1 + 1/x & \text{if } x < 0 \end{cases}$, show that $f'(x) = g'(x)$ on their domains. Can we conclude from the AT that $f - g$ is constant?
   (g) *Without* trigonometry, prove the identity $\arcsin \frac{x-1}{x+1} = 2\arctan\sqrt{x} - \frac{\pi}{2}$ on its domain.
   (h) Compute the limits and explain rigorously:
   • $\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$; • $\lim_{x\to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$.
   (i) If f'' is continuous, show that $f''(x) = \lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$.

2a) $\tan(\alpha_n) = \frac{1}{n} \Rightarrow \alpha_n = \arctan(\frac{1}{n})$

Appendix 0: Indeterminates and Determinates

APPLY: $\lim_{x\to\square}(f(x)+g(x))$, $\lim_{x\to\square} cf(x)$, $\lim_{x\to\square} f(g(x))$, $\lim_{x\to\square} f(x)^{g(x)}$, etc.

Why are some expressions *in*determinates? What are the *de*terminates equal to?

Operation	Indeterminates	Determinates
sum	$\infty + (-\infty)$	$\infty + c$
difference	$\infty - \infty$	$\infty - c$
$\times$ constant	$\infty \cdot 0$	$\infty \cdot c$ for $c \neq 0$
product	$\infty \cdot 0$	$\infty \cdot \infty$
division	$\frac{0}{0}, \frac{\pm\infty}{\pm\infty}$	$\frac{d}{\infty}, \frac{\infty}{0^+}, \frac{\infty}{0^-}, \frac{\infty}{c}$ $(c \neq 0)$
exponentiation	$0^0, \infty^0, 1^\infty$	$1^1, 1^0, 0^1, 0^\infty, \infty^1, \infty^\infty$

Appendix 1: Key Takeaways and Review with Extensions

- **(PST 1)** Reduce a more general theorem to a special case of it by creating an object (e.g., function) that satisfies the conditions of the special case.
- **(PST 2)** To find the limit L of a recurrence sequence, first show that it is bounded and monotone (or split it into an increasing and a decreasing subsequences). Conclude that it is convergent and hit the recurrence equation with $\lim_{n\to\infty}$ on both sides to solve for L.
- **(Strategy)** Real-life problem $\xrightarrow{\text{translate}}$ Calculus problem $\xrightarrow{\text{solve}}$ Calculus answer $\xrightarrow{\text{translate back}}$ Real-life answer
- **(Applications)** To apply a theorem, we must check that its hypothesis is satisfied in the particular problem and then state its conclusion without further proof.
- **(Implications)** Consider "$A \Rightarrow B$" (if A then B).
  1. Its **converse** is "$B \Rightarrow A$".
  2. Its **contrapositive** is "not $B \Rightarrow$ not A".
  3. Its **inverse** is "not $A \Rightarrow$ not B".
- **(Sequence Limit)** A sequence $\{x_n\}$ is **convergent** with limit L if for every $\varepsilon > 0$ there is N such that $x_n \overset{\varepsilon}{\approx} L$ for any $n \ge N$; i.e., $|L - x_n| < \varepsilon$ for $n \ge N$. In such a case, we write $\lim_{n\to\infty} x_n = L$.
- **(Supremum)** For a nonempty set S of real numbers that is bounded above, the *least upper bound* for S in $\mathbb{R}$ is called the *supremum* of S and is denoted $\sup S$.
- **(Infimum)** For a nonempty set S of real numbers that is bounded below, the *greatest lower bound* for S in $\mathbb{R}$ is called the *infimum* of S and is denoted $\inf S$.
- **(Definitions)** If $f : D \subset \mathbb{R} \to \mathbb{R}$, then f has a:
  1. **global maximum** $f(c)$ if $f(c) \ge f(x)\ \forall x \in D$.
  2. **local maximum** $f(c)$ if $f(c) \ge f(x)$ for all x in an *open* interval containing c: $x \in (c-\delta, c+\delta)$.
  3. **critical number** $c \in D$ if $f'(c) = 0$ or $f'(c) \nexists$.
  4. **slant asymptote** $y = ax + b$ for $x \to \infty$ if $\lim_{x\to\infty}[f(x) - (ax+b)] = 0$.
- **(Continuity)** f is continuous at c if $\lim_{x\to c} f(x) = f(c)$.
- $\boxed{\textbf{(Derivative) } f \text{ is } \textbf{differentiable} \text{ at } a \text{ if the limit } L \text{ exists and is finite: } L = \lim_{x\to a}\frac{f(x) \quad f(a)}{x-a} (= f'(a)). \text{ Alternatively, } f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.}$
- **(Squeeze Theorem, ST)** If $f(x) \le g(x) \le h(x)$ for all x in an open interval that contains a (except possibly at a) and $\lim f(x) = \lim h(x) = L$, then $\lim g(x) = L$.
- **(2nd Derivative Test)** If $f'(c) = 0$ and
  1. $f''(c) > 0$, then $f(c)$ is a local minimum;
  2. $f''(c) < 0$, then $f(c)$ is a local maximum.
- **(Rolle's Theorem, RT)** If f is continuous on $[a,b]$ and differentiable on (a,b), and $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a,b)$.
- **(Mean Value Theorem, MVT)** If f is continuous on $[a,b]$ and differentiable on (a,b), then some tangent slope is equal to the total secant slope: $f'(c) - \frac{f(b)-f(a)}{b-a}$ for some $c \in (a,b)$.
- **(Cauchy's Mean Value Theorem, CMVT)** If f and g are continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) \neq 0$ for all $x \in (a,b)$, then for some $c \in (a,b)$ we have $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$.
- **(Constant Function Theorem, CFT)** If $f'(x) = 0$ on $[a,b]$, then f is constant on (a,b).
- **(Antiderivatives Theorem, AT)** If $f'(x) = g'(x)$ on $[a,b]$, then $f(x) = g(x) + c$ for some constant c.
- **(Fermat's Theorem, FT)** If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.
- **(Intermediate Value Theorem, IVT)** If f is continuous on $[a,b]$ and N is an intermediate value for f (i.e., $f(a) \le N \le f(b)$ or $f(a) \ge N \ge f(b)$), then f attains the value N somewhere on $[a,b]$: $f(c) - N$ for some $c \in [a,b]$.
- **(Extreme Value Theorem, EVT)** If f is continuous on a closed finite interval $[a,b]$, then f attains a global maximum $f(c)$ and a global minimum value $f(d)$ at some numbers $c, d \in [a,b]$.
- **(Closed Interval Method, CIM)** To find the *global* maximum and minimum values of a continuous function f on a closed interval $[a,b]$:
  1. Find the values of f at its critical numbers in (a,b).
  2. Find the values of f at the endpoints of $[a,b]$.
  3. The largest of the values from Steps 1 and 2 is the global maximum value; the smallest of these values is the global minimum value.
- **(CIM on an open interval)** If f is defined and continuous on an open interval and has no critical points, then it has *no* local and *no* global extrema.
- **(L'Hopital's Rule, LH)**

1a) Prove that if f is diff at $x=a$, then it's continuous there

Know: $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a)$ ← exists and is finite

Want: $\lim_{x\to a} f(x) = f(a)$ OR $\lim_{x\to a}[f(x) - f(a)] = 0$

$\left[\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a)\right] \lim_{x\to a}(x-a)$

$\lim_{x\to a}(x-a) \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a) \cdot \lim_{x\to a}(x-a)$

XLL ⇒ $\lim_{x\to a} \cancel{(x-a)} \cdot \frac{(f(x)-f(a))}{\cancel{(x-a)}} = f'(a) \cdot \lim_{x\to a}(x-a)$

$\lim_{x\to a}[f(x) - f(a)] = f'(a) \cdot 0$

$-$LL $\lim_{x\to a} f(x) - \lim_{x\to a} f(a) = 0$

$\lim_{x\to a} f(x) = \lim_{x\to a} f(a) = f(a) \Rightarrow f$ is cont. at $x=a$ □

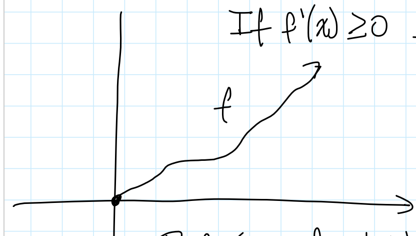
Limits + Calc Recall: Stewart's Calculus

2b) $\arctan(x) \ge x - x^3/3$

$f(x) = \arctan(x) - x + x^3/3 \overset{?}{\ge} 0$

$f(0) = \arctan(0) - 0 + 0^3/3 = 0$

If $f'(x) \ge 0$ for $x \in [0,1]$, then $f(x) \ge 0, x \in [0,1]$.



Proof Suppose for contradiction, $f(c) < 0$ for $c \in (0,1]$, then MVT applied to f on $[0,c] \Rightarrow \exists e \in (0,c)$ s.t.

$f'(e) = \frac{f(c)-f(0)}{c-0} < 0$ ↯ since $f'(e) \ge 0$. □

$f'(x) = \frac{1}{1+x^2} - 1 + x^2 = \frac{x^4}{1+x^2} \ge 0$ □.

- $\boxed{\text{(Derivative) } f \text{ is differentiable at } a \text{ if the limit } L \text{ exists and is finite: } L = \lim_{x\to a} \frac{f(x) - f(a)}{x-a} \; (= f'(a)). \text{ Alternatively, } f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.}$
- **(Squeeze Theorem, ST)** If $f(x) \le g(x) \le h(x)$ for all x in an open interval that contains a (except possibly at a) and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.
- **(Monotonicity)** If $f'(x) > 0$ on an interval I, then $f(x)$ is increasing (not necessarily strictly) on I.
- **(1st Derivative Test)** If $f'(x)$ changes its sign at c:
  1. from + to −, then $f(c)$ is a local maximum;
  2. from − to +, then $f(c)$ is a local minimum.

global maximum value; the smallest of these values is the global minimum value.

- **(CIM on an open interval)** If f is defined and continuous on an open interval and has no critical points, then it has *no* local and *no* global extrema.
- **(L'Hopital's Rule, LH)** Let f and g be differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a).

  If $\dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{0}{0}$ or $\dfrac{+\infty}{\pm\infty}$, then $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \lim_{x\to a} \dfrac{f'(x)}{g'(x)}$, as long as the RHS limit exists (or is ∞ or $-\infty$).

3a) A degree n poly. has at most n distinct real roots, $n \ge 1$ (Induction, RT)

Base Case: $n=1$, $ax+b \longrightarrow$ unique real root $x = -b/a$.

IH: Any degree k poly. has at most k distinct real roots, for some $k \ge 1$.

IS: Let f be a degree $k+1$ poly. Suppose for contradiction f has at least $k+2$ roots.

$x_1 < x_2 < \cdots < x_{k+1} < x_{k+2} \Leftarrow$ distinct real roots of f.

$f(x_1) = f(x_2) = 0 \overset{RT}{\Longrightarrow} f'(c_1) = 0$ for some $c_1 \in (x_1, x_2)$

$f(x_2) = f(x_3) = 0 \overset{RT}{\Longrightarrow} f'(c_2) = 0$ for some $c_2 \in (x_2, x_3)$

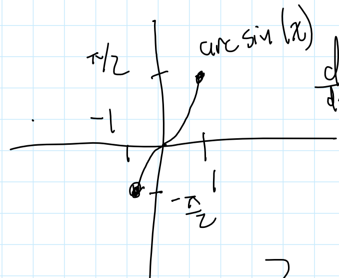
$\vdots$

$f(x_{k+1}) = f(x_{k+2}) = 0 \overset{RT}{\Longrightarrow} f'(c_{k+1}) = 0$ for some $c_{k+1} \in (x_{k+1}, x_{k+2})$

So $f'(x)$ has at least $k+1$ roots but $f'(x)$ is degree k so it has at most k roots (IH) $\lightning$

Conclude

3g) $\frac{d}{dx}\left[\arcsin \frac{x-1}{x+1} = 2\arctan\sqrt{x} - \frac{\pi}{2}\right]$



$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$

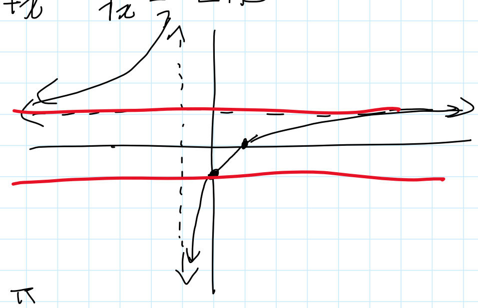
$$LHS = \frac{1}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}} \cdot \frac{(x+1)\cdot 1 - (x-1)\cdot 1}{(x+1)^2}$$

$$= \frac{1}{\sqrt{1 - \frac{(x-1)^2}{(x+1)^2}}} \cdot \frac{2}{(x+1)^2} = \frac{2}{(x+1)\sqrt{(x+1)^2\left(1 - \frac{(x-1)^2}{(x+1)^2}\right)}} = \frac{2}{(x+1)\sqrt{(x+1)^2 - (x-1)^2}} = \frac{2}{(x+1)\sqrt{4x}} = \frac{1}{(x+1)\sqrt{x}}$$

$$RHS = 2 \cdot \frac{1}{1+(\sqrt{x})^2} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{x}} = \frac{1}{1+x} \cdot \frac{1}{\sqrt{x}} = LHS$$

Domain of $\arcsin\left(\frac{x-1}{x+1}\right)$, $\arctan\sqrt{x}$

$\frac{x-1}{x+1} \in [-1, 1]$ $\quad x \ge 0$



$\rightarrow x \ge 0$

$x=0$ $\quad \arcsin\frac{0-1}{0+1} = \arcsin -1 = -\frac{\pi}{2}$

$2\arctan\sqrt{x} - \frac{\pi}{2} = 2\arctan 0 - \frac{\pi}{2} = -\frac{\pi}{2}$ ✓