Discussion 2 10/29/20

Thursday, October 29, 2020 5:07 PM

Inversion in the Plane (Discussion)

Worksheet 4: Property Proofs and Chains of Tangent Figures[1]

Date: 10/29/2020

MATH 74: Transition to Upper-Division Mathematics

with Professor Zvezdelina Stankova, UC Berkeley

Write: clearly. Supply your reasoning in words and/or symbols. Show calculations and relevant pictures.

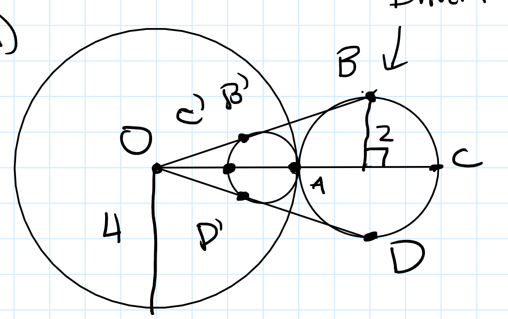
1. **(Experiment)** By experimenting, answer the questions. No proof is required yet, but carefully marked pictures and brief explanations are a must. What happens under inversion $I(O,r)$ to a circle $k_1(O_1,r_1)$ **not** passing through the center O if k_1 is:
   (a) entirely *outside* $k(O,r)$? (b) entirely *inside* k? (c) *intersects* k in two points A and B?
   (d) *externally* tangent to k at point T? (e) *internally* tangent to k at point T?
2. **(The Proof)** Consider inversion $I(O,r)$. For any line l not passing through O, let $OH \perp l$ ($H \in l$), and $I(H) = H_1$. Let k_1 be the circle with diameter OH_1.
   (a) For any point $X \in l$ and let $OX \cap k_1 = Y$. Prove that $OX \cdot OY = r^2$. (*Hint:* Similar $\triangle$s.)
   (b) Let $I(X) = X_1$. Why is $X_1 = Y$? How does this imply that $I(l) \subset k_1$?
   (*Hint:* The distance formula $OX \cdot OX_1 = r^2 \Rightarrow OX_1 = ?$)
3. **(Chain of Tangent Figures)** Draw two parallel lines l_1 and l_2, and circles $k_1(O_1,r_1)$ and $k_2(O_2,r_2)$ between l_1 and l_2 so that l_1 is tangent to k_1 at point A, k_1 is tangent to k_2 at point B, and k_2 is tangent to l_2 at point C. Prove that A, B, and C are colinear (i.e., lie on a line).
   (*Hint:* Similar triangles? Be careful not to assume what you are not given!)
4. **(Geo Shake-&-Bake)** A right $\triangle ABC$ has legs $AC = 3$ and $BC = 4$, and altitude CD to the hypotenuse AB. Find the distance between the incenters O_1 and O_2 of $\triangle ACD$ and $\triangle BCD$ as follows:
   (a) How long are CD, AD, and BD? (*Hint:* Find the area of $\triangle ABC$ in 2 ways.)
   (b) Find the inradii r_1 and r_2 of $\triangle ACD$ and $\triangle BCD$? (*Hint:* Area vs. inradius of a $\triangle$?)
   (c) If T_1 and T_2 are the points of tangency of the two incircles with hypotenuse AB, find the lengths of DT_1 and DT_2. (*Hint:* Review L139.)
   (d) Find the distance between the incenters O_1 and O_2. (*Hint:* What figure is $T_1T_2O_2O_1$?)
   (e) Find the distance from O_1 to vertex C. (*Hint:* PT?)
   (f) Starting with lengths $AC = b$ and $BC = a$, find a formula for O_1O_2.
   (g) List the sequence of steps that led from the original problem to the final result. What extra objects did we have to plot and/or find along the way?

Draw+ Explain how to find O_1, O_2.

Extra Background and Practice: *Famous Points in a* $\triangle$: L141, W141; *Constructing a* $\triangle$: L142, W142

6. **(Fundamentals)** W141: #1, 2, 3, 4*; W142: #1, 2*, 3.
   (*Hint:* In W141 #4, "chase" all angles in the picture, using properties of incenters and right triangles. In W142 #2, how do we move/duplicate an angle, using a straightedge and compasses?)

1d)



$I(C) = C'$ $\quad OC' = \frac{r^2}{OC} = \frac{4^2}{8} = 2$

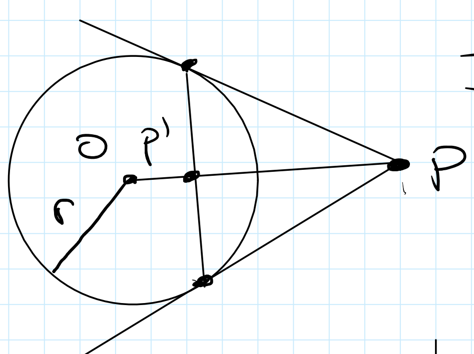
$I(B) = B'$ $\quad OB' = \frac{r^2}{OB} = \frac{16}{\sqrt{6^2+2^2}} = \frac{16}{\sqrt{40}} \approx 2.5$

$I(A) = A'$ $\quad OA' = \frac{r^2}{OA} = \frac{4^2}{4} = 4.$

$A' = A$

Guess $I(\text{circle}) = \text{smaller circle}$

To prove: $I(\text{circle}) \subseteq \text{small circle}$ + surjective (onto)



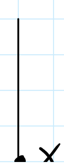
$I(P) = P'$. $\quad OP' \cdot OP = r^2$

Step 1 Draw OP

Step 2 Find $P' \in OP$ st that

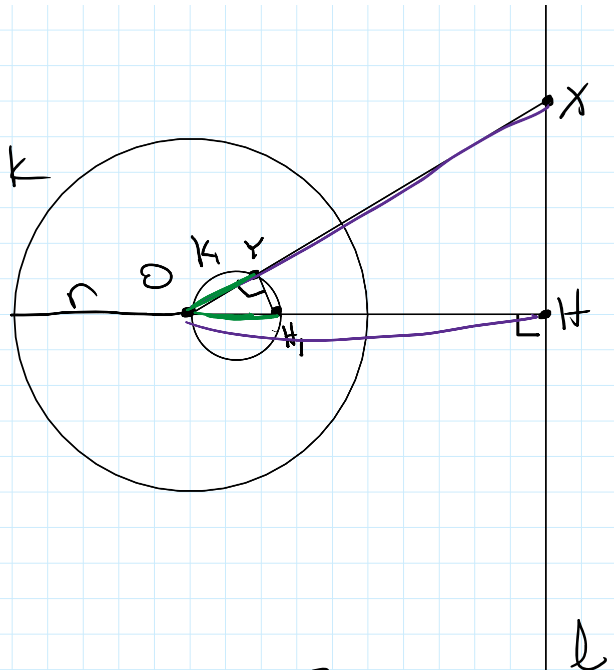
$OP' = \frac{r^2}{OP}$.

2)



$I(H) = H.$

2)



$I(H) = H_1$

k_1 has diameter OH_1.

Want to prove $I(l) = k_1$

First $I(l) \subseteq k_1$

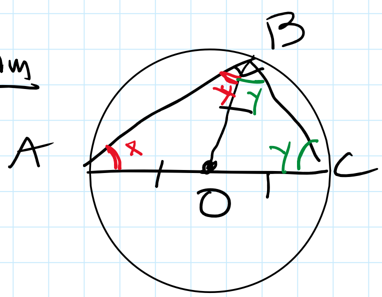
Want to show $I(X) = Y \in k_1$

$\rightarrow \boxed{OX \cdot OY = r^2}$

$\triangle OYH_1 \sim \triangle OHX$ by AA similarity

$\frac{OH_1}{OX} = \frac{OY}{OH} \Rightarrow OX \cdot OY = OH \cdot OH_1 = r^2$ □

Thm

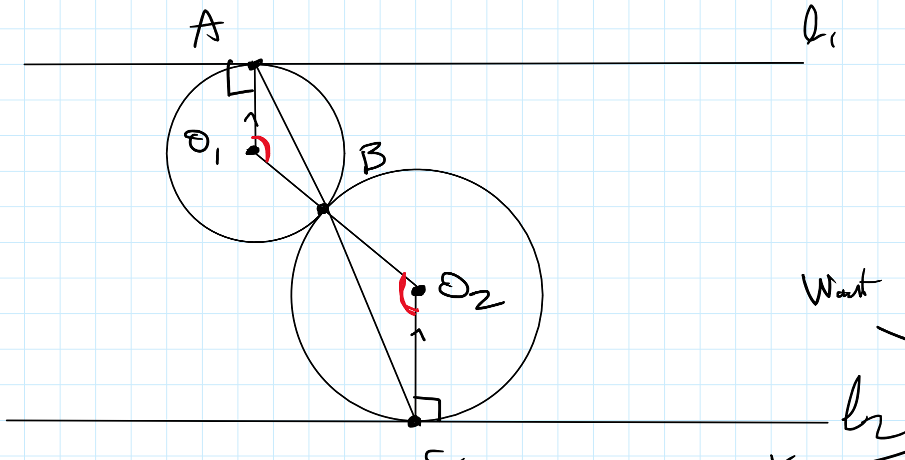


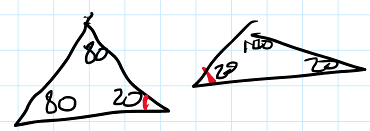
If AC is a diameter, then $\angle B = 90°$

Pf $\triangle AOB$, $\triangle BOC$ are isosceles because $OA = OB = OC =$ radius.

$x + x + y + y = 180° \xrightarrow{\div 2} x + y = 90°$ □

3)





We know O_1BO_2 is a line.

Want: Want to prove ABC is a line.

$\Rightarrow \angle ABC = 180° = \angle ABO_2 + \angle O_2BC$.

Know: $\angle ABO_1 + \angle ABO_2 = 180°$

$\rightarrow$ Want to prove $\angle ABO_1 = \angle CBO_2$

Want

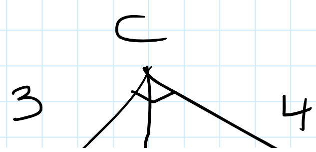
$l_1 \parallel l_2$, $AO_1 \parallel CO_2$

O_1BO_2 is a transversal $\rightarrow$ alt. int. angles are equal

$\angle AO_1B = \angle BO_2C$

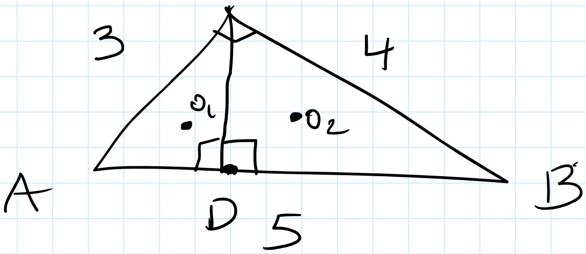
$\triangle AO_1B$, $\triangle BO_2C$ isosceles $\rightarrow$ similar $\rightarrow \angle ABO_1 = \angle CBO_2$ □

4)



$CD = ? = 12/5$

$\text{Area}(ABC) = \frac{AC \cdot BC}{\ } = \frac{3 \cdot 4}{\ } = 6$



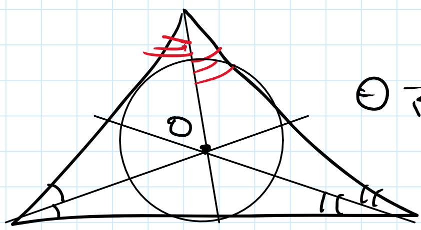
$$\text{Area}(ABC) = \frac{AC \cdot BC}{2} = \frac{3 \cdot 4}{2} = 6$$

$$= \frac{AB \cdot CD}{2} = \frac{5}{2} \cdot CD = 6$$

$$AD = ? = \sqrt{3^2 - \left(\frac{12}{5}\right)^2} = \frac{9}{5}$$

Or $\triangle ADC \overset{AA}{\sim} \triangle ACB$

O_1, O_2 are incenters = intersection of angle bisectors = center of incircle



O is the center of the incircle: L138.pdf