Discussion 2 10/27/20

Tuesday, October 27, 2020 3:08 PM

Inversion in the Plane (Discussion)

Worksheet 2: Tangent Circles and Tangents to Circles[1]

Date: 10/27/2020

MATH 74: Transition to Upper-Division Mathematics
with Professor Zvezdelina Stankova, UC Berkeley

Write: clearly. Supply your reasoning in words and/or symbols. Show calculations and relevant pictures.

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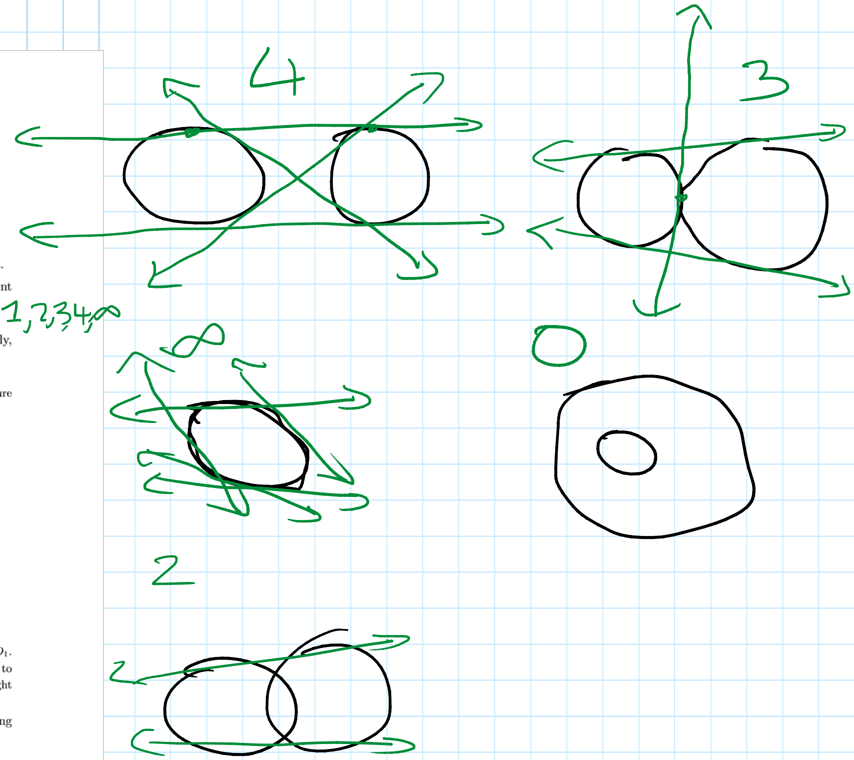
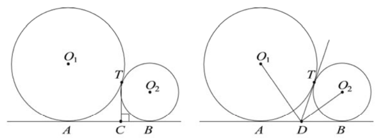
1. **(Common Tangents)** What are *common tangent lines* to two circles? How many common tangent lines can two circles have and why? Explain briefly and supply a picture in each situation. (*Hint:* Consider cases according to how the two circles are positioned with respect to each other.)

0, 1, 2, 3, 4, ∞



2. **(Two Tangent Circles)** Two circles with centers O_1 and O_2 and of radii 7 cm and 5 cm, respectively, are externally tangent at point T.
   (a) How long is their common external tangent segment AB? (*Hint:* Connect all tangency points with the centers of the circles on which they lie. What type of a figure do you see? Apply PT to a carefully constructed right $\triangle O_1O_2X$, where $O_2X \| AB$.)
   (b) Prove that $\angle ATB$ is right. (*Hint:* $O_1A \| O_2B$, and $\triangle O_1AT$ and $\triangle O_2BT$ are isosceles.)
   (c) Find the distance from T to the midpoint of the tangent segment AB.

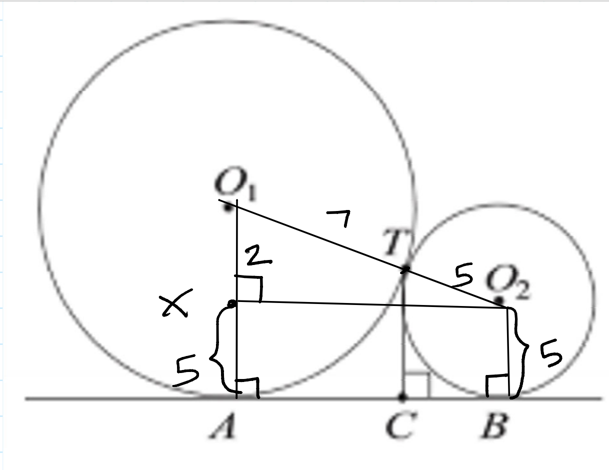


Problems 2a,b,d Problem 2c,e,f

   (d) What is the distance TC from T to their common external tangent segment AB? (*Hint:* As before, connect all relevant points. You know all sides of the resulting right trapezoid ABO_2O_1. By drawing $O_2X \| AB$, create the same right $\triangle O_1O_2X$ as before. But now, with the altitude TC to AB, you have actually created a second, smaller right triangle. What can you say about these two right triangles? How does this help find TC?)
   (e) Let the internal tangent of the two circles intersect the external tangent AB in point D. How long is TD? (*Hint:* Compare TD with some segments on the picture! $AB = ?$)
   (f) Prove that $\angle O_1DO_2 = 90°$. (*Hint:* Connect some points. Two quadrilaterals form a right trapezoid.)

Extra Background and Practice: *Circle Inscribed in a Triangle:* L138-139, W138-139

4. **(Basics)** W138: #1, 2, 3, 4, W139: #1, 2, 3, 4.



Problems 2a,b,d

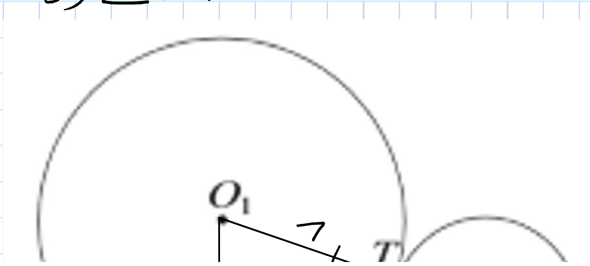
a) ABO_2X is a rectangle so $\overline{AB}=\overline{O_2X}$.

$\overline{O_1X}=\overline{O_1A}-\overline{XA}=7-5=2$

$\overline{O_1O_2}=\overline{O_1T}+\overline{TO_2}=7+5=12$

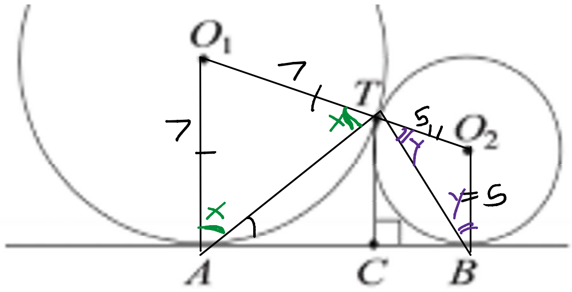
$\overline{AB}=\overline{O_2X}=\sqrt{12^2-2^2}=\sqrt{140}=2\sqrt{35}$

b) $\angle ATB = ?$



$\triangle O_1AT$ and $\triangle O_2TB$ are both isoceles.

$\angle ATB = 180 - x - y = 90°$



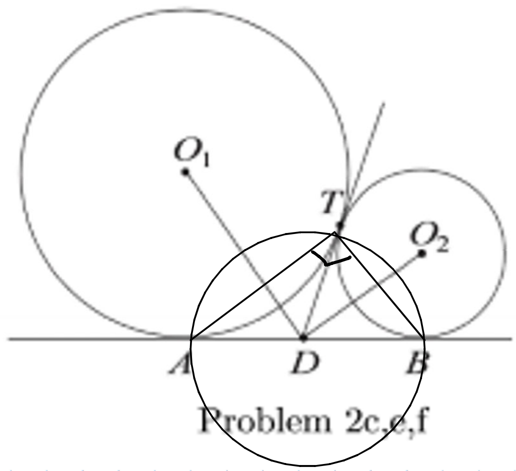
Problems 2a,b,d

$\angle ATB = 180 - x - y = 90^\circ$

$\angle TAB = 90 - x$

$+\angle TBA = 90 - y$

$180^\circ = 360 - 2x - 2y \overset{\div 2}{\Rightarrow} 90 = 180 - x - y$

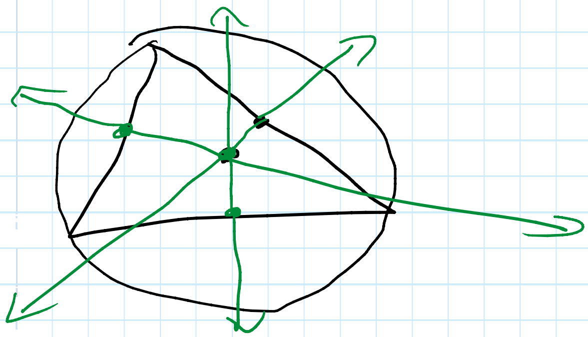


Problem 2c,e,f

c) D is the midpoint of AB. TD = ?

circumcenter = center of circumscribed circle

= intersection of perpendicular bisector

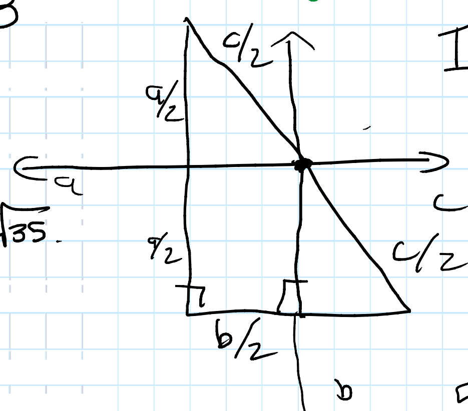


D is midpoint of AB,

→ D is the circumcenter of △TAB

DT, DA, DB are radii of the circumcircle

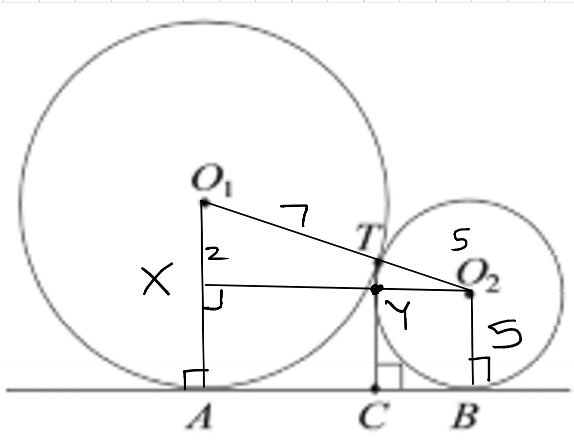
$\overline{DT} = \overline{DA} = \overline{DB} = \frac{1}{2}(2\sqrt{35}) = \sqrt{35}$.



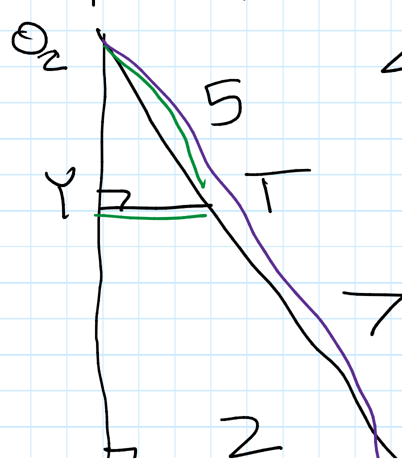
Thm The circumcenter of a right triangle is the midpoint of the hypotenuse.

$\overline{TC} = \overline{TY} + \underset{5}{\overline{YC}} = 5\frac{5}{6}$

(d)



Problems 2a,b,d



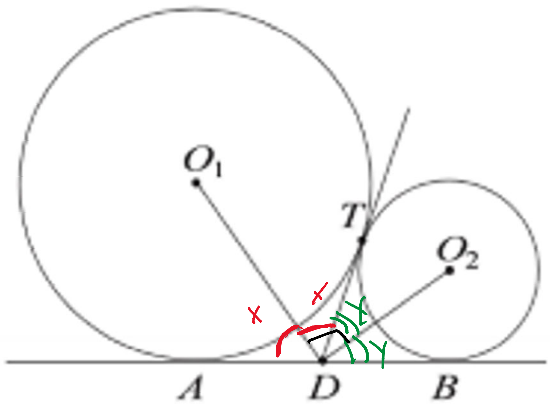
$\triangle O_2YT \sim \triangle O_2XO_1$

$\frac{O_2T}{O_2O_1} = \frac{TY}{O_1X}$

$\frac{5}{12} = \frac{TY}{2}$

$TY = 5/6$

Problems 2a,b,d



Problem 2c,e,f

$TY = 5/6$

(e) D is so that DT is a common tangent.

DT and DB are common tangents to O_2

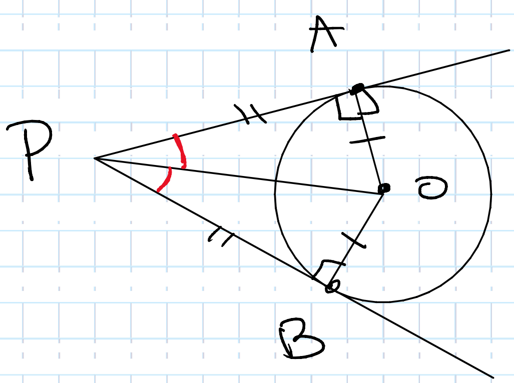
$\Rightarrow DT = DB$

DT and DA are common tangents to O_1

$DT = DA$

$DA = DT = DB \Rightarrow D$ is the midpoint of AB.

$DT = \sqrt{35}$ (from before)



Thm $\triangle PAO \cong \triangle PBO$ by HL $\rightarrow PA = PB$

$\angle OPB = \angle OPA$

(f) $\angle O_1DO_2 = x + y = 90^\circ$

$2x + 2y = 180 \rightarrow x + y = 90$