Discussion 1 10/22/20

Tuesday, October 20, 2020 3:05 PM

Complex Numbers (Discussion)

Worksheet 3: Roots of Unity[1]

Date: 10/22/2020

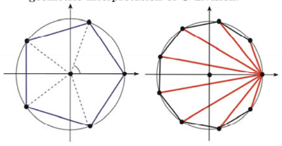
MATH 74: Transition to Upper-Division Mathematics
with Professor Zvezdelina Stankova, UC Berkeley

Read: *Session 8: Complex Numbers. Part II* (vol. II)

- §6. Roots in $\mathbb{C}$ (pp. 196-197)
- §7. Roots of Unity and Regular Polygons (p. 198)

Write: clearly. Supply your reasoning in words and/or symbols. Show calculations and relevant pictures.

1. **(Operating in Polar)** Given three $\mathbb{C}$-numbers: $z_1 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$, $z_2 = -\frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4}i$, and $z_3 = -2i$:
   (a) Calculate *algebraically* the three pairwise products, z_1z_2, z_2z_3, and z_3z_1. Plot them and the original numbers z_1, z_2, and z_3.
   (b) Explain *geometrically* the products, using our geometric interpretation of $\mathbb{C}$-multiplication.
   (c) Explain *geometrically* what the complex ratios $\frac{z_2}{z_1}$, $\frac{z_3}{z_2}$, and $\frac{z_1}{z_3}$ should be using an analogous geometric interpretation of $\mathbb{C}$-division.



2. **(Root Formula)** A non-zero complex number $z = |z|(\cos\theta + i\sin\theta)$ has exactly n complex n^{th}-roots $w_0, w_1, \ldots, w_{n-1}$, given by the formula
$$w_k = \sqrt[n]{|z|}\left(\cos\frac{\theta+2\pi k}{n} + i\sin\frac{\theta+2\pi k}{n}\right),$$
where $k = 0, 1, \ldots, n-1$.

3. **(Roots of i)**
   (a) Use polar coordinates to show that i has exactly two square roots $\sqrt{i}$. First reason geometrically, and then use the Root Formula.
   (b) Repeat for the cube roots $\sqrt[3]{i}$, showing that i has exactly three cube roots.

4. **(Taking Roots)** Consider the equations $w^6 = z$ for $z = 1, -64, i$, and $64i$. Find all $\mathbb{C}$-solutions w for these four equations, and draw pictures.

5. **(Geometric Series Review)** Reprove by MI or otherwise that for any $z \in \mathbb{C}$, $z \neq 1$, and $n \in \mathbb{N}$:
$$1 + z + z^2 + z^3 + \cdots + z^{n-1} = \frac{z^n - 1}{z - 1}.$$

6. **(Primitive Roots of Unity)** Factor the polynomials into *linear* factors with complex roots:
(a) $z^3 - 1$; (b) $z^5 - 1$; (c) $z^n - 1$ for any $n \geq 1$;
(d) $z^4 + z^3 + z^2 + z + 1$; (e) $z^8 + z^7 + \cdots + z + 1$;
(f) $z^{n-1} + z^{n-2} + \cdots + z + 1$ for any $n \geq 2$.

HW

7. **(Vieta's on Root Sums & Products)**
   (a) For the roots $x_1, x_2, \ldots, x_n$ of any polynomial $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ ($a_n \neq 0$), prove two of Vieta's formulas:
   - $x_1 + x_2 + \cdots + x_n = -a_{n-1}/a_n$;
   - $x_1x_2\cdots x_n = (-1)^n a_0/a_n$.

   only 6cf

   (b) For each of the polynomials in #6, find the sum and the product of its roots. (*Warning*: Be careful whether 1 is among the roots or not.)

8. **(Sequence Re-Cycling)** Let $a_{n+1} = a_n - a_{n-1}$ for $n \geq 2$, $a_1 = 1$ and $a_2 = 2$. We proved before by MI the pattern $\{1, 2, 1, -1, -2, -1\}$ for $\{a_n\}$ and the direct formula $a_n = -2\cos\frac{(n+1)\pi}{3}$ for $n \geq 1$.
   (a) To explain the mysterious last formula, find the roots $r_{1,2}$ of $x^2 = x - 1$, their sum and product. Set $a_n = Ar_1^n + Br_2^n$, a linear combination of the powers of these roots, substitute $n = 0, 1$, and solve the resulting linear system for $A = -r_1$ and $B = -r_2$. Finally, use de Moivre's formula to rewrite r_1^{n+1} and r_2^{n+1} and simplify the direct formula for a_n.
   (b) Repeat for the sequence $a_{n+1} = \sqrt{2}a_n - a_{n-1}$ for $n \geq 2$, $a_1 = 1$, and $a_2 = \sqrt{2}$.

1

$z^3 = 1 \to z = 1$

$w_k = \sqrt[n]{|z|}\left(\cos\frac{\theta+2\pi k}{n} + i\sin\frac{\theta+2\pi k}{n}\right)$

$w_k^n = |z|\left(\cos\frac{\theta+2\pi k}{n} + i\sin\frac{\theta+2\pi k}{n}\right)^n$

DeMoivre's

$= |z|\left(\cos(\theta+2\pi k) + i\sin(\theta+2\pi k)\right)$

$= |z|(\cos\theta + i\sin\theta) = z.$

$z^3 = 1 = 1(\cos 0^\circ + i\sin 0^\circ)$

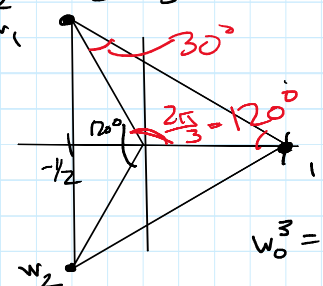
$w_k = \sqrt[3]{1}\left(\cos\frac{2\pi k}{3} + i\sin\frac{2\pi k}{3}\right).$

$w_0 = 1\cdot(\cos 0 + i\sin 0) = 1$

≈ 0.86

$w_1 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = \omega$

$w_2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2} = \omega^2.$



$w_0^3 = w_1^3 = w_2^3 = 1$

6a) $z^3 - 1 = (z - w_0)(z - w_1)(z - w_2)$

$= (z-1)(z-\omega)(z-\omega^2)$, $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ (prim. root of unity)

$x^2 - 5x + 6 = (x-2)(x-3)$

roots: $x = 2, 3$

Factor $z^2 + z + 1 = \frac{z^3 - 1}{z - 1} = \frac{\cancel{(z-1)}(z-\omega)(z-\omega^2)}{\cancel{z-1}} = (z-\omega)(z-\omega^2).$

6b) $z^5 - 1 = (z - w_0)(z - w_1) \cdots (z - w_4)$

Roots $z = w_0, w_1, \ldots, w_4$

$z^5 - 1 = (z-1)(z-w_1)(z-w_2)(z-w_3)(z-w_4)$

$w_k = \sqrt[5]{1\cdot(\cos 0 + i\sin 0)}$

$= 1\cdot\left(\cos\frac{0+2\pi k}{5} + i\sin\frac{0+2\pi k}{5}\right)$

$w_0 = 1 = w_1^5$

$w_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} = \zeta_5 = w_1^6$

6d) $z^4 + z^3 + z^2 + z + 1 = \frac{z^5 - 1}{z - 1} = (z - w_1)(z - w_1^2)(z - w_1^3)\cdot(z - w_1^4)$

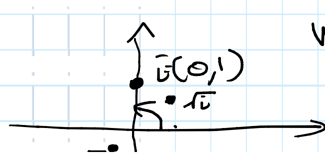
$w_2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5} = w_1^2$

$\vdots$

$w_4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5} = w_1^4$

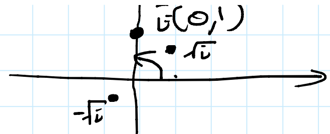
$\sqrt{i}$ $\quad i = |i|(\cos\theta + i\sin\theta)$

$= 1\cdot\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$



$-\sqrt{i}$ $\quad i = |i|(\cos\theta + i\sin\theta)$

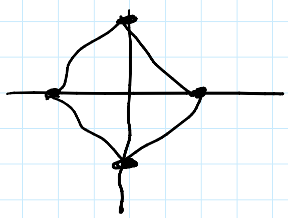
$= 1\cdot\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

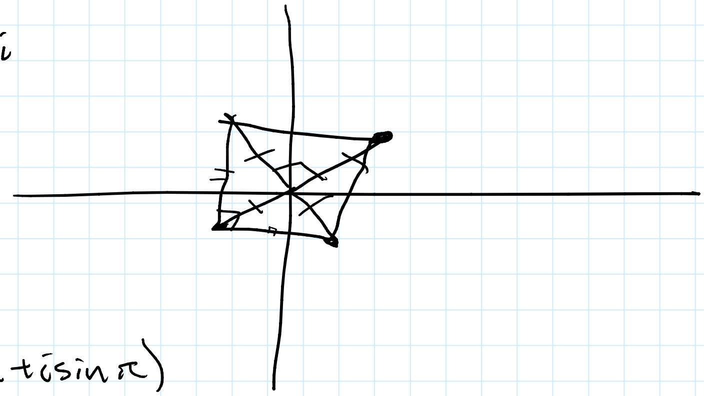


$\sqrt{i}$ $\quad w_0 = \sqrt{1}\left(\cos\frac{\frac{\pi}{2}+2\pi\cdot 0}{2} + i\sin\frac{\frac{\pi}{2}+2\pi\cdot 0}{2}\right) = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \neq w_1^2$

$w_1 = \sqrt{1}\left(\cos\frac{\frac{\pi}{2}+2\pi\cdot 1}{2} + i\sin\frac{\frac{\pi}{2}+2\pi\cdot 1}{2}\right) = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$.

$\sqrt[4]{1} = 1, i, -1, -i$





$-1 = 1\cdot(\cos\pi + i\sin\pi)$

$\sqrt{-1} = \cos\left(\frac{\pi}{2}\right) + i\sin\frac{\pi}{2} = i, -i$

$z^8 = -1$ has no sol'n in $\mathbb{R}$, but 8 sol'n in $\mathbb{C}$.

7a) $a_n x^n + a_{n-1}x^{n-1} + \dots + a_0 = a_n(x-x_1)(x-x_2)(x-x_3)\dots(x-x_n)$

$= a_n\left(x^n + x^{n-1}(-x_1 - x_2 - \dots - x_n) + \dots + (-1)^n x_1x_2\dots x_n\right)$

$= a_n x^n + a_n x^{n-1}(-x_1 - x_2 - \dots - x_n) + \dots + (-1)^n a_n x_1 x_2 \dots x_n$

Equating: $\quad a_{n-1} = a_n(-x_1 - x_2 - \dots - x_n) \Rightarrow x_1 + x_2 + \dots + x_n = -a_{n-1}/a_n$

$a_0 = (-1)^n a_n x_1 x_2 \dots x_n \Rightarrow x_1 x_2 \dots x_n = (-1)^n a_0/a_n$.

$w_0 \dots w_4$ be roots of $z^5 = 1$

What is $\quad w_1 + w_2 + w_3 + w_4 = ? = -a_3/a_4 = -1/1 = -1$.

$z^4 + z^3 + z^2 + z + 1 = (z - w_1)(z - w_2)(z - w_3)(z - w_4)$

$w_1 \cdot w_2 w_3 \cdot w_4 = (-1)^4 \cdot a_0/a_4 = 1\cdot 1/1 = 1$