Discussion 1 10/20/20

Tuesday, October 20, 2020 3:05 PM

Complex Numbers (Discussion)

Worksheet 1: Basic Operations on Complex Numbers[1]

Date: 10/20/2020

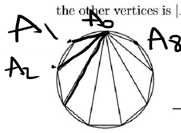
MATH 74: Transition to Upper-Division Mathematics
with Professor Zvezdelina Stankova, UC Berkeley

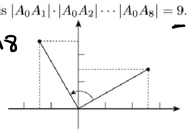
Read: *Session 9: Complex Numbers. Part I* (vol. I, pp. 179-180, 183-189, 191)

- §1. A Problem from Geometry
- §4. Basic Operations on Complex Numbers
- §3. Complex Numbers via Geometry
- §5. Complex Multiplication

Write: clearly. Supply your reasoning in words and/or symbols. Show calculations and relevant pictures.

1. **(Challenge)** Let $A_0A_1 \dots A_8$ be a regular 9-gon (nonagon) inscribed in a unit circle. Prove that the product of the distances from one vertex to each of the other vertices is $|A_0A_1| \cdot |A_0A_2| \cdots |A_0A_8| = 9$.





2. **(Imaginary Vectors)** Plot the complex numbers $z_1 = (-3, 3)$ and $z_2 = (-4, -2)$ as points in the $\mathbb{C}$-plane. Perform the following operations on them geometrically in pictures, and then calculate the same thing algebraically. What geometric shapes did you use in your geometric solutions?
(a) $z_1 + z_2$; (b) $z_1 - z_2$; (c) $0.3z_1$, $-0.5z_2$;
(d) $|0.3z_1|$, $|0.5z_2|$; (e) $|0.3z_1 \pm (-0.5)z_2|$;
(f) $\mathbb{R}\mathrm{e}(z_1) + \mathbb{I}\mathrm{m}(z_2)i$; (h) $\mathbb{I}\mathrm{m}(z_2) - \mathbb{R}\mathrm{e}(z_1)i$.

3. **(Geo-Modulus)** Consider the equations:
(a) $|z| = 2$; (b) $|w - 1| = 2$; (c) $|t + i| = 2$.
Both geometrically and algebraically:
   - Describe all *complex* solutions.
     (*Hint:* In (b), substitute first $z = w - 1$, solve the new equation, and translate your answer.)
   - Find all *real* solutions.

4. **(Geo-Conjugation)** Find for which $z, w \in \mathbb{C}$ and $a, b \in \mathbb{R}$ the following identities are true:
(a) $\bar{z} = z$; (b) $|\bar{z}| = |z|$; (c) $\bar{z} = -z$; (d) $\overline{az} = a\bar{z}$;
(e) $\overline{z + w} = \bar{z} + \bar{w}$; (f) $\overline{az + bw} = a\bar{z} + b\bar{w}$.
Explain both geometrically and algebraically.

5. **(Geo-Inequalities)** Geometrically describe the *complex* solutions of the inequalities:
(a) $|z| \le 2$; (b) $|w - 1| \ge 2$;
(c) $|t + i| > 2$; (d) $|u + 1 + i| < 2$.

6. **(Complex Multiplication)** In the $\mathbb{C}$-plane, plot the complex numbers $z = 2 - 3i$ and $w = 4 + i$. We are interested in the numbers: zw, $|z\bar{w}|$, $\bar{z} \cdot \bar{w}$, $|z\bar{z}|$, $z + \bar{z}$, $w\bar{w}$, $w - \bar{w}$; iz, iw, $|iz|$, and $|-3iw|$.
(a) Perform the indicated operations *algebraically*.
(b) Depict the answers *geometrically* and try to explain them *geometrically* (in words).
(c) Prove that for any $z, w \in \mathbb{C}$:
   - $|iz| = |z|$; • $|zw| = |z| \cdot |w|$; • $\overline{zw} = \bar{z} \cdot \bar{w}$.

7. **(Imaginary Patterns)** What are i^{2019}, i^{2020}, and i^{2021}? Explain in words and devise a general formula for all i^n where $n \in \mathbb{N}$. (*Hint*: "Cycling remainders"? If you can, give a *geometric* explanation.)

8. **(Order of Operations)** Calculate $|w|$ and $\bar{w}$ two ways for the number $w = (1 + i)(1 + 2i)(1 + 5i)$. (*Hint*: Brute-force find the Cartesian form $w = x + yi$ with $x, y \in \mathbb{R}$, or use that both modulus and conjugation preserve complex multiplication.)

9. **(Moduli Palooza)** Find the modulus of

$$\prod_{k=0}^{2020} \left(\frac{1+k+k^2+i}{1+k^2} \right) = \frac{1+i}{1} \cdot \frac{3+i}{2} \cdot \frac{7+i}{5} \cdots \frac{4082421+i}{4080401}.$$

(*Hint:* $|z_0z_1 \cdots z_n| = |z_0||z_1| \cdots |z_n|$. Why?)

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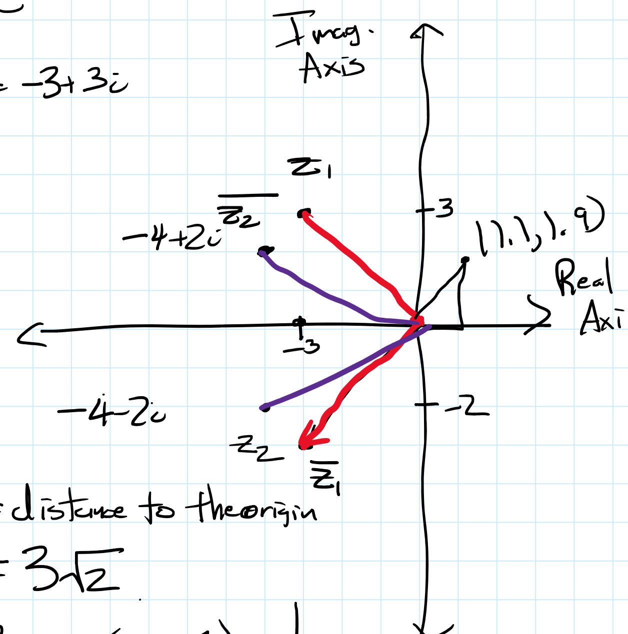
$|a+bi| = \sqrt{a^2+b^2}$

$Re(a+bi) = a$

$Im(a+bi) = b$

$z_1 = -3+3i$



$|z_1|$ = distance to the origin

$= 3\sqrt{2}$

$\underbrace{|0.3z_1 + (-0.5)z_2|}$

$0.3(-3+3i) + (-0.5)(-4-2i)$

$= -0.9 + 0.9i + (2 + 1i)$

$= 1.1 + 1.9i \quad (1.1, 1.9)$

$|1.1 + 1.9i| = \sqrt{1.1^2 + 1.9^2} \approx 2.2$

h. $Im(z_2) - Re(z_1)i$

$= -2 - (-3)i = -2 + 3i$

$\bar{z}$ = complex conjugation

$\overline{a+bi} = a - bi$

a) $\overline{a+bi} = a+bi$? $\to b = 0 \to z \in \mathbb{R}$

b) $|\overline{a+bi}| = |a+bi| \to$ always

c) $\overline{a+bi} = -(a+bi) \to a = 0 \to z \in \mathbb{R}i \; (Re(z) = 0)$

f) $\overline{a(r+si) + b(t+ui)} = a\overline{(r+si)} + b\overline{(t+ui)} \to$ always

$\overline{az+bw} = a\bar{z} + b\bar{w}$

$(z = -\bar{z})(-1)$

$\bar{z} = -z$

$z = 2-3i, \; w = 4+i$

$i = \sqrt{-1} \Rightarrow i^2 = -1$

$iz, \; w\bar{w}, \; zw$

$iz = i(2-3i) = 2i - 3i^2 = 2i - 3(-1) = 3 + 2i$

$iw = i(4+i) = 4i + i^2 = -1 + 4i$

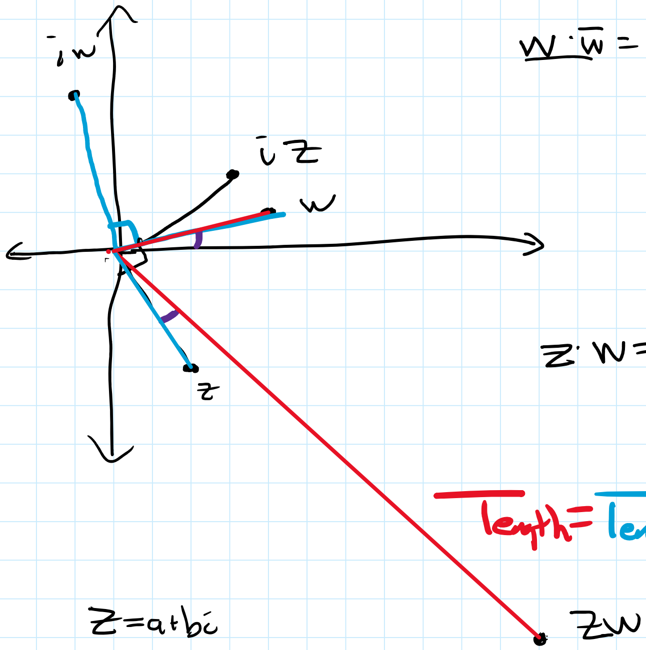
iz, $w\overline{w}$, zw
iw

$iz = i(2-3i) = 2i - 3i^2 = 2i - 3(-1) = 3 + 2i$
$iw = i(4+i) = 4i + i^2 = -1 + 4i$

$\underline{w \cdot \overline{w}} = (4+i)(4-i) = 16 \cancel{-4i + 4i} - i^2$
$= 16 - (-1) = \underline{17}.$

$\underline{|w|} = \sqrt{4^2 + 1^2} = \sqrt{17}$
$|w| = \sqrt{w \cdot \overline{w}}$

$z \cdot w = (2-3i)(4+i) = 8 + 2i - 12i - 3i^2$
$= 8 - 10i - 3(-1)$
$= 11 - 10i$



$\overline{\text{length}} = \overline{\text{length}} \cdot \overline{\text{length}}$

$z = a + bi$

$|iz| = |i(a+bi)| = |ai + bi^2| = |-b + ai| = \sqrt{(-b)^2 + a^2} = \sqrt{a^2 + b^2} = |a + bi| = |z|.$

$\underline{|zw|} = \text{algebra} = \underline{|z|} \cdot \underline{|w|}$

$\underline{\overline{zw} = \overline{z} \cdot \overline{w}}$

$|zw| = \sqrt{(zw)(\overline{zw})} = \sqrt{z \cdot w \cdot \overline{z} \cdot \overline{w}} = \sqrt{z \cdot \overline{z}} \cdot \sqrt{w \cdot \overline{w}} = |z| \cdot |w|.$