MATH 128A Numerical Analysis Discussion Section

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Announcement

- Quiz #1
  - Cover HW #1 (1.1, 1.2, 1.3, 2.1)

Brief Review

- Fixed point method
  - Iterative method
  - Fixed point theorem
- Newton's method
  - Requires $f'(x)$
  - The most effective method
- Secant method
  - Approximate $f'(p_{n-1}) \approx \frac{f(p_{n-1})-f(p_{n-2})}{p_{n-1}-p_{n-2}}$
  - Slower than Newton's method

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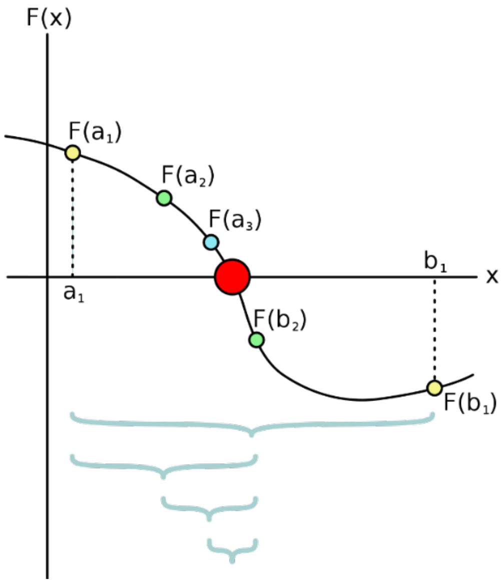
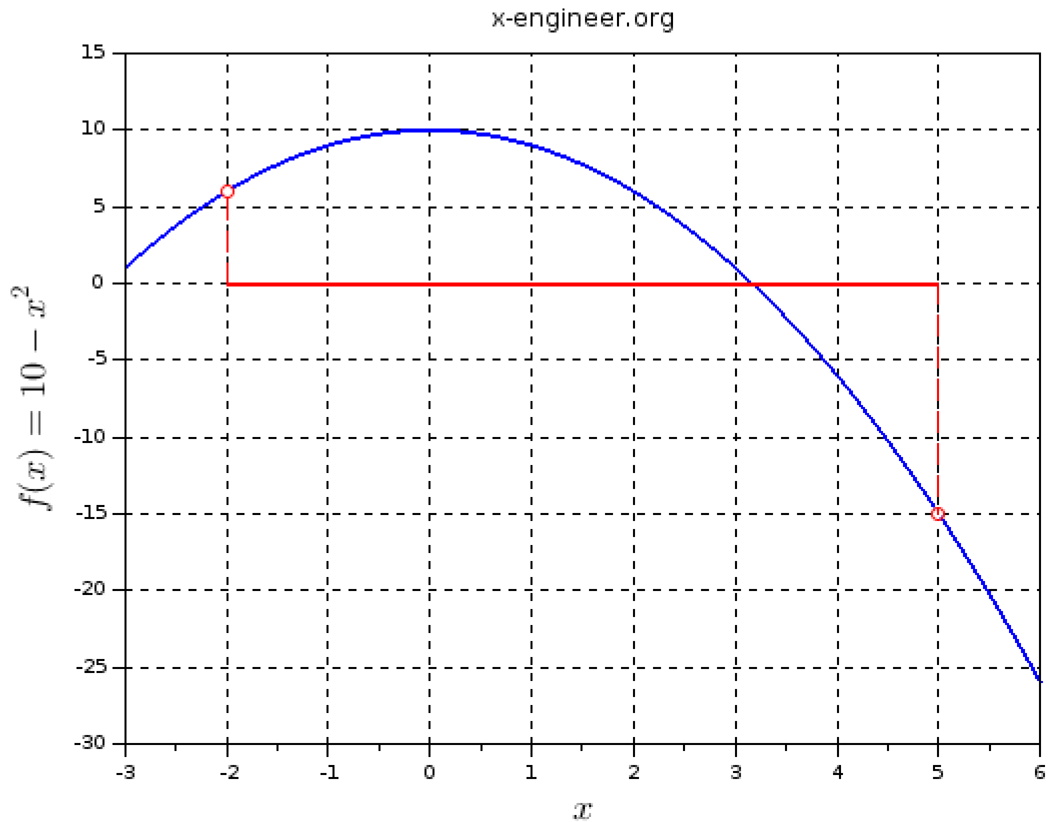
Brief Review

- Bisection method
  - Based on the Intermediate value theorem
  - half -> half -> …
  - Very slow, but reliable(or stable).



Brief Review

- Bisection method



Fixed point method

- Find the root of

$$f(x) = x^2 - x - 1$$

- How?
  - $x = x^2 - 1$
  - $x = 1 + \frac{1}{x}$
  - $x = \sqrt{x + 1}$

Fixed point theorem

(Fixed-Point Theorem)

Let $g \in C[a, b]$ be such that $g(x) \in [a, b]$, for all x in $[a, b]$. Suppose, in addition, that g' exists on (a, b) and that a constant $0 < k < 1$ exists with

$$|g'(x)| \leq k, \quad \text{for all } x \in (a, b).$$

Then for any number p_0 in $[a, b]$, the sequence defined by

$$p_n = g(p_{n-1}), \quad n \geq 1,$$

converges to the unique fixed point p in $[a, b]$. ■

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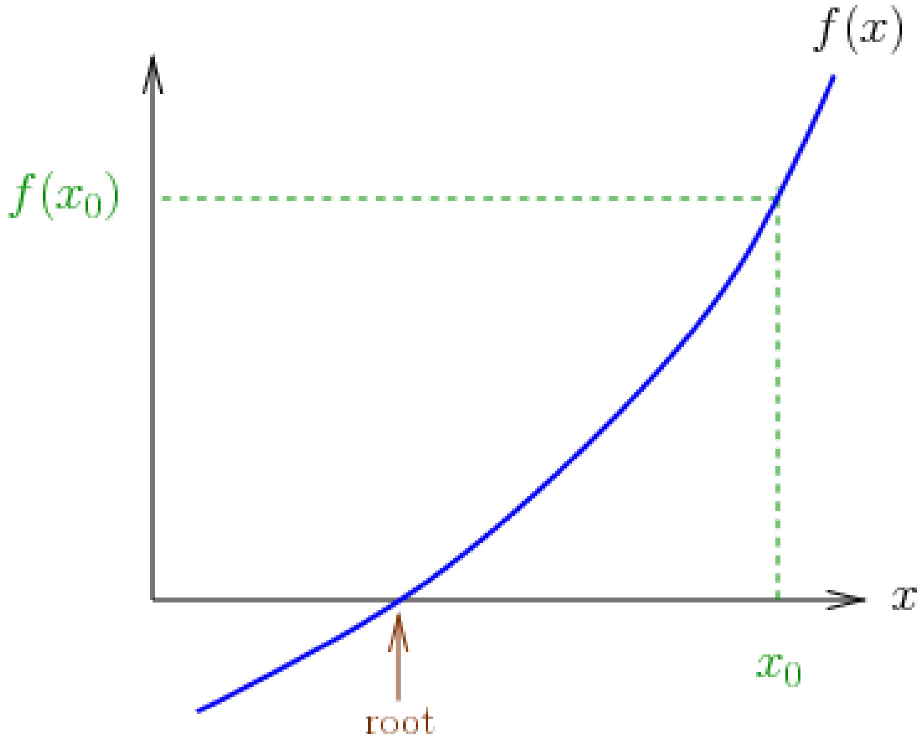
Newton's method



- f(x) is continuous and differentiable near the root
- $x_{n+1} = x_n - \frac{f(x_n)}{f\prime(x_n)}$
- Converging very fast.

Newton's method - cautions

- f(x) is continuous and differentiable
- Stationary points
- Converging to a different root
- Diverging
  - $f(x) = x^3 - 5x$

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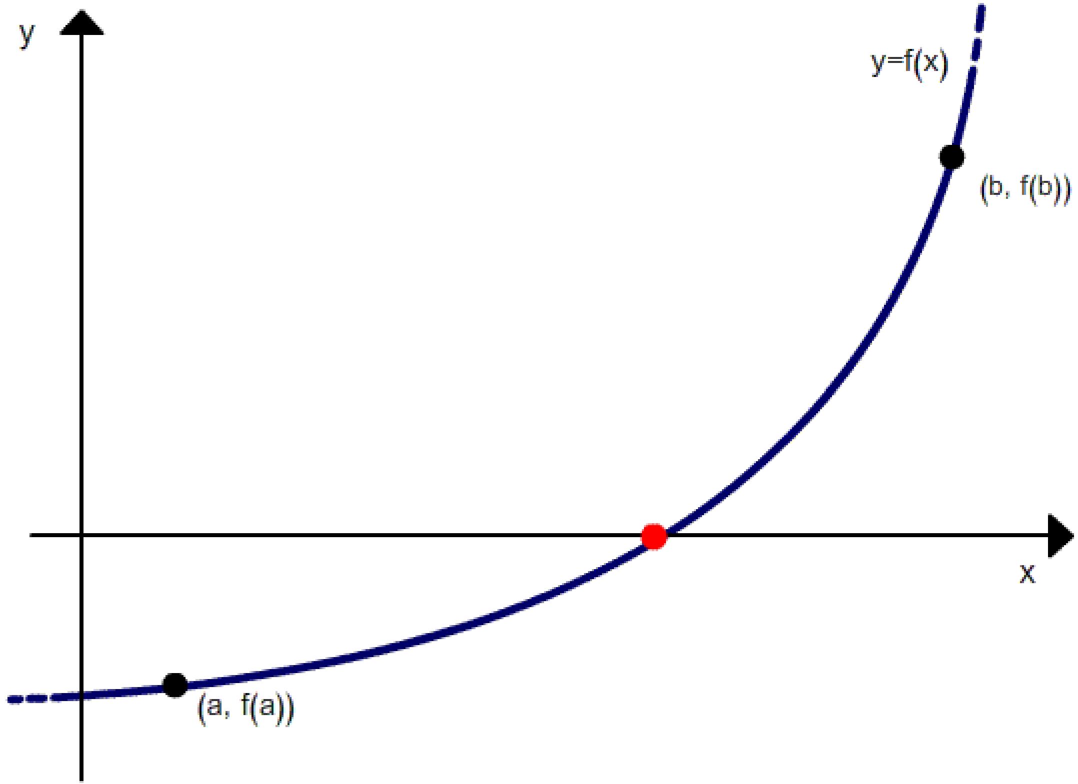
Secant method



- Start with two distinct points
- $x_n = x_{n-1} - \frac{(x_{n-1} - x_{n-2})}{(f(x_{n-1}) - f(x_{n-2}))} f(x_n)$
- Don’t need to calculate derivatives

Order of convergence

- x : Exact root x_n : Numerical root (nth step)
- $e_n := x - x_n$ (error at nth step)
- $\lim\limits_{n \to \infty} \frac{|e_{n+1}|}{|e_n|^{\alpha}} = \gamma$

- $\alpha = 1,\ \gamma < 1$: linear convergence
- $\alpha = 2$: quadratic convergence (γ depends on e_0)

Order of convergence

- Bisection method : Linear, $\gamma = 1/2$

- Newton's method : Quadratic (merely linear if m>1)

- Secant method : $\alpha = \frac{1+\sqrt{5}}{2}$

- Modified Newton's method : Quadratic

Quiz