MATH 128A Numerical Analysis Discussion Section

Raehyun Kim*

* Department of Mathematics, UC Berkeley

Outline

- Runge-Kutta Method
  - Midpoint Euler, RK4
- Multi-step Method
  - Adams-Bashforth / Adams-Moulton
- Predictor-Corrector Method

Runge-Kutta Method

- Runge-Kutta method
  - Similar to the high-order Taylor method but it uses the function values instead of the value of derivatives
  - Approximate the value of derivatives using the Taylor polynomial in two variables
  - (OR) Divide 1 time step into multiple stages

Runge-Kutta Method

- Example
  - Midpoint method

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i)\right), \quad \text{for } i = 0, 1, \ldots, N-1.$$

  - Modified Euler

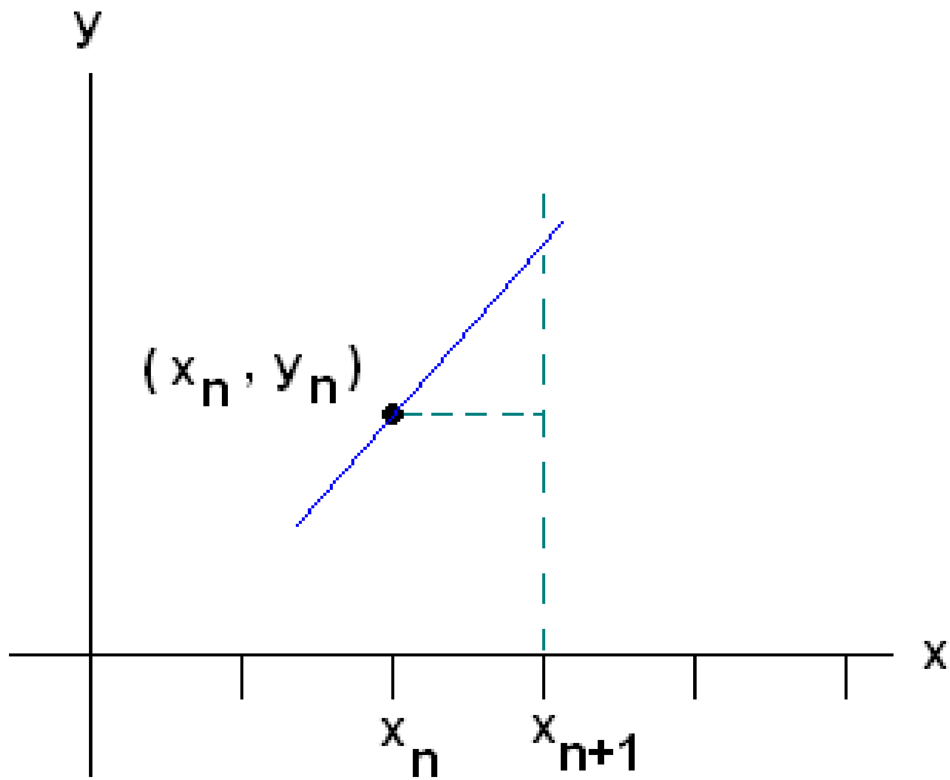
$$w_0 = \alpha,$$

$$w_{i+1} = w_i + \frac{h}{2}[f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))], \quad \text{for} \quad i = 0, 1, \ldots, N-1.$$

Runge-Kutta Method

- Example
  - Mi

$w_($

w_{i+} 

$. . , N - 1.$

  - Mo

w

w_{i+} $1, \ldots, N - 1.$

Runge-Kutta Method

- Example
  - RK-4th order

$$w_0 = \alpha,$$

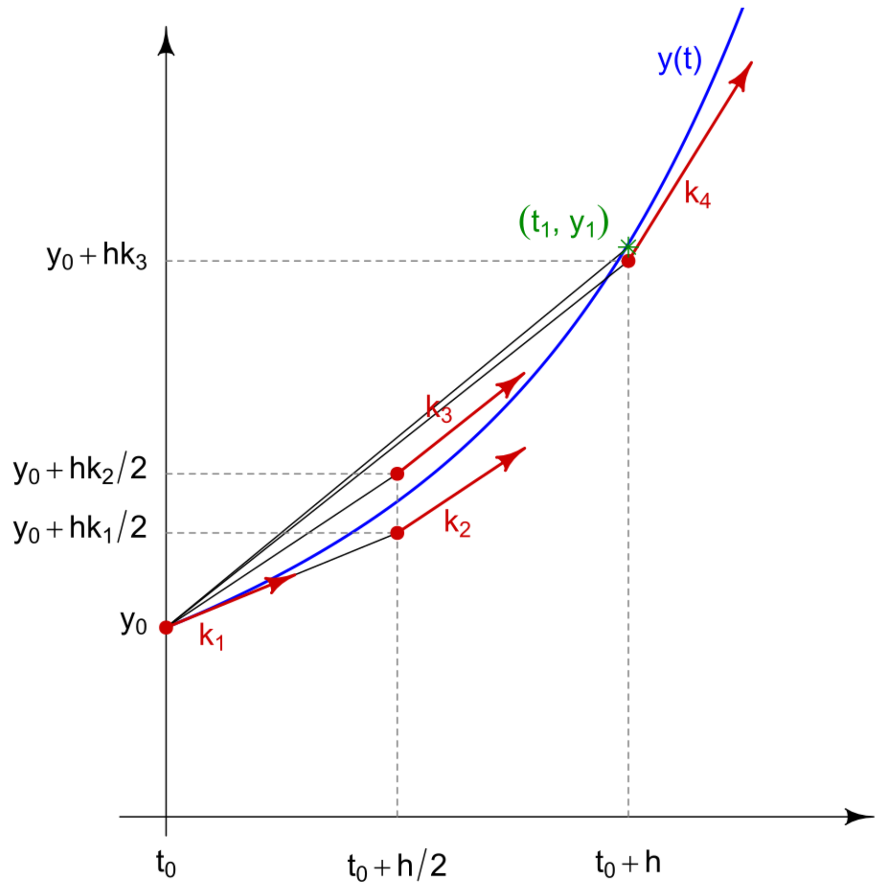
$$k_1 = hf(t_i, w_i),$$

$$k_2 = hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_1\right)$$

$$k_3 = hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_2\right)$$

$$k_4 = hf(t_{i+1}, w_i + k_3),$$

$$w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k$$



Runge-Kutta Method

a. $y' = e^{t-y}, \quad 0 \le t \le 1, \quad y(0) = 1,$ with $h = 0.5$; actual solution $y(t) = \ln(e^t + e - 1)$.

Multi-step method

- Numerical scheme using multiple $f\big(t_k, y(t_k)\big)$
  - (Forward) Euler

$$y_{k+1} = y_k + hf(t_k, y_k)$$

  - Trapezoidal

$$y_{k+1} = y_k + \frac{h}{2}\big(f(t_k, y_k) + f(t_{k+1}, y_{k+1})\big)$$

  - Adams-Bashforth
  - Adams-Moulton

Multi-step method

- Basic idea of the construction
  - Approximate

$$\int_{t_n}^{t_{n+1}} f\big(t, y(t)\big)dt$$

  - Using the interpolation -> LMM
    - Excluding the right endpoint -> explicit (open)
    - Including the right endpoint -> implicit (close)
  - Using the quadrature rule -> RK mthd / GL mthd

Multi-step method

- Adams-Bashforth

$$y_{n+1} = y_n + hf(t_n, y_n), \qquad \text{(This is the Euler method)}$$
$$y_{n+2} = y_{n+1} + h\left(\frac{3}{2}f(t_{n+1}, y_{n+1}) - \frac{1}{2}f(t_n, y_n)\right),$$
$$y_{n+3} = y_{n+2} + h\left(\frac{23}{12}f(t_{n+2}, y_{n+2}) - \frac{16}{12}f(t_{n+1}, y_{n+1}) + \frac{5}{12}f(t_n, y_n)\right),$$
$$y_{n+4} = y_{n+3} + h\left(\frac{55}{24}f(t_{n+3}, y_{n+3}) - \frac{59}{24}f(t_{n+2}, y_{n+2}) + \frac{37}{24}f(t_{n+1}, y_{n+1}) - \frac{9}{24}f(t_n, y_n)\right),$$
$$y_{n+5} = y_{n+4} + h\left(\frac{1901}{720}f(t_{n+4}, y_{n+4}) - \frac{2774}{720}f(t_{n+3}, y_{n+3}) + \frac{2616}{720}f(t_{n+2}, y_{n+2}) - \frac{1274}{720}f(t_{n+1}, y_{n+1}) + \frac{251}{720}f(t_n, y_n)\right).$$

(Wikipedia)

Multi-step method

- Adams-Moulton

$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1})$, This is the backward Euler method

$y_{n+1} = y_n + \frac{1}{2}h\left(f(t_{n+1}, y_{n+1}) + f(t_n, y_n)\right)$, This is the trapezoidal rule

$$y_{n+2} = y_{n+1} + h\left(\frac{5}{12}f(t_{n+2}, y_{n+2}) + \frac{2}{3}f(t_{n+1}, y_{n+1}) - \frac{1}{12}f(t_n, y_n)\right),$$

$$y_{n+3} = y_{n+2} + h\left(\frac{9}{24}f(t_{n+3}, y_{n+3}) + \frac{19}{24}f(t_{n+2}, y_{n+2}) - \frac{5}{24}f(t_{n+1}, y_{n+1}) + \frac{1}{24}f(t_n, y_n)\right),$$

$$y_{n+4} = y_{n+3} + h\left(\frac{251}{720}f(t_{n+4}, y_{n+4}) + \frac{646}{720}f(t_{n+3}, y_{n+3}) - \frac{264}{720}f(t_{n+2}, y_{n+2}) + \frac{106}{720}f(t_{n+1}, y_{n+1}) - \frac{19}{720}f(t_n, y_n)\right).$$

(Wikipedia)

Multi-step method

4. Use each of the Adams-Bashforth methods to approximate the solutions to the following initial-value problems. In each case, use starting values obtained from the Runge-Kutta method of order four. Compare the results to the actual values.

a. $y' = \dfrac{2 - 2ty}{t^2 + 1}$, $0 \le t \le 1$, $y(0) = 1$, with $h = 0.1$ actual solution $y(t) = \dfrac{2t + 1}{t^2 + 1}$.

Runge-Kutta-Fehlberg method

- Runge-Kutta-Fehlberg method
  - Combine two RK methods to achieve better accuracy
  - From its design, it becomes more efficient than arbitrary combination of RK methods.

Runge-Kutta-Fehlberg method

- Runge-Kutta-Fehlberg method

4. Use the Runge-Kutta-Fehlberg method with tolerance $TOL = 10^{-6}$, $hmax = 0.5$, and $hmin = 0.05$ to approximate the solutions to the following initial-value problems. Compare the results to the actual values.

a. $y' = \frac{2-2ty}{t^2+1}$, $0 \le t \le 3$, $y(0) = 1$; actual solution $y(t) = (2t+1)/(t^2+1)$.

Variable Step-Size Multistep Methods

2. Use the Adams Variable Step-Size Predictor-Corrector Algorithm with $TOL = 10^{-4}$ to approximate the solutions to the following initial-value problems:

a. $y' = (y/t)^2 + y/t, \quad 1 \le t \le 1.2, \quad y(1) = 1$, with $hmax = 0.05$ and $hmin = 0.01$.