MATH 128A Numerical Analysis Discussion Section

Raehyun Kim*

* Department of Mathematics, UC Berkeley

Outline

- Numerical Integration
  - Trapezoidal/Simpson's rule
  - Degree of accuracy or precision
- Composite Numerical Integration
  - Composite Trapezoidal/Simpson's rule
  - Round-Off Error Stability
- Romberg Integration
  - Extrapolation
- Adaptive Simpson
- Gaussian Quadrature

Numerical Integration

- Trapezoidal rule
  - Using linear Lagrange polynomial $P(x)$ to approximate the given function $f(x)$
  - $P(x)$ interpolates $f(x_0), f(x_1)$

$$\int_a^b f(x)\,dx = \frac{h}{2}[f(x_0) + f(x_1)] - \frac{h^3}{12} f''(\xi).$$

Numerical Integration

- Simpson's rule
  - Using quadratic Lagrange polynomial $P(x)$ to approximate the given function $f(x)$
  - $P(x)$ interpolates $f(x_0), f(x_1), f(x_2)$ where $x_1 = \frac{x_0+x_2}{2}$

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] - \frac{h^5}{90} f^{(4)}(\xi).$$

Composite Numerical Integration

- Composite Trapezoidal rule
  - Using Trapezoidal rule on the meshed domain.

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] - \frac{b-a}{12}h^2 f''(\mu).$$

Composite Numerical Integration

- Composite Simpson's rule
  - Using Simpson's rule on the meshed domain.

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(a) + 2\sum_{j=1}^{(n/2)-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(b)\right] - \frac{b-a}{180}h^4 f^{(4)}(\mu).$$

Romberg Integration

- Extrapolation
  - Using extrapolation to achieve higher accuracy
  - Luckily, the remainder series of the Trapezoidal/Simpson's rule are even power series.
    -> accuracy increases twice faster than normal case

$$
\begin{aligned}
\mathbf{R}_{1,1} &= \frac{h_0}{2}\left(f(a)+f(b)\right) = \frac{b-a}{2}\left(f(a)+f(b)\right) \quad \left(\stackrel{def}{=} \mathcal{N}_1(h_0)\right), \\
\mathbf{R}_{2,1} &= \frac{h_1}{2}\left(f(a)+f(b)+2f(a+h_1)\right) \\
&= \frac{1}{2}\left(\mathbf{R}_{1,1} + h_0 f(a+h_1)\right), \quad \left(\stackrel{def}{=} \mathcal{N}_1\left(\frac{h_0}{2}\right)\right) \\
\vdots \quad & \quad \vdots \\
\mathbf{R}_{k,1} &= \frac{1}{2}\left(\mathbf{R}_{k-1,1} + h_{k-2}\sum_{j=1}^{2^{k-2}} f(a+(2j-1)h_{k-1})\right) \quad \left(\stackrel{def}{=} \mathcal{N}_1\left(\frac{h_0}{2^{k-1}}\right)\right)
\end{aligned}
$$

Adaptive Quadrature Method

- Recursive Simpson's rule
  - Using the formula

$$\left|\int_a^b f(x)\,dx - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right)\right| \approx \frac{1}{16}\left(\frac{h^5}{90}\right) f^{(4)}(\xi)$$

$$\approx \frac{1}{15}\left|S(a,b) - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right)\right|$$

  - If the integral is less than the given tol, then stop. If not, then apply above formula to each half-interval

Gaussian Quadrature

- Legendre polynomial
  - Legendre polynomial $P_n(x)$ is a monic polynomial s.t. $\int_{-1}^{1} P_n(x)p(x) = 0$ for $\forall p(x) \in P_{n-1}[x]$

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = x^2 - \frac{1}{3},$$

$$P_3(x) = x^3 - \frac{3}{5}x, \quad \text{and} \quad P_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}.$$

Gaussian Quadrature

- Gaussian Quadrature

This permits Gaussian quadrature to be applied to any interval $[a, b]$, because

$$\int_a^b f(x)\,dx = \int_{-1}^1 f\left(\frac{(b-a)t+(b+a)}{2}\right)\frac{(b-a)}{2}\,dt. \tag{4.41}$$