MATH 128A Numerical Analysis Discussion Section

Raehyun Kim*

* Department of Mathematics, UC Berkeley

Outline

- Midterm review

Announcement

- Midterm : Next Wed. / Regular class
  - Allow cheat sheet on one side, on size A paper. No limit on font size.
  - Practice exam will be uploaded

Sample midterm – Fixed point

6. Consider the iteration

$$x_{k+1} = \alpha x_k + \beta, \quad k = 0, 1, \cdots,$$

with $|\alpha| < 1$.

(a) Assume that the iteration converges, what is the limit?

(b) What is the order of convergence of this iteration?

(a) xk -> x. Substitute and check the condition of x. you can use simple induction

(b) Use the definition of order of convergence.

Sample midterm – Fixed point

3. Consider the iteration

$$x_{k+1} = x_k/2 + 1/x_k, \quad k = 0, 1, \cdots.$$

Assume that $x_0 > 0$ and that the iteration converges.

(a) What number does the iteration converge to?

(b) What is the order of convergence?

(a) xk -> x. Substitute and check the condition of x. you can use simple induction

(b) Use the definition of order of convergence.

Sample midterm – Fixed point

1. Consider the iteration

$$x_{k+1} = (x_k + 1/x_k)/2, \quad k = 0, 1, \cdots.$$

Assume that $x_0 > 0$ and that the iteration converges.

(a) What number does the iteration converge to?

(b) What is the order of convergence?

(a) xk -> x. Substitute and check the condition of x. you can use simple induction

(b) Use the definition of order of convergence.

Sample midterm – Fixed point

1. (a) Show that the cubic equation $2x^3 - 6x + 1 = 0$ has a real root in the interval $[0 \quad 1/2]$. Perform one step of Bisection method with this interval.

   (b) Reformulate the above equation as

$$x = \frac{2x^3 + 1}{6}.$$

   Define the fixed point iteration (FPI) based on this equation, and show that FPI convergences for any initial guess in $[0 \quad 1/2]$.

(b) Use Fixed point thm

(Fixed-Point Theorem)

Let $g \in C[a, b]$ be such that $g(x) \in [a, b]$, for all x in $[a, b]$. Suppose, in addition, that g' exists on (a, b) and that a constant $0 < k < 1$ exists with

$$|g'(x)| \leq k, \quad \text{for all } x \in (a, b).$$

Then for any number p_0 in $[a, b]$, the sequence defined by

$$p_n = g(p_{n-1}), \quad n \geq 1,$$

converges to the unique fixed point p in $[a, b]$. ■

Sample midterm – Root finding

5. Let $f(x) = x^2 - 2$.

   (a) Given $p_0 = 0$, use Newton's method to find p_1.

   (b) Show that $f(x)$ has a root in $[0, 2]$.

   (c) Given the initial interval $[0, 2]$, perform one step of bisection.

(a) Newton's method

(b) Try intermediate value theorem(IVT). Use extreme value theorem(EVT) if IVT doesn't work

(c) Bisection method

Sample midterm – Root finding

2. Show that the function $f(x) = x - 2^{-x}$ has a root in $[0, 1]$, and perform one step of Bisection towards finding the root.

(a) Try intermediate value theorem(IVT)

(b) Use extreme value theorem(EVT) if IVT doesn't work

(c) Bisection method

Sample midterm – Interpolation

4. Find a polynomial $P(x)$ of degree at most 2 such that

$$P(0) = 0, \quad P(-1) = P(1) = 1.$$

(a) Several ways. Lagrange polynomial, Newton's divided difference, Derive equations and solve it.

Sample midterm – Interpolation

2. Let $x_0 < x_1 < x_2$. Find a second degree polynomial $P(x)$ such that

$$P(x_0) = f_0, \quad P(x_1) = f_1, \quad \text{and} \quad P'(x_2) = f_2'.$$

(a) Several ways. Lagrange polynomial, Newton's divided difference, Derive equations and solve it.

Sample midterm – Interpolation

4. Let $x_0 < x_1 < x_2$. Show that there is a unique polynomial $P(x)$ of degree at most 3 such that

$$P(x_j) = f(x_j), \quad j = 0, 1, 2, \quad \text{and} \quad P'(x_1) = f'(x_1).$$

Give an explicit formula for $P(x)$.

(a) Several ways. Lagrange polynomial, Newton's divided difference, Derive equations and solve it.

Sample midterm – Interpolation

2. A natural cubic spline S on $[0,2]$ is defined by

$$S(x) = \begin{cases} S_0(x) = 1 + 2x - x^3, & \text{if} \quad 0 \le x \le 1, \\ S_1(x) = 2 + b(x-1) + c(x-1)^2 + d(x-1)^3, & \text{if} \quad 1 \le x \le 2. \end{cases}$$

Find b, c and d.

(a) Use the required conditions of the cubic spline interpolation.