MATH 128A Numerical Analysis Discussion Section

Raehyun Kim*

* Department of Mathematics, UC Berkeley

Announcement

- Quiz #1
  - Published on the gradescope MATH 128A 102/103
  - Regrade request is available until Sep. 20

6. Find $\max_{a \le x \le b} |f(x)|$ for the following functions and intervals.

a. $f(x) = 2x/(x^2+1)$, $[0, 2]$

b. $f(x) = x^2\sqrt{(4-x)}$, $[0, 4]$

c. $f(x) = x^3 - 4x + 2$, $[1, 2]$

d. $f(x) = x\sqrt{(3-x^2)}$, $[0, 1]$

Use the Bisection method to find solutions, accurate to within 10^{-5} for the following problems.

a. $3x - e^x = 0$ for $1 \le x \le 2$

b. $2x + 3\cos x - e^x = 0$ for $0 \le x \le 1$

c. $x^2 - 4x + 4 - \ln x = 0$ for $1 \le x \le 2$ and $2 \le x \le 4$

d. $x + 1 - 2\sin \pi x = 0$ for $0 \le x \le 0.5$ and $0.5 \le x \le 1$

Brief Review

- Order of Convergence
  - Modified Newton's method
  - Another measurement
- Accelerating Convergence
  - Aitken's Δ^2 Method
  - Steffensen's Method
- Lagrange Interpolation Polynomial
  - $p_{\mathrm{n},i}(x_j) = \delta_{i,j}$

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Order of convergence

- p : Exact root p_n : Numerical root (nth step)
- $e_n := p - p_n$ (error at nth step)
- $\lim_{n \to \infty} \frac{|e_{n+1}|}{|e_n|^{\alpha}} = \gamma$
- $\alpha = 1,\ \gamma < 1$: linear convergence
- $\alpha = 2$: quadratic convergence (γ depends on e_0)

Order of convergence

- Fixed point iteration (Thm 2.8/2.9)
  - Converge linearly if
    - FPI($g(x)$) converges to fixed point p
    - $g(x)$ is continuously differentiable with $0 < |g'(p)| < 1$
  - Converge at least quadratically if
    - FPI($g(x)$) converges to fixed point p
    - $g'(x)$ is continuously differentiable with $\mathrm{g}'(\mathrm{p}) = 0$

Order of convergence

- Bisection method : Linear, $\gamma = 1/2$

- Newton's method : Quadratic (if the root is simple. O.w. merely linear)

- Secant method : $\alpha = \frac{1+\sqrt{5}}{2}$

- Modified Newton's method : Quadratic

Newton's method

- f(x) is continuous and differentiable

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- Converging very fast.

- Q. What if $f'(p) = 0$?
  A. Can not guarantee fast convergence.
  Indeed, it merely linearly converges.

Modified Newton's method

- f(x) is continuous and differentiable
- multiplicity m of root x
  - Simply $f(a) = f'(a) = \cdots = f^{(m-1)}(a) = 0$
- $x_{n+1} = x_n - \frac{\boldsymbol{m} f(x_n)}{f'(x_n)}$
- Quadratic convergent

Modified Newton's method

- f(x) is continuous and differentiable
- multiplicity m of root x
  - Simply $f(a) = f'(a) = \cdots = f^{(m-1)}(a) = 0$
- $x_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{f'(x_n)^2 - f(x_n)\boldsymbol{f''(x_n)}}$
- Quadratically convergent but impractical

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Aitken's Δ^2 Method

$$\widehat{p_n} = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}$$

- Accelerate the convergence of a sequence that is linearly convergent.
- How to use?

  Calculate p_n until the error is $\sqrt{Tol}$ and apply Aitken's Δ^2 Method. Profit!

Steffensen's Method

$$p_0^{(i)} = \{\Delta^2\}\left(p_0^{(i-1)}\right)$$
$$p_1^{(i)} = g\left(p_0^{(i)}\right)$$
$$p_2^{(i)} = g\left(p_1^{(i)}\right)$$
$$p_0^{(i+1)} = \{\Delta^2\}\left(\boldsymbol{p_0^{(i)}}\right)$$

- Accelerate the order of convergence of the given fixed point method using Aitken's method

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Lagrange polynomial

- Polynomials s.t. $p_{\mathrm{n},i}(x_j) = \delta_{i,j}$ for given $\{x_j\}_{j=0,\ldots,n}$

$$p_{\mathrm{n},i}(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

- Interpolation result

$$Q(x) = \sum f(x_i) p_{n,i}(x)$$