Lectures on Random Matrices

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August 16, 2012

1 Introduction

Disordered systems in quantum mechanics are modeled by Schrödinger operators with random potentials. As a classical example, consider the Anderson model describing electron propagation in a disordered environment. The associated Schrödinger operator is of the form $H = -\Delta + \lambda V$, where the potential V is random and the parameter λ represents the strength of disorder. The energy levels of the electron are given by the eigenvalues of the operator H and due to the randomness of the potential, we are mostly interested in their statistical properties. According to the universality conjecture for random Schrödinger operators, there are two distinctive regimes depending on the disorder strength λ. In the strong disorder regime, the eigenfunctions are localized and the local spectral statistics is Poisson. In the weak disorder regime, the eigenfunctions are delocalized and a repulsive potential governs the interaction between eigenvalues. In the lattice approximation of the Schrödinger operator $-\Delta + V$ is replaced with a large symmetric sparse matrix with random diagonal entries.

Wigner proposed to study the statistics of eigenvalues of large random matrices as a model for the energy levels of heavy nuclei. For a Wigner ensemble we take a large hermitian (or symmetric) $N \times N$ matrix $[h_{ij}]$ where $\{h_{ij} : i \leq j\}$ are independent identically distributed random variables of mean zero and variance N^{-1}. The central question for Wigner ensemble is the universality conjecture which asserts that the local statistics of the eigenvalues are independent of the distributions of the entries as N gets large. This local statistics can be calculated when the entry distribution is chosen to be Gaussian. The density of eigenvalues in large N limit is given by the celebrated Wigner semicircle law in the interval $[-2, 2]$. Joint distribution of eigenvalues away from the edges ± 2 has a determinantal structure and is obtained from a sine kernel. The sine kernel is replaced with the Airy kernel near the edges ± 2 after a rescaling of the eigenvalues. The largest eigenvalue obeys a different universality law and is governed by the Tracy-Widom distribution.

It is a remarkable fact that many of the universality laws discovered in the theory of random matrices appear in a variety of different models in statistical mechanics. A prominent example is the planar random growth models which belong to Kardar-Parisi-Zhang universality class. In these models, a stable phase grows into an unstable phase through aggregation. The rough boundary separating different phases is expected to obey a central limit theorem and its universal law is conjectured to be the Tracy-Widom distribution. This has been rigorously established for two models; simple exclusion process and Hammersley process. Another surprising example is the Riemann ζ-function. It is conjectured that after appropriate rescaling, the zeros of the ζ-function, $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$, lying on the vertical line $Re\ s = 1/2$, have the same local statistics as the eigenvalues of a Wigner ensemble .

2 Wigner Ensemble and Semicircle Law

We start with the description of our main model. Consider a $N \times N$ matrix $H = H_N = [h_{ij}]$ which is either symmetric $h_{ij} = h_{ji} \in \mathbb{R}$ or Hermitian $h_{ij} = \bar{h}_{ji} \in \mathbb{C}$. The matrix H is called a *Wigner matrix (ensemble)* if $\{h_{ij} : i < j\}$ and $\{h_{ii} : i\}$ are two sets of independent identically distributed random variables. We always assume that H is centered; $\mathbb{E}h_{ij} = 0$ for all i and j. As we discussed in the introduction, we are primarily interested in the behavior of H_N as $N \to \infty$. Let us write $\lambda_1^N \leq \lambda_2^N \leq \dots \lambda_N^N$ for the eigenvalues of the matrix H_N. (When there is no danger of confusion, we simply write λ_i for λ_i^N.) Observe

$$\mathbb{E}N^{-1} \sum_i \lambda_i^2 = N^{-1} Tr\ H^2 = \mathbb{E}N^{-1} \sum_{i,j} |h_{ij}|^2 = \mathbb{E}\left[(N-1)|h_{12}|^2 + |h_{11}|^2\right].$$

To have the left-hand side of order one, we assume that

$$N\mathbb{E}h_{ij}^2 = 1 \text{ for } i \neq j, \quad N\mathbb{E}h_{ii}^2 = 2, \tag{2.1}$$

in the case of symmetric H and we assume

$$N\mathbb{E}h_{ij}^2 = 1, \tag{2.2}$$

for all i and j in the case of Hermitian H. Note that $2\mathbb{E}h_{12}^2 = \mathbb{E}h_{11}^2$ in the symmetric case. This is of no significance and is assumed to simplify some explicit formulas we derive later when all h_{ij}s are Gaussian random variables. Under the assumption (2.1), we expect $\lambda_i^N = O(1)$ and hope that the empirical measure

$$\rho_N = \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i^N},$$

to be convergent as $N \to \infty$. We have the following celebrated theorem of Wigner.

Theorem 2.1 *For a Wigner matrix,*

$$\lim_{N \to \infty} \rho_N(dx) = \rho(dx) = \frac{1}{2\pi} \sqrt{(4 - x^2)^+}\ dx, \tag{2.3}$$

in probability, where the convergence occurs in weak sense.

We need to develop some tools before we get to the proof of Theorem 2.1.

As our first step, we try to establish the regularity of the eigenvalues as the the matrix varies. For this we use the standard norm $\|A\| = \sqrt{trA^2} = (\sum_{i,j} |a_{ij}|^2)^{1/2}$ that comes from the inner product $< A, B >= tr(AB) = \sum_{i,j} a_{ij}\bar{b}_{ij}$. Let us write $\lambda_1(A) \leq \cdots \leq \lambda_i(A) \leq \cdots \leq \lambda_N(A)$ for the eigenvalues of a symmetric or Hermitian $N \times N$ matrix A. The following inequality of *Hoffman* and *Wielandt* shows the Lipschitzness of $\lambda(A) = (\lambda_1(A), \dots, \lambda_N(A))$.

Lemma 2.1 *For every symmetric A and B,*

$$\sum_i |\lambda_i(A) - \lambda_i(B)|^2 \le \|A - B\|^2 = tr(A - B)^2.$$

Proof. Note that since $\sum_i \lambda_i(A)^2 = trA^2$, it suffice to show

$$trAB \le \sum_i \lambda_i(A)\lambda_i(B).$$

Write D_A for the diagonal matrix which has the eigenvalues $\lambda_1(A), \dots, \lambda_N(A)$ on its main diagonal. Without loss of generality, we assume that $A = D_A$. We then find an orthogonal matrix $U = [u_{ij}]$ that diagonalize B. We have

$$\begin{aligned} trAB = trD_A U^T D_B U &= \sum_{i,j} \lambda_i(A)\lambda_j(B)u_{ij}^2 \\ &\le \sup \left\{ \sum_{i,j} \lambda_i(A)\lambda_j(B)w_{ij} : W = [w_{ij}] \text{ is a doubly stochastic matrix} \right\}. \end{aligned}$$

It remains to show that the supremum is attained at the identity matrix. To see this, write $\bar{W}$ for a maximizer. Inductively we show that we can switch to a maximizer $\hat{W}$ such that $\hat{w}_{ii} = 1$ for $i = 1, \dots, k$. We only verify this for $k = 1$ because the general case can be done in the same way. Indeed if $\bar{w}_{11} < 1$, then we can find i and j such that $\bar{w}_{1j}, \bar{w}_{i1}$ are nonzero. Set $r = \min\{\bar{w}_{1j}, \bar{w}_{i1}\}$ and we switch from $\bar{W}$ to $\hat{W}$ by changing only the entries at positions $11, 1j, i1$, and ij by $\hat{w}_{11} = \bar{w}_{11} + r$, $\hat{w}_{ij} = \bar{w}_{ij} + r$, $\hat{w}_{1j} = \bar{w}_{1j} - r$, and $\hat{w}_{i1} = \bar{w}_{i1} - r$. We claim that $\hat{W}$ is also a maximizer because

$$\sum_{i,j} \lambda_i(A)\lambda_j(B)(\hat{w}_{ij} - \bar{w}_{ij}) \ge r(\lambda_1(A) - \lambda_i(A))(\lambda_1(B) - \lambda_j(B)) \ge 0.$$

If $\hat{w}_{11} = 1$, then we are done. Even if $\hat{w}_{11} = 1$ fails, the matrix $\hat{W}$ is better than $\bar{W}$ in the sense that $\bar{W}$ has one more 0 entry on either the first row or column. Repeating the same procedure to $\hat{W}$, either we get 1 on the position 11 or we produce one more 0 on the first row or column. It is clear that after we apply the above procedure at most $2(N-1)$ times, we obtain 1 for the position 11. This completes the proof. □

To motivate our second tool, let us mention that a standard trick for analyzing a symmetric/Hermitian operator H is by studying its resolvent $(H - z)^{-1}$. The trace of resolvent is of particular interest because of its simple relation with the eigenvalues. Indeed

$$S_N(z) := N^{-1}Tr(H_N - z)^{-1} = N^{-1}\sum_{i=1}^{N}(\lambda_i - z)^{-1} = \int \frac{\rho_N(dx)}{x - z}, \tag{2.4}$$

and this is well-defined for $z \in \mathbb{C} - \mathbb{R}$. We then recognize that the right-hand side is the *Stieltjes transform* of the empirical measure ρ_N. This suggests an analytical way of studying the sequence ρ_N, namely we study the asymptotic behavior of S_N as N gets large. For any bounded measure μ, define

$$S(\mu, z) := \int \frac{\mu(dx)}{x - z}, \tag{2.5}$$

for $z \in \mathbb{C} - \mathbb{R}$. Note that $S(\mu, z)$ is analytic in z and is almost the Cauchy integral (the factor $(2\pi i)^{-1}$ is missing) associated with the measure μ defined on $\mathbb{R}$. Here are some basic facts about Stieltjes transform.

Lemma 2.2

- *(i)* $|S(\mu, z)| \leq \mu(\mathbb{R})/|Im\ z|$.
- *(ii) If* $\sup_n \mu_n(\mathbb{R}) < \infty$ *and* $\lim_{n\to\infty} \mu_n = \mu$ *vaguely, then* $\lim_{n\to\infty} S(\mu_n, z) = S(\mu, z)$ *for every* $z \in \mathbb{C} - \mathbb{R}$.
- *(iii) We have*

$$\lim_{\varepsilon\to 0} \frac{1}{\pi} Im\ S(\mu, \alpha + i\varepsilon)\ d\alpha = \mu(d\alpha), \tag{2.6}$$

  weakly.
- *(ii) If* $S(\mu, z) = S(\nu, z)$ *for all* $z \in \mathbb{C} - \mathbb{R}$, *then* $\mu = \nu$.
- *(v) If* $\lim_{n\to\infty} S(\mu_n, z) = S(z)$ *exists for every* $z \in \mathbb{C} - \mathbb{R}$, *then* $S(z) = S(\mu, z)$ *for some measure* μ *and* $\lim_{n\to\infty} \mu_n = \mu$ *vaguely.*

Proof. The proofs of (i) and (ii) are obvious and (iii) implies (iv). As for (iii), we certainly have

$$\frac{1}{\pi} Im\ S(\mu, \alpha + i\varepsilon)\ d\alpha = (\mu * C_\varepsilon)(d\alpha), \tag{2.7}$$

where C_ε is the Cauchy density

$$C_\varepsilon(\alpha) = \varepsilon^{-1} C_1(\alpha/\varepsilon) = \frac{1}{\pi} \frac{\varepsilon}{\alpha^2 + \varepsilon^2}.$$

Now it is clear that for any bounded continuous f,

$$\lim_{\varepsilon\to 0} \int f d(\mu * C_\varepsilon) = \lim_{\varepsilon\to 0} \int (f * C_\varepsilon) d\mu = \int f d\mu,$$

by Bounded Convergence Theorem and because C_ε is an approximation to identity.

We now turn to (iv). Let μ be any vague limit of μ_n. By part (ii), we have that $S(z) = S(\mu, z)$. Hence all limit points coincide and $\mu_n \to \mu$ vaguely. $\square$

Our goal is proving Theorem 2.1 and for this we try to calculate the large N limit of $S_N(z)$. This would be particularly simple when the random variables h_{ij}'s are Gaussian. In this case the matrix H is called a *Gaussian Wigner ensemble* (GWE). In the case of GWE, we first want to reduce the convergence of $\{S_N(z)\}$ to that of $\{\mathbb{E}S_N(z)\}$. For this we appeal to a suitable concentration inequality. First observe that we may represents a symmetric Wigner matrix H as a vector $H = (h_{ij} : i \leq j) \in \mathbb{R}^M$, with $M = N(N+1)/2$. We then assert that $S_N(z) = F(H)$ is a Lipschitz function for which the Lipschitz constant can be bounded with the aid of Lemma 2.1:

$$\begin{aligned}
|F(H) - F(H')| &= \frac{1}{N}\sum_i \left[(\lambda_i(H) - z)^{-1} - (\lambda_i(H') - z)^{-1}\right] \\
&\leq \frac{|Im\ z|^{-2}}{N}\sum_i |\lambda_i(H) - \lambda_i(H')| \\
&\leq |Im\ z|^{-2}\left(\frac{1}{N}\sum_i (\lambda_i(H) - \lambda_i(H'))^2\right)^{\frac{1}{2}} \\
&\leq \frac{|Im\ z|^{-2}}{\sqrt{N}}\|H - H'\| \leq \frac{2|Im\ z|^{-2}}{\sqrt{N}}\left(\sum_{i\leq j}(h_{ij} - h'_{ij})^2\right)^{1/2}.
\end{aligned}$$

Hence, if we regard $S_N(z)$ as a function $F : \mathbb{R}^M \to \mathbb{R}$, then for its Lipschitz constant $Lip(F)$, we have

$$Lip(F) \leq 2|Im\ z|^{-2}/\sqrt{N}. \tag{2.8}$$

We now would like to bound

$$|F(H) - \mathbb{E}F(H)|, \tag{2.9}$$

for a centered Gaussian $H = (h_{ij} : i \leq j) \in \mathbb{R}^M$, where each coordinate h_{ij} has a variance of order $O(N^{-1})$. For this we use *Logarithmic Sobolev Inequality(LSI).* We say that a probability measure μ satisfies LSI(a), if for every probability density function f,

$$\int f \log f\ d\mu \leq a \int |\nabla \sqrt{f}|^2 d\mu. \tag{2.10}$$

By Herbst Lemma, LSI implies a sub-Gaussian tails estimate and this in turn implies a concentration inequality.

Lemma 2.3 *(Herbst) Let μ be a probability measure on $\mathbb{R}^M$ which satisfies LSI(a). Then for any Lipschitz function F with $\int F\,d\mu = 0$,*

$$\int e^{tF} d\mu \le \exp\left(\frac{1}{4} a t^2 Lip(F)^2\right). \tag{2.11}$$

Proof. First assume that F is continuously differentiable. Choose $f = e^{tF}/Z(t)$ with $Z(t) = \int e^{tF} d\mu$ in (2.10) to assert

$$t\frac{Z'}{Z} - \log Z \le \frac{a}{4Z}\int t^2 |\nabla F|^2 e^{tF} d\mu \le \frac{1}{4} a t^2 (Lip\ F)^2.$$

Hence

$$\frac{d}{dt}\frac{\log Z(t)}{t} \le \frac{a}{4}(Lip\ F)^2.$$

From this and $\lim_{t\to 0}(\log Z(t))/t = \int F d\mu = 0$, we deduce that $\log Z(t) \le at^2(Lip\ F)^2/4$. This is exactly (2.11) when $F \in C^1$. Extension to arbitrary Lipschitz functions is done by approximations. □

Remark 2.1 We may apply Chebyshev Inequality to assert that if μ satisfies LSI(a) and F is any Lipschitz function, then

$$\mu\left\{F - \int F\,d\mu \ge r\right\} \le \exp\left(-\frac{r^2}{aLip(F)^2}\right). \tag{2.12}$$

From this and an analogous inequality for $-F$, we deduce

$$\mu\left\{\left|F - \int F\,d\mu\right| \ge r\right\} \le 2\exp\left(-\frac{r^2}{aLip(F)^2}\right). \tag{2.13}$$

□

On account of Lemma 2.3, we wish to have a *LSI* for the Gaussian measures. The following exercise would prepare us for such an inequality.

Exercise 2.1.

- (i) Show that $\max_a(ab - e^a) = b\log b - b$.
- (ii) Show that for any probability density f,

$$\int f\log f d\mu = \sup_g\left(\int fg d\mu - \log\int e^g d\mu\right) = \sup\left\{\int fg d\mu : \int e^g d\mu \le 1\right\}.$$

- (iii) Show that the function $(a,b) \mapsto (\sqrt{a} - \sqrt{b})^2$ and the functional $f \mapsto \int |\nabla\sqrt{f}|^2 d\mu$ are convex.

□

The following two classical lemmas give simple recipe for establishing LSI for many important examples.

Lemma 2.4 *If μ_i satisfies $LSI(a_i)$ for $i = 1, \dots, k$, then the product measure $\mu = \mu_1 \times \dots \times \mu_k$ satisfies $LSI(a)$ for $a = \max_i a_i$.*

Proof. Take any non-negative C^1 function $f(x_1, \dots, x_k)$ with $\int f d\mu = 1$ and set

$$f_i(x_i, x_{i+1}, \dots, x_k) = \int f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_k) \mu_1(dx_1) \dots \mu_{i-1}(dx_{i-1}),$$
$$\mu^i(dx_i, \dots, x_k) = \mu_i(dx_i) \times \dots \times \mu_k(dx_k), \quad x^i = (dx_i, \dots, x_k)$$

Note that $f_1 = f$ and $f_{k+1} = 1$. We have that the entropy $\int f \log f d\mu$ equals

$$\int f \log \frac{f_1 \dots f_k}{f_2 \dots f_{k+1}} \, d\mu = \sum_{i=1}^k \int f \log \frac{f_i}{f_{i+1}} \, d\mu = \sum_{i=1}^k \int f_i \log \frac{f_i}{f_{i+1}} \, d\mu^i$$
$$= \sum_{i=1}^k \int \left(\int \frac{f_i(x^i)}{f_{i+1}(x^{i+1})} \log \frac{f_i(x^i)}{f_{i+1}(x^{i+1})} \, \mu_i(dx_i) \right) f_{i+1}(x^{i+1}) \mu^{i+1}(dx^{i+1})$$
$$\leq \sum_{i=1}^k a_i \int \left(\int \left| \nabla_i \sqrt{\frac{f_i}{f_{i+1}}} \right|^2 d\mu^i \right) f_{i+1} d\mu^{i+1} = \sum_{i=1}^k a_i \int \left(\int \left| \nabla_i \sqrt{f_i} \right|^2 d\mu^i \right) d\mu^{i+1}$$
$$\leq \sum_{i=1}^k a_i \int |\nabla_{x_i} \sqrt{f}|^2 d\mu \leq (\max_i a_i) \sum_{i=1}^k \int |\nabla_{x_i} \sqrt{f}|^2 d\mu,$$

where for the first inequality we used $\int f_i / f_{i+1} d\mu_i = 1$ and for the second inequality we used Exercise 2.4(iii) and Jensen's inequality. We are done. □

Theorem 2.2 *The probability measure $\gamma(dx) = e^{-V(x)} dx$, $x \in \mathbb{R}^M$, satisfies $LSI(4c)$ provided that the second derivative $D^2 V$ satisfies $D^2 V(x) \geq c^{-1} I$ for every x.*

Proof. The idea of the proof goes back to Bakry and Emery. Let us write $T^t = e^{tL}$ for the semigroup associated with the generator $L = \Delta - \nabla V \cdot \nabla$. Note that the measure γ is reversible with the generator L, simply because $L = -\nabla^* \nabla$, where $\nabla^* = -\nabla + \nabla V$ is the adjoint of ∇ with respect to γ. Moreover,

$$2\Gamma_1(f, g) := L(fg) - fLg - gLf = 2\nabla f \cdot \nabla g,$$
$$2\Gamma_2(f, g) := L\Gamma_1(f, g) - \Gamma_1(Lf, g) - \Gamma_1(f, Lg) = 2 \sum_{i,j} f_{x_i x_j} g_{x_i x_j} + 2(D^2 V) \nabla f \cdot \nabla g,$$
$$Lf/f = L \log f + |\nabla \log f|^2, \quad \int \Gamma_1(f, g) \, d\gamma = \int \nabla f \cdot \nabla g \, d\gamma = -\int f L g \, d\gamma.$$

Now if f is a probability density with respect to γ and $f_t = T_t f$, $h(t) = \int f_t \log f_t d\gamma$, then

$$\begin{aligned}
h'(t) &= \int (\log f_t) L f_t \; d\gamma = -\int \nabla f_t \cdot \nabla \log f_t \; d\gamma = -\int \Gamma_1(f_t, \log f_t) \; d\gamma, \\
h''(t) &= -\int (\Gamma_1(Lf_t, \log f_t) + \Gamma_1(f_t, Lf_t/f_t)) \; d\gamma \\
&= \int \big(Lf_t \cdot L\log f_t - \Gamma_1(f_t, L\log f_t) - \Gamma_1(f_t, |\nabla \log f_t|^2)\big) \; d\gamma \\
&= -\int \big(2f_t\Gamma_1(\log f_t, L\log f_t) + \Gamma_1(f_t, |\nabla \log f_t|^2)\big) \; d\gamma \\
&= \int \big(-2f_t\Gamma_1(\log f_t, L\log f_t) + f_t L|\nabla \log f_t|^2)\big) \; d\gamma \\
&= \int f_t \Gamma_2(\log f_t, \log f_t) \; d\gamma \geq \int (D^2V)\nabla f_t \cdot \nabla f_t \; f_t^{-1} d\gamma \\
&\geq c^{-1} \int \Gamma_1(f_t, f_t) f_t^{-1} \; d\gamma = -c^{-1} h'(t).
\end{aligned}$$

Hence

$$\int |\nabla f|^2/f \; d\gamma = -h'(0) \geq h'(t) - h'(0) \geq c^{-1}(h(0) - h(t)),$$

and this implies $LSI(4c)$ for γ provided that we can show that $\lim_{t\to\infty} h(t) = 0$ for a subsequence. To see this, first observe that if $g_t = \sqrt{f_t}$, then $\int g_t^2 d\gamma = 1$ and $\int_0^\infty \int |\nabla g_t|^2 \; d\gamma dt < \infty$. Hence for some $t_n \to \infty$, we have that $\int |\nabla g_{t_n}|^2 \; d\gamma \to 0$ as $n \to \infty$. From this, we deduce that $g_{t_n} \to 1$ in $L^2(\gamma)$ by Rellich's theorem. Hence $f_t \to 0$ almost everywhere along a subsequence. Note that if we assume that f is bounded, then $\{f_t\}$ is uniformly bounded in t and we may use the Bounded Convergence Theorem to deduce that $\lim_{t\to\infty} h(t) = 0$ for a subsequence. This implies LSI in the case of the bounded f. The general f can be treated by a truncation. For example, for every ℓ, choose a smooth non-decreasing function ϕ_ℓ such that $\phi_\ell(f) = f$, for $f \leq \ell$, $\phi_\ell(f) = \ell + 1$, for $f \geq \ell + 2$, $\phi' \leq 1$ everywhere, and $\phi_\ell(f) \geq (\ell + 1)f/(\ell + 2)$, for $f \leq \ell + 2$. Given a density function f, we set $f^\ell = \phi_\ell(f)$ and apply LSI to f^ℓ. We then send $\ell \to \infty$ to establish LSI for arbitrary f. □

As an immediate consequence of Lemma 2.5, the law of $(h_{ij} : i \leq j)$ satisfies $LSI(a)$ for a constant $a = O(N^{-1})$ in the case of a Gaussian ensemble. This allows us to give a short proof of Theorem 2.1 in the Gaussian case.

Proof of Theorem 2.1 (Symmetric Gaussian Case). Let us write $s_N(z)$ for $\mathbb{E}S_N(z)$ where $S_N(z) = S(z, \rho_n)$. By (2.8), (2.13) and Theorem 2.2,

$$\mathbb{P}\left\{|S_N(z) - s_n(z)| > \delta\right\} \leq 2\exp\left(-\frac{1}{32}(Im\ z)^4 N^2 \delta^2\right). \tag{2.14}$$

We now concentrate on the convergence of the sequence $\{s_N\}$. Write $G(z,H) = [g_{ij}(z,H)] = (H-z)^{-1}$. We certainly have

$$(H-z)^{-1} + z^{-1} = z^{-1}H(H-z)^{-1}.$$

Hence

$$\begin{aligned}
s_N(z) &= -z^{-1} + z^{-1}N^{-1}\mathbb{E}\ tr(H(H-z)^{-1}) = -z^{-1} + z^{-1}N^{-1}\sum_{i,j}\mathbb{E}g_{ij}(z,H)h_{ij} \\
&= -z^{-1} + z^{-1}N^{-2}\sum_{i\neq j}\mathbb{E}\frac{\partial g_{ij}(z,H)}{\partial h_{ij}} + 2z^{-1}N^{-2}\sum_{i}\mathbb{E}\frac{\partial g_{ii}(z,H)}{\partial h_{ii}} \\
&= -z^{-1} - z^{-1}N^{-2}\sum_{i\neq j}\mathbb{E}\left(g_{ij}(z,H)^2 + g_{ii}(z,H)g_{jj}(z,H)\right) - 2z^{-1}N^{-2}\sum_{i}\mathbb{E}g_{ii}(z,H)^2 \\
&= -z^{-1} - z^{-1}N^{-2}\sum_{i,j}\mathbb{E}\left(g_{ij}(z,H)^2 + g_{ii}(z,H)g_{jj}(z,H)\right) \\
&= -z^{-1} - z^{-1}s_N(z)^2 + Err_1 + Err_2,
\end{aligned}$$

where we used the elementary identities

$$\begin{aligned}
&\int xf(x)(2\pi\sigma)^{-1/2}\exp\left(-x^2/(2\sigma)\right) = \int \sigma f'(x)(2\pi\sigma)^{-1/2}\exp\left(-x^2/(2\sigma)\right), \\
&\frac{dG(z,H)}{dh_{ij}} = -G(z,H)[\mathbb{1}((k,l)=(i,j) \text{ or } (k,l)=(j,i))]_{k,l}G(z,H),
\end{aligned}$$

for the third and fourth equalities, and

$$\begin{aligned}
Err_1 &= z^{-1}(s_N(z)^2 - \mathbb{E}S_N(z)^2) = -z^{-1}\mathbb{E}(S_N(z) - s_N(z))^2, \\
Err_2 &= -z^{-1}N^{-2}\mathbb{E}\ tr(H-z)^{-2}.
\end{aligned}$$

We wish to show that $Err_i \to 0$, as $N\to\infty$ for $i=1,2$. From (2.14),

$$|Err_1| \le |z|^{-1}\int_0^\infty 4r\exp\left(-\frac{1}{32}(Im\ z)^2N^2r^2\right)dr = 64|z|^{-1}(Im\ z)^{-2}N^{-2}. \tag{2.15}$$

From $|\lambda_i - z|^{-2} \le (Im\ z)^{-2}$ we deduce

$$|Err_2| \le |z|^{-1}(Im\ z)^{-2}N^{-1}. \tag{2.16}$$

From this and (2.15) we deduce that if $\lim_N s_N(z) = s(z)$ along a subsequence, then $s(z)$ satisfies

$$s(z)^2 + zs(z) + 1 = 0. \tag{2.17}$$

This equation has two solutions of the form

$$s(z) = \frac{1}{2}(-z + \sqrt{z^2 - 4}). \tag{2.18}$$

The property $(Im\ z)ImS_N(z) > 0$ for $z \in \mathbb{C} - \mathbb{R}$, implies that if $Im\ z > 0$, then $Im\ s(z) \geq 0$. This allows to select a unique solution of (2.17) when $Im\ z > 0$, namely for the square root we require $Im\ \sqrt{z^2 - 4} > 0$. It remains to identify $s(z)$ given by (2.17) as the Stieltjes transform of the semi-circle law. This will be done in Lemma 2.5. $\square$

Lemma 2.5 *The Stieltjes transform of* $\rho(x)dx = \frac{1}{2\pi}\sqrt{(4 - x^2)^+}\ dx$ *is given by* (2.18).

Proof. On account of (2.6), we need to evaluate

$$\lim_{\varepsilon \to 0} \frac{1}{\pi} Im\ s(x + i\varepsilon) = \lim_{\varepsilon \to 0} \frac{1}{2\pi} Im\ \sqrt{(x + i\varepsilon)^2 - 4} = \lim_{\varepsilon \to 0} \frac{1}{2\pi} Im\ \sqrt{x^2 - 4 - \varepsilon^2 + i2\varepsilon x}.$$

Write $A = x^2 - 4 - \varepsilon^2$, $\delta = \varepsilon x$, and $\sqrt{A + i2\delta} = a + ib$, so that $a^2 - b^2 = A$ and $ab = \delta$. Hence $b^4 + Ab^2 - \delta^2 = 0$, and

$$b^2 = \frac{1}{2}\left(-A \pm \sqrt{A^2 + 4\delta^2}\right).$$

Since $b^2 > 0$, the root with positive sign is acceptable. Note that since $\varepsilon > 0$, we take a square root of $z^2 - 4$ for which $Im\ s(z) > 0$. This simply requires that $b > 0$. Now if $|x| > 2$, then for small ε we also have that $A > 0$, and as a result $\lim_{\varepsilon \to 0} b^2 = 0$. On the other hand, if $|x| < 2$, then $A < 0$ for small ε and $\lim_{\varepsilon \to 0} b^2 = 4 - x^2$. This completes the proof. $\square$

Remark 2.2. Wigner's original proof of the semi-circle law involves calculating the moments of $\rho_N(dx)$ and passing to the limit. In fact the limiting moments are given by Catalan numbers. To see this observe

$$\begin{aligned} m_{2n} :&= \int_{-2}^{2} x^{2n} \rho(x) dx = (2\pi)^{-1} 2^{2(n+1)} \int_{-1}^{1} x^{2n} \sqrt{1 - x^2} dx \\ &= (2\pi)^{-1} 2^{2(n+1)} \int_{-\pi/2}^{\pi/2} \sin^{2n}\theta\ \cos^2\theta\ d\theta. \end{aligned}$$

On the other hand, since the sequence $a_n = \int_{-\pi/2}^{\pi/2} \sin^{2n}\theta\ d\theta$ satisfies

$$a_n - a_{n+1} = \int_{-\pi/2}^{\pi/2} \sin^{2n}\theta\ \cos^2\theta\ d\theta = \frac{1}{2n+1} \int_{-\pi/2}^{\pi/2} \frac{d}{d\theta}(\sin^{2n+1}\theta) \cos\theta\ d\theta = \frac{a_{n+1}}{2n+1},$$

we have

$$a_n = \frac{2n-1}{2n} a_{n-1} = \cdots = \frac{2n-1}{2n} \frac{2n-3}{2n-2} \cdots \frac{1}{2} a_0 = \frac{(2n)!}{(2^n n!)^2} \pi.$$

Hence,

$$m_{2n} = (2\pi)^{-1} 2^{2(n+1)} \frac{a_{n+1}}{2n+1} = \frac{(2n)!}{(n+1)(n!)^2}. \tag{2.19}$$

That is, m_{2n} is the nth Catalan number. From (2.19) and $m_{2n+1} = 0$ we deduce that if $|z| > 2$, then

$$S(z,\rho) = -z^{-1} \int \sum_{n=0}^{\infty} z^{-2n} x^{2n} \rho(x)dx = -z^{-1} \sum_{n=0}^{\infty} z^{-2n} m_{2n}.$$

With some work, we can see that this sum coincides with (2.18). □

Proof of Theorem 2.1 (General Symmetric Case). Step 1. As our first step, we argue that for a small error, we may assume that $h_{ii} = 0$, for all i and that for a constant c, we have $|h_{ij}| \le c/\sqrt{N}$, for all i and j. To see this, let us write H' for the matrix we obtain from H by replacing h_{ii} with 0 and h_{ij} with

$$\left[h_{ij}1\!\!1\left(\sqrt{N}|h_{ij}| \le \ell\right) - \sqrt{N}m_\ell\right] / \left(\sqrt{N}\sigma_\ell\right),$$

where

$$m_\ell = \mathbb{E}\sqrt{N}h_{ij}1\!\!1(\sqrt{N}|h_{ij}| \le \ell), \quad \sigma_\ell^2 = \mathbb{E}\left(\sqrt{N}h_{ij}1\!\!1(\sqrt{N}|h_{ij}| \le \ell) - m_\ell\right)^2.$$

We write $S'_N(z) = N^{-1}tr(H'-z)^{-1}$. By Lemma 2.1,

$$\sum_i (\lambda_i(H) - \lambda_i(H'))^2 \le \sum_i h_{ii}^2 + \sum_{i \ne j} \left[h_{ij} - \left(h_{ij}1\!\!1(\sqrt{N}|h_{ij}| \le \ell) - m_\ell)\sigma_\ell^{-1}\right)\right]^2.$$

From this we deduce

$$\begin{aligned}
\mathbb{E}\left(S_N(z) - S'_N(z)\right)^2 &\le (Im\ z)^{-4}\mathbb{E}\left(N^{-1}\sum_i |\lambda_i(H) - \lambda_i(H')|\right)^2 \\
&\le N^{-1}\mathbb{E}\left\{\sum_i h_{ii}^2 + \mathbb{E}\sum_{i \ne j}\left[h_{ij} - \left(h_{ij}1\!\!1(\sqrt{N}|h_{ij}| \le \ell) - m_\ell)\sigma_\ell^{-1}\right)\right]^2\right\}.
\end{aligned}$$

Hence,

$$\mathbb{E}\left(S_N(z) - S'_N(z)\right)^2 \le 2N^{-1} + \mathbb{E}\left[\bar{h} - \left(\bar{h}1\!\!1(|\bar{h}| \le \ell) - m_\ell)\sigma_\ell^{-1}\right)\right]^2. \tag{2.20}$$

Note that the right-hand side goes to 0 if N and $\ell \to \infty$.

Step 2. Recall that $G = (H - z)^{-1} = [g_{ij}(H)]$ and we are interested in $\sum_i g_{ii}$ when $Im\ z > 0$. Let us find a formula relating $g_{ii}(H)$ to $g_{ii}(H^{(i)})$ where $H^{(i)}$ is the matrix we obtain from H by deleting the i-th row and column. Let us write $G^{(i)}$ for $(H^{(i)} - z)^{-1}$. First we derive a formula for g_{11}. Indeed, writing (h_{11}, a_1) for the first row of H and (g_{11}, b_1) for the first row of G and using $(H - z)G = I$ leads to the system of equations

$$(h_{11} - z)g_{11} + a_1 \cdot b_1 = 1,$$
$$g_{11}a_1^t + \left(H^{(1)} - z\right) b_1^t = 0.$$

To solve this system for g_{11}, we first use the second equation to assert that $b^t = -g_{11}(H^{(1)} - z)^{-1}a^t$ and using this in the first equation yields

$$g_{11} = \left(h_{11} - z - a_1 G^{(1)} \cdot a_1\right)^{-1}.$$

In general,

$$g_{ii} = \left(h_{ii} - z - a_i G^{(i)} \cdot a_i\right)^{-1}, \tag{2.21}$$

where we are writing a_i for the i-th row of H with h_{ii} deleted. By Step 1, we may assume that $h_{ii} = 0$ for all i. hence

$$S_N(z) = -N^{-1} \sum_i \left(z + a_i G^{(i)} \cdot a_i\right)^{-1}. \tag{2.22}$$

Step 3. It is clear that for our goal we need to argue that $a_i G^{(i)} \cdot a_i$ is close to S_N. In fact we first try to show that $a_i G^{(i)} \cdot a_i$ is close to $trG^{(i)}$. This is not surprising at all; if we write $\mathbb{E}_i$ for the expected value with respect to the variables $(h_{ij} : j \leq i)$, then since a_i is independent of $G^{(i)}$,

$$\mathbb{E}_i a_i G^{(i)} \cdot a_i = \sum_{k,\ell \neq i} g_{kl}^{(i)} \mathbb{E}_i h_{ik} h_{i\ell} = \sum_{k \neq i} g_{kk}^{(i)} \mathbb{E}_i h_{ik}^2 = N^{-1} trG^{(i)} =: S_N^{(i)}(z).$$

Recall that $Im\ z > 0$ and we can readily show that $Im\ S_N^{(i)}(z) \geq Im\ z > 0$. Hence

$$\left| z + S_N^{(i)}(z) \right|^{-1} \geq (Im\ z)^{-1}.$$

On the other hand, since $H^{(i)}$ is diagonalizable by a orthogonal matrix $U^{(i)}$, the matrix $G^{(i)}$ is diagonalizable by the matrix $V = (U^{(i)})$. Denote the eigenvalues of $H^{(i)}$ by $\mu_1, \ldots, \mu_{N-1}$. Write $\hat{D}$ for the diagonal matrix which has $(\mu_i - z)^{-1}$ for the entries on the main diagonal. We have $G^{(i)} = V^* \hat{D} V$, which implies

$$a_i G^{(i)} \cdot a_i = (a_i V)^* \hat{D} (a_i V) = \sum_i (\mu_i - z)^{-1} w_i^2,$$

where w_i's are the components of the vector $a_i V$. From this, we can readily deduce

$$\left|z + a_i G^{(i)} \cdot a_i\right|^{-1} \geq (Im\ z)^{-1}.$$

Putting all pieces together we learn

$$\left|\left(z + a_i G^{(i)} \cdot a_i\right)^{-1} - \left(z + S_N^{(i)}(z)\right)^{-1}\right| \leq (Imz)^{-2} \left|a_i G^{(i)} \cdot a_i - S_N^{(i)}(z)\right| = (Imz)^{-2} |E_i|,$$

where

$$E_i = a_i G^{(i)} \cdot a_i - S_N^{(i)}(z),$$

Hence

$$\left|S_N(z) + N^{-1} \sum_i \left(z + S_N^{(i)}(z)\right)^{-1}\right| \leq (Im\ z)^{-2} N^{-1} \sum_i |E_i|. \tag{2.23}$$

Note that we can write $E_i = Err_i + Err'_i$, where

$$Err_i = \sum_{k,\ell \neq i,, k \neq \ell} g_{kl}^{(i)} h_{ik} h_{i\ell},$$
$$Err'_i = \sum_{k \neq i} g_{kk}^{(i)} \left(h_{ik}^2 - N^{-1}\right).$$

Further, using $\mathbb{E} h_{ij} = 0$ and the independence of $h_{ik}, k \neq i$, from $G^{(i)}$,

$$\begin{aligned}
\mathbb{E}(Err_i)^2 &= 2\mathbb{E} \sum_{k,\ell \neq i, k \neq \ell} \left(g_{kl}^{(i)}\right)^2 (h_{ik} h_{i\ell})^2 = 2N^{-2} \mathbb{E} \sum_{k,\ell \neq i} \left(g_{kl}^{(i)}\right)^2 \leq 2N^{-2} \mathbb{E} \sum_{k,\ell \neq i} \left(g_{kl}^{(i)}\right)^2 \\
&\leq 2N^{-2} \mathbb{E}\ tr \left(H^{(i)} - z\right)^{-2} \leq 2(Imz)^{-2} N^{-1}, \\
\mathbb{E}(Err'_i)^2 &= \mathbb{E} \sum_{k \neq i,\ k \neq l} \left(g_{kk}^{(i)}\right)^2 \left(h_{ik}^2 - N^{-1}\right)^2 \leq c_1 N^{-2} \mathbb{E} \sum_{k \neq i,\ k \neq l} \left(g_{kk}^{(i)}\right)^2 \leq c_1 (Imz)^{-2} N^{-1}.
\end{aligned}$$

From this and (2.23) we deduce

$$\mathbb{E}\left(S_N(z) + N^{-1} \sum_i \left(z + S_N^{(i)}(z)\right)^{-1}\right)^2 \leq c_2 (Imz)^{-4} N^{-1}. \tag{2.24}$$

Step 4. It remains to show that we can replace $S_N^{(i)}$ with S_N in (2.23) for a small error. Note that $S_N = trG$ and $S_N^{(i)} = trG^{(i)}$ with G and $G^{(i)}$ of different sizes. Let us write $\hat{H}^{(i)}$ for a matrix we obtain from H by replacing its i-th row and column with 0. We also write

$\hat{G}^{(i)} = (\hat{H}^{(i)} - z)^{-1}$. We can readily show that in terms of eigenvalues, the matrix $\hat{H}^{(i)}$ has the same eigenvalues as $H^{(i)}$ plus a 0 eigenvalue. As a result

$$\left| S_N^{(i)} - N^{-1} tr \hat{G}^{(i)} \right| \leq N^{-1} |Imz|^{-1}. \tag{2.25}$$

Moreover, by Lemma 2.1,

$$\begin{aligned} N^{-1}|tr\hat{G}^{(i)} - trG| &\leq |Imz|^{-1} N^{-1} \sum_j \left| \lambda_j(H) - \lambda_j\left(\hat{H}^{(i)}\right) \right| \\ &\leq |Imz|^{-1} \left[N^{-1} \sum_j \left(\lambda_j(H) - \lambda_j\left(\hat{H}^{(i)}\right)\right)^2 \right]^{1/2} \\ &\leq |Imz|^{-1} \left(2N^{-1} \sum_{j \neq i} h_{ij}^2 \right)^{1/2}. \end{aligned}$$

Therefore

$$\mathbb{E}\left(N^{-1} |tr\hat{G}^{(i)} - trG| \right)^2 \leq 2(Imz)^{-2} N^{-1}.$$

From this, (2.25) and (2.3) we deduce that if $s(z)$ is a limit point of $S_N(z)$, then $s(z)$ satisfies

$$s(z) + (z + s(z))^{-1} = 0.$$

From this we deduce that s is given by (2.18). This completes the proof. □

Exercise 2.2.

- (i) Verify Lemma 2.1 for Hermitian matrices.
- (ii) Establish Theorem 2.1 in the case of Hermitian Wigner ensembles.

3 Gaussian Ensembles GOE and GUE

In this section we derive an explicit formula for the eigenvalues in the case of a Gaussian Wigner ensemble. Using this formula, we can find the law governing the correlation and the gap between eigenvalues in the large N limit.

Consider a symmetric Gaussian Wigner ensemble $H = [h_{ij}] = N^{-1/2}\tilde{H}$. The law of $h_{ij} = N^{-1/2}\tilde{h}_{ij}$ is given by

$$\begin{aligned} (2\pi)^{-1/2}\sqrt{N} e^{-Nh_{ij}^2/2} dh_{ij} &= (2\pi)^{-1/2} e^{-\tilde{h}_{ij}^2/2} d\tilde{h}_{ij}, \\ \sqrt{2}(2\pi)^{-1/2}\sqrt{N} e^{-Nh_{ii}^2/4} dh_{ii} &= \sqrt{2}(2\pi)^{-1/2} e^{-\tilde{h}_{ii}^2/4} d\tilde{h}_{ii}, \end{aligned}$$

in the case of $i \neq j$ and $i = j$ respectively. This leads to the formula

$$
\begin{aligned}
\mathbb{P}_N^1(dH) &= 2^{-N/2}(2\pi)^{-N(N+1)/4}N^{N(N+1)/4}\exp\left(-Ntr H^2/4\right)\prod_{i\le j} dh_{ij} \\
&= 2^{-N/2}(2\pi)^{-N(N+1)/4}\exp\left(-tr\tilde{H}^2/4\right)\prod_{i\le j} d\tilde{h}_{ij}, \qquad (3.1)
\end{aligned}
$$

for the law of $H = N^{-1/2}\tilde{H}$. We note that the measure $\mathbb{P}_N^1(dH)$ is invariant with respect to an orthogonal conjugation U^tHU, with U any orthogonal matrix. For this reason the measure $d\mathbb{P}_N^1$ is known as a *Gaussian orthogonal ensemble* or in short GOE.

In the Hermitian case, the diagonal entries $h_{ii} = N^{-1/2}\tilde{h}_{ii}$ are real and distributed as

$$(2\pi)^{-1/2}\sqrt{N}e^{-Nh_{ii}^2/2}dh_{ii} = (2\pi)^{-1/2}e^{-\tilde{h}_{ii}^2/2}d\tilde{h}_{ii},$$

and off-diagonal entries $h_{ij} = x_{ij} + iy_{ij} = N^{-1/2}\tilde{h}_{ij} = N^{-1/2}(\tilde{x}_{ij} + i\tilde{y}_{ij})$ with $\mathbb{E}x_{ij}^2 = \mathbb{E}y_{ij}^2 = 1/(2N)$, $\mathbb{E}|h_{ij}|^2 = 1/N$ are distributed according to

$$(\pi)^{-1}Ne^{-N|h_{ij}|^2/2}dh_{ij} = (\pi)^{-1}e^{-|\tilde{h}_{ij}|^2/2}d\tilde{h}_{ij},$$

where by dh_{ij} and $d\tilde{h}_{ij}$ we mean $dx_{ij}dy_{ij}$ and $d\tilde{x}_{ij}d\tilde{y}_{ij}$. As a result, the law of $H = N^{-1/2}\tilde{H}$ is given by

$$
\begin{aligned}
\mathbb{P}_N^2(dH) &= 2^{-N/2}\pi^{-N^2/2}N^{N^2/2}\exp\left(-Ntr H^2/2\right)\prod_{i\le j} dh_{ij} \\
&= 2^{-N/2}\pi^{-N^2/2}\exp\left(-tr\tilde{H}^2/2\right)\prod_{i\le j} d\tilde{h}_{ij}. \qquad (3.2)
\end{aligned}
$$

We note that the measure $\mathbb{P}_N^2(dH)$ is invariant with respect to an unitary conjugation U^*HU, with U any unitary matrix. For this reason the measure $d\mathbb{P}_N^2$ is known as a *Gaussian unitary ensemble* or in short GUE.

Exercise 3.1. Consider the inner product $< H, H' >= 2\sum_{i\neq j} h_{ij}h'_{ij} + \sum_i h_{ii}h'_{ii}$ on the space of symmetric/Hermitian matrices. Given an orthogonal/unitary matrix, define the linear operator T by $T(H) = U^*HU$. Show that T is an isometry for $< \cdot, \cdot >$. From this deduce that the Lebesgue measure $\prod_{i\le j} dh_{ij}$ is invariant under the map T. (This implies that $\mathcal{P}_N^\beta$ is invariant under an orthogonal/unitary conjugation.) □

Since any symmetric (respectively Hermitan) matrix H can be expressed as UDU^t with D diagonal and U orthogonal (respectively unitary), we may try to find the joint law of (D, U) when H is a GOE (respectively GUE). First we need to come up with a unique representation $H = UDU^*$. This is easily done if we know that the eigenvalues of H are distinct. We then insist that the entries on the main diagonal of D are given by $\lambda_1(H) <$

$\cdots < \lambda_N(H)$. Once this is assumed on the eigenvalues, we almost have a unique choice for U because the columns of U are the eigenvectors. As we will see later, we can arrange for U to have nonzero entries and if we assume that all diagonal entries are positive, then we have a unique choice for U. Here is our main theorem in this section.

Theorem 3.1

- *(i) With probability one with respect to $\mathbb{P}_N^\beta$, all eigenvalues are distinct and all eigenvectors have nonzero coordinates. Hence, with probability one with respect to $\mathbb{P}_N^\beta$, there is a unique representation $H = UDU^*$ with $D = diag[\lambda_1(H), \dots, \lambda_N(H)]$, $\lambda_1(H) < \cdots < \lambda_N(H)$, $U = [u_{ij}]$, with $u_{ij} \neq 0$, $u_{ii} > 0$ for all i and j and U is orthogonal (respectively unitary) if $\beta = 1$ (respectively $\beta = 2$). The space of such matrices U is denoted by $\mathcal{U}_2^\beta$.*

- *(ii) In the representation $H = UDU^*$ of part (i), the variables D and U are independent. The law of U is given by taking the unique normalized Haar measure of the space of orthogonal (respectively unitary) matrices when $\beta = 1$ (respectively $\beta = 2$), and projecting it onto the space $\mathcal{U}_2^\beta$. The law of $(\lambda_1(H), \dots, \lambda_N(H))$ is given by*

$$Z_N(\beta)^{-1} \mathbb{1}(\lambda_1 < \cdots < \lambda_N) |\Delta(\lambda_1, \dots, \lambda_N)|^\beta \exp\left(-\beta N \sum_{i=1}^N \lambda_i^2/4\right) \prod_{i=1}^N d\lambda_i, \tag{3.3}$$

where

$$\Delta(\lambda_1, \dots, \lambda_N) = \prod_{i<j} (\lambda_i - \lambda_j),$$

is the Vandermonde determinant and $Z_N(\beta)$ is the normalizing constant and is given by

$$Z_N(\beta)^{-1} = (2\pi)^{-N/2} \left(2^{-1}\beta\right)^{\beta N(N-1)/4 + N/2} N^{N/2 + N(N-1)\beta/4} \prod_{i=1}^N \frac{\Gamma(\beta/2)}{\Gamma(j\beta/2)}. \tag{3.4}$$

Here $\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx$.

Remark 3.1. What Theorem 3.1 says is that

$$\mathcal{P}_N^\beta(dH) = \mu_N^\beta(d\lambda_1, \dots, d\lambda_N) \nu_N^\beta(dU),$$

with μ_N^β given by (3.3) and (3.4), and $\nu_N(dU)$ the Haar measure of the space of unitary/orthogonal matrices. Since $trH^2 = \sum_i \lambda_i^2$, the non-trivial claim of part (ii) is the equality

$$\prod_{i \le j} dh_{ij} = C_N(\beta)^{-1} \mathbb{1}(\lambda_1 < \cdots < \lambda_N) |\Delta(\lambda_1, \dots, \lambda_N)|^\beta \prod_{i=1}^N d\lambda_i \; \nu_N^\beta(dU),$$

for a constant C_N. We note that when $\beta = 1$, the left hand side is the Lebesgue measure of $\mathbb{R}^{N(N+1)/2}$, whereas μ_N^1 is a measure on a manifold of dimension $N(N-1)/2$. □

First we show that the eigenvalues are distinct almost surely for a GUE or GOE. To have an idea, let us examine this when N=1. In this case, we simply need to make sure that the quadric equation

$$\lambda^2 - (h_{11} + h_{22})\lambda + h_{11}h_{22} - |h_{12}|^2 = 0,$$

has two distinct solutions. For this the discriminant must be non-zero. That is,

$$(h_{11} + h_{22})^2 + 4|h_{12}|^2 - 4h_{11}h_{22} \neq 0.$$

In the case of GOE, we have $(h_{11}, h_{22}, h_{12}) \in \mathbb{R}^3$ and the discriminant vanishes on a two dimensional surface which is of zero Lebesgue measure. Hence almost surely eigenvalues are distinct. We want to generalize this argument for general N. For this we will define a discriminant that is a polynomial in the entries of H and vanishes if and only if H has non distinct eigenvalues. This immediately implies that almost surely the eigenvalues are distinct because of the following straight forward fact.

Exercise 3.2. Let $p(x_1, \dots, x_k)$ be a nonzero polynomial of k variables. Show that the zero set of p is of zero Lebesgue measure in $\mathbb{R}^k$. □

We are now ready to prove

Lemma 3.1 *The set of symmetric matrices $H = (h_{ij} : i \leq j) \in \mathbb{R}^{N(N+1)/2}$ with distinct eigenvalues is of full Lebesgue measure.*

Proof. Consider the characteristic polynomial $p(\lambda) = det(H - \lambda) = a_N\lambda^N + \dots + a_1\lambda + a_0$. the coefficients $a_i's$ are all homogeneous polynomials of the entries of H. The matrix H has distinct eigenvalues iff $p(\lambda)$ and $q(\lambda) = p'(\lambda) = b_m\lambda^m + \dots + b_1\lambda + b_0$, $m = N - 1$, $b_j = (j+1)a_{j+1}$, have no common eigenvalue. We define the discriminant of p by

$$D(p) = a_N^{2N-1} \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^2,$$

and this can by as the determinant of a *Sylvester matrix.* More precisely,

$$D(p) = (-1)^{N(N-1)/2} R(p, q)$$

where the resultant $R(p, q)$ is the determinant of the Sylvester matrix $S(p, q)$ and vanishes iff p and q have a common root. In fact $R(p, q)$ is a polynomial in the coefficients of p and q and hence a polynomial in entries of H. We now apply Exercise 3.2 to deduce the lemma. □

We are now ready to give the proof of the first part of Theorem 3.1.

Proof of Theorem 3.1(i). Write $H = UDU^t$ with $D = diag[\lambda_1(H), \dots, \lambda_N(H)]$, $\lambda_1(H) < \dots < \lambda_N(H)$, and U an orthogonal matrix. Note that the columns $u^1, \dots, u^N$ of U are the eigenvectors of U. We would like to show that for every r, all components of the vector u^r are nonzero with probability one.

Pick an eigenvalue $\lambda = \lambda_r(H)$ and set $A = H - \lambda$. Write B for the adjoint of A. Since H is symmetric, the adjoint of A is the same as the cofactor of A and its entries are given by $b_{ij} = (-1)^{i+j} \det A^{(ij)}$, where A^{ij} is the matrix we obtain from A by deleting the i-th row and the j-th column. We certainly have

$$AB = [Ab^1, \dots, Ab^N] = (\det A)I = 0,$$

where $b^1, \dots, b^N$ denote the columns of B. Hence $Ab^i = 0$ for all i and since $\lambda = \lambda_r$ is an eigenvalue of multiplicity 1, we deduce that for every i, there exists a scalar c_i such that $b^i = c_i u^r$. We wish to show that $u_{ir} \neq 0$ for all i and for this it suffices to show that $b_{ii} \neq 0$. But $b_{ii} = \det(H^{(ii)} - \lambda_r) = 0$ means that the matrices H and $H^{(ii)}$ have a common root. This is equivalent to asserting that the resultant $R(H, H^{(ii)}) = 0$. This is a nonzero polynomial in the entries of H. Hence, using Exercise 3.2, we learn that $b_{ii} = 0$ occurs only for a set of matrices H of zero Lebesgue measure. Thus almost surely all entries u_{ij} are nonzero. Finally, since each column u^r is an eigenvector, we can arrange to have $u_{ii} > 0$ and this condition uniquely determines U. □

To derive (3.3) and (3.4), we need to study the Jacobian of the map $H \mapsto (D, U)$. For this, let us first parametrize the space of unitary (respectively orthogonal) matrices $\mathcal{U}^2$ (respectively $\mathcal{U}^1$) in a smooth fashion. Let us write $\mathcal{U}_1^2$ (respectively $\mathcal{U}_1^1$) for the set of unitary (respectively orthogonal) matrices $U = [u_{ij}]$ such that $u_{ij} \neq 0$. We also write $\mathcal{U}_2^\beta$ for the set of $U = [u_{ij}] \in \mathcal{U}_1^\beta$ such that $u_{ii} > 0$ for all i. Evidently $\mathcal{U}_1^\beta$ is an open subset of $\mathcal{U}^\beta$ and $\dim \mathcal{U}^\beta = \beta N(N-1)/2$. We now give a smooth parametrization for a nice subset of $\mathcal{U}_2^\beta$. To this end, let us define the map $\Gamma : \mathcal{U}^\beta \to \mathbb{R}^{\beta N(N-1)/2}$, by $\Gamma(U) = (u_{ij}/u_{ii} : i < j)$. This map gives such a smooth parametrization we are looking for. For this, let us consider a nice subset $\mathcal{U}_3^\beta$ of $\mathcal{U}_2^\beta$ on which Γ is injective. To be more precise, set $\mathcal{U}_3^\beta$ to be the set of matrices $U = [u_{ij}] \in \mathcal{U}_2^\beta$ such that $\det[u_{ij}]_{i,j=1}^k \neq 0$, for every $k \in \{2, 3, \dots, N\}$. We have the following lemma.

Lemma 3.2
- *(i) The map $\Gamma : \mathcal{U}_3^\beta \to \mathbb{R}^{\beta N(N-1)/2}$ is injective with smooth inverse.*
- *(ii) $\Gamma(\mathcal{U}_3^\beta)$ is of full measure in $\mathbb{R}^{\beta N(N-1)/2}$.*
- *(iii) The matrix $U \in \mathcal{U}^\beta$ in the representation $H = UHU^*$ belongs to $\mathcal{U}_3^\beta$ with probability one with respect to $\mathbb{P}_N^\beta$.*

Proof of (i). We only discuss the case $\beta = 1$ because the proof in the case of $\beta = 2$ is identical. We need to learn how to determine $(u_{ij} : i \geq j)$ from our knowledge of $\Gamma(U)$. Write $v_{ij} := u_{ij}/u_{ii}$. Note that we only need to determine $(v_{ij} : i \geq j)$ because the condition $\sum_j u_{ij}^2 = 1$ means

$$u_{ii}^{-2} = 1 + \sum_{j:i<j} v_{ij}^2 + \sum_{j:i>j} v_{ij}^2.$$

To determine $(u_{ij}/u_{ii} : i \geq j)$ from $\Gamma(U)$, we use the fact that the rows of U are mutually orthogonal. This can be achieved inductively. Suppose that we already know $(v_{ij} : 1 \leq r, 1 \leq j \leq N)$ and we wish to determine $(v_{ij} : i = r+1, 1 \leq j \leq N)$. Since $\Gamma(U)$ is known by assumption, we also know t $(v_{ij} : i = r+1, i \leq j)$. Hence we only need to determine r many unknowns, namely $(v_{ij} : i = r+1, i > j)$. This is done by setting up a system of r linear equations. The fact that the $r+1$-th row is orthogonal to the first r rows of U yield the desired equations. In order to have a solution to these equation, we need

$$\det[v_{ij}]_{i,j=1}^{r} = \left(\prod_{i=1}^{r} u_{ii}\right)^{-1} \det[u_{ij}]_{i,j=1}^{r} \neq 0. \tag{3.5}$$

This is the case because $U \in \mathcal{U}_3^\beta$. Evidently, the inverse is smooth.

Proof of (ii). Note that $v = (v_{ij} : i < j) \in \Gamma(\mathcal{U}_3^\beta)$ if v_{ij} is nonzero and (3.5) is valid. We note that all v_{ij} with $i > j$ can be expressed as a ratio of two nonzero polynomials of v and that the left-hand side of (3.5) can be expressed as a ratio of two non zero polynomials of v. Hence we may apply Exercise 3.2 because to assert that the range of Γ is of full measure.

Proof of (iii). We are going to formulate a property about H which implies (iii) and is proved in just the same way we showed that the unitary matrix in the statement of Theorem 3.1(i) has nonzero entries. More precisely, given a matrix A of size $N \times N$, and $r \leq N$, write $\mathcal{N}_r$ for the set of subsets $I \subseteq \{1, \dots, N\}$ of size r and $M = N!/[r!(N-r)!]$ for the number of such subsets. We then define a $M \times M$ matrix $\Lambda_r(A) = [\det A_{I,J}]$ with $I, J \in \mathcal{N}_r$. Evidently $\Lambda_r(A^*) = \Lambda_r(A)^*$ and that Λ_r of the identity matrix is the identity matrix. Also, if $D = diag[\lambda_1, \dots, \lambda_N]$ is a diagonal matrix, then $\Lambda_r(D)$ is diagonal with the II entry given by the product of λ_i over $i \in I$. A celebrated formula of *Cauchy-Binet* asserts that $\Lambda_r(AB) = \Lambda_r(A)\Lambda_r(B)$. Hence our representation $H = UDU^*$ implies that $\Lambda_r(H) = \Lambda_r(U)\Lambda_r(D)\Lambda_r(U)^*$. Note that this is the analogous representation of the symmetric/Hermitian $\Lambda_r(H)$ because $\Lambda_r(U)$ is unitary and $\Lambda_r(D)$. As in the proof of Theorem 3.1(i), we can find a polynomial of entries of $\Lambda_r(H)$ that vanishes iff $\Lambda_r(H)$ does not have distinct eigenvalues or $\Lambda_r(U)$ has a zero entry. Since the entries of $\Lambda_r(H)$ are polynomials of entries of H, we end up with a polynomial of enries of H which vanishes if $\Lambda_r(H)$ has a zero entry. We are done. $\square$

Proof of Theorem 3.1(ii). Step 1. Set $X = \{\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N) : \lambda_1 \leq \cdots \leq \lambda_N\}$, and let us write $\mathcal{H}^1$ for the space of symmetric matrices and $\mathcal{H}^2$ for the space of Hermitian matrices. Define $\Phi : X \times \mathcal{U}^\beta \to \mathcal{H}^\beta$ by $\Phi(\boldsymbol{\lambda}, U) = U diag[\lambda_1, \dots, \lambda_N]U^*$. The pullback of the measure $\mathbb{P}_N^\beta$ under Φ is denoted by $\mathbb{Q}_N^\beta(d\boldsymbol{\lambda}, dU) = \mathbb{Q}_N^\beta(\boldsymbol{\lambda}, dU)\mu_N(d\boldsymbol{\lambda})$. Since the measure $\mathbb{P}_N^\beta$ is invariant with respect to the conjugation $H \mapsto WHW^*$, we deduce that the conditional measure $\mathbb{Q}_N^\beta(\boldsymbol{\lambda}, dU)$ is invariant with respect to the left multiplication $U \mapsto WU$. Hence the conditional measure $\mathbb{Q}_N^\beta(\boldsymbol{\lambda}, dU)$ must be the normalized Haar measure γ_N^β of $\mathcal{U}^\beta$. By the uniqueness of the Haar measure, we learn that $\mathbb{Q}_N^\beta(d\boldsymbol{\lambda}, dU) = \gamma_N^\beta(dU)\mu_N(d\boldsymbol{\lambda})$. Note that by part (i) of the theorem, we know that γ_N^β is concentrated on matrices U of nonzero entries, i.e. $\gamma_N^\beta(\mathcal{U}_1) = 1$. Let us write $\pi : \mathcal{U}_1^\beta \to \mathcal{U}_2^\beta$ for the projection onto $\mathcal{U}_2^\beta$. More precisely, given a matrix $U \in \mathcal{U}_1^\beta$, we may multiply each column u^i by a unique number c_i with $|c_i| = 1$ to produce $\pi(U) \in \mathcal{U}_2^\beta$. The push forward of γ_N^β under π is denoted by ν_N^β. Evidently if $\hat{\Phi}$ denotes the restriction of Φ to the set $X \times \mathcal{U}_2^\beta$, then $\nu_N^\beta(dU)\mu_N(d\boldsymbol{\lambda})$ is the pullback of $\mathbb{P}_N^\beta$ with respect to the injective transformation $\hat{\Phi}$.

Step 2. We now study the measure $\mu_N(d\boldsymbol{\lambda})$. Define $\Psi : \mathbb{R}^N \times \Gamma(\mathcal{U}_3^\beta) \to \mathcal{H}^\beta$ by $\Psi(\boldsymbol{\lambda}, \mathbf{v}) = \Gamma(\mathbf{v})^{-1} diag[\lambda_1, \dots, \lambda_N]\Gamma(\mathbf{v})^{-1*}$, where Γ is as in Lemma 3.3 and $\mathbf{v} = (v_{ij} : i < j)$. If we write $H = [h_{ij}] = \Psi(\boldsymbol{\lambda}, \mathbf{v})$, then

$$\prod_{i \le j} dh_{ij} = C_N(\beta)^{-1} \mathbb{1}(\lambda_1 < \cdots < \lambda_N) |\det D\Psi(\boldsymbol{\lambda}, \mathbf{v})| \prod_{i=1}^N d\lambda_i \ \hat{\mu}_N^\beta(d\mathbf{v}),$$

where $\hat{\nu}_N$ is the pullback of ν_N under the map Γ. Hence

$$\mu_N(d\boldsymbol{\lambda}) = C_N(\beta)^{-1} \mathbb{1}(\lambda_1 < \cdots < \lambda_N) \left(\int |\det D\Psi(\boldsymbol{\lambda}, \mathbf{v})| \ \hat{\mu}_N^\beta(d\mathbf{v}) \right) \prod_{i=1}^N d\lambda_i$$

To complete the proof, it suffices to show

$$\det D\Psi(\boldsymbol{\lambda}, \mathbf{v}) = \Delta(\boldsymbol{\lambda})^\beta f(\mathbf{v}), \tag{3.6}$$

for some function f, and that the normalizing constant is given by (3.4). To achieve this, let us directly calculate

$$dH = \sum_i \frac{\partial h_{ij}}{\partial \lambda_i} d\lambda_i + \sum_{k<l} \frac{\partial h_{ij}}{\partial v_{kl}} dv_{kl}.$$

Note that when $\beta = 2$, then v_{kl} are complex numbers. We certainly have

$$dH = (dU)DU^* + UD(dU^*) + U^*(dD)U, \text{ or } U^*(dH)U = U^*(dU)D + D(dU^*)U + dD.$$

From this and $(dU^*)U + U^*(dU) = 0$, we deduce

$$U^*(dH)U = AD - DA + dD, \text{ or } dH = U[AD - DA + dD]U^* =: UBU^*,$$

where $A = U^*(dU) = [a_{ij}]$. But $AD - DA = [a_{ij}(\lambda_i - \lambda_j)]$. So, $dh_{ij} = \sum_{k,l} u_{ik} b_{kl} \bar{u}_{jl}$ with $b_{ii} = d\lambda_i$ and $b_{ij} = a_{ij}(\lambda_i - \lambda_j)$ when $i \ne j$. Note that a_{ij} is a 1-form independent of $\boldsymbol{\lambda}$ for each i and j. Moreover, since $A^* + A = 0$, we have that $a_{ji} = -\bar{a}_{ij}$. When $\beta = 1$, as we calculate the $\wedge_{i \le j} dh_{ij}$, we simply get

$$\Delta(\boldsymbol{\lambda}) \wedge_{i=1}^N d\lambda_i \wedge \alpha,$$

where α is a $N(N-1)/2$-form in $\mathbf{v}$ with coefficients independent of $\boldsymbol{\lambda}$. In fact α is simply $f(\mathbf{v}) \wedge_{i<j} a_{ij}$, for a function f. When $\beta = 2$, we first write $h_{ij} = k_{ij} + ik'_{ij}$ with k_{ij} and k'_{ij} the real and imaginary parts of h_{ij}. Now

$$(\wedge_i dh_{ii}) \wedge (\wedge_{i<j} dk_{ij}) \wedge \left(\wedge_{i<j} dk'_{ij}\right) = \Delta(\boldsymbol{\lambda})^2 \wedge_{i=1}^N d\lambda_i \wedge \alpha,$$

where α is a $N(N-1)$-form in $\mathbf{v}$ with coefficients independent of $\boldsymbol{\lambda}$. Here we are using the fact that as we take the real and imaginary part of α, the factor $\lambda_i - \lambda_j$ is repeated. This completes the proof of the Theorem except for the proof of (3.3). We establish (3.3) in Lemma 3.3 below. □

Remark 3.2. The quick way of verifying (3.6) when $\beta = 1$ is by observing that since $h_{ij} = \sum_k u_{ik}u_{jk}\lambda_k$, the partial derivative $\partial h_{ij}/\partial \lambda_k$ is independent of $\boldsymbol{\lambda}$ and that the partial derivative $\partial h_{ij}/\partial v_r$ is linear in $\boldsymbol{\lambda}$. As a result, $\det D\Psi(\boldsymbol{\lambda}, \mathbf{v})$ is a polynomial in $\boldsymbol{\lambda}$ of dimension at most $N(N-1)/2$ with coefficients which may depend on $\mathbf{v}$. On the other hand if $\lambda_i = \lambda_j$ for some $i \neq j$, then $\det D\Psi(\boldsymbol{\lambda}, \mathbf{v})$ must vanish. To see this, observe that if $\det D\Psi(\boldsymbol{\lambda}, \mathbf{v})$ does not vanish for such $\boldsymbol{\lambda}$, then by Inverse Mapping Theorem, $\Psi(\boldsymbol{\lambda}, \mathbf{v})$ would be invertible near such $\boldsymbol{\lambda}$ and this is not the case. Hence the polynomial $\Delta(\boldsymbol{\lambda})$ must divide $\det D\Psi(\boldsymbol{\lambda}, \mathbf{v})$. Since this is of dimension at most $N(N-1)/2$, we are done. This argument does not work when $\beta = 2$; we would get that $\Delta(\boldsymbol{\lambda})$ must divide $\det D\Psi(\boldsymbol{\lambda}, \mathbf{v})$ but we need $\Delta(\boldsymbol{\lambda})^2$ to divide $\det D\Psi(\boldsymbol{\lambda}, \mathbf{v})$. □

Lemma 3.3 *We have*

$$(3.7) \qquad \frac{1}{N!}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |\Delta(\boldsymbol{\lambda})|^\beta \exp\left(-\sum_{i=1}^N \lambda_i^2/2\right) \prod_{i=1}^N d\lambda_i = (2\pi)^{N/2} \prod_{i=1}^N \frac{\Gamma(i\beta/2)}{\Gamma(\beta/2)}.$$

Proof. The left-hand side of (3.7) equals

$$\begin{aligned}
&\lim_{\ell\to\infty} \frac{1}{N!}\int_{-\sqrt{2\ell}}^{\sqrt{2\ell}} \cdots \int_{-\sqrt{2\ell}}^{\sqrt{2\ell}} |\Delta(\boldsymbol{\lambda})|^\beta \prod_{i=1}^N \left(1 - \frac{\lambda_i^2}{2\ell}\right)^\ell d\lambda_i \\
&= \lim_{\ell\to\infty} (2\ell)^{N/2+\beta N(N-1)/4}\ \frac{1}{N!}\int_{-1}^{1} \cdots \int_{-1}^{1} |\Delta(\boldsymbol{\lambda})|^\beta \prod_{i=1}^N \left(1-\lambda_i^2\right)^\ell d\lambda_i \\
&= \lim_{\ell\to\infty} (2\ell)^{N/2+\beta N(N-1)/4} 2^{2\ell+N+\beta N(N-1)/2} S_N(\ell+1, \ell+1, \beta),
\end{aligned}$$

where S_N is the Selberg integral:

$$(3.8) \qquad S_N(a,b,\beta) = \frac{1}{N!}\int_0^1 \cdots \int_0^1 |\Delta(\boldsymbol{\lambda})|^\beta \prod_{i=1}^N \lambda_i^{a-1}(1-\lambda_i)^{b-1} d\lambda_i.$$

According to a celebrated result of Selberg,

$$S_N(a,b,\beta) = \prod_{i=1}^{N} \frac{\Gamma(a+(i-1)\beta/2)\Gamma(b+(i-1)\beta/2)\Gamma(i\beta/2)}{\Gamma(a+b+(N+i-2)\beta/2)\Gamma(\beta/2)}. \tag{3.9}$$

Hence, the left-hand side of (3.7) equals

$$\lim_{\ell\to\infty} (2\ell)^{N/2+\beta N(N-1)/4} 2^{2\ell+N+\beta N(N-1)/2} \prod_{i=1}^{N} \frac{\Gamma(\ell+1+(i-1)\beta/2)^2\Gamma(i\beta/2)}{\Gamma(2\ell+2+(N+i-2)\beta/2)\Gamma(\beta/2)}.$$

Note that by Stirling's formula

$$\lim_{n\to\infty} \Gamma(n) e^{-n\log n + n}\sqrt{\frac{n}{2\pi}} = 1.$$

As a result

$$\begin{aligned}\frac{\Gamma(\ell+1+A)^2}{\Gamma(2\ell+2+B)} &\approx \frac{(\ell+1+A)^{2(\ell+1+A)}e^{-2A+B}\sqrt{2}\sqrt{2\pi}}{(2\ell+2+B)^{2\ell+2+B}\sqrt{\ell}} \\ &\approx \ell^{2A-B}\frac{\left[1+A/(\ell+1)\right]^{2(\ell+1+A)}e^{-2A+B}\sqrt{2}\sqrt{2\pi}}{\left[1+B/(2\ell+2)\right]^{2\ell+2+B}\sqrt{\ell}} \\ &\approx \sqrt{2}\sqrt{2\pi}\ell^{2A-B-\frac{1}{2}},\end{aligned}$$

as $\ell\to\infty$. Therefore, the left-hand side of (3.7) equals

$$\lim_{\ell\to\infty} (2\ell)^{N/2+\beta N(N-1)/4} 2^{2\ell+N+\beta N(N-1)/2} \prod_{i=1}^{N} \frac{\sqrt{2}\sqrt{2\pi}\ell^{(i-1)\beta/-(N+i-2)\beta/2-\frac{1}{2}}\Gamma(i\beta/2)}{\Gamma(\beta/2)}.$$

From this we can readily deduce (3.7). □

4 Correlations and Edge Distributions for Gaussian Ensembles

In this section, we first derive an explicit formula for the r-point correlations of Gaussian ensembles and use this formula to find the gap distributions in large N limit. We treat GUE first because our formulas would be simpler when $\beta = 2$. For our purposes, let us look at the eigenvalues $\bar{x} = (x_1,\dots,x_N) = \sqrt{N}\boldsymbol{\lambda}$ of $\bar{H} = \sqrt{N}H$ and we do no longer insist on ordering of the eigenvalues. So the law of $\bar{x}$ with respect to GUE is given by

$$\hat{\mu}_N(d\mathbf{x}) = (Z_N')^{-1}\Delta(\mathbf{x})^2 \exp\left(-\sum_i x_i^2/2\right)\prod_i dx_i,$$

where the normalizing constant is simply given by

$$Z'_N = N!(2\pi)^{N/2} \prod_{n=0}^{N-1} n!. \tag{4.1}$$

In our first result, we derive an explicit formula for the marginals of $\hat{\mu}_N$ and even give a new proof of (4.1). To discover such a formula, recall that $\Delta(\bar{x}) = \det[x_i^{j-1}]_{i,j=1}^N$ and by adding multiples of the i-th columns to the j-th columns for $i < j$, we learn that $\Delta(\bar{x}) = \det[P^{j-1}(x_i)]_{i,j=1}^N$ for any collection of monic polynomials P_j such that the degree of P_j is j. Hence, for any collection of positive constants $(c_j : j \in \mathbb{N})$, we may write

$$\Delta(\mathbf{x})^2 \exp\left(-\sum_i x_i^2/2\right) = \left(\prod_{i=1}^{N-1} c_i^2\right)\left(\det\left[c_{j-1}^{-1} P_{j-1}(x_i) e^{-x_i^2/4}\right]_{i,j=1}^N\right)^2.$$

For the $(x_1, \dots, x_r)$ marginals, we wish to integrate out the variables $x_{r+1}, \dots, x_N$. To have a simple outcome, perhaps we set

$$\psi_i(x) = c_i P_i(x) e^{-x^2/4},$$

so that

$$\Delta(\mathbf{x})^2 \exp\left(-\sum_i x_i^2/2\right) = \left(\prod_{i=0}^{N-1} c_i^2\right)\left(\det\left[\psi_{j-1}(x_i)\right]_{i,j=1}^N\right)^2, \tag{4.2}$$

and require

$$\int \psi_i(x)\psi_j(x)dx = 0, \text{ if } i \neq j, \qquad \int \psi_i^2(x)dx = 1. \tag{4.3}$$

Equivalently

$$\int P_i(x)P_j(x)e^{-x^2/2}dx = 0, \text{ if } i \neq j, \qquad c_i^2 = \int P_i^2(x)e^{-x^2/2}dx. \tag{4.4}$$

Hence, we may try to find an orthogonal basis for $L^2(e^{-x^2/2}dx)$ consisting of the polynomials $\{P_i : i = 0, 1, \dots\}$. In fact the first condition in (4.4) is satisfied if

$$\int P_i(x)x^j e^{-x^2/2}dx = 0, \text{ for } \quad j < i. \tag{4.5}$$

This would be the case if $P_i(x)e^{-x^2/2}$ is an exact $i - th$ derivative of a function and the celebrated Hermite polynomials given by

$$P_i(x) = (-1)^i e^{x^2/2} \frac{d^i}{dx^i} e^{-x^2/2}, \tag{4.6}$$

certainly satisfy (4.5). Moreover

$$c_i^2 = \int P_i^2(x)e^{-x^2/2}dx = \int P_i(x)x^i e^{-x^2/2}dx = i! \int e^{-x^2/2}dx = i!\sqrt{2\pi}. \tag{4.7}$$

We are now ready to state our first result.

Theorem 4.1 *The r-dimensional marginals of $\hat{\mu}_N$ are given by $p_N^{(r)}(x_1, \ldots, x_r) \prod_{i=1}^r dx_i$ with*

$$p_N^{(r)}(x_1, \ldots, x_r) = \frac{(N-r)!}{N!} \det \left[\sum_{k=1}^N \psi_{k-1}(x_i)\psi_{k-1}(x_j) \right]_{i,j=1}^r . \tag{4.8}$$

(When $r = 1$, we simply have that $p_N^{(1)}(x_1) = N^{-1}K_N^2(x_1)$, where $K_N(x) = \sum_{k=1}^N \psi_{k-1}^2(x)$.)

Proof. From (4.2)

$$\begin{aligned} p_N^{(r)}(x_1, \ldots, x_r) &= \frac{1}{Z_N'} \int \cdots \int \Delta(\mathbf{x})^2 \exp\left(-\sum_i x_i^2/2\right) \prod_{\ell=r+1}^N dx_\ell \\ &= \frac{1}{Z_N''} \int \cdots \int \left(\det\left[\psi_{j-1}(x_i)\right]_{i,j=1}^N\right)^2 \prod_{\ell=r+1}^N dx_\ell \\ &= \frac{1}{Z_N''} \int \cdots \int \sum_{\sigma,\tau \in S_N} \varepsilon(\sigma)\varepsilon(\tau) \prod_{i=1}^N \psi_{\sigma(i)-1}(x_i)\psi_{\tau(i)-1}(x_i) \prod_{\ell=r+1}^N dx_\ell, \end{aligned} \tag{4.9}$$

where S_N denotes the set of permutations of $\{1, \ldots, N\}$, $\varepsilon(\sigma)$ is the sign of the expression $\prod_{i<j}(\sigma(i) - \sigma(j))$, and

$$Z_N'' = Z_N' \left(\prod_{i=0}^{N-1} c_i^2 \right)^{-1} . \tag{4.10}$$

We note that if $\sigma(i) \neq \tau(i)$ for some $i > r$, the dx_i integration would be 0 in (4.9); otherwise the integral is 1. Hence the nonzero contributions in (4.9) come from pairs (σ, τ) such that

$$\sigma(i) = \tau(i) \text{ for } i > r, \quad \text{and } \{\sigma(1), \ldots, \sigma(r)\} = \{\tau(1), \ldots, \tau(r)\}.$$

For such a pair, let order the elements of $\{\sigma(1), \ldots, \sigma(r)\}$ as $\alpha_1 < \ldots, \alpha_r$. Note that if we fix $\alpha_1 < \cdots < \alpha_r$, then the restriction of σ and τ can be regarded as two permutations σ' and τ' and the there are $(N-r)!$ choices for the the restriction of σ or τ to the complement of $\{\alpha_1, \ldots, \alpha_r\}$. As a result,

$$p_N^{(r)}(x_1, \ldots, x_r) = \frac{(N-r)!}{Z_N''} \sum_{1 \le \alpha_1 < \cdots < \alpha_r \le N} \sum_{\sigma', \tau' \in S_r} \varepsilon(\sigma')\varepsilon(\tau') \prod_{i=1}^{r} \psi_{\alpha_{\sigma(i)}-1}(x_i)\psi_{\alpha_{\tau(i)}-1}(x_i)$$

$$(4.11) \qquad = \frac{(N-r)!}{Z_N''} \sum_{1 \le \alpha_1 < \cdots < \alpha_r \le N} \left(\det \left[\psi_{\alpha_j - 1}(x_i)\right]_{i,j=1}^{r}\right)^2.$$

Let us write $A = [\psi_{j-1}(x_i)]_{i,j=1}^{N}$. Recall that by Cauchy-Binnet's formula

$$(4.12) \qquad \Lambda_r(AA^t) = \Lambda_r(A)\Lambda_r(A)^t =: B.$$

In fact for the index set $I = \{1, \ldots, r\}$, we have $b_{II} = \sum_J (\det A_{IJ})^2$, where the summation is over all index sets $J \subset \{1, \ldots, N\}$ of size r and recall that $A_{IJ} = [a_{ij}]_{i \in I, j \in J}$. But this sum is exactly (4.11). From this and (4.12) we deduce

$$(4.13) \qquad p_N^{(r)}(x_1, \ldots, x_r) = \frac{(N-r)!}{Z_N''} \det \left[\sum_{k=1}^{N} \psi_{k-1}(x_i)\psi_{k-1}(x_j)\right]_{i,j=1}^{r}.$$

It remains to verify $Z_N'' = N!$. This is obvious because in the calculations (4.9) and (4.11) we could have chosen $r = 0$ and integrate out all variables. For $r = 0$, the left hand side of (4.11) is simply 1 and the right-hand side is $N!/Z_N''$, completing the proof of (4.8). ☐

Remark 4.1. In our proof of Theorem 4.1, we managed to give a new proof of (4.1). In fact (4.1) is an immediate consequence of $Z_N'' = N!$, (4.10) and (4.7). ☐

To analyze the r-point correlations of GUE in large N limit, we need to study the kernel

$$K_N(x, y) = \sum_{k=0}^{N-1} \psi_k(x)\psi_k(y) = (2\pi)^{-1/2} \sum_{k=0}^{N-1} (k!)^{-1} P_k(x)P_k(y)e^{-(x^2+y^2)/4},$$

that is the kernel associated with the projection onto the span of $\{\psi_1, \ldots, \psi_{N-1}\}$. Let us state some useful properties of Hermite polynomials.

Lemma 4.1

- *(i)* $P_{k+1}(x) = xP_k(x) - P_k'(x)$.
- *(ii)* $xP_k(x) = P_{k+1}(x) + kP_{k-1}(x)$.
- *(iii)* $P_k''(x) - xP_k'(x) = -kP_k(x)$.
- *(iv) (Christoffel-Darboux Formula) For* $x \neq y$,

$$\sum_{k=0}^{N-1} \frac{P_k(x)P_k(y)}{k!} = \frac{P_N(x)P_{N-1}(y) - P_{N-1}(x)P_N(y)}{(N-1)!(x-y)}.$$

Proof. (*i*) This follows from differentiating $de^{-x^2/2}/dx^k = (-1)^k P_k(x)e^{-x^2/2}$.

(*ii*) Let us write $< f >$ for $\int f(x)e^{-x^2/2}dx$. Since P_k is an orthogonal basis for $L^2(e^{-x^2/2}dx)$, we have

$$xP_k(x) = \sum_\ell \frac{< xP_k(x)P_\ell(x) >}{< P_\ell^2(x) >} P_\ell(x). \tag{4.14}$$

By (4.5), the only nonzero terms are when $\ell = k-1, k, k+1$. Again by (4.5),

$$\begin{aligned} &< xP_k(x)P_{k-1}(x) > =< P_k(x)\,(xP_{k-1}(x)) >=< P_k(x)^2 >, \\ &< xP_k(x)P_{k+1}(x) > =< P_{k+1}(x)^2 >, \\ &\quad < xP_k(x)P_k(x) > = 0, \end{aligned}$$

where for the third line we used the fact that P_k^2 is even. From this (4.14) and and (4.7) we deduce (ii).

(*iii*) From (i) and (ii) we deduce that $P_k'(x) = kP_{k-1}(x)$. On the other hand, from differentiating (i),

$$P_k''(x) - xP_k'(x) = P_k(x) - P_{k+1}'(x) = P_k(x) - (k+1)P_k(x) = -kP_k(x).$$

(*iv*) Note that $K_N(x,y)$ is the projection kernel for the space spanned by $\{\psi_0, \dots, \psi_N\}$ and behaves like the δ-function as $N \to \infty$. Hence K_N becomes singular when $x = y$ in large N limit. Let us multiply K_N by $x - y$ and use (ii) to get an expression in terms of the Hermite polynomials. Indeed by (ii), the expression $(x-y)P_k(x)P_k(y)/(k!)$ equals

$$(k!)^{-1}\left[P_{k+1}(x)P_k(y) + kP_{k-1}(x)P_k(y) - P_{k+1}(y)P_k(x) - kP_{k-1}(y)P_k(x)\right] = X_{k+1} - X_k,$$

where

$$X_{k+1} = (k!)^{-1}\left[P_{k+1}(x)P_k(y) - P_{k+1}(y)P_k(x)\right].$$

This completes the proof of (iii). □

Remark 4.2. From part (iii) we know that P_k is the eigenfunction of the Ornstein-Uhlenbeck operator $L = d^2/dx^2 - xd/dx$ associated with the eigenvalue $-k$. In fact ψ_k is also an eigenfunction for the Schrödinger operator $d^2/dx^2 - x^2/4$ associated with the eigenvalue $-k - 1/2$. Here is the reason,

$$\psi_k''(x) = c_k^{-1}\left[P_k''(x) - \frac{1}{2}P_k(x) - xP_k'(x) + \frac{x^2}{4}P_k(x)\right] = -\left(k + \frac{1}{2}\right)\psi_k(x) + \frac{x^2}{4}\psi_k(x).$$

□

Lemma 4.1 (iv) yields

$$K_N(x,y) = \sqrt{N}\frac{\psi_N(x)\psi_{N-1}(y) - \psi_{N-1}(x)\psi_N(y)}{x-y}, \tag{4.15}$$

when $x \neq y$. The limit of this as $y \to x$ yields

$$K_N(x,x) = \sqrt{N}\left(\psi'_N(x)\psi_{N-1}(x) - \psi'_{N-1}(x)\psi_N(x)\right), \tag{4.16}$$

But $x_i = \sqrt{N}\lambda_i$ are unnormalized eigenvalues and if we expressed our marginals in terms of $\lambda)i$'s we obtain

$$\begin{aligned} p_N^{(r)}(x_1,\ldots,x_r)dx_1\ldots dx_r &= N^{r/2}p_N^{(r)}(\sqrt{N}\lambda_1,\ldots,\sqrt{N}\lambda_r)d\lambda_1\ldots d\lambda_r \\ &= \frac{(N-r)!}{N!}\det\left[N^{1/2}K_N(\sqrt{N}\lambda_i,\sqrt{N}\lambda_j)\right]_{i,j=1}^{r} d\lambda_1\ldots d\lambda_r. \end{aligned} \tag{4.17}$$

Let us focus on eigenvalues near the energy $E=0$. Since the typical distance between two consecutive eigenvalues is of order $O(N^{-1})$, we may multiply the eigenvalues by N to get $\alpha_i = N\lambda_i = \sqrt{N}x_i$. In terms of $\alpha_1,\ldots,\alpha_N$,

$$\begin{aligned} p_N^{(r)}(x_1,\ldots,x_r)dx_1\ldots dx_r &= N^{-r/2}p_N^{(r)}(\alpha_1/\sqrt{N},\ldots,\alpha_r/\sqrt{N})d\alpha_1\ldots d\alpha_r \\ &= \frac{(N-r)!}{N!}\det\left[N^{-1/2}K_N(\alpha_i/\sqrt{N},\alpha_j/\sqrt{N})\right]_{i,j=1}^{r} d\alpha_1\ldots d\alpha_r. \end{aligned} \tag{4.18}$$

For r-correlations, we are interested in observables that involve exactly r many particles. Since there are exactly $N(N-1)\ldots(N-r+1)$ many r particles, we consider

$$\begin{aligned} \hat{p}_N^{(r)}(\alpha_1,\ldots,\alpha_r) &= \frac{N!}{(N-r)!}N^{-r/2}p_N^{(r)}(\alpha_1/\sqrt{N},\ldots,\alpha_r/\sqrt{N}) \\ &= \det\left[N^{-1/2}K_N(\alpha_i/\sqrt{N},\alpha_j/\sqrt{N})\right]_{i,j=1}^{r}. \end{aligned}$$

Hence we need to study the large N limit of

$$\hat{p}_N^{(r)}(\alpha_1,\ldots,\alpha_r) = \det\left[\hat{K}_N(\alpha_i,\alpha_j)\right]_{i,j=1}^{r}, \tag{4.19}$$

where

$$\hat{K}_N(\alpha_1,\alpha_2) = \sqrt{N}\frac{\psi_N(\alpha_1/\sqrt{N})\psi_{N-1}(\alpha_2/\sqrt{N}) - \psi_{N-1}(\alpha_1/\sqrt{N})\psi_N(\alpha_2/\sqrt{N})}{\alpha_1-\alpha_2}, \tag{4.20}$$

when $\alpha_1 \neq \alpha_2$. Moreover,

$$\hat{K}_N(\alpha,\alpha) = \psi'_N(\alpha/\sqrt{N})\psi_{N-1}(\alpha/\sqrt{N}) - \psi'_{N-1}(\alpha/\sqrt{N})\psi_N(\alpha/\sqrt{N}). \tag{4.21}$$

Theorem 4.2 yields the correlations in N large limit.

Theorem 4.2 *For every $r \geq 2$, the r-dimensional marginals densities $\hat{p}_N^{(r)}$ converge to*

$$p^{(r)}(\alpha_1, \ldots, \alpha_r) = \det\left[\hat{K}(\alpha_i, \alpha_j)\right]_{i,j=1}^r, \tag{4.22}$$

where $\hat{K}(\alpha_1, \alpha_2) = \sin(\pi(\alpha_1 - \alpha_2))/(\pi(\alpha_1 - \alpha_2))$.

Theorem 4.2 is an immediate consequence of Lemma 4.2.

Lemma 4.2 *We have $\lim_{N\to\infty} \hat{K}_N = \hat{K}$, locally uniformly, where $\hat{K}_N$ was defined in* (4.20) *and K was defined in Theorem 4.2.*

Proof. First observe that if $f(x) = N^{1/4}\psi_N(x/\sqrt{N})$ and $g(x) = N^{1/4}\psi_{N-1}(x/\sqrt{N})$, then

$$\hat{K}_N(\alpha_1, \alpha_2) = g(\alpha_2)\int_0^1 f'(t\alpha_1 + (1-t)\alpha_2)dt - f(\alpha_2)\int_0^1 g'(t\alpha_1 + (1-t)\alpha_2)dt. \tag{4.23}$$

On the other hand, we may use Lemma 4.1(i) to write

$$\psi_k'(x) = -\frac{x}{2}\psi_k(x) + \sqrt{k}\psi_{k-1}(x).$$

This would allow us to replace all the derivatives in (4.23) with expressions involving ψ_k's. Hence for the Lemma, we only need to study the asymptotic behavior of $N^{1/4}\psi_k(x/\sqrt{N})$ for $k = N, N-1, N-2$. This will be carried out in Lemma 4.3. □

Lemma 4.3 *We have*

$$\lim_{N\to\infty}\left|N^{1/4}\psi_n(x/\sqrt{N}) - \pi^{-1/2}\cos\left(x - \frac{n\pi}{2}\right)\right| = 0,$$

locally uniformly, where $n = N - \ell$ for a fixed ℓ.

Proof. First note

$$(-1)^n\frac{d^n}{dx^n}e^{-x^2/2} = (-1)^n\frac{d^n}{dx^n}\int(2\pi)^{-1/2}e^{-ix.\xi}e^{-\xi^2/2}d\xi = \int(2\pi)^{-1/2}(i\xi)^n e^{-ix.\xi}e^{-\xi^2/2}d\xi.$$

Hence

$$\begin{aligned}
N^{1/4}\psi_n(x/\sqrt{N}) &= (2\pi)^{-3/4}(n!)^{-1/2}e^{x^2/(4N)}N^{1/4}\int(i\xi)^n e^{-ix\xi/\sqrt{N}}e^{-\xi^2/2}d\xi\\
&= (2\pi)^{-3/4}(n!)^{-1/2}e^{x^2/(4N)}N^{n/2+3/4}\int(i\xi)^n e^{-ix\xi}e^{-N\xi^2/2}d\xi\\
&\approx (2\pi)^{-3/4}(N!)^{-1/2}N^{N/2+3/4}\int\left(\xi e^{-\xi^2/2}\right)^N i^n\xi^{n-N}e^{-ix\xi}d\xi\\
&\approx (2\pi)^{-1}e^{N/2}N\int\left(\xi e^{-\xi^2/2}\right)^N i^n\xi^{n-N}e^{-ix\xi}d\xi\\
&= (2\pi)^{-1}e^{N/2}N^{1/2}\int\left(\xi e^{-\xi^2/2}\right)^N Re\left(i^n e^{-ix\xi}\right)\xi^{n-N}d\xi,
\end{aligned}$$

where we used $e^{x^2/N} \approx 1$ for the third line, the Stirling's formula $N! \approx N^{N+1/2}e^{-N}\sqrt{2\pi}$ for the last line and use the fact that ψ_n is real for the last line. We now argue that the integrand is an even function of ξ. To see this, observe that if

$$f(\xi) = Re\left(i^n e^{-ix\xi}\right) = \cos\left(x\xi - \frac{n\pi}{2}\right),$$

then $f(-\xi) = (-1)^n f(\xi)$. As a result, the function $f(\xi)\xi^n$ is even and

$$N^{1/4}\psi_n(x/\sqrt{N}) \approx 2(2\pi)^{-1}e^{N/2}N^{1/2}\int_0^\infty \left(\xi e^{-\xi^2/2}\right)^N \cos\left(x\xi - \frac{n\pi}{2}\right)\xi^{n-N}d\xi,$$

$$= \pi^{-1}e^{N/2}N^{1/2}\int_0^\infty F(\xi)^N G(\xi)d\xi, \tag{4.24}$$

where

$$F(\xi) = \xi e^{-\xi^2/2}, \quad G(\xi) = \cos\left(x\xi - \frac{n\pi}{2}\right)\xi^{n-N}.$$

Note that $n - N = -\ell$ is constant and the function G is independent of N. We now apply the Laplace's method to find the asymptotic of (4.24). Note that $\max F = e^{-1/2}$ and it is achieved at $\xi = 1$. Near $\xi = 1$, the function $F(\xi) = \exp(\log\xi - \xi^2/2)$ looks like

$$e^{-1/2}e^{-(\xi-1)^2}.$$

Since $G(1) = \cos(xi - (n\pi)/2)$, we deduce

$$\int_0^\infty F(\xi)^N G(\xi)d\xi \approx e^{-N/2}\cos\left(x - \frac{n\pi}{2}\right)\int_{|\xi-1|\le\delta} e^{-N(\xi-1)^2}d\xi \approx \sqrt{\pi}e^{-N/2}\cos\left(x - \frac{n\pi}{2}\right)N^{-1/2}.$$

This and (4.24) complete the proof. □

Theorem 4.2 deals with the eigenvalues near the origin. More generally we may look at the eigenvalues near an energy level E. For $E \in (-2, 2)$, we expect to have the same scaling. Since the gaps between particles are inversely proportional to the density, it is more convenient to rescale as

$$\lambda_i = E + \frac{\alpha_i}{N\rho(E)}, \quad x_i = \sqrt{N}E + \frac{\alpha_i}{\sqrt{N}\rho(E)}$$

where $\rho(E) = (2\pi)^{-1}\sqrt{4 - E^2}$. Since, $dx_i = N^{-1/2}\rho(E)^{-1}d\alpha_i$, we define

$$\hat{p}_N^{(r)}(\alpha_1, \dots, \alpha_r; E) = \frac{N!}{(N-r)!}N^{-r/2}\rho(E)^{-r}p_N^{(r)}\left(\sqrt{N}E + \frac{\alpha_1}{\rho(E)\sqrt{N}}, \dots, \sqrt{N}E + \frac{\alpha_r}{\rho(E)\sqrt{N}}\right)$$

$$= \det\left[N^{-1/2}\rho(E)^{-1}K_N\left(\sqrt{N}E + \frac{\alpha_i}{\rho(E)\sqrt{N}}, \sqrt{N}E + \frac{\alpha_j}{\rho(E)\sqrt{N}}\right)\right]_{i,j=1}^r.$$

The generalization of Theorem 4.2 in this case is Theorem 4.3.

Theorem 4.3 *Assume that $E \in (-2,2)$. For every $r \geq 2$, the r-dimensional marginals densities $\hat{p}_N^{(r)}(\cdot;E)$ converge to*

$$p^{(r)}(\alpha_1,\dots,\alpha_r) = (r!)^{-1}\det\left[K(\alpha_i,\alpha_j)\right]_{i,j=1}^r, \tag{4.25}$$

where $K(\alpha_1,\alpha_2) = \sin(\alpha_1-\alpha_2)/(\alpha_1-\alpha_2)$.

Note that Theorem 4.3 becomes Theorem 4.2 when $E=0$ because $\rho(0)=\pi^{-1}$. To prove Theorem 4.3, define

$$K_N(\alpha_1.\alpha_2;E) = \rho(E)^{-1}N^{-1/2}K_N\left(\sqrt{N}E+\frac{\alpha_1}{\rho(E)\sqrt{N}},\sqrt{N}E+\frac{\alpha_2}{\rho(E)\sqrt{N}}\right).$$

We now need to show

Lemma 4.4 *Assume that $E\in(-2,2)$. We have $\lim_{N\to\infty}K_N(\cdot;E)=K$, locally uniformly, where K was defined in Theorem 4.3.*

As in the proof of Lemma 4.2, it suffices to show

Lemma 4.5 *Assume that $E\in(-2,2)$. We have*

$$\lim_{N\to\infty}\left|N^{1/4}\psi_n\left(\sqrt{N}E+\frac{x}{\rho(E)\sqrt{N}}\right)-\rho(E)\cos\left(x-\frac{n\pi}{2}\right)\right|=0,$$

locally uniformly, where $n=N-\ell$ for a fixed ℓ.

If we try to mimic the proof of Lemma 4.3, we run into a difficulty because of the appearance of the factor $\exp(iNE\xi/\rho(E))$. More precisely, The function F in (4.24) now takes the form

$$\xi e^{-\xi^2/2}e^{iNE\xi/\rho(E)}.$$

For such a function F, the method of Laplace is no longer available and we need to apply the so-called *steepest descent* to handle an oscillatory F. Before explaining this method, let us discuss the behavior of eigenvalues near the edge for which the same method may be used.

We now turn to the eigenvalues correlation near the edges. By Semicircle Law we expect to have

$$\sharp\{\lambda_i:\lambda_i\geq 2-\varepsilon\}\approx\frac{N}{2\pi}\int_{2-\varepsilon}^2\sqrt{4-x^2}\,dx=\frac{2}{3\pi}N\varepsilon^{3/2}.$$

To have finitely many eigenvalues in $(2-\varepsilon,\infty)$, we choose $\varepsilon=O(N^{-3/2})$. This suggests setting $\lambda_i=2+\alpha_i'N^{-2/3}$, or $x_i=2\sqrt{N}+\alpha_i'N^{-1/6}$ and looking at

$$\begin{aligned}p_N^{(r)}(x_1,\dots,x_r)dx_1\dots dx_r &= N^{-r/6}p_N^{(r)}(2\sqrt{N}+\alpha_1'N^{-1/6},\dots,2\sqrt{N}+\alpha_r'N^{-1/6})d\alpha_1'\dots d\alpha_r'\\ &=\frac{(N-r)!}{N!}\det\left[N^{-1/6}K_N(2\sqrt{N}+\alpha_i'N^{-1/6},2\sqrt{N}+\alpha_j'N^{-1/6})\right]_{i,j=1}^r d\alpha_1'\dots d\alpha_r'.\end{aligned} \tag{4.26}$$

Again since we are interested in observables of any r particles, we consider

$$\begin{aligned}\tilde{p}_N^{(r)}(\alpha'_1, \ldots, \alpha'_r) &= N^{-r/6} p_N^{(r)}(2\sqrt{N} + \alpha'_1 N^{-1/6}, \ldots, 2\sqrt{N} + \alpha'_r N^{-1/6}) \\ &= \det \left[\tilde{K}_N(\alpha'_i, \alpha'_j)\right]_{i,j=1}^{r}.\end{aligned}$$

where

$$\tilde{K}_N(\alpha_1, \alpha_2) = N^{-1/6} K_N(2\sqrt{N} + \alpha_1 N^{-1/6}, 2\sqrt{N} + \alpha_2 N^{-1/6}).$$

Theorem 4.4 *For every $r \geq 2$, the r-dimensional marginals densities $\tilde{p}_N^{(r)}$ converge to*

$$\tilde{p}^{(r)}(\alpha_1, \ldots, \alpha_r) = \det \left[\tilde{K}(\alpha_i, \alpha_j)\right]_{i,j=1}^{r}, \tag{4.27}$$

where

$$\tilde{K}(\alpha_1, \alpha_2) = \frac{Ai(\alpha_1)Ai'(\alpha_2) - Ai'(\alpha_1)Ai(\alpha_2)}{\alpha_1 - \alpha_2}.$$

and $Ai(x) = \pi^{-1} \int_0^\infty \cos(t^3/3 + xt)dt$ is the Airy function.

The main ingredient for the proof of Theorem 4.4 is Lemma 4.6 below.

Lemma 4.6 *For every positive C,*

$$\lim_{N\to\infty} \sup_{z\in\mathbb{C}, |z|\leq C} \left| N^{1/12}\psi_N\left(2\sqrt{N} + \frac{z}{N^{1/6}}\right) - Ai(z)\right| = 0.$$

Here by $Ai(z)$ we mean

$$\lim_{\ell\to\infty} \frac{1}{2\pi} \int_{-\ell}^{\ell} e^{-it^3/3 - izt} dt.$$

Observe that if $z \in \mathbb{R}$, then

$$\lim_{\ell\to\infty} \frac{1}{2\pi} \int_{-\ell}^{\ell} e^{-it^3/3-izt}dt = \lim_{\ell\to\infty} \frac{1}{2\pi} \int_{-\ell}^{\ell} \cos(-t^3/3 - zt)dt = \lim_{\ell\to\infty} \frac{1}{\pi} \int_0^{\ell} \cos(t^3/3 + zt)dt,$$

which is the definition we gave previously for the Airy function.

Proof of Theorem 4.4. Using Lemma 4.1(i) and (ii), we have that $P'_N = NP_{N-1}$. As a result

$$\psi'_N(x) = -\frac{x}{2}\psi_N(x) + \sqrt{N}\psi_{N-1}(x).$$

From this we learn

$$\begin{aligned}K_N(\alpha_1, \alpha_2) &= \frac{\psi_N(\alpha_1)\psi'_N(\alpha_2) - \psi_N(\alpha_2)\psi'_N(\alpha_1)}{\alpha_1 - \alpha_2} - \frac{1}{2}\psi_N(\alpha_1)\psi_N(\alpha_2), \\ \tilde{K}_N(\alpha_1, \alpha_2) &= \frac{Ai_N(\alpha_1)Ai'_N(\alpha_2) - Ai_N(\alpha_2)Ai'_N(\alpha_1)}{\alpha_1 - \alpha_2} - \frac{1}{2N^{2/3}} Ai_N(\alpha_1)Ai_N(\alpha_2).\end{aligned}$$

where

$$Ai_N(\alpha) = N^{1/12}\psi_N(2\sqrt{N} + \alpha N^{-1/6})$$

We are done if we can show

$$\lim_{N\to\infty} Ai_N = Ai, \quad \lim_{N\to\infty} N^{-1/6}Ai'_N = Ai',$$

locally uniformly. Since Ai_N and Ai are entire function, it suffices to establish the first limit and this is the content of Lemma 4.6. □

As we mentioned earlier, for the proof of Lemma 4.3, the Laplace's method does not apply and we need to appeal to the method of steepest descent. The same comment applies to the proof of Lemma 4.6. Before embarking on the proof of Lemma 4.6, let us explain what the method of steepest descent is and why is needed here. Recall that if we have two real-valued functions F and G and N is large, then the main contribution in the integral $\int_I e^{NF} G d\xi$, comes from points $\xi \in I$ at which F takes its largest value. To simplify the presentation, let us assume that the maximum of F in I is achieved at a single point ξ_0 in the interior of I and that $F''(\xi_0) < 0$. Then the method of Laplace is applicable and yields

$$\int_I e^{NF} G d\xi \approx e^{NF(\xi_0)}\sqrt{2\pi}(-NF''(\xi_0))^{-1/2}G(\xi_0). \tag{4.28}$$

Now imagine that the function $F = A + iB$ that appears on the left-hand side of (4.28) is complex valued. Our first guess would be that the main contribution in this integral comes from points ξ_0 at which the amplitude $|e^F| = e^A$ takes its maximum value. Near such a point ξ_0,

$$F(\xi) \approx F(\xi_0) + iB'(\xi_0)(\xi - \xi_0) + \frac{1}{2}(A''(\xi_0) + iB''(\xi_0))(\xi - \xi_0)^2.$$

Hence

$$\begin{aligned}\int_{|\xi-\xi_0|\le\delta} e^{NF} G d\xi &\approx G(\xi_0)\exp\left[N\left(F(\xi_0) - \frac{B'(\xi_0)^2}{2a_0}\right)\right]\int \exp\left[\frac{Na_0}{2}\left(\xi + 2i\frac{B'(\xi_0)}{a_0}\right)^2\right] \\ &= G(\xi_0)\exp\left[N\left(F(\xi_0) - \frac{B'(\xi_0)^2}{2a_0}\right)\right]\left(\frac{2\pi}{-Na_0}\right)^{1/2}\end{aligned}$$

where $a_0 = A''(\xi_0) + iB''(\xi_0)$, we have taken the standard branch of square root, and have used the fact that $Re\ a_0 < 0$ (see Exercise 4.1 below). Note that if $B'(\xi_0) \neq 0$, then the phase e^{iB}, changes the exponential term $e^{NF(\xi_0)}$ and it is not clear that the integral near ξ_0 is giving the dominant contribution. This problem would not arise if $B'(\xi_0) = 0$ i.e. ξ_0 is a stationary phase point. If F happens to be an analytic function, then it is more convenient to think of the integral as a contour integral with the contour given by a parametrization of I. For our purposes, we assume that $I = (-\infty, \infty)$. The point is that now the condition $A'(\xi_0) = B'(\xi_0) = 0$ simply means that $F'(\xi_0) = 0$ and near ξ_0,

$$F(\xi) \approx F(\xi_0) + \frac{1}{2}F''(\xi_0)(\xi - \xi_0)^2.$$

The only problem is that if we insist on finding a point ξ_0 at which $F'(\xi_0) = 0$, the point ξ_0 may not lies on the real axis. On the other hand, we may apply Cauchy's formula to deform our contour γ to pass through ξ_0 and we try to choose our deformed contour so that the along this contour Laplace's method applies and the main contribution comes from the ξ_0-nearby points. This method is also called *saddle point* method because if we set $z = x + iy = \xi - \xi_0$, then $z^2 = x^2 - y^2 + i2xy$ and 0 is a saddle critical point for the functions $x^2 - y^2$ and $2xy$. So, in principle, we try to deform our contour to pass through a saddle point and we do this so that along γ, the phase stays stationary as much as possible while amplitude reaches its largest value. Since F is analytic, the level sets of $Re\ F$ are perpendicular to the level sets of $Im\ F$. So, moving along $Im\ F = c$ near ξ_0 would do the job. In other words, we start with a nearby valley of ReF, move along a level set of $Im\ F$ to reach ξ_0 and continue along a *steepest descent* path to keep the *phase stationary*. To have a simple example, imagine that we want to study the large N limit of $\int_{-\infty}^{\infty} e^{iNx^2} dx$. The analytic function $F(z) = iz^2 = -2xy + i(x^2 - y^2)$ has its only critical point at 0.

Exercise 4.1 Let a and z_0 be two complex numbers with $Re\ a > 0$. Show

$$\int_{-\infty}^{\infty} e^{-a(\xi+z_0)^2/2} d\xi = \sqrt{\frac{2\pi}{a}},$$

where we take the standard branch of square root for $\sqrt{a}$. *Hint :* Write the integral as an integral over a line in $\mathbb{C}$ that passes through $z_0\sqrt{a}$ and makes the angle $arg\ \sqrt{a}$ with the x-axis. Then use Cauchy's theorem to replace this line with the x-axis. □

Proof of Lemma 4.6. As in the proof of Lemma 4.3, for $w = 2N^{1/2} + zN^{-1/6}$,

$$\psi_N(w) = -i(2\pi)^{-3/4}(N!)^{-1/2} e^{w^2/4} \int_{-i\infty}^{i\infty} \xi^N e^{\xi^2/2 - w\xi} d\xi.$$

Two large exponents appear in the integrand, N and w. Since they are not of the same order, we try to replace the contour of integration $i\mathbb{R}$ with the tilted line $L = \{w\zeta : \zeta \in i\mathbb{R}\}$. Note that w has a large real part and $iw \in L$. So, the line L makes a small angle with the imaginary axis. Now if we apply Cauchy's formula, for such a replacement we need to make sure that

$$\lim_{\ell \to \infty} \int_{S_\ell} \xi^N e^{\xi^2/2 - w\xi} d\xi = 0,$$

where S_ℓ is the line segment $\{\mp y + (\pm\ell)i : 0 \le y \le \beta\ell\}$ where β is the tangent of the angle between L and the imaginary axis and β is small when N is large. This follows from the fact that for a constant c_N,

$$\left|\int_{S_\ell} \xi^N e^{\xi^2/2 - w\xi} d\xi\right| \le (\sqrt{2}\ell)^N \int_0^{\beta\ell} e^{-(\ell^2 - y^2)/2 + c_N \ell} dy \to 0,$$

as $\ell \to \infty$. Hence

$$\psi_N(w) = -i(2\pi)^{-3/4}(N!)^{-1/2} e^{w^2/4} w^{N+1} \int_{-i\infty}^{i\infty} \zeta^N e^{w^2(\zeta^2/2 - \zeta)} d\zeta. \tag{4.29}$$

Since $w^2 = 4N + O(N^{-1/3})$, we see that the integrand is now of the form $\exp[NR(\zeta) + O(N^{2/3})]$ for the function $R(\zeta) = \log\zeta + 2\zeta^2 - 4\zeta$. Since $R'(\zeta) = (2\zeta - 1)^2$, the function R has a single critical point $1/2$ and our contour $i\mathbb{R}$ does not pass through this critical point. We once more apply Cauchy's formula to replace $i\mathbb{R}$ in our integral with $i\mathbb{R} + 1/2$. This is possible because by $Re\ w > 0$, the integration over the line segment $\{\pm\ell i + x : 0 \le x \le 1/2\}$ goes to 0 as $\ell \to \infty$. As a result

$$\begin{aligned}
\psi_N(w) &= -i(2\pi)^{-3/4}(N!)^{-1/2} e^{-w^2/8} w^{N+1} \int_{-i\infty}^{i\infty} \left(\zeta + \frac{1}{2}\right)^N e^{w^2(\zeta^2 - \zeta)/2} d\zeta \\
&= -i(2\pi)^{-3/4}(N!)^{-1/2} e^{-w^2/8} \left(\frac{w}{2}\right)^{N+1} \int_{-i\infty}^{i\infty} (\zeta + 1)^N e^{(w/2)^2(\zeta^2/2 - \zeta)} d\zeta \\
&= -i(2\pi)^{-3/4}(N!)^{-1/2} e^{-w^2/8} \left(\frac{w}{2}\right)^{N+1} \int_{-i\infty}^{i\infty} e^{(w/2)^2 F(\zeta) + N' \log(1+\zeta)} d\zeta,
\end{aligned}$$

where $F(\zeta) = \log(1+\zeta) + \zeta^2/2 - \zeta$ and $N' = N - (w/2)^2 = O(N^{-1/3})$. Note that now the only critical point of F is 0 and the contour $i\mathbb{R}$ does pass through this critical point. Note that by Stirling's formula,

$$\begin{aligned}
(N!)^{-1/2} e^{-w^2/8} \left(\frac{w}{2}\right)^{N+1} &\approx (2\pi)^{-1/4} N^{1/4} \left(1 + \frac{z}{2N^{2/3}}\right)^{N+1} \exp\left(-\frac{z}{2N^{2/3}}\right) \\
&\approx (2\pi)^{-1/4} N^{1/4},
\end{aligned}$$

uniformly over z satisfying $|z| \le C$. As a result,

$$N^{1/12}\psi_N(w) \approx N^{1/3} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{(w/2)^2 F(\zeta) + N' \log(1+\zeta)} d\zeta. \tag{4.30}$$

Note that $Re\ F(it) = \frac{1}{2}(\log(1+t^2) - t^2)$ is negative and attains its maximum value at $t = 0$. But $Im\ F(it) = tan^{-1}t - t$ is nonzero and results in an oscillatory integral. As the saddle point method suggests, we now try to deform the contour $i\mathbb{R}$ to η so that along η, F is real and negative. For example try a curve γ which solves the equation $F(\gamma(t)) = -t$ for $t > 0$. This would replace $i\mathbb{R}^+$. For the rest $i\mathbb{R}^-$, we use $-\bar{\gamma}$. Let us first find such a curve γ. Observe that near 0, the function F looks like $\zeta^3/3 - \zeta^4/4 + \dots$. If we take the wedge $W = \{\rho e^{i\theta} : \theta \in [\pi/3, \pi/2]\}$, then F behaves nicely on the boundary of W. In fact, $F(it) = -it^3/3 - t^4/4 + \dots$, $Re\ F(it) < 0$ is negative for $t > 0$ and $Im\ F(it)$ is also negative with $Im\ F(it) + t$ bounded by $\pi/2$. Also observe that $F(te^{i\pi/3}) = -t^3/3 + t^4 e^{i4\pi/3}/4 + \dots$ lies fully in the second quadrant. We wish to show that the function $F(\zeta) + t$ has a unique root in the wedge W. To see this take a large r and look at the set $W_r = \{a \in W : |a| \le r\}$. We note that since for large $|\zeta|$, the function F is almost $\zeta^2/2$, the function F maps the circular boundary of W to an almost circular arc that crosses $\mathbb{R}^-$. From all this, it is not hard to deduce that the boundary of $F(W)$ winds around points in the interior of $F(W)$ once. In particular, for every $t > 0$, and sufficiently large r, the boundary of $F(W)$ winds around $-t$ once. Since F is analytic, this winding number equals the number of roots of $F + t$. Hence, there is a unique solution $\gamma(t)$ with $F(\gamma(t)) = -t$. In fact for the same reason, F^{-1} is well-defined and analytic in the interior of $F(W)$. So, $\gamma(t) = F^{-1}(-t)$ is an analytic function for $t > 0$. It is not hard to see that γ is continuous at 0 and $\gamma(0) = 0$ because $F(\zeta) = 0$ has only one solution $\zeta = 0$. Moreover, since $|F(\zeta)| = O(|\zeta|^2)$ for large z and $F(\zeta) = \zeta^3/3 + \dots$ near 0, we learn

$$\gamma(t) = O(t^{1/2}),\ \text{as } t \to \infty, \quad \gamma(t) = e^{i\pi/3}(3t)^{1/3} + O(t^{4/3}),\ \text{as } t \to 0. \tag{4.31}$$

Since the contour γ lies inside W, we can readily show that the integration over $i\mathbb{R}^+$ in (4.30) can be replaced with γ. The proof is very similar to what we used in the beginning of the proof. Hence

$$\begin{aligned} I^+ &:= N^{1/3} \int_0^{i\infty} e^{(w/2)^2 F(\zeta) + N' \log(1+\zeta)} d\zeta = N^{1/3} \int_0^{\infty} e^{-(w/2)^2 t} (1 + \gamma(t))^{N'} \gamma'(t) dt \\ &= N^{-2/3} \int_0^{\infty} e^{-(w/2)^2 t/N} (1 + \gamma(t/N))^{N'} \gamma'(t/N) dt. \end{aligned}$$

Observe that since $F(\gamma(t)) = -t$, we have that $\gamma' = -\gamma^{-2}(1 + \gamma)$. From this and (4.31) we deduce,

$$\gamma'(t) = O(t^{-1/2}),\ \text{as } t \to \infty, \quad \gamma'(t) = e^{i\pi/3}(3t)^{-2/3} + O(t^{1/3}),\ \text{as } t \to 0. \tag{4.32}$$

Using (4.30) and (4.31) we learn

$$(1 + \gamma(t/N))^{N'} \approx \exp\left(-z(3t)^{1/3} e^{\pi i/3}\right), \quad N^{-2/3}\gamma'(t/N) \approx e^{i\pi/3}(3t)^{-2/3}. \tag{4.33}$$

To pass to the limit, we use dominated convergence; observe that for large N,

$$\begin{aligned} &\left| e^{-(w/2)^2 t/N} \right| \le e^{-t/4}, \quad N^{-2/3} |\gamma'(t/N)| \le c_1 \max\left(t^{-2/3}, N^{-1/6} t^{-1/2}\right) \le c_1 \left(t^{-2/3} + 1\right), \\ &|\log(1 + \gamma(t/N))| \le c_1 t^{1/3} N^{-1/3}, \quad |1 + \gamma(t/N)|^{N'} \le e^{c_2 t^{1/3}}, \end{aligned}$$

for constants c_1 and c_2. This allows us to pass to the limit $N \to \infty$ to deduce

$$I^+ \approx \int_0^\infty \exp\left(-t - z(3t)^{1/3}e^{\pi i/3} + i\pi/3\right)(3t)^{-2/3}dt.$$

Replacing t with $s^3/3$ yields

$$I^+ \approx \int_0^\infty \exp\left(-s^3/3 - zse^{\pi i/3} + i\pi/3\right)ds. \tag{4.34}$$

Since $F(\bar\gamma(t)) = -t$, for the integration over $i\mathbb{R}^-$, we use $\bar\gamma$ and reverse time. As the result,

$$\begin{aligned} I^- :&= N^{1/3}\int_{-i\infty}^0 e^{(w/2)^2F(\zeta)+N'\log(1+\zeta)}d\zeta = -N^{-2/3}\int_0^\infty e^{-(w/2)^2t/N}(1+\bar\gamma(t/N))^{N'}\bar\gamma'(t/N)dt \\ &=\approx \int_0^\infty \exp\left(-s^3/3 - zse^{-\pi i/3} - i\pi/3\right)ds. \end{aligned}$$

From this, (4.30) and (4.32) we conclude

$$N^{1/12}\psi_N(w) \approx \frac{1}{2\pi i}\int_0^\infty e^{-s^3/3}\left[\exp\left(-zse^{\pi i/3} + i\pi/3\right) - \exp\left(-zse^{-\pi i/3} - i\pi/3\right)\right]ds. \tag{4.35}$$

It remains to show that the right hand side is the Airy function. First introduce a contour C that consists of two rays emanating from the origin and making angles $\pm\pi/6$ with the imaginary axes. The contour C is oriented so that the imaginary part goes from $-\infty$ to ∞ as we move along C. Clearly the right-hand side of (4.35) equals

$$\frac{1}{2\pi i}\int_C e^{\zeta^3/3 - z\zeta}d\zeta.$$

We note that this integral is absolutely convergent. We may deform the contour C to $i\mathbb{R}$. However the resulting integral is no longer absolutely convergent convergent and as a result the right-hand side of (4.35) equals

$$\lim_{\ell\to\infty}\frac{1}{2\pi i}\int_{-i\ell}^{i\ell} e^{\zeta^3/3 - z\zeta}d\zeta = \lim_{\ell\to\infty}\frac{1}{2\pi}\int_{-\ell}^{\ell} e^{-it^3/3 - izt}dt.$$

□

Remark 4.3. As we have seen in the proof of Lemma 4.6,

$$Ai(z) = \frac{1}{2\pi i}\int_C e^{\zeta^3/3 - z\zeta}d\zeta, \tag{4.36}$$

where C is a contour that consists of two rays emanating from the origin and making angles $\pm\pi/6$ with the imaginary axes. Since this integral is absolutely convergent, we can differentiate under the integral sign to obtain

$$Ai''(z) = \frac{1}{2\pi i}\int_C \zeta^2 e^{\zeta^3/3 - z\zeta} d\zeta = zAi(z) + \frac{1}{2\pi i}\int_C \frac{d}{d\zeta} e^{\zeta^3/3-z\zeta} d\zeta.$$

Hence

$$Ai''(z) = zAi(z). \tag{4.37}$$

□

Proof of Lemma 4.5. Set $w = \sqrt{N}E + x/\sqrt{N}$. For simplicity, we assume that $n = N$. Recall (4.29) and again replace the contour $i\mathbb{R}$ with $i\mathbb{R} + 1/2$ to assert

$$\psi_N(w) = -i(2\pi)^{-3/4}(N!)^{-1/2}e^{-w^2/8}\left(\frac{w}{2}\right)^{N+1}\int_{-i\infty}^{i\infty}(\zeta+1)^N e^{(w/2)^2(\zeta^2/2-\zeta)}d\zeta,$$

as in the proof of Lemma 4.6. We write,

$$\psi_N(w) = -i(2\pi)^{-3/4}(N!)^{-1/2}e^{-w^2/8}\left(\frac{w}{2}\right)^{N+1}\int_{-i\infty}^{i\infty}(\zeta+1)^{N-(w/E)^2} e^{(w/E)^2R(\zeta)}d\zeta,$$

where $R(\zeta) = \log(\zeta+1) + (E/2)^2(\zeta^2/2 - \zeta)$. we note that R' has exactly two simple roots at $\pm it_0$ with $t_0 = \sqrt{4-E^2}/E$. Set

$$X(\zeta) = (\zeta+1)^N e^{(w/2)^2(\zeta^2/2-\zeta)},$$

and observe

$$|X(\pm it_0)| = (t_0^2+1)^{N/2}e^{-(w/2)^2t_0^2/2}.$$

Moreover, by Stirling's formula,

$$\begin{aligned}-i(2\pi)^{-3/4}(N!)^{-1/2}e^{-w^2/8}\left(\frac{w}{2}\right)^{N+1}|X(\pm it_0)| &\approx -(2\pi i)^{-1}(E/2)N^{1/4}e^{-x/E}\left(1+\frac{x}{EN}\right)^{N+1}\\ &\approx -(2\pi i)^{-1}(E/2)N^{1/4}.\end{aligned}$$

As a result,

$$N^{1/4}\psi_N(w) \approx -\frac{E\sqrt{N}}{2\pi i}|X(\pm it_0)|^{-1}\int_{-i\infty}^{i\infty}(\zeta+1)^{N-(w/E)^2}e^{(w/E)^2R(\zeta)}d\zeta,$$

□

Exercise 4.2.

- (i) Use Laplace's method to establish Stirling's formula $\Gamma(s) \approx (2\pi)^{-1/2} s^{s-1/2} e^{-s}$, as $s \to \infty$.
- (ii) Use saddle point method to establish Lemma 4.5. *Hint:* Use (4.29) and observe that the corresponding function $R(\zeta) = \log \zeta + E^2(\zeta^2/2 - \zeta)$ has two simple roots when $E \in (-2, 2)$. (This explain the different scaling in Lemmas 4.5 and 4.6.)

□

As our last topic in this section, we try to find the law of the largest eigenvalue. The tightness of the rescaled last eigenvalue follows from a result of Ledoux:

Lemma 4.7 *There exist positive constants C_0 and C_1 such that for every t,*

$$\limsup_{N\to\infty} \mathbb{P}_N \left(\max_i \lambda_i \geq 2 + tN^{-2/3} \right) \leq e^{-C_0 t}, \tag{4.38}$$

$$\mathbb{P}_N \left(\max_i \lambda_i \geq 2e^{tN^{-2/3}} \right) \leq C_1 e^{-2C_0 t}.$$

We postpone the proof of Lemma 4.7 for later.

Since the joint distribution of eigenvalues is given by a determinant and the size of our matrix gets large, Fredholm determinant should be relevant. Let us review a well-know formula for determinant that is even meaningful for trace-class operators and behaves well as N gets large.

Lemma 4.8 *For a $N \times N$ matrix $A = [a_{ij}]$,*

$$\det(I - A) = 1 + \sum_{k=1}^{N-1} (-1)^k \sum_{1 \leq \alpha_1 < \cdots < \alpha_k \leq N} \det[a_{\alpha_i \alpha_j}]_{i,j=1}^k. \tag{4.39}$$

Proof. By direct expansion

$$\begin{aligned}
\det(I - A) &= \sum_{\sigma \in S_N} \varepsilon(\sigma) \prod_{i=1}^N (\delta_{i\sigma(i)} - a_{i\sigma(i)}) \\
&= 1 + \sum_{k=1}^N (-1)^k \sum_{\sigma \in S_N} \varepsilon(\sigma) \sum_{1 \leq \alpha_1 < \cdots < \alpha_k \leq N} \prod_{i \neq \alpha_1, \ldots \alpha_k} \delta_{i\sigma(i)} \prod_{s=1}^k a_{\alpha_s \sigma(\alpha_s)} \\
&= 1 + \sum_{k=1}^N (-1)^k \sum_{1 \leq \alpha_1 < \cdots < \alpha_k \leq N} \sum_{\sigma \in S_N} \varepsilon(\sigma) 1\!\!1(\sigma(i) = i, \text{ for } i \neq \alpha_1, \ldots \alpha_k) \prod_{s=1}^k a_{\alpha_s \sigma(\alpha_s)} \\
&= 1 + \sum_{k=1}^N (-1)^k \sum_{1 \leq \alpha_1 < \cdots < \alpha_k \leq N} \sum_{\tau \in S_k} \varepsilon(\tau) \prod_{i=1}^k a_{\alpha_i \alpha_{\tau(i)}} \\
&= 1 + \sum_{k=1}^N (-1)^k \sum_{1 \leq \alpha_1 < \cdots < \alpha_k \leq N} \det[a_{\alpha_i \alpha_j}]_{i,j=1}^k.
\end{aligned}$$

□

Theorem 4.5 *For every t,*

$$\lim_{N\to\infty} \mathbb{P}_N \left(\max_i \lambda_i \leq 2 + tN^{-2/3}\right) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_t^{\infty} \cdots \int_t^{\infty} \det[\tilde{K}(\alpha_i', \alpha_j')]_{i,j=1}^k \prod_{i=1}^k d\alpha_i'.$$

Proof. Recall $\lambda_i = 2 + \alpha_i' N^{-2/3}$, or $x_i = 2\sqrt{N} + \alpha_i' N^{-1/6}$. Evidently

$$\mathbb{P}_N \left(\max_i \lambda_i \leq 2 + tN^{-2/3}\right) = \mathbb{P}_N \left(\max_i x_i \leq 2\sqrt{N} + tN^{-1/6}\right).$$

Pick a large positive t' and set $w = 2\sqrt{N} + tN^{-1/6}$ and $w' = 2\sqrt{N} + t'N^{-1/6}$. By Lemma 4.7,

$$\lim_{N\to\infty} \left| \mathbb{P}_N \left(\max_i x_i \leq w\right) - \mathbb{P}_N \left(x_i \notin (w, w') \text{ for } i = 1, \ldots, N\right) \right| \leq e^{-C_0 t'}. \tag{4.40}$$

On the other hand, we use Theorem 4.1 to assert that the expression

$$\mathbb{P}_N \left(x_i \notin (w, w'), i = 1, \ldots, N\right) \tag{4.41}$$

equals

$$\begin{aligned}
&\frac{1}{N!} \int_{[w,w']^c} \cdots \int_{[w,w']^c} \left(\det\left[\psi_{j-1}(x_i)\right]_{i,j=1}^N\right)^2 \prod_{i=1}^N dx_i \\
&= \frac{1}{N!} \int_{[w,w']^c} \cdots \int_{[w,w']^c} \sum_{\sigma,\tau\in S_N} \varepsilon(\sigma)\varepsilon(\tau) \prod_{i=1}^N \psi_{\sigma(i)-1}(x_i)\psi_{\tau(i)-1}(x_i) dx_i \\
&= \int_{[w,w']^c} \cdots \int_{[w,w']^c} \sum_{\sigma\in S_N} \varepsilon(\sigma) \prod_{i=1}^N \psi_{i-1}(x_i)\psi_{\sigma(i)-1}(x_i) dx_i \qquad (4.42) \\
&= \sum_{\sigma\in S_N} \varepsilon(\sigma) \prod_{i=1}^N \int_{[w,w']^c} \psi_{i-1}(x)\psi_{\sigma(i)-1}(x) dx \\
&= \det\left[\int_{[w,w']^c} \psi_{i-1}(x)\psi_{j-1}(x) dx\right]_{i,j=1}^N .
\end{aligned}$$

From this, $\int \psi_i\psi_j = \delta_{ij}$, and Lemma 4.7, we dduce that the expression (4.41) equals

$$\begin{aligned}
&\det\left[\delta_{ij} - \int_w^{w'} \psi_{i-1}(x)\psi_{j-1}(x)dx\right]_{i,j=1}^N \\
&= 1 + \sum_{k=1}^N (-1)^k \sum_{1\le\alpha_1<\cdots<\alpha_k\le N} \det\left[\int_w^{w'} \psi_{\alpha_i-1}(x)\psi_{\alpha_j-1}(x)dx\right]_{i,j=1}^k \\
&= 1 + \sum_{k=1}^N \frac{(-1)^k}{k!}\int_w^{w'}\cdots\int_w^{w'} \sum_{1\le\alpha_1<\cdots<\alpha_k\le N} \left(\det\left[\psi_{\alpha_i-1}(x_j)\right]_{i,j=1}^k\right)^2 \prod_{j=1}^k dx_j \\
&= 1 + \sum_{k=1}^N \frac{(-1)^k}{k!}\int_w^{w'}\cdots\int_w^{w'} \det\left[K_N(x_i,x_j)\right]_{i,j=1}^k \prod_{j=1}^k dx_j,
\end{aligned}$$

where for the third equality, we applied the calculation (4.42) in reversed order and for the last equality we used Cauchy-Binet's formula and the fact that if $A = [\psi_{j-1}(x_i)]_{i,j=1}^N$, then $AA^t = [K_N(x_i,x_j)]_{i,j=1}^N$. (Compare with (4.12).) The change of variables $x_i = 2\sqrt{N} + \alpha_i' N^{-1/6}$, $i = 1,\dots,N$ and (4.40) yield that the expression (4.40) equals

$$1 + \sum_{k=1}^N \frac{(-1)^k}{k!}\int_t^{t'}\cdots\int_t^{t'} \det\left[\tilde{K}_N(\alpha_i',\alpha_j')\right]_{i,j=1}^k \prod_{j=1}^k d\alpha_j'. \tag{4.43}$$

As in the proof of Theorem 4.4, we know that $\lim_{N\to\infty}\tilde{K}_N = \tilde{K}$ locally uniformly. We want to use this to pass to the limit term by term in (4.43). For this, we need to bound each term of the sum. By Hadamard's inequality (see Lemma 4.9 below),

$$\int_t^{t'}\cdots\int_t^{t'} \det\left[\tilde{K}_N(\alpha_i',\alpha_j')\right]_{i,j=1}^k \prod_{j=1}^k d\alpha_j' \le (t'-t)^k \max_{a,b\in[t,t']} |K_N(a,b)|^k k^{k/2}.$$

From this and the fact that $\tilde{K}_N \to \tilde{K}$ locally uniformly as $N\to\infty$, we deduce

$$\int_t^{t'}\cdots\int_t^{t'} \det\left[\tilde{K}_N(\alpha_i',\alpha_j')\right]_{i,j=1}^k \prod_{j=1}^k d\alpha_j' \le e^{ck}k^{k/2}.$$

This would allow us to replace $\tilde{K}_N$ with $\tilde{K}$ in (4.43) as $N\to\infty$ because $\sum_k e^{ck}k^{k/2}/(k!) < \infty$. From this and (4.40) we learn

$$\begin{aligned}
\lim_{N\to\infty}\mathbb{P}_N\left(\max_i \lambda_i \le 2 + tN^{-2/3}\right) =& 1 + \sum_{k=1}^\infty \frac{(-1)^k}{k!}\int_t^{t'}\cdots\int_t^{t'} \det[\tilde{K}(\alpha_i',\alpha_j')]_{i,j=1}^k \prod_{i=1}^k d\alpha_i' \\
&+ Error(t'),
\end{aligned}$$

where $|Error(t')| \leq e^{-Ct'}$. We finally need to send t' to infinity. To replace the upper limit in the integrals with ∞, we need to make sure that $\tilde{K}(a,b)$ decays sufficiently fast for large values of a and b. This will be carried out in Lemma 4.10 below. $\square$

Lemma 4.9 *(Hadamard) For a $k \times k$ matrix $A = [a_{ij}]$, with columns $a^1, \dots, a^k$,*

$$|\det A| \leq \prod_{i=1}^{k} |a^i| \leq k^{k/2} \max_{i,j} |a_{ij}|.$$

Proof. Without loss of generality, we may assume that $|a^i| = 1$ for $i = 1, \dots, k$. Write $t_1, \dots, t_k$ for the eigenvalues $B = [b_{ij}] = A^t A$. Since $b_{ii} = |a_i|^2$,

$$(\det A)^2 = \det B = \prod_{i=1}^{k} t_i \leq \left(k^{-1}(t_1 + \cdots + t_k)\right)^k = \left(k^{-1} tr\ B\right)^k = 1.$$

Hence,

$$|\det A| \leq \prod_{i=1}^{k} |a^i|.$$

Finally observe that $|a^i| \leq \sqrt{k} \max_{i,j} |a_{ij}|$. $\square$

Lemma 4.10 *For every $a > 0$,*

$$\sup_{x,y \geq a} |\tilde{K}(x,y)|(x^{-1/2} + y^{-1/2}) e^{4(x^{3/2}+y^{3/2})/3} < \infty.$$

Proof. We have

$$\begin{aligned}
\tilde{K}(\alpha_1, \alpha_2) &= \frac{Ai(\alpha_1)Ai'(\alpha_2) - Ai'(\alpha_1)Ai(\alpha_2)}{\alpha_1 - \alpha_2} \\
&= \frac{Ai(\alpha_1) - Ai(\alpha_2)}{\alpha_1 - \alpha_2} Ai'(\alpha_2) - Ai(\alpha_1)\frac{Ai'(\alpha_2) - Ai'(\alpha_1)}{\alpha_1 - \alpha_2} \\
&= Ai'(\alpha_2) \int_0^1 Ai'(t\alpha_1 + (1-t)\alpha_2)dt - Ai(\alpha_1) \int_0^1 Ai''(t\alpha_1 + (1-t)\alpha_2)dt.
\end{aligned}$$

Hence we need to bound Ai, Ai' and Ai''. For this, it suffices to show that as $x \to \infty$,

$$Ai(x) \approx (4\pi)^{-1/2} x^{-1/4} e^{-\frac{2}{3}x^{3/2}}, \tag{4.44}$$

$$Ai'(x) \approx (4\pi)^{-1/2} x^{1/4} e^{-\frac{2}{3}x^{3/2}}, \tag{4.45}$$

$$Ai''(x) \approx (4\pi)^{-1/2} x^{3/4} e^{-\frac{2}{3}x^{3/2}}. \tag{4.46}$$

We only establish (4.44) because the proofs of (4.45) and (4.46) are similar. (Also (4.44) implies (4.46) by (4.37).). For (4.44), recall

$$Ai(x) = \frac{1}{2\pi i}\int_C e^{\zeta^3/3 - x\zeta} d\zeta.$$

where the contour C consists of two rays emanating from the origin and making angles $\pm\pi/6$ with the imaginary axes. The contour C is oriented so that the imaginary part goes from $-\infty$ to ∞ as we move along C. A change of variable $\zeta = \sqrt{x}\eta$ yields

$$Ai(x) = \frac{\sqrt{x}}{2\pi i}\int_C e^{x^{3/2}(\eta^3/3 - \eta)} d\eta.$$

On the portion of C with positive imaginary part, $\eta^3 = (te^{i\pi/3})^3 = -t^3$. So we expect to have a decay of order $O(e^{-x^{3/2}})$. Hence, let us make a change of variable $\eta = \alpha + 1$ to obtain

$$Ai(x) = e^{-x^{3/2}}\frac{\sqrt{x}}{2\pi i}\int_{C'} e^{x^{3/2}(\alpha^3/3 + \alpha^2)} d\alpha,$$

where C' is C shifted from the origin to -1. Write $C' = C_1 + C_2$ where C_1 is the portion of C that lies on the right side of the imaginary axis. C_1 is parametrized as $\alpha = \pm i\sqrt{3} + te^{\pm i\pi/3}$. One readily checks

$$Re(\alpha^3/3 + \alpha^2) = -t^3/3 - 5t^2/4 - 3t - 3.$$

Hence,

$$\left| e^{-x^{3/2}}\frac{\sqrt{x}}{2\pi i}\int_{C'} e^{x^{3/2}(\alpha^3/3+\alpha^2)} d\alpha \right| \le e^{-4x^{3/2}}\frac{\sqrt{x}}{\pi}\int_0^\infty e^{-3x^{3/2}t} dt \le c_1 x^{-1} e^{-4x^{3/2}},$$

for a positive constant c_1. On the other hand, we may deform C_2 to the interval $[-\sqrt{3}i, \sqrt{3}i]$ to assert

$$\begin{aligned}\sqrt{x}e^{-x^{3/2}}\frac{1}{2\pi i}\int_{C_2} e^{x^{3/2}(\alpha^3/3+\alpha^2)} d\alpha &= \sqrt{x}e^{-x^{3/2}}\frac{1}{2\pi}\int_{-\sqrt{3}}^{\sqrt{3}} e^{-x^{3/2}(it^3/3+t^2)} dt \\ &= x^{-1/4}e^{-x^{3/2}}\frac{1}{2\pi}\int_{-\sqrt{3}x^{3/4}}^{\sqrt{3}x^{3/4}} e^{ix^{-3/4}t^3/3 - t^2} dt \\ &\approx x^{-1/4}e^{-x^{3/2}}(2\sqrt{\pi})^{-1},\end{aligned}$$

by the dominated convergence theorem. This completes the proof of (4.44). □

It remains to prove Lemma 4.7.

Proof of Lemma 4.7. Step1. First observe that it suffices to prove the second inequality in (4.38). This inequality is established with the help of Chebyshev's inequality;

$$\mathbb{P}_N\left(\max_i \lambda_i > 2e^{tN^{-2/3}}\right) \le 2^{-2k}e^{2tN^{-2/3}k}\mathbb{E}_N\sum_i \lambda_i^{2k}. \tag{4.47}$$

Let us write

$$\mathbb{E}_N N^{-1}\sum_i \lambda_i^{2k} = \frac{(2k)!}{(k+1)(k!)^2}A_N(k),$$

so that $\lim_{N\to\infty} A_N(k) = 1$ by Semi-circle Law and (2.19). In fact we will show that for a positive constant c_0,

$$A_N(k) \le e^{c_0k^3N^{-2}}. \tag{4.48}$$

Assuming this for now and using (4.47), we obtain

$$\mathbb{P}_N\left(\max_i \lambda_i > 2e^{tN^{-2/3}}\right) \le 2^{-2k}e^{-2tN^{-2/3}k}Ne^{c_0k^3N^{-2}}\frac{(2k)!}{(k+1)(k!)^2}. \tag{4.49}$$

On the other hand by Stirling's formula,

$$\frac{(2k)!}{(k+1)(k!)^2} \approx \frac{2^{2k}}{\sqrt{\pi}\sqrt{k}(k+1)} \approx \frac{2^{2k}}{\sqrt{\pi}k^{3/2}}.$$

This and (4.49) imply

$$\mathbb{P}_N\left(\max_i \lambda_i > 2e^{-tN^{-2/3}}\right) \le c_1e^{2tN^{-2/3}k}e^{c_0k^3N^{-2}}Nk^{-3/2}.$$

Choosing $k = [N^{2/3}]$ in this inequality yields the second inequality in (4.38).

Step 2. It remains to establish (4.48). We first derive a formula for the moment generating functions of the eigenvalues. In view of Theorem 4.1 and (4.16),

$$\begin{aligned}
\mathbb{E}_N N^{-1}\sum_{i=1}^N e^{t\sqrt{N}\lambda_i} &= \int p_N^{(1)}(x)e^{tx}dx \\
&= N^{-1/2}\int \left(\psi_N'(x)\psi_{N-1}(x)-\psi_N(x)\psi_{N-1}'(x)\right)e^{tx}dx \\
&= -t^{-1}\int \frac{d}{dx}\left(\psi_N'(x)\psi_{N-1}(x)-\psi_N(x)\psi_{N-1}'(x)\right)e^{tx}dx \\
&= -t^{-1}\int \left(\psi_N''(x)\psi_{N-1}(x)-\psi_N(x)\psi_{N-1}''(x)\right)e^{tx}dx \\
&= t^{-1}\int \psi_N(x)\psi_{N-1}(x)e^{tx}dx \\
&= \frac{1}{t\sqrt{2\pi}N!}\int P_N(x)P_{N-1}(x)e^{-x^2/2+tx}dx \\
&= \frac{e^{t^2/2}}{t\sqrt{2\pi}N!}\int P_N(x)P_{N-1}(x)e^{-(x-t)^2/2}dx \\
&= \frac{e^{t^2/2}}{t\sqrt{2\pi}N!}\int P_N(x+t)P_{N-1}(x+t)e^{-x^2/2}dx,
\end{aligned}$$

where we used integration by parts and Remark 4.2 for the third and fifth equality. On the other hand, by Lemma 4.1 (i) and (ii), we know that $P_k'(x) = kP_{k-1}$. Hence

$$P_N(x+t) = \sum_{k=0}^N \binom{N}{k} t^k P_{N-k}(x),$$

by Taylor's expansion. As a result,

$$\int P_N(x+t)P_{N-1}(x+t)e^{-x^2/2}dx = \sqrt{2\pi}\sum_{k=1}^N (N-k)!\binom{N}{k}\binom{N-1}{k-1}t^{2k-1}.$$

Replacing t with $t/\sqrt{N}$ yields,

$$\mathbb{E}_N N^{-1}\sum_{i=1}^N e^{t\lambda_i} = e^{t^2/(2N)}\left[1+\sum_{k=1}^{N-1}\frac{(N-1)\dots(N-k)}{(k+1)!k!N^k}t^{2k}\right] = B_N(t^2/N), \tag{4.50}$$

where $B_N(t) = e^{t/2}C_N(t)$ with

$$C_N(t) = 1+\sum_{k=1}^{N-1}\frac{(N-1)\dots(N-k)}{(k+1)!k!}t^k.$$

Step3. Observe that for $k \geq 1$,

$$\left(t\frac{d^2}{dt^2} + (t+2)\frac{d}{dt} - (N-1)\right) t^k = k(k+1)t^{k-1} - (N-k-1)t^k.$$

Hence

$$\begin{aligned}\left(t\frac{d^2}{dt^2} + (t+2)\frac{d}{dt} - (N-1)\right) C_N(t) &= 1 - N + \sum_{k=1}^{N-1} \frac{(N-1)\dots(N-k)}{(k-1)!k!} t^{k-1} \\ &\quad - \sum_{k=1}^{N-1} \frac{(N-1)\dots(N-k-1)}{(k+1)!k!} t^k \\ &= \sum_{k=2}^{N-1} \frac{(N-1)\dots(N-k)}{(k-1)!k!} t^{k-1} \\ &\quad - \sum_{k=1}^{N-2} \frac{(N-1)\dots(N-k-1)}{(k+1)!k!} t^k \\ &= 0.\end{aligned}$$

As a result,

$$\left(t\frac{d^2}{dt^2} + 2\frac{d}{dt} - (N + t/4)\right) B_N(t) = 0.$$

Writing $B_N(t) = \sum_0^\infty a_k t^k$ yields

$$(k+1)(k+2)a_{k+1} - Na_k - a_{k-1}/4 = 0, \quad 2a_1 - Na_0 = 0, \tag{4.51}$$

for $k \geq 1$. By (4.50) and the definition of $A_N(k)$,

$$B_N(t^2/N) = \sum_{k=0}^\infty N^{-k} a_k t^{2k} = \sum_{k=0}^\infty \frac{A_N(k)}{(k+1)!k!} t^{2k}.$$

Therefore,

$$a_k = \frac{N^k}{(k+1)!k!} A_N(k).$$

This and (4.51) imply a formula of Harer and Zagier:

$$A_N(k+1) = A_N(k) + \frac{k(k+1)}{4N^2} A_N(k-1). \tag{4.52}$$

Final Step. From (4.52) we deduce that $A_N(k)$ is increasing in k and

$$A_N(k+1) = A_N(k) + \frac{k(k+1)}{4N^2} A_N(k-1) \leq \left(1 + \frac{k(k+1)}{4N^2}\right) A_N(k).$$

As a result,

$$A_N(k) \leq \prod_{\ell=2}^{k}\left(1+\frac{\ell(\ell-1)}{4N^2}\right) \leq \exp\left(\sum_{\ell=2}^{k}\frac{\ell(\ell-1)}{4N^2}\right).$$

From this, we can readily deduce (4.48). This completes the proof of the Lemma. □

5 Dyson Brownian Motion

In this section we study the matrix-valued process $H_N(t) = H(t) = [h_{ij}(t)]$ where $(h_{ij}(t) : i \geq j)$ are independent Brownian motions and $H(t)$ is either symmetric or Hermitian for every t. In the symmetric case, $\mathbb{E}_N h_{ij}(t)^2 = tN^{-1}$ for $i \neq j$ and $\mathbb{E}_N h_{ii}(t)^2 = 2tN^{-1}$. In the Hermitian case, $\mathbb{E}_N h_{ii}(t)^2 = tN^{-1}$ and for $i \neq j$, h_{ij} is a complex-valued Brownian motion with $Re\ h_{ij}$ independent from $Im\ h_{ij}$, and $\mathbb{E}_N Re\ h_{ii}(t)^2 = \mathbb{E}_N Im\ h_{ii}(t)^2 = tN^{-1}$. We refer to the process $H(t)$ as the Dyson Brownian motion (DBM). Dyson derived a stochastic differential equations for eigenvalues and eigenvectors of H. Before embarking on this derivation, we recall two fundamental facts from Stochastic Calculus. In what follows all processes are assumed to be continuous in time t. Given a filtration $\mathcal{F}_t$, we say an adapted process $X(t)$ is a *(local) semimartingale* if $X(t) - X(0) = M(t) + A(t)$ where both M and A are adopted, M is a (local) martingale and A is a process of bounded variation. Given a martingale, we write $[M](t)$ for the unique process of bounded variation $A(t)$ such that $M(t)^2 - A(t)$ is a martingale. If M and $\tilde{M}$ are two martingale, then $[M, \tilde{M}] = 4^{-1}([M + \tilde{M}] - [M] - [\tilde{M}])$ so that $M\tilde{M} - [M, \tilde{M}]$ is a martingale. If X and $\tilde{X}$ are two semimartingales with martingale parts M and $\tilde{M}$, then we define $[X, \tilde{X}]$ to be $[M, \tilde{M}]$. Here are two fundamental results from stochastic calculus:

Proposition 5.1
- *(i)(Ito) Given a (local) semimartingale $X = (X_1, \dots, X_d)$ and a C^2 function f, we have*

$$df(X(t)) = \nabla f(X(t)) \cdot dX(t) + \frac{1}{2}\sum_{i,j=1}^{d} f_{x_i x_j}(X(t))d[X_i, X_j](t).$$

- *(ii)(Ito) Let $B(t) = (B_1(t), \dots, B_d(t))$ be a standard Brownian motion, and assume that continuous functions $a(t,x) = (a_1(t,x), \dots, a_d(t,x))$ and $\sigma(t,x) = [\sigma_{ij}(t,x)]_{i,j=1}^d$ satisfy*

$$\sup_{0\leq t\leq T} |a(t,x) - a(y,t)| \leq c_0|x-y|, \qquad \sup_{0\leq t\leq T} \|\sigma(t,x) - \sigma(y,t)\| \leq c_0|x-y|,$$
$$\sup_{0\leq t\leq T} |a(t,x)| \leq c_0(|x|+1), \qquad \sup_{0\leq t\leq T} \|\sigma(t,x)\| \leq c_0(|x|+1),$$

for a constant c_0. Then the stochastic differential equation

$$dX = a(t, X)dt + \sigma(t, X)dB,$$

has a unique solution in $[0, T]$.

- *(iii)(Levy) Let $M = (M_1, \dots, M_d)$ be a continuous martingale with $[M_i, M_j](t) = \delta_{ij} t$. Then M is a standard d-dimensional Brownian motion.*

We now carry out some formal calculation to derive Dyson's formula for the eigenvalues and eigenvectors of H. Write $H = UDU^*$ where U is orthogonal/unitary and $D = diag[\lambda_1, \dots, \lambda_N]$ is a diagonal matrix with $\lambda_1 < \cdots < \lambda_N$. For our formal calculations, let us pretend that $H \mapsto (U, \boldsymbol{\lambda})$ is smooth. Define a martingale $\hat{H}$ by $\hat{H}(0) = 0$, and

$$d\hat{H} = U^*(dH)U. \tag{5.1}$$

Let us consider the symmetric case first so that

$$d\hat{h}_{ij} = \sum_{k,l} u_{ki} u_{lj} dh_{kl} = \sum_k u_{ki} u_{kj} dh_{kk} + \sum_{k<l} (u_{ki} u_{lj} + u_{li} u_{kj}) \, dh_{kl}.$$

From this and Ito's formula we learn

$$\begin{aligned} d[\hat{h}_{ij}, \hat{h}_{i'j'}](t) &= N^{-1} \left(2 \sum_k u_{ki} u_{kj} u_{ki'} u_{kj'} + \sum_{k<l} (u_{ki} u_{lj} + u_{li} u_{kj})(u_{ki'} u_{lj'} + u_{li'} u_{kj'}) \right) dt \\ &= N^{-1} \left(\sum_k u_{ki} u_{kj} u_{ki'} u_{kj'} + \sum_{k<l} (u_{ki} u_{lj} u_{ki'} u_{lj'} + u_{li} u_{kj} u_{li'} u_{kj'}) \right) dt \\ &\quad + N^{-1} \left(\sum_k u_{ki} u_{kj} u_{ki'} u_{kj'} + \sum_{k<l} (u_{ki} u_{lj} u_{li'} u_{kj'} + u_{li} u_{kj} u_{ki'} u_{lj'}) \right) dt \\ &= N^{-1} \sum_{k,l} u_{ki} u_{lj} u_{ki'} u_{lj'} dt + N^{-1} \sum_{k,l} u_{ki} u_{lj} u_{li'} u_{kj'} dt. \end{aligned}$$

As a result,

$$d[\hat{h}_{ij}, \hat{h}_{i'j'}](t) = N^{-1} \left(\delta_{ii'} \delta_{jj'} + \delta_{ij'} \delta_{ji'} \right) dt. \tag{5.2}$$

From this and Levy's theorem (Proposition 5.1(ii)), we deduce that $\hat{H}(t)$ is also a DBM. In summary,

$$dH = U(d\hat{H})U^*, \tag{5.3}$$

where $\hat{H}$ is DBM. On the other hand, since U and $\boldsymbol{\lambda}$ are semimartingales, and $H = UDU^*$, we know

$$dH = (dU)DU^* + UD(dU^*) + U(dD)U^* + \left[\sum_k \lambda_k d[u_{ik}, u_{jk}]\right]_{ij=1}^N . \tag{5.4}$$

Here we used Ito's formula as in Lemma 5.1(i) for the function $f(x_1, x_2, x_3) = x_1 x_2 x_3$:

$$d(X_1X_2X_3) = X_2X_3dX_1 + X_1X_2dX_3 + X_1X_3dX_2 + X_1d[X_2, X_3] + X_2d[X_3, X_1] + X_3d[X_1, X_2],$$

where X_1, X_2, X_3 are of the form $u_{ik}, u_{jk}, \lambda_k$. Note that from Section 2 we know that $\boldsymbol{\lambda}$ is independent of U, so the bracket of an entry of U with an eigenvalue is 0. From (5.3) and (5.4) we deduce

$$d\hat{H} = U^*(dU)D + D(dU^*)U + dD + U^*\left[\sum_k \lambda_k d[u_{ik}, u_{jk}]\right]_{ij=1}^N U. \tag{5.5}$$

On the other hand, since $U^*U = I$,

$$U^*(dU) + (dU^*)U + \left[\sum_k d[u_{ki}, u_{kj}]\right]_{ij=1}^N = 0. \tag{5.6}$$

The entries of the matrix $U^*(dU) = Adt + dV$ are semimartingales with $A = [a_{ij}]$ of finite variation and $V = [v_{ij}]$ a matrix-valued martingale. Taking the martingale parts of the both sides of (5.6), we learn that V is skew-symmetric. We now take the martingale parts of the both sides of (5.5) and use skew-symmetry of V to deduce that when $i \neq j$,

$$d\hat{h}_{ij} = \lambda_j dv_{ij} + \lambda_i dv_{ji} = (\lambda_j - \lambda_i)dv_{ij} \quad \text{or} \quad dv_{ij} = (\lambda_j - \lambda_i)^{-1} d\hat{h}_{ij}.$$

We now try to determine A. From (5.6) we know

$$(A + A^*)dt = -\left[\sum_k d[u_{ki}, u_{kj}]\right]_{ij=1}^N .$$

On the other hand, since $dU = UAdt + UdV$,

$$\begin{aligned} d[u_{ki}, u_{kj}] &= d\left[\sum_\ell u_{k\ell} dv_{\ell i}, \sum_\ell u_{k\ell} dv_{\ell j}\right] \\ &= \mathbb{1}(i = j)N^{-1} \sum_{\ell:\ell\neq i} u_{k\ell}^2 (\lambda_\ell - \lambda_i)^{-2} dt - \mathbb{1}(i \neq j)N^{-1} u_{ki}u_{kj}(\lambda_j - \lambda_i)^{-2} dt. \end{aligned} \tag{5.7}$$

As a result,

$$A + A^* = -\left[\delta_{ij} N^{-1} \sum_{\ell:\ell\neq i} (\lambda_\ell - \lambda_i)^{-2}\right]_{ij=1}^N . \tag{5.8}$$

Later we will show that in fact the matrix A is symmetric and hence

$$A = -\frac{1}{2N}\left[\delta_{ij} \sum_{\ell:\ell\neq i} (\lambda_\ell - \lambda_i)^{-2}\right]_{ij=1}^N . \tag{5.9}$$

In summary, the matrix U solves the stochastic differential equation

$$dU = UAdt + UdV, \tag{5.10}$$

with $V^* + V = 0$, $dv_{ij} = (\lambda_j - \lambda_i)^{-1} d\hat{h}_{ij}$, whenever $i \neq j$, and A is given by (5.9).

We now try to determine an equation for $\boldsymbol{\lambda}$. By ,(5.10)

$$d[u_{ik}, u_{jk}] = d\left[\sum_\ell u_{i\ell} dv_{\ell k}, \sum_\ell u_{j\ell} dv_{\ell k}\right] = N^{-1} \sum_{\ell:\ell\neq k} u_{i\ell} u_{j\ell} (\lambda_k - \lambda_\ell)^{-2} dt.$$

From this we deduce

$$U^* \left[\sum_k \lambda_k d[u_{ik}, u_{jk}]\right]_{ij=1}^N U = N^{-1}\left[\delta_{ij} \sum_{k:k\neq i} \lambda_k (\lambda_k - \lambda_i)^{-2}\right]_{ij=1}^N dt. \tag{5.11}$$

From this, (5.5) and (5.9) we learn

$$d\hat{h}_{ii} = 2a_{ii}\lambda_i + d\lambda_i + N^{-1} \sum_{k\neq i} \lambda_k (\lambda_k - \lambda_i)^{-2} dt.$$

Hence,

$$d\lambda_i = N^{-1} \sum_{k\neq i} (\lambda_i - \lambda_k)^{-1} dt + d\hat{h}_{ii}.$$

On the other hand, if we take the finite-variation parts of off-diagonal entries of both sides of (5.5), we obtain

$$0 = a_{ij}\lambda_j + a_{ji}\lambda_i,$$

because the matrix of (5.11) is diagonal. Since by (5.8), the matrix $A + A^*$ is also diagonal, we deduce that $a_{ij} = 0$ if $i \neq j$, confirming the claim (5.9).

Exercise 5.1 Let H be a DBM and define a matrix-valued process K by $dK = dH - Kdt/2$. Choose an orthogonal/unitary matrix U so that $U^*KU = D$ where D is diagonal. We write $\lambda_1 \leq \cdots \leq \lambda_N$ for the eigenvalues of K. define $\hat{H}$ and $\hat{K}$ by $d\hat{H} = U^*(dH)U$ and $d\hat{K} = U^*(dK)U$. Show that $\hat{H}$ is a DBM and that $d\hat{K} = d\hat{H} - Ddt/2$. Show that $dU = UdV + Adt$ with V and A as before. Derive

$$d\lambda_i = \frac{1}{N}\sum_{k\neq i}(\lambda_i - \lambda_k)^{-1}dt - \frac{1}{2}\lambda_i dt + d\hat{h}_{ii}. \tag{5.12}$$

for the evolution of $\boldsymbol{\lambda}$. □

We are now ready to state and prove Dyson's theorem. For Theorem 5.1, we consider a general DBM where $H(0)$ is simply a symmetric/Hermitian matrix.

Theorem 5.1 *Let $H(t)$ be a DBM. Then the eigenvalues and eigenvectors of $H(t)$ satisfy*

$$d\lambda_i = N^{-1}\sum_{k:k\neq i}(\lambda_k - \lambda_i)^{-1}dt + d\hat{h}_{ii}, \tag{5.13}$$

$$dU = UAdt + UdV, \tag{5.14}$$

where A is given by (5.8) *and $V = [v_{ij}]$ is a skew-symmetric matrix satisfying $dv_{ij} = (\lambda_j - \lambda_i)^{-1}d\hat{h}_{ij}$ for $i \neq j$, with $\hat{H}$ a DBM.*

Proof. This is our strategy: we first prove the existence of a (unique) solution to (5.13) and (5.14). We then use $\boldsymbol{\lambda}$ and U to construct H by $H = UDU^*$. We then show that in fact the resulting H is DBM. Since we already know what the law of the eigenvalues and eigenvectors of the constructed H are, we deduce that the equations (5.13) and (5.14) are correct.

Step 1. First we assume that the eigenvalues of $H(0)$ are distinct and construct a unique solution to the equation (5.12). To do so, first we replace the drift with a smooth function; define ψ_ε by $\psi_\varepsilon(r) = r^{-1}$ if $|r| \leq \varepsilon$ and $\psi_\varepsilon(r) = \varepsilon^{-2}r$ if $|r| \geq \varepsilon$. Consider the equation

$$d\lambda_i = N^{-1}\sum_{k:k\neq i}\psi_\varepsilon(\lambda_i - \lambda_k)dt + d\hat{h}_{ii}. \tag{5.15}$$

By Ito's theorem, the equation (5.14) has a unique solution. This solution is denoted by $\boldsymbol{\lambda}^\varepsilon(t)$. Let $\tau(\varepsilon)$ be the first time, $|\lambda_i^\varepsilon(t) - \lambda_j^\varepsilon| = \varepsilon$ for some $i \neq j$. Note that $\boldsymbol{\lambda}^\varepsilon(t)$ does solve (5.12) so long as $t < \tau(\varepsilon)$. For the same reason, if $\varepsilon < \varepsilon'$, then $\tau(\varepsilon) \geq \tau(\varepsilon')$. Hence, if $\tau = \lim_{\varepsilon\to 0}\tau(\varepsilon)$, then we have a unique solution to (5.12) up to time τ. As a result, we only need to show that $\tau = \infty$ almost surely.

As before, let us write $\beta = 1$ in the symmetric case and $\beta = 2$ in the Hermitian case. Observe that $\boldsymbol{\lambda}$ is a diffusion with the generator

$$\mathcal{L} = (\beta N)^{-1}\left(\Delta - \nabla W \cdot \nabla\right),$$

where

$$W(\boldsymbol{\lambda}) = -\beta \sum_{i \neq j} \log |\lambda_i - \lambda_j|.$$

(Here and below by a summation over $i \neq j$ we mean a summation over unordered distinct pairs of i and j, so alternatively we can take the summation over $i < j$.) To show that $\tau(\varepsilon)$ is large for small ε, we use strong Markov property

$$\mathbb{E}_N f(\boldsymbol{\lambda}(t \wedge \tau(\varepsilon)) = \mathbb{E}_N f(\boldsymbol{\lambda}(0)) + \mathbb{E}_N \int_0^{t\wedge\tau(\varepsilon)} f(\boldsymbol{\lambda}(s))ds, \tag{5.16}$$

for a function f that is bounded below and we have an uniform upper bound for $\mathcal{L}f$. We start from the quadratic function $q(\boldsymbol{\lambda}) = |\boldsymbol{\lambda}|^2$;

$$\mathcal{L}q(\boldsymbol{\lambda}) = 2/\beta + 2N^{-1} \sum_{i \neq j} \lambda_i (\lambda_i - \lambda_j)^{-1} = 2/\beta + N - 1. \tag{5.17}$$

The function q doesn't do the job for us because it is not large at the stopping time $\tau(\varepsilon)$. However the potential W is large at $\tau(\varepsilon)$. Moreover

$$\mathcal{L}W(\boldsymbol{\lambda}) = R + (1-\beta) \sum_i \left(\sum_{j:j\neq i} (\lambda_i - \lambda_j)^{-1} \right)^2,$$

where

$$\begin{aligned}
R &= \sum_{i\neq j} (\lambda_i - \lambda_j)^{-2} - \sum_i \left(\sum_{j:j\neq i} (\lambda_i - \lambda_j)^{-1} \right)^2 \\
&= \sum_{i\neq j\neq k} (\lambda_i - \lambda_j)^{-1}(\lambda_i - \lambda_k)^{-1} \\
&= 2^{-1} \sum_{i\neq j\neq k} \left[(\lambda_j - \lambda_i)^{-1}(\lambda_j - \lambda_k)^{-1} + (\lambda_k - \lambda_j)^{-1}(\lambda_k - \lambda_i)^{-1} \right] \\
&= -2^{-1} \sum_{i\neq j\neq k} (\lambda_i - \lambda_j)^{-1}(\lambda_i - \lambda_k)^{-1}.
\end{aligned}$$

From the equality of the third term with the last term we learn that R is 0 and as a result,

$$\mathcal{L}W(\boldsymbol{\lambda}) = (1-\beta) \sum_i \left(\sum_{j:j\neq i} (\lambda_i - \lambda_j)^{-1} \right)^2 \leq 0. \tag{5.18}$$

Finally we set $f = N(q + 4N) + \beta^{-1} W$. From (5.17) and and (5.18) we deduce

$$\mathcal{L}f \leq N(2/\beta + N - 1). \tag{5.19}$$

On the other hand, we use the elementary fact $x^2+4 \geq 2\log(1+|x|)$, to assert

$$\begin{aligned} f(\boldsymbol{\lambda}) &\geq 2N\sum_i \log(1+|\lambda_i|) - \sum_{i\neq j}\log|\lambda_i-\lambda_j| \\ &\geq \sum_{i,j}\log\left((1+|\lambda_i|)(1+|\lambda_j|)\right) - \sum_{i\neq j}\log|\lambda_i-\lambda_j| \\ &\geq \sum_{i\neq j}\left(-\log|\lambda_i-\lambda_j|\right)^+ . \end{aligned}$$

Using this and (5.19) for (5.16) yield

$$\begin{aligned} |\log\varepsilon|\ \mathbb{P}_N(\tau(\varepsilon)\leq t) \leq \mathbb{E}_N f(\boldsymbol{\lambda}(t\wedge\tau(\varepsilon))) &\leq f(\boldsymbol{\lambda}(0)) + \mathbb{E}_N\int_0^{t\wedge\tau(\varepsilon)} f(\boldsymbol{\lambda}(s))ds \\ &\leq f(\boldsymbol{\lambda}(0)) + tN(2/\beta+N-1). \end{aligned}$$

Hence,

$$\mathbb{P}_N(\tau(\varepsilon)\leq t) \leq [f(\boldsymbol{\lambda}(0)) + tN(2/\beta+N-1)]\,|\log\varepsilon|^{-1}.$$

As a result,

$$\mathbb{P}_N(\tau(e^{-\ell^2})\leq t\ i.o) = 0,$$

and this in turn implies

$$\lim_{\varepsilon\to 0}\tau(\varepsilon) = 0,$$

almost surely.

Step 2. Using the constructed $\boldsymbol{\lambda}$ of the first step, we solve the equation (5.14) with the initial condition $U(0)$ that is orthogonal/unitary and satisfies $U(0)^*H(0)U(0) = D(0)$ with $D(0)$ a diagonal matrix. We note that the Brownian motions we use in (5.14) are independent of those used to construct $\boldsymbol{\lambda}$, and that in the interval $[0,\tau(\varepsilon)]$ all the $|\lambda_i-\lambda_j|^{-1}$ are bounded by ε^{-1}. As a result, (5.14) has a unique solution in the interval $[0,\tau(\varepsilon)]$.

We claim that $U(t)$ is orthogonal/unitary for $t>0$. To see this, observe that if $B=U^*U$, then $U^*dU = BdV + BAdt$. On the other hand

$$dB = d(U^*U) = U^*(dU) + (dU^*)U + \left[\sum_k d[u_{ki},u_{kj}]\right]_{ij=1}^N,$$

and since

$$\begin{aligned} \sum_k d[u_{ki},u_{kj}] &= \mathbb{1}(i=j)N^{-1}\sum_k\sum_{\ell:\ell\neq i} u_{k\ell}^2(\lambda_\ell-\lambda_i)^{-2}dt - \mathbb{1}(i\neq j)N^{-1}\sum_k u_{ki}u_{kj}(\lambda_j-\lambda_i)^{-2}dt \\ &= \mathbb{1}(i=j)N^{-1}\sum_{\ell:\ell\neq i} b_{\ell\ell}(\lambda_\ell-\lambda_i)^{-2}dt - \mathbb{1}(i\neq j)N^{-1}b_{ij}(\lambda_j-\lambda_i)^{-2}dt, \end{aligned}$$

as in (5.7), and $V + V^* = 0$, $B^* = B$, $A^* = A$, we deduce

$$dB = BdV - (dV)B + (BA + AB)dt$$
$$+ N^{-1}\left[\mathbb{1}(i = j) \sum_{\ell:\ell \neq i} b_{\ell\ell}(\lambda_\ell - \lambda_i)^{-2} - \mathbb{1}(i \neq j) b_{ij}(\lambda_j - \lambda_i)^{-2} \right] dt.$$

This stochastic differential equation has $B = I$ as a solution. By uniqueness, $B(t) = I$ in $[0, \tau(\varepsilon)]$ for every $\varepsilon > 0$. Thus $U^*(t)U(t)$ for every t.

We now set $H' = UDU^*$ where $D = diag[\lambda_1, \ldots, \lambda_N]$. Evidently $H'(0) = H(0)$. We define $\hat{H}'$ by $d\hat{H}' = U^*(dH')U$, $\hat{H}'(0) = 0$. We wish to show that $\hat{H} = \hat{H}'$. Observe that the equations (5.4) and (5.5) are all valid for H' and $\hat{H}'$. From this and $U^* dU = dV + Adt$, we can readily deduce that $\hat{h}_{ij} = \hat{h}'_{ij}$ for $i \neq j$. We also use (5.10) and (5.13) to deduce that $\hat{h}_{ii} = \hat{h}'_{ii}$. Hence $\hat{H} = \hat{H}'$. From this we learn that $\hat{H}'$ is DBM. Since $dH' = U(d\hat{H}')U^* = U(d\hat{H})U^*$, we deduce that $H' = H$. This completes the proof when the eigenvalues of $H(0)$ are distinct.

Step 3. Let $H(t)$ be a DBM with $H(0)$ any symmetric matrix. We claim that $H(\delta)$ has distinct eigenvalues for any δ positive. This is because by parabolic regularity, the law of $H(\delta) = [h_{ij}(\delta)]_{ij}$ is absolutely continuous with respect to the Lebesgue measure $\prod_{i \leq j} dh_{ij}$. Since the set symmetric matrices for which with non-distinct eigenvalues is of 0 Lebesgue measure, the matrix $H(\delta)$ has distinct eigenvalues almost surely for any δ positive. If $\Omega(\varepsilon, \delta)$ denotes the set of matrices H whose eigenvalues satisfy $|\lambda_i - \lambda_j| \geq \varepsilon$ whenever $i \neq j$, then on this set, the process $\boldsymbol{\lambda}(t)$ satisfies (5.13) for $t \geq \delta$ almost surely. Since $\mathbb{P}_N(\Omega(\varepsilon, \delta)) \to 1$ as $\varepsilon \to 0$, we deduce that the process $\boldsymbol{\lambda}(t)$ satisfies (5.13) for $t \geq \delta$ almost surely. Finally we send $\delta \to 0$ to complete the proof. $\square$

We note that the invariant measure for DBM takes the form $\prod_{i \neq j} |\lambda_i - \lambda_j|^\beta \prod_i d\lambda_i$. For our purposes we prefer to work with the Ornstein-Uhlenbeck variant of DBM that was introduced in Exercise 5.1. We note that the diffusion (5.12) has a generator of the form

$$\mathcal{L}f = (\beta N)^{-1} (\Delta - \nabla V \cdot \nabla), \tag{5.20}$$

where now V is given by

$$V(\boldsymbol{\lambda}) = \beta N |\lambda|^2/4 - \beta \sum_{i \neq j} \log |\lambda_i - \lambda_j|. \tag{5.21}$$

The invariant measure for the generator $\mathcal{L}$ takes the form $Z_N^{-1} e^{-V} d\boldsymbol{\lambda}$ which is exactly the Gaussian ensemble. In the next section, we will see how this can be used to establish the universality of the Wigner ensemble.

Exercise 5.2.

- (i) Verify Theorem 5.1 when $\boldsymbol{\lambda}$ satisfies (5.12) instead of (5.13).

- (ii) Note that near the boundary $\eta_i = \lambda_{i+1} - \lambda_i = 0$ of the domain $\Lambda_N = \{\boldsymbol{\lambda} : \lambda_1 < \dots \le \lambda_N\}$, the operator $\mathcal{L}$ has the form

$$\mathcal{L} = 2(\beta N)^{-1}\frac{d^2}{d\eta_i^2} + 2(N\eta_i)^{-1}\frac{d}{d\eta_i} + \mathcal{L}',$$

where $\mathcal{L}'$ is a non-singular operator near the boundary $\eta_i = 0$. Motivated by this consider the Bessel process

$$\mathcal{A} = 2^{-1}\frac{d^2}{dx^2} - \beta(2x)^{-1}\frac{d}{dx},$$

and show that if initially $x(0) > 0$, then $x(t)$ never crosses 0 if and only if $\beta \ge 1$. *Hint:* Use a function similar to f of Step 1 of the proof of Theorem 5.1.

□

6 Universality

Recently universality for Wigner ensembles has been established by two different methods. First approach has been initiated by Erdös, Schlein and Yau and based on DBM and LSI. The second approach was employed by Tao and Vu and analyze the differentiability of the eigenvalues as a function of the matrix H. This section is devoted to the first approach. Theorem 6.1 appeared in [ESY]. This Theorem is the analog of Theorem 4.3 for non-Gaussian ensembles.

Theorem 6.1 *Assume that $E \in (-2, 2)$. For every $r \ge 2$, let $\hat{p}_N^{(r)}(\cdot; E)$ denote the r-dimensional marginals densities of a Wigner ensemble H as in Theorem 4.3. Assume that the probability density of the entries of H have a sub-exponential decay. Then the averaged correlation function*

$$\frac{1}{2a}\int_{-a}^{a} \hat{p}_N^{(r)}(\alpha_1, \dots, \alpha_r; E + a)da$$

converges to

$$p^{(r)}(\alpha_1, \dots, \alpha_r) = (r!)^{-1} \det\left[K(\alpha_i, \alpha_j)\right]_{i,j=1}^{r}, \qquad (6.1)$$

as $N \to \infty$, where $K(\alpha_1, \alpha_2) = \sin(\alpha_1 - \alpha_2)/(\alpha_1 - \alpha_2)$.

We first start with a variant of Theorem 2.2 that works for the domain $\Lambda_N = \{\boldsymbol{\lambda} : \lambda_1 < \dots \le \lambda_N\}$. Let us write $S(f) = S_\mu(f) = \int f \log f d\mu$ for the entropy with respect to the measure μ and $D(f) = D_\mu(f) = 4\int(\nabla\sqrt{f})^2 d\mu = \int(\nabla f)^2 f^{-1} d\mu$.

Theorem 6.2 *Assume that the function V' is given by*

$$V'(\boldsymbol{\lambda}) = V(\boldsymbol{\lambda}) + A(\boldsymbol{\lambda}),$$

where V is as in (5.21)*, $A : \mathbb{R}^N \to \mathbb{R}$ is a C^2 function, and the second derivative D^2V' satisfies $D^2V'(x) \geq c^{-1}I$ for every x. Then the probability measure $\mu(d\boldsymbol{\lambda}) = Z^{-1}e^{-V'(\boldsymbol{\lambda})}d\boldsymbol{\lambda}$, $\boldsymbol{\lambda} \in \Lambda_N$, satisfies LSI$(4c)$. Moreover, if $\mathcal{A}' = \Delta - \nabla V' \cdot \nabla$ and $g_t = e^{t\mathcal{A}}g$, then*

$$\frac{dD(g_t)}{dt} \leq -\int (D^2V')\nabla g_t \cdot \nabla g_t \; g_t^{-1} \; d\mu. \tag{6.2}$$

Needless to say that the proof of Theorem 6.2 is almost identical to the proof of Theorem 2.2 except that some care is needed when we integrate by parts. For Theorem 6.2, we need to make sure that the no contribution is coming from the boundaries $\lambda_i = \lambda_{i+1}$, $i = 1, \dots, N$, each time we integrate by parts. Since the difference between Theorem 2.2 and 6.2 is technical, let us delay its proof for now and see how it can help us with the issue of the universality. In the view of Theorem 4.3, what we need to show for Theorem 6.1 is that for a large N, Wigner ensemble can be replaced with a Gaussian ensemble. From now on, we refer to $H(t)$ of Exercise 5.1 or a solution of (4.12) as DBM'. Note that for GOE or GUE we already know that Theorem 6.1 is true by Theorem 4.3, and that $H(t)$ approaches a Gaussian ensemble in the large t limit. Basically we want to prove Theorem 6.1 for a Wigner ensemble $H(0)$ whereas we already know it is true for $H(\infty)$ in a DBM'. An important observation is that for the finite dimensional marginals near an energy level E, we only need to compare the law $\boldsymbol{\lambda}$ locally with a Gaussian ensemble. Even though the global equilibrium is approximated for a large t, it is plausible that a local equilibrium is reached much faster. It fact Erdös et al in [ESY] show that such a local equilibrium is already reached at a short time of order $N^{-\varepsilon}$ for some $\varepsilon > 0$. They then show that $H(N^{-\varepsilon})$ is sufficiently close to $H(0)$ so that the conclusion of Theorem 6.1 is valid for a general ensemble under a mild condition on the law of its entries. Before embarking on the details of the work [ESY], let us outline the steps of the proof.

- (i) We switch from the potential V to a new potential $V' = V + A$ which confines the eigenvalues to a small neighborhood to those values predicted by the semi-circle law. This new potential induces a new infinitesimal generator $\mathcal{L}'$ and a reversible diffusion $\boldsymbol{\lambda}'(t)$ which reaches its equilibrium measure μ' much faster than DBM'.

- (ii) Via Bakry-Emery type result and LSI, we use (i) to show that the law of $\boldsymbol{\lambda}'(t)$ is sufficiently close to its equilibrium measure μ' for a time of order N^{ε} so that for the marginals of eigenvalue gaps $\lambda'_{i+1} - \lambda'_i$ we can switch to μ'.

- (iii) We show how a universality for the finite dimensional marginals of eigenvalue gaps $\lambda'_{i+1} - \lambda'_i$ leads to a similar universality result for the original DBM' process $\boldsymbol{\lambda}(t)$ with $t = N^{-\varepsilon}$, for some positive $\varepsilon > 0$.

- (iv) We show how a universality for gap distributions imply the universality for finite dimensional marginals.

- (v) Finally we show that for N large, the corresponding marginals of the variables $\boldsymbol{\lambda}(0)$ and $\boldsymbol{\lambda}(N^{-\varepsilon})$ are close.

Let us first review what Theorem 6.2 offers in the case of a global equilibrium. To this end, let us assume that the law of $\boldsymbol{\lambda}(0)$ is given by

$$f_0(\boldsymbol{\lambda})\mu_N^\beta(d\boldsymbol{\lambda}) = Z_N(\beta)^{-1} f_0(\boldsymbol{\lambda}) e^{-V_\beta(\boldsymbol{\lambda})} d\boldsymbol{\lambda},$$

with $V = V_\beta$ as is (5.21). Hence the variable $\boldsymbol{\lambda}(t)$ is distributed according to $f_t d\mu_N^\beta$, where f_t satisfies

$$\frac{df_t}{dt} = \mathcal{L} f(t), \quad f(0) = f_0,$$

with $\mathcal{L}$ given by (5.20). We have

$$\begin{aligned} D^2 V(\boldsymbol{\lambda}) v \cdot v &= 2^{-1}\beta N |v|^2 + \beta \sum_{i \neq j} (\lambda_i - \lambda_j)^{-2} v_i^2 - 2 \sum_{i \neq j} (\lambda_i - \lambda_j)^{-2} v_i v_j \\ &= 2^{-1}\beta N |v|^2 + \beta \sum_{i \neq j} (\lambda_i - \lambda_j)^{-2} (v_i - v_j)^2. \end{aligned} \tag{6.3}$$

Hence, μ_N^β satisfies $LSI(2/(N\beta))$:

$$S(g) \leq 2(N\beta)^{-1} D(g),$$

for every density function g. This gives a very fast convergence to the equilibrium measure if we take a diffusion with generator $\mathcal{A} = \Delta - \nabla V \cdot \nabla$. But for our DBM', the process is slowed down by a factor $(\beta N)^{-1}$ More precisely, if $g_t = e^{t\mathcal{A}}$, then $f_t = g_{(\beta N)^{-1} t}$ because $\mathcal{L} = (\beta N)^{-1}\mathcal{A}$. In fact from

$$\frac{dS(f_t)}{dt} = -D(f_t),$$

we learn

$$S(f_t) \leq e^{-2t} S(f_0),$$

which shows that if we choose $t >> 1$ independently of N, then a global equilibrium is already reached.

We recall a celebrated inequality of Csisza and Kullback.

Lemma 6.1 *For every μ-probability denisty g,*

$$\left(\int |g-1| d\mu \right)^2 \leq 2 S_\mu(g).$$

Using Lemma 6.2, we can assert

$$\int |f_t - 1| d\mu_N \leq e^{-t}(2S(f_0))^{1/2}.$$

The Bakry-Emery's argument also yields

$$\frac{dD(f_t)}{dt} \leq 0.$$

As a result,

$$\frac{t}{2} D(f(t)) \leq \int_{t/2}^{t} D(f_s) ds = \frac{1}{4}\left(S(f_{t/2}) - S(f_t)\right) \leq \frac{1}{4} S(f_{t/2}).$$

Hence

$$D(f_t) \leq (2t)^{-1} e^{-2t} S(f_0).$$

It is worth mentioning that the inequalities (6.2) and (6.4) also yield

$$(\beta N)^{-1} \int_0^\infty \int \sum_{i \neq j} (\lambda_i - \lambda_j)^{-2} \left(\frac{\partial f_t}{\partial \lambda_i} - \frac{\partial f_t}{\partial \lambda_j}\right)^2 f_t^{-1} d\mu_N \, dt \leq D(f_0). \tag{6.4}$$

We now embark on the proof of Theorem 6.1 by articulating the first step we briefly described in the outline of the proof. Roughly, we have a situation that can be compared with a metastable system associated with a potential that has many unstable local equilibriums. If the system is already in one of the unstable well and stays there for a while, we may as well replace the potential with a convex one that coincides with the original one inside that particular well. While the system is in the well, there is no difference between the original potential and the modified potential. The advantage of this replacement is that for the new potential we may apply our LSI trick to get a bound on the distance between the non-equilibrium state and the corresponding local equilibrium.

According to Theorem 2.1, the density of eigenvalues is given by ρ of (2.3). We wish to confine the eigenvalues to those values that can be constructed from ρ very accurately. More precisely, choose numbers $\gamma_1 < \cdots < \gamma_N$ such that

$$\int_{\gamma_i}^{\gamma_{i+1}} \rho(x) dx = N^{-1}, \quad \gamma_N = 2. \tag{6.5}$$

We note that the empirical measure associated with the deterministic sequence $\{\gamma_i\}_{i=1}^N$ approximates ρ with a small error of size $O(N^{-1})$. More precisely, if $\Gamma(x) = N^{-1}|\{i : \gamma_i \leq x\}|$, then

$$\left|\Gamma(x) - \int_{-2}^{x} \rho(y) dy\right| \leq \frac{1}{N}, \tag{6.6}$$

for every x. We now choose a potential $V' = V + A$ with

$$A(\boldsymbol{\lambda}) = N\delta^{-2}\sum_{i=1}^{N}(\lambda_i - \gamma_i)^2. \tag{6.7}$$

We now consider a diffusion $\boldsymbol{\lambda}'(t)$ generated by $\mathcal{L}' = (N\beta)^{-1}(\Delta - \nabla V' \cdot \nabla)$. What the additional potential does in practice is confining each eigenvalue λ_i to a (random) interval of size $O(\delta)$ about the value γ_i. Let us write $\mu'_N(d\boldsymbol{\lambda}) = (Z'_N)^{-1}e^{-V'}d\boldsymbol{\lambda}$, where Z'_N is the normalizing constant. Set $S'(g) = S_{\mu'_N}(g)$ and $D'(g) = D_{\mu'_N}(g)$. As an immediate consequence of Theorem 6.2, we have the following bounds:

Lemma 6.2 *Let $g_t = e^{t\mathcal{L}'}g_0$. Then*

$$\begin{aligned}
&S'(g_t) \le e^{-2\delta^{-2}t}S'(g_0),\\
&\int |g_t - 1|d\mu'_N \le e^{-\delta^{-2}t}(2S'(g_0))^{1/2},\\
&D'(g_t) \le (2t)^{-1}e^{-2\delta^{-2}t}S'(g_0),\\
&(\beta N)^{-2}\int_0^\infty \int \sum_{i\neq j}(\lambda_i - \lambda_j)^{-2}\left(\frac{\partial g_t}{\partial \lambda_i} - \frac{\partial g_t}{\partial \lambda_j}\right)^2 g_t^{-1}\ d\mu'_N\ dt \le D(g_0).
\end{aligned} \tag{6.8}$$

For Step (ii), we now want to use Lemma 6.2 to prove a universality-type result for gap distribution for a measure $d\nu = g_0 d\mu'_N$ for which the entropy $S'(g)$ and the entropy production $D'(g)$ is uniformly bounded.

Lemma 6.3 *For every smooth function $J : \mathbb{R}^r \to \mathbb{R}$ of compact support, there exists a constant $C_0 = C_0(J)$ such that for every $\alpha > 0$,*

$$\left|\int \hat{J}(\boldsymbol{\lambda})(g_0 - 1)d\mu'_N\right| \le C_0 N^{-1/2}(D'(g_0)\alpha)^{1/2} + C_0(S'(g_0))^{1/2}e^{-\alpha\delta^{-2}},$$

where

$$\hat{J}(\boldsymbol{\lambda}) = \frac{1}{N}\sum_{i=1}^{N-m_r} J(N(\lambda_i - \lambda_{i+m_1}), \dots, N(\lambda_{i+m_{r-1}} - \lambda_{i+m_r})),$$

for $m_1 < m_2 < \dots, m_r$.

Proof. By Lemma 5.1,

$$\int |\hat{J}||g_\alpha - 1|d\mu'_N \le c_0 e^{-\delta^{-2}\alpha}(2S'(g_0))^{1/2}. \tag{6.9}$$

To ease the notation, we write ∂_i for differentiation with respect to λ_i. We also write J_k, $j = 1, \ldots, r$ for the derivative of J with respect to its k-th argument. We have

$$\left|\int \hat{J}\ (g_\alpha - g_0)\ d\mu'_N\right| = \left|\int_0^\alpha \int \hat{J}\mathcal{L}' g_s\ d\mu'_N\ ds\right| = \left|(\beta N)^{-1} \int_0^\alpha \int \sum_i \partial_i \hat{J}\ \partial_i g_s\ d\mu'_N\ ds\right|$$

$$= \left|\int_0^\alpha \int \frac{1}{\beta N} \sum_{k=1}^r \sum_{i=1}^{N-r} J_k(N(\lambda_i - \lambda_{i+m_1}), \ldots, N(\lambda_{i+m_{r-1}} - \lambda_{i+m_r}))(\partial_{i+m_k} g_s - \partial_{i+m_{k+1}} g_s)\ d\mu'_N\ ds\right|.$$

We simplify the notation by writing

$$J_k^i = J_k(N(\lambda_i - \lambda_{i+m_1}), \ldots, N(\lambda_{i+m_{r-1}} - \lambda_{i+m_r})),$$

and apply Schwartz Inequality to obtain

$$\left|\int \hat{J}\ (g_\alpha - g_0)\ d\mu'_N\right| \leq \frac{1}{\beta N} \left|\int_0^\alpha \int \sum_{k=1}^r \sum_{i=1}^{N-r} (J_k^i)^2 (\lambda_{i+m_k} - \lambda_{i+m_{k+1}})^2 g_s\ d\mu'_N\ ds\right|^{1/2}$$

$$\cdot \left|\int_0^\alpha \int \sum_{k=1}^r \sum_{i=1}^{N-r} (\partial_{i+m_k} g_s - \partial_{i+m_{k+1}} g_s)(\lambda_{i+m_k} - \lambda_{i+m_{k+1}})^{-2} g_s^{-1}\ d\mu'_N\ ds\right|^{1/2}$$

$$\leq c_1 N^{-1/2} (D'(g_0)\alpha)^{1/2},$$

where we used (6.8) and the fact that J is of compact support. From this and (6.9) we deduce the Lemma. □

We now turn to Step (iii) of the outline. Namely, we wish to obtain the gap distribution of $\boldsymbol{\lambda}(t)$ for small t in large N limit. Writing $f_t\ d\mu_N = f'_t\ d\mu'_N$ for the law of $\boldsymbol{\lambda}(t)$, we would like to bound $S'(f'_t)$ and $D'(f'_t)$ so that we can apply Lemma 6.3 for $g_0 = f'_t$.

Lemma 6.4 *If*

$$A = \sup_t \left|\int \sum_i (\lambda_i - \gamma_i)^2 f_t d\mu_N\right|,$$

then

$$S'(f'_t) \leq e^{-\beta\delta^{-2}t} S(f_0) + 2N\delta^2 tA, \quad D'(f'_t) \leq 4\beta N t^{-1} S'(f'_{t/2}) + 12N^2\delta^{-4}A. \tag{6.10}$$

Proof. The main ingredient is a formula of Yau [Y] that we first derive. We would like to have an expression for the time derivative of $S'(f'_t)$. This is the relative entropy of the measure $f_t d\mu_N$ with respect to $d\mu'_N$. More precisely, if

$$\psi(\boldsymbol{\lambda}) = z_N^{-1} \exp\left(-N\delta^{-2} \sum_{i=1}^N (\lambda_i - \gamma_i)^2\right),$$

then we can write

$$S'(f'_t) = \int f_t \log f'_t \ d\mu_N = \int f_t \log \frac{f_t}{\psi} \ d\mu_N = \int \phi(f_t, \psi) \ d\mu_N,$$

where $\phi(a,b) = a\log(ab^{-1})$. We note that ϕ is a convex function. We now have

$$\begin{aligned}
\partial_t S'(f'_t) &= \int \partial_t f_t \ \phi_a(f_t,\psi) \ d\mu_N = \int \mathcal{L} f_t \ \phi_a(f_t,\psi) \ d\mu_N \\
&= \int \left[\mathcal{L} f_t \ \phi_a(f_t,\psi) + \mathcal{L}\psi \ \phi_b(f_t,\psi)\right] \ d\mu_N - \int \mathcal{L}\psi \ \phi_b(f_t,\psi) \ d\mu_N \\
&= -(\beta N)^{-1} \int \left[\nabla f_t \cdot \nabla \phi_a(f_t,\psi) + \nabla\psi \cdot \nabla\phi_b(f_t,\psi)\right] \ d\mu_N - \int \mathcal{L}\psi \ \phi_b(f_t,\psi) \ d\mu_N \\
&= -(\beta N)^{-1} \int \frac{|\nabla f'_t|^2}{f'_t} \ d\mu_N - \int \mathcal{L}\psi \ \phi_b(f_t,\psi) \ d\mu_N,
\end{aligned}$$

where for the last equality we used the calculation

$$\nabla f_t \cdot \nabla \phi_a(f_t,\psi) + \nabla\psi \cdot \nabla\phi_b(f_t,\psi) = \nabla(f'_t\psi) \cdot \nabla f'_t (f'_t)^{-1} - \nabla\psi \cdot \nabla f'_t = |\nabla f'_t|^2 (f'_t)^{-1}\psi.$$

On the other hand,

$$\begin{aligned}
-\int \psi \ \mathcal{L}\phi_b(f_t,\psi) \ d\mu_N = \int \mathcal{L} f'_t \ d\mu'_N &= \int \mathcal{L}' f'_t \ d\mu'_N + 2\beta^{-1}\delta^{-2} \int \sum_i (\lambda_i - \gamma_i) \frac{\partial f'_t}{\partial \lambda_i} \ d\mu'_N \\
&= 2\beta^{-1}\delta^{-2} \int \sum_i (\lambda_i - \gamma_i) \frac{\partial f'_t}{\partial \lambda_i} \ d\mu'_N \\
&\le 2^{-1}(\beta N)^{-1} D'(f'_t) + 2\beta^{-1} N \delta^{-4} \int \sum_i (\lambda_i - \gamma_i)^2 f'_t d\mu'_N.
\end{aligned}$$

As a result,

$$\partial_t S'(f'_t) \le -2^{-1}(\beta N)^{-1} D'(f'_t) + 2\beta^{-1} N \delta^{-4} A. \tag{6.11}$$

From this and LSI

$$S'(f'_t) \le 2^{-1}\delta^2 N^{-1} D'(f'_t),$$

we deduce

$$\partial_t S'(f'_t) \le -\beta^{-1}\delta^{-2} S'(f'_t) + 2\beta^{-1} N \delta^{-4} A.$$

This immediately implies the first inequality in (6.10). For the second inequality in (6.10), first observe

$$2D'(f'_t) = 2\int \left|\nabla \frac{f_t}{\psi}\right|^2 f_t^{-1}\psi^2 d\mu_N \ge D(f_t) - 2\int |\nabla\psi|^2 f_t d\mu_N.$$

From this and (6.11) we deduce

$$\partial_t S'(f'_t) \le -2^{-1}(\beta N)^{-1} D(f_t) + 6\beta^{-1} N \delta^{-4} A. \tag{6.12}$$

We integrate both sides of (6.12) from $t/2$ to t to assert

$$\frac{t}{2} D(f_t) \le \int_{t/2}^{t} D(f_s)ds \le 2\beta N S'(f'_{t/2}) + 4N^2\delta^{-4}At,$$

where we use the monotonicity of $D(f_t)$ that is a consequence of Theorem 6.2. From this

$$D'(f'_t) = \int \left|\nabla \frac{f_t}{\psi}\right|^2 f_t^{-1}\psi^2 d\mu_N \le 2D(f_t) + 2\int |\nabla\psi|^2 f_t d\mu_N,$$

we deduce

$$D'(f'_t) \le 4\beta t N S'(f'_{t/2}) + 12N^2\delta^{-4}At.$$

This implies the second inequality in (6.10). □

We now discuss Step (iii) of the outline. Namely how an average of the correlation as in Theorem 6.1 can be expressed as an average we encountered in Lemma 6.3. First observe that by definition,

$$\frac{1}{2a}\int_{-a}^{a}\int J(\alpha_1,\dots,\alpha_r)\hat{p}_N^{(r)}(\alpha_1,\dots,\alpha_r;E+a)da, \tag{6.13}$$

equals

$$\frac{1}{2a}\int_{E-a}^{E+a}\int \sum_{i_1\ne i_2\ne\cdots\ne i_r} J(N\rho(E)(\lambda_{i_1}-E'),\dots,N\rho(E)(\lambda_{i_r}-E'))f_t(\boldsymbol{\lambda})\mu_N(d\boldsymbol{\lambda})dE'.$$

By symmetry we may assume that $i_1<\cdots<i_r$ and by rewriting $i_1=i$, $i_2=i+m_1,\dots,i_r=i+m_{r-1}$ with $0<m_1<m_2<\cdots<m_{r-1}$, we may rewrite (6.13) as $\sum_{\mathbf{m}\in M_r} X(\mathbf{m})$, where

$$X(\mathbf{m}) = \int_{E-a}^{E+a}\int\sum_i \tilde{J}(N(\lambda_i-E'),N(\lambda_i-\lambda_{i+m_1})\dots,N(\lambda_{i+m_{r-2}}-\lambda_{i+m_{r-1}}))f_t(\boldsymbol{\lambda})\mu_N(d\boldsymbol{\lambda})dE',$$

with

$$\tilde{J}(u_1,\dots,u_r) = \frac{r!}{2a}J(\rho(E)u_1,\rho(E)(u_2-u_1),\dots,\rho(E)(u_r-u_{r-1})),$$
$$M_r = \{\mathbf{m}=(m_1,\dots,m_{r-1})\in\mathbb{N}^{r-1}: 0<m_1<m_2<\cdots<m_{r-1}\}.$$

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