Solutions to Problem Set 13

10.2.10 The roots of the auxiliary equation are $r_1 = \sqrt{-\lambda}$ and $r_2 = -\sqrt{-\lambda}$ We consider separately the cases where λ is negative, zero or positive.

When $\lambda < 0$ then r_1 and r_2 are real and distinct, and the general solution is

$$y(x) = c_1 e^{(\sqrt{-\lambda})x} + c_2 e^{-(\sqrt{-\lambda})x}$$

so that

$$y' = \sqrt{-\lambda} c_1 e^{(\sqrt{-\lambda})x} - \sqrt{-\lambda} c_2 e^{-(\sqrt{-\lambda})x}.$$

The first initial condition gives us that

$$0 = y'(0) = \sqrt{-\lambda} c_1 - \sqrt{-\lambda} c_2$$

so that $c_1 = c_2$. Using the second initial condition and substituting c_1 for c_2 and π for x in the above expression for y gives

$$0 = y(\pi) = c_1 e^{(\sqrt{-\lambda})\pi} + c_1 e^{-(\sqrt{-\lambda})\pi} = c_1(e^{(\sqrt{-\lambda})\pi} + e^{-(\sqrt{-\lambda})\pi})$$

which can only happen if c_1 (and therefore c_2) is equal to 0 since $\sqrt{-\lambda}$ is real. Thus there are no non-trivial solutions y when $\lambda < 0$.

When $\lambda = 0$ then $r_1 = r_2 = 0$, and the general solution is $y(x) = c_1 + c_2 x$ with $y'(x) = c_2$. Thus the initial conditions tell us that $c_2 = 0$ and $0 = y(\pi) = c_1 + 0 = c_1$. Therefore there are no non-trivial solutions when $\lambda = 0$.

Lastly consider $\lambda > 0$ Then $r_1 = i\sqrt{\lambda}$ and $r_2 = -i\sqrt{\lambda}$ are non-real and distinct, and the general solution is

$$y = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x)$$

so that

$$y' = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda} x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda} x).$$

The first initial condition tells us that $0 = y'(0) = c_2\sqrt{\lambda}$, and since $\lambda \neq 0$ we infer that $c_2 = 0$. This together with the second initial condition tell us that

$$0 = y(\pi) = c_1 \cos(\sqrt{\lambda}\pi).$$

Ignoring the trivial solution given by $c_1 = 0$, we infer that $\cos(\sqrt{\lambda}\pi) = 0$ or equivalently that $\sqrt{\lambda}\pi = (2n+1)\frac{\pi}{2}$ for n any integer. That is, $\lambda = \frac{(2n+1)^2}{4}$ for n any non-negative integer. Setting $\lambda_n = \frac{(2n+1)^2}{4}$ and substituting this into the equation for y we get

$$y_n = c_1 \cos(\sqrt{\lambda_n} x) + c_2 \sin(\sqrt{\lambda_n} x) = c_1 \cos\left(\frac{(2n+1)}{2} x\right).$$

Thus for any constant c_1 and any non-negative integer n we have that this y_n is an eigenfunction for the eigenvalue λ_n, and that these are the only solutions.

10.2.12 The roots of the auxiliary equation are $r_1 = \sqrt{-\lambda}$ and $r_2 = -\sqrt{-\lambda}$ We consider separately the cases where λ is negative, zero or positive.

When $\lambda < 0$, r_1 and r_2 are real and distinct. The general solution is

$$y(x) = c_1 e^{(\sqrt{-\lambda})x} + c_2 e^{-(\sqrt{-\lambda})x}$$

so that

$$y' = \sqrt{-\lambda} c_1 e^{(\sqrt{-\lambda})x} - \sqrt{-\lambda} c_2 e^{-(\sqrt{-\lambda})x}.$$

The first initial condition tells us that

$$0 = y'(0) = \sqrt{-\lambda} c_1 - \sqrt{-\lambda} c_2$$

so that $c_1 = c_2$. The initial condition then tells us that

$$0 = y'(\frac{\pi}{2}) = \sqrt{-\lambda} c_1 e^{(\sqrt{-\lambda})\frac{\pi}{2}} - \sqrt{-\lambda} c_1 e^{-(\sqrt{-\lambda})\frac{\pi}{2}}$$

so that if we ignore the trivial solution given by $c_1 = 0$ we get

$$e^{(\sqrt{-\lambda})\frac{\pi}{2}} = e^{-(\sqrt{-\lambda})\frac{\pi}{2}}$$

and thus $e^{(\sqrt{-\lambda})\pi} = 1$ and so $\lambda = 0$, a contradition. Thus we conclude that $c_1 = c_2 = 0$ and there are no non-trivial solutions when $\lambda < 0$.

When $\lambda = 0$, we have the repeated root $r_1 = r_2 = 0$ and thus the general solution is $y(x) = c_1 + c_2 x$ with $y' = c_2 = 0$ using either initial condition. And as a matter of fact it's easy to check that any constant c_2 gives a solution $y = c_2$ when $\lambda = 0$.

Lastly if $\lambda > 0$ then $r_1 = \sqrt{-\lambda} = i\sqrt{\lambda}$ and $r_2 = -\sqrt{-\lambda} = -i\sqrt{\lambda}$ are non-real and distinct, giving a general solution

$$y = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x)$$

with

$$y' = -c_1 \sqrt{\lambda} \sin(\sqrt{\lambda} x) + c_2 \sqrt{\lambda} \cos(\sqrt{\lambda} x).$$

The first initial condition tell us that $0 = y'(0) = c_2\sqrt{\lambda}$ so that $c_2 = 0$. The second initial condition then implies

$$0 = y'(\frac{\pi}{2}) = -c_1 \sqrt{\lambda} \sin(\sqrt{\lambda}\frac{\pi}{2}).$$

Ignoring the trivial solution given by $c_1 = 0$ we infer that $\sin(\sqrt{\lambda}\frac{\pi}{2}) = 0$ or equivalently that $(\sqrt{\lambda}\frac{\pi}{2}) = n\pi$ for any integer n. This implies that $\lambda = 4n^2$. Letting $\lambda_n = 4n^2$ we get the corresponding solution

$$y_n = c_1 \cos(\sqrt{\lambda_n} x) = c_1 \cos(2nx).$$

Thus for any constant c_1 and any non-negative integer n we have this f_n as an eigenfunction for the eigenvalue λ_n. Note that when $n = 0$, we have $y_n = y_0 = c_1$. So the solutions found in the $\lambda = 0$ case are also solutions for the $\lambda > 0$ case, and these are all the solutions.

10.2.18 Substituting 3 for β and π for L in the solution to the heat flow problem (Eq. 11, pg 635) we get

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-3n^2 t} \sin nx.$$

Letting $t = 0$ we get

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin nx.$$

Setting this equal to $f(x) = \sin 4x + 3\sin 6x - \sin 10x$ we find that $c_4 = 1$, $c_6 = 3$, $c_{10} = -1$, and all other $c_i = 0$. Plugging these values in to the series for $u(x,t)$ gives

$$\begin{aligned} u(x,t) &= e^{-3(4)^2 t} \sin 4x + 3e^{-3(6)^2 t} \sin 6x - e^{-3(10)^2 t} \sin 10x \\ &= e^{-48t} \sin 4x + 3e^{-108t} \sin 6x - e^{-300t} \sin 10x. \end{aligned}$$

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10.2.22 Substituting 3 for α and π for L in the solution to the vibrating string problem (Eq. 24, pg 638) we get

$$u(x,t) = \sum_{n=1}^{\infty} [a_n \cos 3nt + b_n \sin 3nt] \sin nx.$$

Evaluating u at $t = 0$ and setting the result equal to the given function f gives

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin nx = \sin x - \sin 2x + \sin 3x.$$

Thus $a_1 = 1$, $a_2 = -1$, $a_3 = 1$, and all other $a_i = 0$.

Next we compute

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} [-3na_n \sin 3nt + 3nb_n \cos 3nt] \sin nx$$

and evaluating this at $t = 0$ and setting the result equal to $g(x)$ we get

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} 3nb_n \sin nx = 6\sin 3x - 7\sin 5x.$$

This tells us that $6 = 3(3)b_3$, $-7 = 3(5)b_5$ so that $b_3 = \frac{2}{3}$, $b_5 = -\frac{7}{15}$ and all other $b_i = 0$.

Plugging this information about the constants back into the formula for $u(x,t)$ gives

$$u(x,t) = \cos 3t \sin x - \cos 6t \sin 2x + (\cos 9t + \frac{2}{3}\sin 9t)\sin 3x - \frac{7}{15}\sin 15t \sin 5x.$$

10.3.2 f is the product of two odd functions and therefore is even. (See 7(b) below)

10.3.4 $f(-1) = \sin(0) = 0$ which is not equal to either $\pm f(1) = \pm \sin(2)$. Thus f is neither odd nor even.

10.3.7

(a) If f, g are even then

$$(fg)(-x) = f(-x)g(-x) = f(x)g(x) = (fg)(x)$$

so fg is even.

(b) If f, g are odd then

$$(fg)(-x) = f(-x)g(-x) = (-f(x))(-g(x)) = f(x)g(x) = (fg)(x)$$

so fg is even.

(c) If f is even and g is odd then

$$(fg)(-x) = f(-x)g(-x) = (f(x))(-g(x)) = -f(x)g(x) = -(fg)(x)$$

so fg is odd.

10.3.14 The function f is piecewise continuous on $[-\pi, \pi]$ so we let $T = \pi$ in the definition of Fourier Series and compute the coefficients a_n and b_n.

For n a non-negative integer we have

$$\begin{aligned} a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx &= \frac{1}{\pi}\int_{-\pi}^{0} (x+\pi)\cos(nx)dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos(nx)dx \\ &= \int_{-\pi}^{0}\cos(nx)dx + \frac{1}{\pi}\int_{-\pi}^{0} x\cos(nx)dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos(nx)dx \\ &= \int_{-\pi}^{0}\cos(nx)dx + \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)dx = \int_{-\pi}^{0}\cos(nx)dx. \end{aligned}$$

For $n = 0$ we then have $a_0 = \pi$ and for $n \neq 0$ we have

$$a_n = \int_{-\pi}^{0}\cos(nx)dx = \left[\frac{1}{n}\sin(nx)\right]_{-\pi}^{0} = \frac{1}{n}\sin(n\pi) = 0$$

Next we see that for any positive integer n we have

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\int_{-\pi}^{0}(x+\pi)\sin(nx)dx + \frac{1}{\pi}\int_{0}^{\pi} x\sin(nx)dx$$

$$= \int_{-\pi}^{0} \sin(nx)dx + \frac{1}{\pi}\int_{-\pi}^{0} x\sin(nx)dx + \frac{1}{\pi}\int_{0}^{\pi} x\sin(nx)dx$$

$$= \int_{-\pi}^{0} \sin(nx)dx + \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)dx = \int_{-\pi}^{0} \sin(nx)dx + \frac{2}{\pi}\int_{0}^{\pi} x\sin(nx).$$

We will compute these two integrals separately.

$$\int_{-\pi}^{0} \sin(nx)dx = \left[-\frac{1}{n}\cos(nx)\right]_{-\pi}^{0} = -\frac{1}{n} + \frac{1}{n}\cos(n\pi) = \frac{1}{n}(-1+(-1)^n),$$

which is 0 for even n and $\frac{-2}{n}$ for odd n.

$$\frac{2}{\pi}\int_{0}^{\pi} x\sin(nx) = \frac{2}{\pi n^2}\left[\sin(nx) - nx\cos(nx)\right]_0^{\pi} = \frac{2}{\pi n^2}\left(-n\pi\cos(n\pi)\right) = \frac{2}{n}(-1)^{n+1},$$

which is $-\frac{2}{n}$ for even n and $\frac{2}{n}$ for odd n.
Adding these two expressions together we get b_n is $-\frac{2}{n}$ for even n and $b_n = 0$ for odd n, which we can express as $b_n = -\frac{1}{n}(1+(-1)^n)$.
Having computed the coefficients, we can now express the Fourier series for f as

$$\frac{\pi}{2} - \sum_{n=1}^{\infty} \frac{1}{n}(1+(-1)^n)\sin(nx).$$

10.3.26 Let $m \neq n$ be positive integers. We consider three cases: $\cos\cos$, $\sin\sin$, and $\cos\sin$.

(a) For $\cos\cos$ we use the identity $\cos A\cos B = \frac{1}{2}(\cos(A+B)+\cos(A-B))$

$$\int_{-1}^{1} \cos\frac{(2n-1)\pi}{2}x\cos\frac{(2m-1)\pi}{2}x\ dx =$$

$$\frac{1}{2}\int_{-1}^{1} \cos\left(\frac{(2n-1)\pi}{2}x + \frac{(2m-1)\pi}{2}x\right) + \cos\left(\frac{(2n-1)\pi}{2}x - \frac{(2m-1)\pi}{2}x\right) dx =$$

$$\frac{1}{2}\int_{-1}^{1} \cos(\pi(n+m-1)x) + \cos(\pi(n-m)x)\ dx.$$

Now we know that $n+m-1 \neq 0$ since n, m are positive integers. Since in addition $n \neq m$ we see that this last integral evaluates to

$$\frac{1}{2}\left[\frac{1}{\pi(n+m-1)}\sin(\pi(n+m-1)x) + \frac{1}{\pi(n-m)}\sin(\pi(n-m)x)\right]_{-1}^{1} = 0.$$

Thus we have shown orthogonality in the $\cos\cos$ case.

(b) For the sin sin case we use the identity $\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$

$$\int_{-1}^{1} \sin \frac{(2n-1)\pi}{2} x \sin \frac{(2m-1)\pi}{2} x \, dx =$$

$$\frac{1}{2}\int_{-1}^{1} \cos\left(\frac{(2n-1)\pi}{2}x - \frac{(2m-1)\pi}{2}x\right) - \cos\left(\frac{(2n-1)\pi}{2}x + \frac{(2m-1)\pi}{2}x\right) dx =$$

$$\int_{-1}^{1} \cos(\pi(n-m)x) - \cos(\pi(n+m-1)x) dx,$$

which is again zero for the same reasons as in the cos cos case. Thus we have shown orthogonality in the sin sin case.

(c) For the cos sin case we use the identity $\cos A \sin B = \frac{1}{2}(\sin(A+B) - \sin(A-B))$. Then

$$\int_{-1}^{1} \cos \frac{(2n-1)\pi}{2} x \sin \frac{(2m-1)\pi}{2} x \, dx$$

$$= \frac{1}{2}\int_{-1}^{1} \sin\left(\frac{(2n-1)\pi}{2}x + \frac{(2m-1)\pi}{2}x\right) - \sin\left(\frac{(2n-1)\pi}{2}x - \frac{(2m-1)\pi}{2}x\right) dx$$

$$= \frac{1}{2}\int_{-1}^{1} \sin(\pi(n+m-1)x) - \sin(\pi(n-m)x) \, dx.$$

Since both terms of the integrand are odd functions of x we see that the integral evaluates to zero. Thus we have shown orthogonality in the cos sin case. It can also be shown that $\int_{-1}^{1} \cos^2 \frac{(2n-1)\pi}{2} x \, dx$ and $\int_{-1}^{1} \sin^2 \frac{(2n-1)\pi}{2} x \, dx$ equal 1. For instance in the case of sin:

$$\int_{-1}^{1} \sin^2\left(\frac{(2n-1)\pi}{2}x\right) dx = \frac{1}{2}\int_{-1}^{1} (1 - \cos(\pi(2n-1)x)) = \frac{1}{2}\left[x - \frac{\sin((2n-1)\pi x)}{(2n-1)\pi}\right]_{-1}^{1} = 1,$$

and the case for cos is similar.

10.3.27 We first need a way of enumerating the functions in the preceding problem. We can do this as follows. For n a positive integer we define

$f_{2n-1} = \cos\left(\frac{(2n-1)\pi}{2}x\right)$, $f_{2n} = \sin\left(\frac{(2n-1)\pi}{2}x\right)$,

Examining equation (20) (pg 652) for the coefficients c_m, and noticing that the system in the previous problem is *orthonormal*, we see that

$$c_m = \frac{\int_{-1}^{1} f(x) f_m(x) dx}{1} = \int_{0}^{1} f_m(x) dx.$$

In particular

$$c_{2n-1} = \int_{0}^{1} f_{2n-1}(x) dx = \int_{0}^{1} \cos\left(\frac{(2n-1)\pi}{2}x\right) dx = \left[\frac{2}{(2n-1)\pi} \sin\left(\frac{(2n-1)\pi}{2}x\right)\right]_{0}^{1}$$

$$= \frac{2}{(2n-1)\pi} \sin\left(\frac{(2n-1)\pi}{2}\right) = \frac{2}{(2n-1)\pi}(-1)^{n+1}$$

Also,

$$c_{2n} = \int_0^1 f_{2n}(x)dx = \int_0^1 \sin\left(\frac{(2n-1)\pi}{2}x\right)dx$$

$$= \left[-\frac{2}{(2n-1)\pi}\cos\left(\frac{(2n-1)\pi}{2}x\right)\right]_0^1 = \frac{2}{(2n-1)\pi}.$$

Thus the orthogonal expansion for f is

$$\sum_{m=1}^{\infty} c_m f_m = \sum_{n=1}^{\infty} [c_{2n-1}f_{2n-1} + c_{2n}f_{2n}] = \sum_{n=1}^{\infty} \frac{2}{(2n-1)\pi}\left[(-1)^{n+1}f_{2n-1} + f_{2n}\right]$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)}\left[(-1)^{n+1}\cos\left(\frac{(2n-1)\pi}{2}x\right) + \sin\left(\frac{(2n-1)\pi}{2}x\right)\right].$$

10.4.4 Pictures omitted, but compare Figure 10.11 on page 662. We also give expressions for the restrictions of $\tilde{f}(x)$, f_o, and f_e to the interval $[-\pi, \pi]$:

$$\tilde{f}(x) = \begin{cases} \pi - x & 0 < x < \pi \\ -x & -\pi < x < 0 \end{cases};$$

$$f_o = \begin{cases} \pi - x & 0 < x < \pi \\ -x - \pi & -\pi < x < 0 \end{cases};$$

$$f_e = \begin{cases} \pi - x & 0 < x < \pi \\ x + \pi & -\pi < x < 0 \end{cases}.$$

10.4.6 The function f is piecewise continuous on the interval $[0, \pi]$, so we let $T = \pi$ in the definition of Fourier sine series and get $\sum_{n=1}^{\infty} b_n \sin(nx)$ where

$$b_n = \frac{2}{\pi}\int_0^{\pi} \cos(x)\sin(nx)dx = \frac{2}{\pi}\left[\frac{\sin(x)\sin(nx) + n\cos(x)\cos(nx)}{1-n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-n\cos(n\pi) - n}{1-n^2}\right] = \frac{2n}{\pi(1-n^2)}(\cos(n\pi) - 1) = \frac{2n}{\pi(1-n^2)}((-1)^n - 1).$$

Plugging this in give us

$$\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{n}{(1-n^2)}((-1)^n - 1)\sin(nx).$$

10.4.12 The function f is piecewise continuous on the interval $[0, \pi]$, so we let $T = \pi$ in the definition of Fourier cosine series and get $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$. Where

$$a_0 = \frac{2}{\pi} \int_0^{\pi} (1 + x) dx = \frac{2}{\pi} \left[x + \frac{x^2}{2} \right]_0^{\pi} = 2 + \pi,$$

and for $n \neq 0$

$$a_n = \frac{2}{\pi} \int_0^{\pi} \cos(nx) dx + \frac{2}{\pi} \int_0^{\pi} x \cos(nx) dx.$$

It is easy to see that this first term evaluates to zero. Thus for $n \neq 0$ we have

$$a_n = \frac{2}{\pi} \int_0^{\pi} x \cos(nx) dx = \frac{2}{\pi} \left[\frac{nx \sin(nx) + \cos(nx)}{n^2} \right]_0^{\pi}$$

$$= \frac{2}{\pi n^2} (\cos(n\pi) - 1) = \frac{2}{\pi n^2} ((-1)^n - 1)$$

Plugging in these constants gives us

$$1 + \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} ((-1)^n - 1) \cos(nx).$$

10.4.18 We have $\beta = 5$, $L = \pi$. Thus the heat function is

$$u(x, t) = \sum_{n=1}^{\infty} c_n e^{-5n^2 t} \sin(nx).$$

Plugging in $t = 0$ gives

$$u(x, 0) = \sum_{n=1}^{\infty} c_n \sin(nx) = f(x)$$

Thus we need to find the coefficients for the Fourier sine series for f. These are given by

$$b_n = \frac{2}{\pi} \int_0^{\pi} x(\pi - x) \sin(nx) dx$$

$$= \frac{2}{\pi} \left[\frac{1}{n^3} ((n^2 x(x - \pi) - 2) \cos(nx) + n(\pi - 2x) \sin(nx)) \right]_0^{\pi}$$

$$= \frac{-4}{\pi n^3} (\cos(n\pi) - 1) = \frac{4}{\pi n^3} (1 - (-1)^n).$$

Thus the heat function is

$$u(x, t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^3} (1 - (-1)^n) e^{-5n^2 t} \sin(nx).$$

10.4.19 Again we have $\beta = 5$, $L = \pi$, so the heat function is

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-5n^2 t} \sin(nx),$$

and plugging in $t = 0$ gives

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin(nx) = f(x).$$

Thus we need to find the coefficients for the Fourier sine series for f. These are given by

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}(-x)\sin(nx)dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi}(x-\pi)\sin(nx)dx =$$

$$-\frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\sin(nx)dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} x\sin(nx)dx - 2\int_{\frac{\pi}{2}}^{\pi}\sin(nx)dx =$$

$$-\frac{2}{\pi n^2}[\sin(nx) - nx\cos(nx)]_0^{\frac{\pi}{2}} + \frac{2}{\pi n^2}[\sin(nx) - nx\cos(nx)]_{\frac{\pi}{2}}^{\pi} + \frac{2}{n}[\cos(nx)]_{\frac{\pi}{2}}^{\pi}$$

$$= -\frac{4}{\pi n^2}\sin(\frac{n\pi}{2})$$

So the heat function is

$$u(x,t) = \sum_{n=1}^{\infty} -\frac{4}{\pi n^2}\sin(\frac{n\pi}{2})e^{-5n^2 t}\sin(nx),$$

To simplify this a bit we notice that when n is even the nth term of the series is zero, so we use the index $2k-1$ instead.

$$u(x,t) = \frac{-4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\sin(\frac{(2k-1)\pi}{2})e^{-5(2k-1)^2 t}\sin((2k-1)x),$$

the advantage being that we can replace the first sine factor by $(-1)^{k+1}$ and then pull the minus sign back in to get

$$u(x,t) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}(-1)^k e^{-5(2k-1)^2 t}\sin((2k-1)x)$$

10.5.4 From the equation (14) (pg 668) with $\beta = 2$ and $L = 1$ we get the heat function

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-2(n\pi)^2 t}\cos(n\pi x).$$

Letting $t = 0$ we get

$$u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) = x(1-x).$$

So we must find the Fourier cosine series for $x(1-x)$ with $T = 1$.
For $n = 0$ we get $a_0 = 2\int_0^1 x(1-x)dx = \frac{1}{3}$. For nonzero n we have

$$a_n = 2\int_0^1 x(1-x)\cos(n\pi x)dx =$$

$$\frac{2}{\pi^3 n^3}\left[(2-\pi^2 n^2(x-1)x)\sin(\pi n x) - \pi n(2x-1)\cos(\pi n x)\right]_0^1$$

$$= \frac{2}{\pi^3 n^3}\left[-\pi n(-1)^n - \pi n\right] = \frac{2}{\pi^2 n^2}((-1)^{n+1} - 1).$$

Thus the heat function becomes

$$u(x,t) = \frac{1}{6} + \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}((-1)^{n+1} - 1)e^{-2(n\pi)^2 t}\cos(n\pi x).$$