Math 54 Homework 12 Solution

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Question 9.1.12

Let $x_0 = x, x_1 = x', x_2 = y$, and $x_3 = y'$, we see that

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -3 & -2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Question 9.4.4

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 1 & 0 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Question 9.4.8

Let $x_0 = y, x_1 = y'$, and $x_2 = y''$, then

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ cos(t) \end{pmatrix}$$

Question 9.4.14

Suppose we have some linear combination $c_1 \begin{pmatrix} te^{-t} \\ e^{-t} \end{pmatrix} + c_2 \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Evaluating the above equation at $t = 0$ we have $c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$. But $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are linearly independent, so $c_1 = c_2 = 0$, and hence $\begin{pmatrix} te^{-t} \\ e^{-t} \end{pmatrix}$ and $\begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}$ are linearly independent.

Question 9.4.18

Suppose we have some linear combination $c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} t \\ 0 \\ t \end{pmatrix} + c_3 \begin{pmatrix} t^2 \\ 0 \\ t^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Then we see that $c_1 + c_2 t + c_3 t^2 = 0$. But $1, t, t^2$ are linearly independent on $(-\infty, \infty)$ (this is homework question 6.2.25 with $r = 0$), and so $c_1 = c_2 = c_3 = 0$, which implies that $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} t \\ 0 \\ t \end{pmatrix}, \begin{pmatrix} t^2 \\ 0 \\ t^2 \end{pmatrix}$ are linearly independent.

Question 9.4.22

Consider the matrix $X = \begin{pmatrix} e^t & sin(t) & -cos(t) \\ e^t & cos(t) & sin(t) \\ e^t & -sin(t) & cos(t) \end{pmatrix}$. At the point $t = 0$, we have $X(0) = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ which has determinant 2, and so X has linearly independent columns. A fundamental matrix is $X =$

$\begin{pmatrix} e^t & sin(t) & -cos(t) \\ e^t & cos(t) & sin(t) \\ e^t & -sin(t) & cos(t) \end{pmatrix}$, and the general solution is $c_1 \begin{pmatrix} e^t \\ e^t \\ e^t \end{pmatrix} + c_2 \begin{pmatrix} sin(t) \\ cos(t) \\ -sin(t) \end{pmatrix} + c_3 \begin{pmatrix} -cos(t) \\ sin(t) \\ cos(t) \end{pmatrix}$

Question 9.4.27

$$X' = \begin{pmatrix} -6e^{-t} & 6e^{-2t} & 6e^{3t} \\ e^{-t} & -2e^{-2t} & 3e^{3t} \\ 5e^{-t} & -2e^{-2t} & 3e^{3t} \end{pmatrix} = \begin{pmatrix} 0 & 6 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 6e^{-t} & -3e^{-2t} & 2e^{3t} \\ -e^{-t} & e^{-2t} & e^{3t} \\ -5e^{-t} & e^{-2t} & e^{3t} \end{pmatrix} = AX$$

$$X^{-1}(t) = \begin{pmatrix} 0 & \frac{1}{4}e^{t} & -\frac{1}{4}e^{t} \\ -\frac{1}{5}e^{2t} & \frac{4}{5}e^{2t} & -\frac{2}{5}e^{2t} \\ \frac{1}{5}e^{-3t} & \frac{9}{20}e^{-3t} & \frac{3}{20}e^{-3t} \end{pmatrix}$$

$x(t) = X(t)X^{-1}(0)x(0)$ is the solution to the given initial value problem, so we have

$$\begin{aligned} x(t) &= X(t)X^{-1}(0)x(0) \\ &= \begin{pmatrix} 6e^{-t} & -3e^{-2t} & 2e^{3t} \\ -e^{-t} & e^{-2t} & e^{3t} \\ -5e^{-t} & e^{-2t} & e^{3t} \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{5} & \frac{4}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{9}{20} & \frac{3}{20} \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 6e^{-t} & -3e^{-2t} & 2e^{3t} \\ -e^{-t} & e^{-2t} & e^{3t} \\ -5e^{-t} & e^{-2t} & e^{3t} \end{pmatrix} \begin{pmatrix} -\frac{1}{4} \\ -\frac{1}{5} \\ -\frac{1}{20} \end{pmatrix} \\ &= -\frac{1}{4} \begin{pmatrix} 6e^{-t} \\ -e^{-t} \\ -5e^{-t} \end{pmatrix} - \frac{1}{5} \begin{pmatrix} -3e^{-2t} \\ e^{-2t} \\ e^{-2t} \end{pmatrix} - \frac{1}{20} \begin{pmatrix} 2e^{3t} \\ e^{3t} \\ e^{3t} \end{pmatrix} \end{aligned}$$

Question 9.5.22

For the matrix $A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{pmatrix}$, the eigenvalues are $r_1 = 2$ corresponding to the eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $r_2 = 1$ corresponding to the eigenvector $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$, so a fundamental matrix is

$$X = \begin{pmatrix} e^{2t} & -e^{2t} & e^{t} \\ 0 & e^{2t} & e^{t} \\ e^{2t} & 0 & 3e^{t} \end{pmatrix}$$

Question 9.5.34

The eigenvalues for the matrix $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ are $r_1 = -1$ corresponding to eigenvectors $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $r_2 = 2$ corresponding to eigenvectors $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Put $X(t) = \begin{pmatrix} -e^{-t} & -e^{-t} & e^{2t} \\ e^{-t} & 0 & e^{2t} \\ 0 & e^{-t} & e^{2t} \end{pmatrix}$, and solving $X(0)\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix}$ gives $c_1 = 3, c_2 = -1, c_3 = 1$. So the solution to the initial value problem is

$$3 \begin{pmatrix} -e^{-t} \\ e^{-t} \\ 0 \end{pmatrix} - \begin{pmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{pmatrix} + \begin{pmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{pmatrix}$$

Question 9.6.4

The eigenvalues are $r_1 = 2, r_2 = 2+i, r_3 = 2-i$ with eigenvectors $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 5 \\ -2-i \\ 5 \end{pmatrix}, \begin{pmatrix} 5 \\ -2+i \\ 5 \end{pmatrix}$ respectively, so we see that the general solution is

$$c_1 \begin{pmatrix} 0 \\ e^{2t} \\ -e^{2t} \end{pmatrix} + c_2 \begin{pmatrix} 5e^{2t}cos(t) \\ -2e^{2t}cos(t) + e^{2t}sin(t) \\ 5e^{2t}cos(t) \end{pmatrix} + c_3 \begin{pmatrix} 5e^{2t}sin(t) \\ -2e^{2t}sin(t) - e^{2t}cos(t) \\ 5e^{2t}sin(t) \end{pmatrix}$$

Question 9.6.8

The eigenvalues are $r_1 = 1, r_2 = -1, r_3 = 2+3i, r_4 = 2-3i$ with eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2+3i \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2-3i \end{pmatrix}$$

respectively, and thus a fundamental matrix looks like

$$X = \begin{pmatrix} e^t & e^-t & 0 & 0 \\ e^t & -e^{-t} & 0 & 0 \\ 0 & 0 & e^{2t}cos(3t) & e^{2t}sin(3t) \\ 0 & 0 & 2e^{2t}cos(3t) - 3e^{2t}sin(3t) & 2e^{2t}sin(3t) + 3e^{2t}cos(3t) \end{pmatrix}$$

Question 9.6.14

The eigenvalues are $2, 1+i, 1-i$ with eigenvectors $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$ respectively, so a fundamental matrix looks like

$$X = \begin{pmatrix} 0 & e^t cos(t) & e^t sin(t) \\ e^{2t} & 0 & 0 \\ 0 & e^t sin(t) & -e^t cos(t) \end{pmatrix}$$

Now $X^{-1}(0) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and $X^{-1}(-\pi) = \begin{pmatrix} 0 & e^{2\pi} & 0 \\ -e^{\pi} & 0 & 0 \\ 0 & 0 & e^{\pi} \end{pmatrix}$.

So for part (a), $x = X(t)X^{-1}(0)x(0) = X(t) \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2e^t cos(t) + e^t sin(t) \\ 2e^{2t} \\ -2e^t sin(t) - e^t cos(t) \end{pmatrix}$

Similarly, for part (b), $x = X(t)X^{-1}(-\pi)x(-\pi) = X(t) \begin{pmatrix} e^{2\pi} \\ 0 \\ e^{\pi} \end{pmatrix} = \begin{pmatrix} e^{\pi} e^t sin(t) \\ e^{2\pi} e^{2t} \\ -e^{\pi} e^t cos(t) \end{pmatrix}$