SOLUTIONS TO HOMEWORK #11, MATH 54 SECTION 001, SPRING 2012

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Beware of typos. Congratulations to you if you find solutions that are better than mine!

1. Ex. 4.5.2(b): *Given that* $y_1(t) = (1/4)\sin 2t$ *is a solution to* $y'' + 2y' + 4y = \cos 2t$ *and that* $y_2(t) = t/4 - 1/8$ *is a solution to* $y'' + 2y' + 4y = t$*, use the superposition principle to find solutions to the following:*
(b) $y'' + 2y' + 4y = 2t - 3\cos 2t$.

Solution. By the superposition principle, $2\left(\frac{t}{4} - \frac{1}{8}\right) - 3\left(\frac{1}{4}\sin 2t\right) = \boxed{\frac{t}{2} - \frac{1}{4} - \frac{3}{4}\sin 2t}$ solves the equation. □

2. Ex. 4.5.6: *In Problems 3-8, a nonhomogeneous equation and a particular solution are given. Find a general solution for the equation.*
$y'' + 5y' + 6y = 6x^2 + 10x + 2 + 12e^x, \qquad y_p(x) = e^x + x^2$

Solution. To find a general solution to an inhomogeneous equation, you add a particular solution to the inhomogeneous equation to the general solution to the homogeneous equation.

The homogeneous equation is $y'' + 5y' + 6y = 0$. Its auxiliary polynomial is $x^2 + 5x + 6 = (x+2)(x+3)$, and its roots are -2 and -3. The general solution to the homogeneous equation is $c_1e^{-2x} + c_2e^{-3x}$. The general solution to the inhomogeneous equation is $\boxed{c_1e^{-2x} + c_2e^{-3x} + e^x + x^2 \text{ for any numbers } c_1, c_2}$. □

3. Ex. 4.5.10: *In Problems 9-16 decide whether the method of undetermined coefficients together with superposition can be applied to find a particular solution of the given equation. Do not solve the equation.*
$y'' - y' + y = (e^t + t)^2$

Solution. Undetermined coefficients and the superposition principle can only be applied to equations of the form $ay'' + by' + cy = g(t)$, where $g(t)$ is a sum of functions of the form (polynomial)e^{rt}, (polynomial)$e^{\alpha t}\cos\beta t$, or (polynomial)$e^{\alpha t}\sin\beta t$.

Because $(e^t + t)^2 = e^{2t} + 2te^t + t^2 = 1 \cdot e^{2t} + (2t)e^{1t} + t^2e^{0t}$, $\boxed{\text{undetermined coefficients can be used}}$. □

4. Ex. 4.5.18: *In Problems 17-22, find a general solution to the differential equation.*
$y'' - 2y' - 3y = 3t^2 - 5$

Solution. When solving $ay'' + by' + cy =$ (degree d polynomial)e^{rt} by undetermined coefficients, the form of the particular solution is:

$$y_p(t) = t^{\#\text{ of times } r \text{ is a root of } ax^2+bx+c}(\text{degree } d \text{ polynomial with undetermined coefficients})e^{rt}.$$

For this problem,

$$3t^2 - 5 = (3t^2 - 5)e^{0t} = (\text{degree 2 polynomial})e^{0t}.$$

The auxiliary polynomial is $x^2 - 2x - 3$. 0 is not a root of the auxiliary polynomial, so we pick:

$$y_p(t) = t^0(At^2 + Bt + C)e^{0t} = At^2 + Bt + C.$$

Then:

$$y_p'(t) = 2At + B, \qquad y_p''(t) = 2A,$$

so

$$\begin{aligned} y_p'' - 2y_p' - 3y_p &= 2A - 2(2At + B) - 3(At^2 + Bt + C) \\ &= -3At^2 + (-4A - 3B)t + (2A - 2B - 3C). \end{aligned}$$

Since this is to equal $3t^2 - 5$, we have $-3A = 3$, so $A = -1$. Next, $-4A - 3B = 0$, so $B = -\frac{4}{3}A = \frac{4}{3}$. Finally, $2A - 2B - 3C = -5$, so

$$C = \tfrac{2}{3}A - \tfrac{2}{3}B + \tfrac{5}{3} = -\tfrac{2}{3} - \tfrac{8}{9} + \tfrac{5}{3} = \tfrac{1}{9}.$$

That means

$$y_p(t) = -t^2 + \tfrac{4}{3}t + \tfrac{1}{9}.$$

To find a general solution to an inhomogeneous equation, you add a particular solution to the inhomogeneous equation to the general solution to the homogeneous equation.

The homogeneous equation is $y'' - 2y' - 3y = 0$. Its auxiliary polynomial is $x^2 - 2x - 3 = (x+1)(x-3)$, and its roots are -1 and 3. The general solution to the homogeneous equation is $c_1e^{-t} + c_2e^{3t}$. The general solution to the inhomogeneous equation is $\boxed{c_1e^{-t} + c_2e^{3t} - t^2 + \tfrac{4}{3}t + \tfrac{1}{9} \text{ for any numbers } c_1, c_2}$. □

5. Ex. 4.5.28: *In Problems 23-30, find the solution to the initial value problem.*
$y'' + y' - 12y = e^t + e^{2t} - 1; \qquad y(0) = 1, \qquad y'(0) = 3$

Solution. When solving $ay'' + by' + cy = (\text{degree } d \text{ polynomial})e^{rt}$ by undetermined coefficients, the form of the particular solution is:

$$y_p(t) = t^{\#\text{ of times } r \text{ is a root of } ax^2+bx+c}(\text{degree } d \text{ polynomial with undetermined coefficients})e^{rt}.$$

If $g(t)$ is a sum of different terms, then by the superposition principle, the form of the particular solution is the sum of individual forms.

In this case, the auxiliary polynomial is $x^2 + x - 12 = (x+4)(x-3)$ and its roots are -4 and 3. Since $e^t + e^{2t} - 1 = 1e^{1t} + 1e^{2t} + (-1)e^{0t}$, and none of 1, 2, and 0 are roots of the auxiliary polynomial, we pick:

$$y_p(t) = t^0Ae^{1t} + t^0Be^{2t} + t^0Ce^{0t} = Ae^t + Be^{2t} + C.$$

Then:

$$y_p'(t) = Ae^t + 2Be^{2t}, \qquad y_p''(t) = Ae^t + 4Be^{2t},$$

so:

$$y_p'' + y_p' - 12y_p = (Ae^t + 4Be^{2t}) + (Ae^t + 2Be^{2t}) - 12(Ae^t + Be^{2t} + C) = (-10A)e^t + (-6B)e^{2t} + (-12C).$$

For this to equal $e^t + e^{2t} - 1$, we need $-10A = 1$, $-6B = 1$, and $-12C = -1$. Then $A = -\frac{1}{10}$, $B = -\frac{1}{6}$, and $C = \frac{1}{12}$. Therefore:

$$y_p(t) = -\tfrac{1}{10}e^t - \tfrac{1}{6}e^{2t} + \tfrac{1}{12}.$$

Since the roots of the auxiliary polynomial are -4 and 3, the general solution to the homogeneous equation is $c_1e^{-4t}+c_2e^{3t}$, so the general solution to the inhomogeneous equation is $y(t) = c_1e^{-4t}+c_2e^{3t}-\frac{1}{10}e^t-\frac{1}{6}e^{2t}+\frac{1}{12}$.

Then since $y(0) = 1$, we have $c_1 + c_2 - \frac{1}{10} - \frac{1}{4} + \frac{1}{12} = 1$, so:

$$c_1 + c_2 = 1 + \tfrac{1}{10} + \tfrac{1}{6} - \frac{1}{12} = \tfrac{60}{60} + \tfrac{6}{60} + \tfrac{10}{60} - \tfrac{5}{60} = \tfrac{71}{60}.$$

Next,

$$y'(t) = -4c_1e^{-4t} + 3c_2e^{3t} - \tfrac{1}{10}e^t - \tfrac{1}{3}e^{2t}.$$

Since $y'(0) = 3$, we have $-4c_1 + 3c_2 - \frac{1}{10} - \frac{1}{3} = 3$, so

$$-4c_1 + 3c_2 = 3 + \tfrac{1}{10} + \tfrac{1}{3} = \tfrac{90}{30} + \tfrac{3}{30} + \tfrac{10}{30} = \tfrac{103}{30} = \tfrac{206}{60}.$$

So we solve the system $c_1 + c_2 = \frac{71}{60}$, $-4c_1 + 3c_2 = \frac{206}{60}$ for c_1, c_2 as follows:

$$\begin{bmatrix} 1 & 1 & \frac{71}{60} \\ -4 & 3 & \frac{206}{60} \end{bmatrix} \xrightarrow{R_2 \to R_2 + 4R_1} \begin{bmatrix} 1 & 1 & \frac{71}{60} \\ 0 & 7 & \frac{490}{60} \end{bmatrix} \xrightarrow{R_2 \to (1/7)R_2} \begin{bmatrix} 1 & 1 & \frac{71}{60} \\ 0 & 1 & \frac{7}{6} \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & \frac{1}{60} \\ 0 & 1 & \frac{7}{6} \end{bmatrix}.$$

Therefore, $c_1 = \frac{1}{60}$ and $c_2 = \frac{7}{6}$, so the answer is $\boxed{\frac{1}{60}e^{-4t} + \frac{7}{6}e^{3t} - \frac{1}{10}e^t - \frac{1}{6}e^{2t} + \frac{1}{12}}$. □

6. Ex. 4.5.34: *In Problems 31-36, determine the form of a particular solution for the differential equation. Do not solve.*
$y'' + 5y' + 6y = \sin t - \cos 2t$

Solution. When solving

$$ay'' + by' + cy = e^{\alpha t}\Big((\text{degree } d_1 \text{ polynomial})\cos\beta t + (\text{degree } d_2 \text{ polynomial})\cos\beta t\Big)$$

by undetermined coefficients, the form of the particular solution is as follows: Let d be the larger of d_1 and d_2. Then:

$$y_p(t) = t^{\#\text{ of times } \alpha+\beta i \text{ is a root of } ax^2+bx+c} e^{\alpha t}\Big((\text{degree } d \text{ polynomial})\cos\beta t + (\text{degree } d \text{ polynomial})\sin\beta t\Big).$$

If $g(t)$ is a sum of different terms, then by the superposition principle, the form of the particular solution is the sum of individual forms.

For this problem:

$$\sin t - \cos 2t = e^{0t}\Big(0\cos 1t + 1\sin 1t\Big) + e^{0t}\Big((-1)\cos 2t + 0\sin 2t\Big).$$

The auxiliary polynomial is $x^2+5x+6 = (x+2)(x+3)$, which has roots -2 and -3. In particular, $0+1i = i$ and $0+2i = 2i$ are not roots of the auxiliary polynomial, so the form of a particular solution is:

$$y_p(t) = t^0 e^{0t}\Big(A\cos 1t + B\sin 1t\Big) + t^0 e^{0t}\Big(C\cos 2t + D\sin 2t\Big) = \boxed{A\cos t + B\sin t + C\cos 2t + D\sin 2t}$$

$\boxed{\text{for some numbers } A,\ B,\ C,\ D}$. □

7. Ex. 4.6.10: *In Problems 1-10, find a general solution to the differential equation using the method of variation of parameters.*

$y'' + 4y' + 4y = e^{-2t}\ln t$

Solution. To use variation of parameters to solve $ay'' + by' + cy = g(t)$, first let $\{y_1, y_2\}$ be a basis for the solutions to the homogeneous equation $ay''+by'+cy = 0$. Then let v_1 and v_2 be any particular antiderivatives of

$$-\frac{g(t)y_2(t)}{a[y_1(t)y_2'(t) - y_1'(t)y_2(t)]}, \qquad \frac{g(t)y_1(t)}{a[y_1(t)y_2'(t) - y_1'(t)y_2(t)]},$$

respectively. Then $v_1(t)y_1(t) + v_2(t)y_2(t)$ is a particular solution to the inhomogeneous equation.

In this case, the homogeneous equation is $y''+4y'+4y = 0$. The auxiliary polynomial is $x^2+4x+4 = (x+2)^2$, and it has roots $-2, -2$. Therefore, the general solution to the homogeneous equation is $c_1e^{-2t}+c_2te^{-2t}$. That means we can pick:

$$y_1 = e^{-2t}, \qquad y_2 = te^{-2t}.$$

Then:

$$y_1y_2' - y_1'y_2 = e^{-2t}(e^{-2t} - 2te^{-2t}) - (-2e^{-2t})e^{-2t} = (e^{-4t} - 2te^{-4t}) + 2te^{-4t} = e^{-4t}.$$

Therefore, v_1 is any antiderivative of $-\frac{te^{-2t}(e^{-2t}\ln t)}{1e^{-4t}} = -t\ln t$, i.e. $v_1 = \int(-t\ln t)\,dt$. By using integration by parts, i.e.:

$$\int u\,dv = uv - \int v\,du$$

where:

$$u = \ln t, \qquad dv = -t\,dt, \qquad du = \tfrac{1}{t}\,dt, \qquad v = -\tfrac{1}{2}t^2$$

we get:

$$\int(-t\ln t)\,dt = -\tfrac{1}{2}t^2\ln t - \int\left(-\tfrac{1}{2}t^2\right)\tfrac{1}{t}\,dt = -\tfrac{1}{2}t^2\ln t + \int \tfrac{1}{2}t\,dt = -\tfrac{1}{2}t^2\ln t + \tfrac{1}{4}t^2 + C.$$

Since v_1 is any particular antiderivative, we pick $C = 0$, so that $v_1 = -\frac{1}{2}t^2\ln t + \frac{1}{4}t^2$.

Similarly, v_2 is an antiderivative of $\frac{e^{-2t}(e^{-2t}\ln t)}{1e^{-4t}} = \ln t$, i.e. $v_2 = \int \ln t\,dt$. By using integration by parts:

$$u = \ln t, \qquad dv = dt, \qquad du = \tfrac{1}{t}\,dt, \qquad v = t$$

we get

$$\int \ln t\,dt = t\ln t - \int t\tfrac{1}{t}\,dt = t\ln t - \int dt = t\ln t - t + C.$$

We pick $C = 0$ again, so that $v_2 = t\ln t - t$.

Then a particular solution to the inhomogeneous equation is:

$$\begin{aligned} v_1 y_1 + v_2 y_2 &= \left(-\tfrac{1}{2}t^2 \ln t + \tfrac{1}{4}t^2\right)e^{-2t} + (t \ln t - t)te^{-2t} \\ &= \left(-\tfrac{1}{2}t^2 e^{-2t} \ln t + \tfrac{1}{4}t^2 e^{-2t}\right) + (t^2 e^{-2t} \ln t - t^2 e^{-2t}) \\ &= \tfrac{1}{2}t^2 e^{-2t} \ln t - \tfrac{3}{4}t^2 e^{-2t}, \end{aligned}$$

so that the general solution is $\boxed{c_1 e^{-2t} + c_2 te^{-2t} + \tfrac{1}{2}t^2 e^{-2t} \ln t - \tfrac{3}{4}t^2 e^{-2t} \text{ for any numbers } c_1 \text{ and } c_2}$. □

8. Ex. 4.6.20: *Use the method of variation of parameters to show that*

$$y(t) = c_1 \cos t + c_2 \sin t + \int_0^t f(s) \sin(t - s)\, ds$$

is a general solution to the differential equation

$$y'' + y = f(t),$$

where $f(t)$ *is a continuous function on* $(-\infty, \infty)$. *[*Hint: *Use the trigonometric identity* $\sin(t - s) = \sin t \cos s - \sin s \cos t$.*]*

Solution. The homogeneous equation is $y'' + y = 0$. The auxiliary polynomial is $x^2 + 1$, and it has roots ± 1. Therefore, the general solution to the homogeneous equation is $c_1 \cos t + c_2 \sin t$. That means we can pick:

$$y_1 = \cos t, \qquad y_2 = \sin t.$$

Then:

$$y_1 y_2' - y_1' y_2 = \cos t(\cos t) - (-\sin t)\sin t = \cos^2 t + \sin^2 t = 1.$$

Therefore, v_1 can be any antiderivative of $-\frac{\sin t f(t)}{1 \cdot 1} = -f(t)\sin t$. In particular we may choose

$$v_1(t) = -\int_0^t f(s) \sin s\, ds.$$

Similarly, v_2 can be any antiderivative of $\frac{\cos t f(t)}{1 \cdot 1} = f(t)\cos t$. In particular, we may choose

$$v_2(t) = \int_0^t f(s) \cos s\, ds.$$

Then a particular solution to the inhomogeneous equation is:

$$\begin{aligned} v_1(t)y_1(t) + v_2(t)y_2(t) &= -\cos t \int_0^t f(s) \sin s\, ds + \sin t \int_0^t f(s) \cos s\, ds \\ &= \int_0^t (-f(s) \sin s \cos t)\, ds + \int_0^t (f(s) \cos s \sin t)\, ds \\ &= \int_0^t f(s)(\sin t \cos s - \cos t \sin s)\, ds = \int_0^t f(s) \sin(t - s)\, ds, \end{aligned}$$

so the general solution is $c_1 \cos t + c_2 \sin t + \int_0^t f(s) \sin(t - s)\, ds$ for any numbers c_1 and c_2. □

9. Ex. 6.1.6: *In Problems 1-6, determine the largest interval* (a, b) *for which Theorem 1 guarantees the existence of a unique solution on* (a, b) *to the given initial value problem.*
$(x^2 - 1)y''' + e^x y = \ln x; \qquad y(3/4) = 1, \qquad y'(3/4) = y''(3/4) = 0$

Solution. $\boxed{\text{Technically speaking, Theorem 1 does not apply at all to this problem.}}$ Theorem 1 says that for an initial value problem of the form:

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = g(x),$$

$y(x_0) = \gamma_0$, $y'(x_0) = \gamma_1$, $\cdots$, $y^{(n-1)}(x_0) = \gamma_{n-1}$, there is a unique solution on any open interval I that contains x_0 and for which $p_1(x)$, $\ldots$, $p_n(x)$, $g(x)$ are all continuous on I.

However, the coefficient of the highest-order derivative in the given initial-value problem is $x^2 - 1$ and not 1, so Theorem 1 does not apply. However, we can divide through by $x^2 - 1$ to get the following initial value problem:

$$y''' + \frac{e^x}{x^2-1}y = \frac{\ln x}{x^2-1}.$$

$y(3/4) = 1$, $y'(3/4) = y''(3/4) = 0$, and Theorem 1 does apply to this problem. Thus, we want the largest open interval containing 3/4 that 0, 0, $\frac{e^x}{x^2-1}$, and $\frac{\ln x}{x^2-1}$ are all continuous on.

The coefficients of y'' and y' are both 0, so they are continuous everywhere. The function $\frac{e^x}{x^2-1}$ is continuous everywhere except where $x^2 - 1 = 0$, i.e. on $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$. The function $\frac{\ln x}{x^2-1}$ is continuous everywhere except where $x \leq 0$ or $x^2 - 1 = 0$, i.e. on $(0, 1) \cup (1, \infty)$. Therefore, the desired largest open interval is $\boxed{(0, 1)}$. □

10. Ex. 6.1.16: *Using the Wronskian in Problems 15-18, verify that the given functions form a fundamental solution set for the given differential equation and find a general solution.*

$y''' - y'' + 4y' - 4y = 0; \qquad \{e^x, \cos 2x, \sin 2x\}$

Solution. Let $y_1(x) = e^x$, $y_2(x) = \cos 2x$, $y_3(x) = \sin 2x$. Then:

$$\begin{aligned}
y_1''' - y_1'' + 4y_1' - 4y_1 &= e^x - e^x + 4e^x - 4e^x = 0,\\
y_2''' - y_2'' + 4y_2' - 4y_2 &= (8\sin 2x) - (-4\cos 2x) + 4(-2\sin 2x) - 4(\cos 2x) = 0,\\
y_3''' - y_3'' + 4y_3' - 4y_3 &= (-8\cos 2x) - (-4\sin 2x) + 4(2\cos 2x) - 4(\sin 2x) = 0.
\end{aligned}$$

Therefore, y_1, y_2, y_3 are all solutions to the third-degree linear homogeneous differential equation $y''' - y'' + 4y' - 4y = 0$.

Next:

$$\begin{aligned}
W[y_1, y_2, y_3](x) &= \begin{vmatrix} e^x & \cos 2x & \sin 2x \\ e^x & -2\sin 2x & 2\cos 2x \\ e^x & -4\cos 2x & -4\sin 2x \end{vmatrix}\\
&= e^x \begin{vmatrix} -2\sin 2x & 2\cos 2x \\ -4\cos 2x & -4\sin 2x \end{vmatrix} - e^x \begin{vmatrix} \cos 2x & \sin 2x \\ -4\cos 2x & -4\sin 2x \end{vmatrix} + e^x \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}\\
&= e^x(8\sin^2 2x + 8\cos^2 2x) - e^x(-4\sin 2x\cos 2x + 4\sin 2x\cos 2x) + e^x(2\cos^2 x + 2\sin^2 x)\\
&= 8e^x + 0 + 2e^x = 10e^x,
\end{aligned}$$

which is not zero anywhere. Therefore, since y_1, y_2, y_3 are all solutions to the same third-degree linear homogeneous differential equation $y''' - y'' + 4y' - 4y = 0$, this means that $\{y_1, y_2, y_3\}$ is linearly independent, and therefore is a fundamental solution set.

This means that all solutions are $\boxed{c_1 e^x + c_2\cos 2x + c_3\sin 2x \text{ for any numbers } c_1, c_2, c_3}$. □

11. Ex. 6.1.20: *In Problems 19-22, a particular solution and a fundamental solution set are given for a nonhomogeneous equation and its corresponding homogeneous equation.* (a) *Find a general solution to the nonhomogeneous equation.* (b) *Find the solution that satisfies the specified initial conditions.*

$xy''' - y'' = -2; \quad y(1) = 2; \quad y'(1) = -1; \quad y''(1) = -4; \quad y_p = x^2; \quad \{1, x, x^3\}$

Solution. (a) We are given that $\{1, x, x^3\}$ is a fundamental solution set to the homogeneous equation, which means that the general solution to the homogeneous equation is $c_1 + c_2 x + c_3 x^3$ for any real c_1, c_2, c_3. To get the general solution to the inhomogeneous equation, you add a particular solution to the general solution of the homogeneous equation, so the general solution to the inhomogeneous equation is $\boxed{y(x) = c_1 + c_2 x + c_3 x^3 + x^2 \text{ for any real } c_1, c_2, c_3}$.

(b) We have:

$$y(x) = c_1 + c_2 x + c_3 x^3 + x^2, \qquad y'(x) = c_2 + 3c_3 x^2 + 2x, \qquad y''(x) = 6c_3 x + 2.$$

Substituting in $x = 1$ and using $y(1) = 2$, $y'(1) = -1$, and $y''(1) = -4$ gives:

$$2 = c_1 + c_2 + c_3 + 1, \qquad -1 = c_2 + 3c_3 + 2, \qquad -4 = 6c_3 + 2.$$

From $-4 = 6c_3 + 2$, we get $c_3 = -1$.

From $-1 = c_2 + 3c_3 + 2$ we get $c_2 = -3c_2 - 3 = 0$.
From $2 = c_1 + c_2 + c_3 + 1$ we get $c_1 = -c_2 - c_3 + 1 = 2$.
Therefore, the solution to the initial value problem is $\boxed{2 - x^3 + x^2}$. □

12. Ex. 6.1.24: *Let* $L[y] := y''' - xy'' + 4y' - 3xy$*,* $y_1(x) := \cos 2x$*, and* $y_2(x) := -1/3$*. Verify that* $L[y_1](x) = x\cos 2x$ *and* $L[y_2](x) = x$*. Then use the superposition principle (linearity) to find a solution to the differential equation:*
(a) $L[y] = 7x\cos 2x - 3x$.
(b) $L[y] = -6x\cos 2x + 11x$.

Solution.

$$L[y_1] = y_1''' - xy_1'' + 4y_1' - 3xy_1 = (8\sin 2x) - x(-4\cos 2x) + 4(-2\sin 2x) - 3x(\cos 2x) = x\cos 2x,$$
$$L[y_2] = y_2''' - xy_2'' + 4y_2' - 3xy_2 = 0 - x\cdot 0 + 4\cdot 0 - 3x\left(-\tfrac{1}{3}\right) = x,$$

as needed.
(a) $7y_1 - 3y_2 = \boxed{7\cos 2x + 1}$ solves $L[y] = 7x\cos 2x - 3x$.
(b) $-6y_1 + 11y_2 = \boxed{-6\cos 2x - \tfrac{11}{3}}$ solves $L[y] = -6x\cos 2x + 11x$. □

13. Ex. 6.1.28: *The set of functions*

$$\{1, \cos x, \sin x, \ldots, \cos nx, \sin nx\},$$

where n *is a positive integer, is linearly independent on every interval* (a, b)*. Prove this in the special case* $n = 2$ *and* $(a, b) = (-\infty, \infty)$*.*

Solution 1. Let $y_1, \ldots, y_5$ be the functions $1, \cos x, \sin x, \cos 2x, \sin 2x$ that we are to show are linearly independent. Then their Wronskian is:

$$\begin{vmatrix} 1 & \cos x & \sin x & \cos 2x & \sin 2x \\ 0 & -\sin x & \cos x & -2\sin 2x & 2\cos 2x \\ 0 & -\cos x & -\sin x & -4\cos 2x & -4\sin 2x \\ 0 & \sin x & -\cos x & 8\sin 2x & -8\cos 2x \\ 0 & \cos x & \sin x & 16\cos 2x & 16\sin 2x \end{vmatrix}.$$

This will be tedious to compute directly, and we only need to know that the Wronskian is nonzero at a single point, so we evaluate it at $x = 0$:

$$\begin{vmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & -1 & 0 & -4 & 0 \\ 0 & 0 & -1 & 0 & -8 \\ 0 & 1 & 0 & 16 & 0 \end{vmatrix} \xrightarrow{\text{Col. 1}} \begin{vmatrix} 0 & 1 & 0 & 2 \\ -1 & 0 & -4 & 0 \\ 0 & -1 & 0 & -8 \\ 1 & 0 & 16 & 0 \end{vmatrix} \xrightarrow{\text{Row 1}} -\begin{vmatrix} -1 & -4 & 0 \\ 0 & 0 & -8 \\ 1 & 16 & 0 \end{vmatrix} - 2\begin{vmatrix} -1 & 0 & -4 \\ 0 & -1 & 0 \\ 1 & 0 & 16 \end{vmatrix}$$

$$\xrightarrow{\text{Row 2}} -(-(-8))\begin{vmatrix} -1 & -4 \\ 1 & 16 \end{vmatrix} - 2(-1)\begin{vmatrix} -1 & -4 \\ 1 & 16 \end{vmatrix}$$

$$= (-6)\begin{vmatrix} -1 & -4 \\ 1 & 16 \end{vmatrix} = (-6)\big((-1)16 - (-4)1\big) = (-6)(-12) = 72.$$

Since the Wronskian is nonzero at at least one point, the functions $\{y_1, \ldots, y_5\}$ are linearly independent.
(In fact, the Wronskian is 72 for all x, not just $x = 0$, but we didn't need to compute it.) □

Solution 2. Let a, b, c, d, e be numbers for which:

$$a + b\cos x + c\sin x + d\cos 2x + e\sin 2x = 0$$

for all real x. If we can show that $a = b = \cdots = e = 0$, then we will be done.
Plugging in $x = 0$ and $x = \pi$ gives:

$$a + b + d = 0, \qquad a - b + d = 0.$$

Together, those two equations imply $b = 0$ and $a + d = 0$, so $d = -a$.

Plugging in $x = \frac{\pi}{2}$ into:

$$a + c\sin x - a\cos 2x + e\sin 2x = 0$$

gives $2a + c = 0$, so that $c = -2a$.

Plugging in $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$ into:

$$a - 2a\sin x - a\cos 2x + e\sin 2x = 0$$

gives $(1-\sqrt{2})a + e = 0$ and $(1-\sqrt{2})a - e = 0$. These two equatons together imply $a = e = 0$. Then $d = -a = 0$ and $c = -2a = 0$. Therefore, all of a, b, c, d, e are zero, and this proves that $\{1, \cos x, \sin x, \cos 2x, \sin 2x\}$ is independent, as needed. ☐

14. Ex. 6.2.10: *In Problems 1-14, find a general solution for the differential equation with x as the independent variable.*

$y''' + 3y'' - 4y' - 6y = 0.$

Solution. The auxiliary polynomial is $x^3 + 3x^2 - 4x - 6$. By the rational roots theorem, the only possible rational roots of this polynomial are ± 1, ± 2, ± 3, and ± 6. We use synthetic division to test which of these numbers are actually roots:

$$\begin{array}{rrrrr} & 1 & 3 & -4 & -6 \\ & & 1 & 4 & 0 \\ 1 & 1 & 4 & 0 & -6 \end{array} \qquad\qquad \begin{array}{rrrrr} & 1 & 3 & -4 & -6 \\ & & -1 & -2 & 6 \\ -1 & 1 & 2 & -6 & 0 \end{array}$$

Therefore, $x = -1$ is a root of the auxiliary polynomial, and the polynomial factors as $x^3 + 3x^2 - 4x - 6 = (x+1)(x^2 + 2x - 6)$. From the quadratic formula, the roots of $x^2 + 2x - 6$ are:

$$\frac{-2 \pm \sqrt{2^2 - 4 \cdot 1(-6)}}{2} = \frac{-2 \pm \sqrt{28}}{2} = -1 \pm \sqrt{7}.$$

Therefore, the three roots of the auxiliary polynomial are -1, $-1+\sqrt{7}$, and $-1-\sqrt{7}$. Therefore, the general solution to the differential equation is $\boxed{c_1 e^{-x} + c_2 e^{(-1+\sqrt{7})x} + c_3 e^{(-1-\sqrt{7})x} \text{ for any numbers } c_1, c_2, c_3}$. ☐

15. Ex. 6.2.18: *In Problems 15-18, find a general solution to the given homogeneous equation.*

$(D-1)^3(D-2)(D^2+D+1)(D^2+6D+10)^3[y] = 0$

Solution. The question is asking to solve the differential equation whose auxiliary polynomial factors as:

$$(x-1)^3(x-2)(x^2+x+1)(x^2+6x+10)^3 = 0.$$

From the quadratic formula, the roots of $x^2 + x + 1$ are:

$$\frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot 1}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i.$$

The roots of $x^2 + 6x + 10$ are:

$$\frac{-6 \pm \sqrt{6^2 - 4 \cdot 1 \cdot 10}}{2} = \frac{-6 \pm \sqrt{-4}}{2} = -3 \pm i.$$

Therefore, the roots of the auxiliary polynomial are, with multiplicity, 1, 1, 1, 2, $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$, $-3 \pm i$, $-3 \pm i$, $-3 \pm i$. Therefore, the general solution to the differential equation is:

$$\boxed{\begin{aligned} y(x) = c_1 e^x + c_2 x e^x + c_3 x^2 e^x + c_4 e^{2x} + c_5 e^{-x/2}\cos\left(\tfrac{\sqrt{3}}{2}x\right) + c_6 e^{-x/2}\sin\left(\tfrac{\sqrt{3}}{2}x\right) + c_7 e^{-3x}\cos x \\ + c_8 e^{-3x}\sin x + c_9 x e^{-3x}\cos x + c_{10} x e^{-3x}\sin x + c_{11} x^2 e^{-3x}\cos x + c_{12} x^2 e^{-3x}\sin x, \\ \text{for any numbers } c_1, \ldots, c_{12}. \end{aligned}}$$ ☐

16. Ex. 6.2.20: *In Problems 19-21, solve the given initial value problem.*

$y''' + 7y'' + 14y' + 8y = 0; \qquad y(0) = 1, \qquad y'(0) = -3, \qquad y''(0) = 13$

Solution. The auxiliary polynomial is $x^3 + 7x^2 + 14x + 8$. By the rational roots theorem, the only possible rational roots of this polynomial are ± 1, ± 2, ± 4, and ± 8. We use synthetic division to test which of these numbers are actually roots:

$$\begin{array}{r|rrrr} & 1 & 7 & 14 & 8 \\ & & 1 & 8 & 22 \\ \hline 1 & 1 & 8 & 22 & 30 \end{array} \qquad \begin{array}{r|rrrr} & 1 & 7 & 14 & 8 \\ & & -1 & -6 & -8 \\ \hline -1 & 1 & 6 & 8 & 0 \end{array}$$

Therefore, $x = -1$ is a root of the auxiliary polynomial, and the polynomial factors as $x^3 + 7x^2 + 14x + 8 = (x+1)(x^2 + 6x + 8) = (x+1)(x+2)(x+4)$. The roots of the auxiliary polynomial are therefore, -1, -2, and -4. The general solution is therefore: $y(x) = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-4x}$.

Then:

$$y'(x) = -c_1 e^{-x} - 2c_2 e^{-2x} - 4c_3 e^{-4x}, \qquad y''(x) = c_1 e^{-x} + 4c_2 e^{-2x} + 16c_3 e^{-4x}.$$

Plugging in $x = 0$ and using $y(0) = 1$, $y'(0) = -3$, and $y''(0) = 13$ gives:

$$c_1 + c_2 + c_3 = 1, \qquad -c_1 - 2c_2 - 4c_3 = -3, \qquad c_1 + 4c_2 + 16c_3 = 13.$$

We solve this system for c_1, c_2, c_3 using Gauss-Jordan elimination:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -2 & -4 & -3 \\ 1 & 4 & 16 & 13 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 + R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -3 & -2 \\ 0 & 3 & 15 & 12 \end{bmatrix} \xrightarrow[R_3 \to \frac{1}{3}R_3]{R_2 \to -R_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & 5 & 4 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 \to \frac{1}{2}R_3} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[R_2 \to R_2 - 3R_3]{R_1 \to R_1 - R_3} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Therefore, $c_1 = 1$, $c_2 = -1$, and $c_3 = 1$. So $y(x) = \boxed{e^{-x} - e^{-2x} + e^{-4x}}$. □

17. Ex. 6.2.25: *Show that the m functions $e^{rx}, xe^{rx}, \ldots, x^{m-1}e^{rx}$ are linearly independent on $(-\infty, \infty)$.* [Hint: *Show that these functions are linearly independent if and only if $1, x, \ldots, x^{m-1}$ are linearly independent.*]

Solution. Pick any positive integer m. Pick m numbers $c_0, \ldots, c_{m-1}$ for which:

$$c_0 e^{rx} + c_1 x e^{rx} + \cdots + c_{m-1} x^{m-1} e^{rx} = 0$$

for all x in $(-\infty, \infty)$. Then multiplying by e^{-rx} shows that:

$$c_0 + c_1 x + \cdots + c_{m-1} x^{m-1} = 0$$

for all x in $(-\infty, \infty)$. This is a polynomial in x, so if any of $c_0, \ldots, c_{m-1}$ were nonzero, then the fundamental theorem of algebra would say that the equation

$$c_0 + c_1 x + \cdots + c_{m-1} x^{m-1} = 0$$

would have at most $m - 1$ zeroes. But we are assuming that

$$c_0 + c_1 x + \cdots + c_{m-1} x^{m-1} = 0$$

for all x, so the polynomial has infinitely many roots. Therefore, all of $c_0, \ldots, c_{m-1}$ are zero, and this is exactly what it means for $e^{rx}, xe^{rx}, \ldots, x^{m-1}e^{rx}$ to be linearly independent on $(-\infty, \infty)$. □