Hamiltonian ODE, Homogenization, and Symplectic Topology

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November 28, 2018

1 Introduction

Hamiltonian systems of ordinary differential equations appear in celestial mechanics to describe the motion of planets. They are also used in statistical mechanics to model the dynamics of particles in a fluid, gas or many other microscopic models. It was known to Liouville that the flow of a Hamiltonian system preserves the volume. Poincaré observed that the the Hamiltonian flows are *symplectic*; they preserve certain *symplectic area* of two dimensional surfaces. Various *Symplectic Rigidity Phenomena* offer ways to take advantage of the symplicity of Hamiltonian flows.

Writing q and p for the position and momentum coordinates respectively, a Hamiltonian function $H(q, p)$ represents the total energy associated with the pair (q, p). We regard a Hamiltonian system associated with H *completely integrable* if there exists a change of coordinates $(q, p) \mapsto (Q, P)$, such that our Hamiltonian system in new coordinates is still Hamiltonian system that is now associated with a Hamiltonian function $\bar{H}(P)$. For completely integrable systems the coordinates of $P = P(q, p)$ are conserved and the set of (q, p) at which $P(q, p)$ takes a fixed vector is an invariant set for the flow of our system. These invariant sets are homeomorphic to tori in many classical examples of completely integrable systems. According to *Kolmogorov-Arnold-Moser (KAM) Theory*, many of the *invariant tori* survive when a completely integrable system is slightly perturbed. *Aubry-Mather Theory* constructs a family of invariant sets provided that the Hamiltonian function is convex in the momentum variable. These invariant sets lie on the graph of the gradient of certain scalar-valued functions. A. Fathi uses *viscosity solutions* of the Hamilton-Jacobi PDE associated with the Hamiltonian function H to construct Aubry-Mather invariant measures. Recently there have been several interesting works to understand the connection between Aubry-Mather Theory and *Symplectic Topology*. The hope is to use tools from Symplectic Topology to construct interesting invariant sets/measures for Hamiltonian systems associated with with non-convex Hamiltonian functions.

Most of the aforementioned works on Hamiltonian systems are done when the Hamiltonian function is defined on the cotangent bundle of a compact manifold. A prime example is when $p, q \in \mathbb{R}^d$, with H periodic in q-variable, so that we may regard H as a function that is defined on $T^*\mathbb{T}^d = \mathbb{T}^d \times \mathbb{R}^d$. To go beyond the periodic case, we may take a Hamiltonian function that is *quasi-periodic* with respect to q. In fact there is a probabilistic generalization of quasi-periodic condition by selecting H randomly according to a probability measure $\mathbb{P}$ that is invariant with respect to spatial shifts: $\tau_a H(q, p) = H(q + a, p)$. As it turns out the Hamiltonian $\bar{H}$ can be obtained from H by a scaling limit that is called *Homogenization*.

In this course we will explore the connection between Hamilton-Jacobi PDE, Homogenization, Hamiltonian ODE and Symplectic Topology.

1.1 Hamiltonian ODE

In Euclidean setting a Hamiltonian system associated with a C^2 Hamiltonian function $H : \mathbb{R}^{2d} \to \mathbb{R}$ is the ODE

$$\dot{x} = X_H(x) := J\nabla H(x), \tag{1.1}$$

where

$$\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},$$

with I denoting the $d \times d$ identity matrix. Writing $x = (q, p)$ with $q, p \in \mathbb{R}^d$, the system (6.1) means

$$\dot{q} = H_p(q,p), \qquad \dot{p} = -H_q(q,p).$$

We write $\phi_t^H(x)$ for the flow of the vector field X_H. More generally, we can define *Hamiltonian vector fields* on any *symplectic manifold*. By a symplectic manifold we mean a pair (M, ω) with M a C^2 manifold, and ω a non-degenerate closed 2-form on M. Given a C^2 function $H : M \to \mathbb{R}$, we define the vector field $X_H = X_H^\omega$ as the unique vector field such that

$$i_{X_H}\omega = -dH.$$

When ω is the standard symplectic form of $\mathbb{R}^{2d}$, namely

$$\omega = \bar{\omega} := \sum_{i=1}^{d} dq_i \wedge dp_i,$$

and $M = \mathbb{R}^{2d}$, we have $X_H^{\bar{\omega}} = J\nabla H$.

Poincaré discovered that the *circulation* of any closed curve does not change along a Hamiltonian flow. More precisely, if $\bar{\lambda} = p \cdot dq$ and $\gamma : \mathbb{S}^1 \to \mathbb{R}^{2d}$ is a C^1 closed curve, then

$$\int_\gamma \left(\phi_t^H\right)^*(\lambda) := \int_{\phi_t^H(\gamma)} \lambda = \int_\gamma \lambda.$$

By Stokes' Theorem,

$$\int_{\phi_t^H(\Gamma)} \bar{\omega} = \int_\Gamma \bar{\omega}.$$

This really means that if $A(x) = \left(d\phi_t^H\right)_x$, then

$$\bar{\omega}\big(A(x)v, A(x)w\big) = \bar{\omega}(v,w), \quad \text{or} \quad A(x)^* J A(x) = J.$$

If fact the Hamiltonian vector field X_H^ω is chosen so that an analogous identity holds for its flow:

$$\left(d\phi_t^H\right)^*\omega = \omega, \quad \text{or} \quad \omega_{\phi_t^H(x)}\big(A(x)v, A(x)w\big) = \omega_x(v,w).$$

Given a vector field X on a manifold M, we write ψ_t^X for its flow. Given C^1 scalar-Valued function $f : M \to \mathbb{R}$, we define its *Lie derivative* with respect to X by

$$\mathcal{L}_X f(x) = \frac{d}{dt} f\big(\psi_t(x)\big)\big|_{t=0} = (df)_x\big(X(x)\big). \tag{1.2}$$

More generally, if $u(x,t) = f\big(\psi_t(x)\big)$, then

$$u_t = \mathcal{L}_X u.$$

From this, we learn that a function $f \in C^1(M;\mathbb{R})$ is *conserved* along the flow of X iff $\mathcal{L}_X f = 0$. In the case of a Hamiltonian vector field $X = X_H$, the Lie derivative $\mathcal{L}_X f$ is the *Poisson bracket* of H and f:

$$\{H, f\} := \mathcal{L}_{X_H} f = (df)(X_H) = -\omega(X_f, X_H) = \omega(X_H, X_f).$$

1.2 Completely Integrable Systems

We may call a Hamiltonian ODE completely integrable if we have a sufficiently explicit formula for its solutions. One strategy to achieve this is by finding enough conservation laws. As it turns out, a Hamiltonian system in $\mathbb{R}^{2d}$ is completely integrable if it has d many independent conservation laws that do not *interact* with each other. Note that if $f_1, \dots, f_k : M \to \mathbb{R}$ are C^2 functions such that $\{H, f_i\} = 0, i = 1, \dots, k$, then the set

$$M_P = \big\{x \in M : \ (f_1, \dots, f_k) = P\big\},$$

is invariant for the flow:

$$x \in M_P \quad \implies \quad \phi_t(x) \in M_P.$$

We recall a classical result of Liouville, and Arnold-Jost.

Theorem 1.1 *Assume that there are C^2 functions $f_1, \dots, f_d : M \to \mathbb{R}$ such that the following conditions hold:*

- $\{f_i, f_j\} = 0$ *for all i and j.*
- *For $P \in \mathbb{R}^d$, the corresponding set M_P is compact.*
- *For each $x \in M_P$, the vectors $X_{f_1}(x), \dots, X_{f_d}(x)$ are linearly independent.*

Then each such M_P is homeomorphic to a d-dimensional torus. Moreover, the motion of X_H on M_P is conjugate to a free motion.

Remark 1.1(i) The latter claim in Theorem 1.1 means that if we think of a torus as $[0,1]^d$ with $0 = 1$, then the motion is given by $x(t) = x + tv (\mathrm{mod}\, 1)$, for some vector $v \in \mathbb{R}^d$. Depending on the vector v, we may have a periodic or *quasi-periodic* orbit. (The latter means that the closure of the orbit is a k-dimension torus for some $k > 1$.)

(ii) The set M_P is an example of a *Lagrangian submanifold*. This means that $\dim M_P = d$ and $\omega(\restriction_{M_P} = 0$. The latter follows from

$$\omega\big(X_{f_i}, X_{f_j}\big) = \{f_i, f_j\} = 0.$$

□

The sketch of Arnold-Jost's theorem is in order. If we define $\phi_t : M \to M, t = (t_1, \dots, t_d) \in \mathbb{R}^d$ by

$$\phi_t(x) = \phi_{t_1}^{f_1} \circ \dots \circ \phi_{t_d}^{f_d},$$

then $\phi_t(M_P) \subseteq M_P$. On the other hand, if we pick some point $a \in M_P$ and set $\varphi(t) = \phi_t(a)$, then $\varphi : \mathbb{R}^d \to M_P$, and the set

$$\Sigma = \big\{t \in \mathbb{R}^d :\ \varphi(t) = \varphi(0) = a\big\},$$

is a subgroup of $(\mathbb{R}^d, +)$. Since M_P is compact, this subgroup is discrete. That is, there are vectors $v_1, \dots, v_d$, such that

$$\Sigma = \big\{n_1 v_1 + \dots + n_d v_d :\ n_1, \dots, n_d \in \mathbb{Z}\big\}.$$

Hence the quotient $\mathbb{R}^d/\Gamma$ is a torus and the map φ yields a homeomorphism $\hat{\varphi} : \mathbb{R}^d/\Gamma \to M_P$. Moreover, assuming that $f_1 = H$, then ϕ_s^H is conjugate to the map $(t_1, \dots, t_d) \mapsto (t_1 + s, \dots, t_d)$. If we use the basis $(v_1, \dots, v_d)$ for $\mathbb{R}^d$, we can then show that ϕ_s^H is conjugate to a free motion.

Writing Q for the coordinates of $\mathbb{R}^d/\Gamma \equiv \mathbb{T}^d$, we have a homeomorphism $\Psi_P = \hat{\varphi} : \mathbb{T}^d \to M_P$. As we vary P, we obtain a map

$$\Psi : T^*\mathbb{T}^d = \mathbb{T}^d \times \mathbb{R}^d \to M.$$

We think of $\Psi(Q, P) = x$ as a parametrization of M. Setting $\bar{H}(P) = H(x) = H\big(\Psi(Q, P)\big)$, for $x \in M_P$, we obtain a new Hamiltonian function $\bar{H} : T^*\mathbb{T}^d \to \mathbb{R}$ that is independent of Q. The motion of $\hat{\phi}_t\big(Q(0), P(0)\big) := (Q(t), P(t))$ may be defined by

$$\hat{\phi}_t := \Psi^{-1} \circ \phi_t^H \circ \Psi.$$

We already know that $Q(t)$ is a free motion and that $P(t) = P(0)$. We may regard this motion as a solution to the Hamiltonian ODE

$$\dot{Q} = \nabla H(P), \qquad \dot{P} = 0.$$

In summary, we have seen that for a completely integrable Hamiltonian ODE, we can find a change of coordinates that turns our system to free motion. That is, there exists a diffeomorphism Ψ such that

$$\phi_t^{\bar{H}} = \Psi^{-1} \circ \phi_t^H \circ \Psi, \qquad \bar{H} = H \circ \Psi, \tag{1.3}$$

for a Hamiltonian function H that is independent of position. Recall that both ϕ_t^H and $\phi_t^{\bar{H}}$ are symplectic. It is no surprise that the change of coordinates map Ψ is also symplectic. As the following Proposition indicates, a symplectic change of coordinates always transforms a Hamiltonian system to another Hamiltonian system.

Proposition 1.1 *Let (M, ω) and (M', ω') be two symplectic manifolds and assume that$\Psi : M' \to M$ is a diffeomorphism such that $\Psi * \omega = \omega'$. Let $H : M \to \mathbb{R}$ be a Hamiltonian function on M, and let ϕ_t^H be the flow of X_H^ω. Then*

$$\hat{\phi}_t := \Psi^{-1} \circ \phi_t^H \circ \Psi,$$

is the flow of the vector field $X_{\bar{H}}^{\omega'}$ for $\bar{H} = H \circ \Psi$.

1.3 Kolmogorov-Arnold-Moser (KAM) Theory

We may take a small perturbation of a completely integrable system and wonder whether or not some of the invariant tori persist. It turns out that for a small perturbation, an invariant torus persists if the *action variable* $\nabla H(P)$ is sufficiently irrational.

Theorem 1.2 *Assume that $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is of the form*

$$H^\varepsilon(q, p) = H^0(p) + \varepsilon K(q, p),$$

with $\det D^2 H_0 \neq 0$. *Then for every $\tau, \gamma > 0$, there exists $\varepsilon_0 = \varepsilon_0(\tau, \gamma) > 0$ such that if $\nabla H^0(p)$ satisfies a* Diophantine condition *of the form*

$$n \in \mathbb{Z}^d \setminus \{0\} \qquad \Longrightarrow \left| n \cdot \nabla H^0(p) \right| \geq \gamma |n|^{-\tau},$$

the vector field X_{H^ε} has a quasi-periodic orbit of velocity $\nabla H^0(p)$, whenever $|\varepsilon| \leq \varepsilon_0$.

It is worth mentioning that if we set

$$D(\gamma, \tau) = \left\{ v \in \mathbb{R}^d : \; |v \cdot n| \geq \gamma |n|^{-\tau} \text{ for all } n \in \mathbb{Z}^d \setminus \{0\} \right\},$$

then the set $D(\tau) = \cup_{\gamma > 0} D(\gamma, \tau)$ is of full measure whenever $\tau > d - 1$. This is because, the complement of $D(\gamma, \tau)$, restricted to a bounded set, has a volume of order $O(\gamma |n|^{-\tau - 1})$, and

$$\sum_{n \neq 0} |k|^{-\tau - 1} < \infty,$$

iff $\tau + 1 > d$.

1.4 Generating Function

Note that a Hamiltonian vector field is very special as it is fully determined by a scalar-valued function, namely its Hamiltonian function. As it turns out, the symplectic maps are also locally determined by scalar-valued functions known as *generating functions*. To explain this, take an $\bar{\omega}$-symplectic map and observe that since $\Psi^*\bar{\omega} = \bar{\omega}$, we can find a scalar-valued function S such that

$$p \cdot dq - P \cdot dQ = dS. \tag{1.4}$$

Normally we think of S as a function of (q, p) or (Q, P). However, it is more convenient to think of S as a function of other pairs. For example under some non-degeneracy assumption (for example if $Q_p(q, p)$ is invertible so that we can locally solve $Q(q, p) = Q$ implicitly for $p = p(q, Q)$), we may regard $S = S(q, Q)$ so that (1.4) implies

$$S_q(q, Q) = p, \quad -S_Q(q, Q) = P, \qquad \Psi\big(q, S_q(q, Q)\big) = \big(Q, -S_Q(q, Q)\big). \tag{1.5}$$

The scalar-valued functions S is an example of generating function for the symplectic map Ψ. Since there are other type of generating functions that we may consider for a symplectic map, let us refere to S as a *generating function of Type I*.

Alternatively, we may set $W = p \cdot q - S$ and regard W as a function of (Q, p) so that (1.4) means

$$W_p(Q, p) = q, \quad W_Q(Q, p) = P, \qquad \Psi\big(W_p(Q, p), p\big) = \big(Q,\ W_Q(Q, p)\big).$$

The function W is another example of a generating function for the symplectic map Ψ and we will refer to it as a *generating function of Type II*. Sometimes, we also consider a generating function $V(q, P)$ that will be referred to as *generating function of Type III*.

If Ψ is the change of coordinates transformation of a completely integrable system, we have

$$\bar{H}(P) = H(q, p) = H\big(q, W_q(q, P)\big).$$

This means that for each fixed P, the function $q \mapsto W(q, P)$ is a solution to a *Hamilton-Jacobi Equation (HJE)* associated with H. Some care is needed here. Recall that Ψ is a diffeomorphism defined on $\mathbb{T}^d \times \mathbb{R}^d$, whereas our H is defined on $\mathbb{R}^d \times \mathbb{R}^d$. If we wish our change of coordinates to work globally, we need $\Psi\big(\mathbb{T}^d \times \mathbb{R}^d\big)$ to be (at least topologically isomorphic) to $\mathbb{T}^d \times \mathbb{R}^d$. In fact if we assume that H is periodic in q, we may regard $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$. Thinking of $\mathbb{T}^d \times \mathbb{R}^d$, as $T^*\mathbb{T}^d$, we interpret $W_q(q, P)$ as a 1-form on the torus for each P. If we write $W(q, P) = q \cdot P + w^P(q)$ and assume that $w^P : \mathbb{T}^d \to \mathbb{R}$, is periodic, then our HJE reads as

$$H\big(q, P + (dw^P)_q\big) = \bar{H}(P). \tag{1.6}$$

We think of $\alpha^P = P + dw^P$ as a closed 1-form that belongs to cohomology class of the constant (closed) form P.

1.5 Weak KAM Theory

In the classical KAM Theory, we consider a small perturbation of a non-degenerate Hamiltonian function $H_0(p)$ that depends on p only. We have learned that the majority of the invariant tori of unperturbed systems persist for a sufficiently small perturbation. However some invariant tori could be destroyed after a small perturbation. In fact Arnold constructed an example of a perturbed integrable system, in which chaotic orbits - resulting from the breaking of unperturbed KAM tori - coexist with the invariant tori of KAM theorem. This phenomenon is known as *Arnold diffusion*. A natural question is whether or not we can construct a family of invariant sets $\left(M_P : P \in \mathbb{R}^d\right)$ for perturbed systems that come from the invariant tori of the unperturbed system and still carry some of their features. Aubry and Mather constructed such family for the so-called *twist maps* (these maps are the analog of Hamiltonian systems when $d = 1$ and time is discrete). The generalization of Aubry-Mather invariant sets to higher dimensions was achieved by Mather, Mane and Fathi. They prove the existence of interesting invariant (action-minimizing) sets, which generalize KAM tori, and which continue to exist even after KAM tori disappearance.

Aubry-Mather Theory replace the smallness condition with *Tonelli Assumption.* We say that a Hamiltonian function $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is Tonelli, if the following conditions are true:

- $H(q, p)$ is convex in p for each q.
- $|p|^{-1}H(q, p) \to \infty$ as $|p| \to \infty$, uniformly in q.

According to Aubry-Mather and Mather-Mane-Fathi Theory, for each P, there exists a constant $\bar{H}(P)$, a Lipschitz function $w^P : \mathbb{T}^d \to \mathbb{R}$, and an invariant measure μ^P for ϕ^H such that

- The function w^P solves the HJE (1.7).
- The support of the measure μ^P is a subset of

$$M_P = \left\{\left(q, P + (dw)_q\right) : \ q \in \mathbb{T}^d\right\}.$$

Note that we only require the function w to be Lipschitz and not everywhere differentiable. This is because the HJE (1.7) does no possess classical solutions in general. One remedy for this is to consider certain generalized solutions. In fact if we consider the so called *viscosity solutions*, then (1.7) always has at least one Lipschitz solution for each P. This was established by Lions, Papanicolaou and Varadhan in 1987. We then modify the definition of M_P with

$$M_P = \left\{\left(q, P + (dw^P)_q\right) : \ q \in \mathbb{T}^d, \ w^P \ \text{ differentiable at } \ q\right\}. \tag{1.7}$$

1.6 From Torus to General Closed Manifolds

We may replace the torus with any sufficiently smooth manifold M in weak KAM theory. Now our Hamiltonian function H is a C^2 function on the cotangent bundle T^*M. The manifold T^*M carries a standard symplectic form $\omega = d\lambda$ with λ defined as

$$\lambda_{(q,p)}\big(a\big) = p_q\big((d\pi)_{(q,p)}a\big),$$

where $\pi : T^*M \to M$ is the projection $\pi(q,p) = q$ to the base point, and its derivative $(d\pi)_{(q,p)} : T_{(q,p)}T^*M \to T_qM$ projects onto tangent vectors. Recall that in the case of a torus, we know that by a result of Lions-Papanicolaou-Varadhan, the (1.7) has at least one solution. This existence result has been extended to arbitrary closed manifold and convex Hamiltonian by Albert Fathi.

Theorem 1.3 *Let M be a smooth closed Riemannian manifold and assume that $H : T^*M \to \mathbb{R}$ is a Tonelli Hamiltonian. Then for every closed form α, there exists a unique constant $\bar{H}(\alpha)$, and a Lipschitz function $w : M \to \mathbb{R}$ such that w satisfies*

$$H\big(q, \alpha_q + (dv)_q\big) = \bar{H}(\alpha), \tag{1.8}$$

in viscosity sense.

Because of the uniqueness of $\bar{H}$, it is clear that if we add an exact form to α, the value of $\bar{H}$ does not change. Abusing the notion slightly, we may define $\bar{H}$ on the space $H^1(M)$ of the cohomology classes of 1-forms and write $\bar{H}([\alpha])$ in place of $\bar{H}(\alpha)$. Alternatively, for each $P \in H^1(M)$, we may fix a representative $\bar{\alpha}^P$ in class P and search for a Lipschitz $w^P : M \to \mathbb{R}$ such that $\alpha^P = \bar{\alpha}^P + dw^P$. Finally the invariant set M^P is defined by

$$M_P = \left\{\big(q, \bar{\alpha}^P_q + (dw^P)_q\big) :\ q \in M,\ w^P \ \text{ differentiable at } \ q\right\}. \tag{1.9}$$

1.7 From Torus to Stochastic Hamiltonian and Homogenization

Weak KAM Theory a la Fathi is based on taking advantage of the HJE (1.7) in order to construct interesting invariant measures for the corresponding Hamiltonian ODE. It turns out that HJE can be used to to model certain *deterministic and stochastic growths*. More precisely, imagine that we have an interface that separates different phases and this interface is represented by a graph of function $u(q,t)$ at time t. Suppose that the growth rate of this interface depends on the position q, and the inclination of the interface u_q. Mathematically speaking, u satisfies a HJE of the form

$$u_t + H\big(q, u_q(q,t)\big) = 0, \tag{1.10}$$

for a Hamiltonian function H. We think of (1.10) as the microscopic equation describing the evolution of the interface. If a large parameter n represents the ratio between the macro and micro scale, then

$$u^n(q,t) = n^{-1}u(nq, tq),$$

is the corresponding macroscopic height above that macro position q at the macro time t. We observe that u^n now solves

$$u^n_t + H^n\big(q, u^n_q(q,t)\big) = 0, \tag{1.11}$$

where

$$H^n(q,p) = (\gamma^n H)(q,p) = H(nq,p).$$

A *homogenization* occurs if the limit

$$\bar{u}(q,t) = \lim_{n\to\infty} u^n(q,t),$$

exists whenever the limit

$$g(q) := \lim_{n\to\infty} u^n(q,0),$$

exists. As it turns out, in many examples of interest, the limit $\bar{u}$ satisfies a simpler HJE of the form

$$\begin{cases} \bar{u}_t + \bar{H}(\bar{u}_q) = 0 \\ \bar{u}(q,0) = g(q). \end{cases} \tag{1.12}$$

In fact we may use (1.7) to guess that when H is periodic in q, then $\bar{H}$ that appears in (1.12) coincides with $\bar{H}$ that appears in (1.7). This is because if w^P is a periodic function that satisfies (1.7), and we choose $u(q,0) = P\cdot q + w^P(q)$ as the initial condition for (1.10), then $u(q,t) = w^P(q) - t\bar{H}(P)$, and

$$\bar{u}(q,t) = \lim_{n\to\infty} u^n(q,t) = P\cdot q - t\bar{H}(P),$$

which solves (1.12).

We may wonder whether a weak KAM Theory can be achieved for $H : \mathbb{R}^{2d} \to \mathbb{R}$ that are not necessarily periodic. Let us write $\mathcal{H}$ for the space of all C^1 Hamiltonian functions $H : \mathbb{R}^{2d} \to \mathbb{R}$ and two group actions on $\mathcal{H}$, namely the spacial translation and scaling; more precisely we set

$$\tau_a H(q,p) = H(q+a,p), \qquad \gamma_n H(q,p) = H(nq,p),$$

for $a \in \mathbb{R}^d$ and $n \in \mathbb{R}^+$. We certainly have

$$\tau_a \circ \tau_b = \tau_{a+b}, \qquad \gamma_m \circ \gamma_n = \gamma_{mn}.$$

We are interested to know for what Hamiltonian $H \in \mathcal{H}$ we have weak KAM Theory and Homogenization. Let us make a comment on bounded continuous functions K of the position variable. For $K : \mathbb{R}^d \to \mathbb{R}$, we define the translation operator $\tau_a K(q) = K(q + a)$ as before. We note that if a function K is periodic in q, then for each p, the set

$$\big\{\tau_a H(\cdot, p) : \ a \in \mathbb{R}^d\big\},$$

is homeomorphic to a d-dimensional torus. In we take a function $\hat{K} : \mathbb{T}^N \to \mathbb{R}$ and take a $N \times d$ matrix A, then the function $K(q) = \hat{K}(Aq)$ yields a *quasi-periodic* function. In fact the orbit of such K,

$$\Gamma(K) := \{\tau_a K : \ a \in \mathbb{R}^d\},$$

is dense in $\mathbb{T}^N$, in the following condition holds:

$$n \in \mathbb{Z}^d \setminus \{0\} \qquad \Longrightarrow \qquad An \neq 0.$$

More generally we call a bounded continuous function $K : \mathbb{R}^d \to \mathbb{R}$ *almost periodic* if the set $\Gamma(K)$ is precompect in $C_b(\mathbb{R}^d)$ with respect to the uniform-norm.

We regard the group $\big\{\tau_a : \ a \in \mathbb{R}^d\big\}$ as a d-dimensional dynamical system on $\mathcal{H}$, and call $\mathbb{P}$ *translation invariant ergodic measure* if the following conditions are met:

- For every Borel set $A \subset \mathcal{H}$, and $a \in \mathbb{R}^d$, we have $\mathbb{P}\big(\tau_a A\big) = \mathbb{P}(A)$.
- If a Borel set A is invariant i.e., $\tau_a A = A$ for all $a \in \mathbb{R}^d$, then $\mathbb{P}(A) \in \{0, 1\}$.

We many wonder whether or not the weak KAM theory or homogenization are applicable to generic Hamiltonian functions in the support of an invariant ergodic measure. The hope is that Birkhoff Ergodic Theorem would make up for the lack of compactness that has played an essential role when we consider a cotangent bundle of a compact manifold in **1.6**.

1.8 Variational Techniques

Homogenization questions and the existence of interesting invariant measures are closely related to the existence of special orbits of the Hamiltonian ODEs. Such existence questions also play central role in several recent developments in symplectic topology. (A prime example is *Floer Homology* that was formulated by Floer in order to treat *Arnold Conjecture.*) Hamilton observed that the minimizers of the action yield solutions to Hamiltonian systems of celestial mechanics. More generally, we may reduce the existence of special orbits of (6.1) to the existence of a critical point for a suitable *action functional.* More precisely, let us write Γ_T for the space of piecewise C^1 functions $x : [0, T] \to T^*M$, and given a Hamiltonian function $H : T^*M \times [0, T] \to \mathbb{R}$, let us define $\mathcal{A}_H : \Gamma^T \to \mathbb{R}$ by

$$\mathcal{A}(\gamma) = \mathcal{A}_H^T(\gamma) = \int_\gamma (\lambda - H \ dt) = \int_0^T \big[\lambda_{\gamma(t)}\big(\dot{\gamma}(t)\big) - H(\gamma(t), t)\big] \ dt. \tag{1.13}$$

The form $\lambda^H = \lambda - H\ dt$ is known as the *Poincaré-Cartan* form. We note that if we regard $d\lambda^H = \omega + dt \wedge dH$ as a form on $T^*M \times \mathbb{R}$, and $\hat{X}_H = \big(X_H, 1\big)$, then

$$i_{\hat{X}_H} d\lambda^H = i_{X_H}\omega + dH = 0.$$

Moreover, if we take a variation of a path with fixed end points: $w : [0,T] \times [0,\delta] \to T^*M$, with

$$w(t,0) = \gamma(t), \quad w(0,\theta) = w(0,0), \quad w(T,\theta) = w(T,0), \quad w_t(t,0) = v(t),$$

then

$$\begin{aligned}\frac{d}{d\theta}\int_{w(\cdot,\theta)} \lambda\big|_{\theta=0} &= \lim_{h\to 0} h^{-1}\left[\int_{w(\cdot,h)}\lambda - \int_{w(\cdot,0)}\lambda\right] = -\lim_{h\to 0} h^{-1}\int_{w([0,T]\times[0,h])}\omega \\ &= -\lim_{h\to 0} h^{-1}\int_0^h\int_0^T \omega_w\big(w_t, w_\theta\big)\ dtd\theta = -\int_0^T \omega_\gamma\big(\dot\gamma, v\big)\ dt.\end{aligned}$$

This in turn implies

$$-\partial\mathcal{A}_H^T(\gamma) = \big(i_{\dot\gamma}\omega + dH\big)_\gamma = \big(i_{\dot\gamma - X_H(\gamma)}\ \omega\big)_\gamma. \tag{1.14}$$

From this we learn that the *critical points* of $\mathcal{A}$ are the orbits of X_H. In fact *critical values* of $\mathcal{A}$ solve the corresponding Hamilton-Jacobi PDE. To explain this, observe that if we write $S(Q,t) = S(Q,t;q)$ for the generating function of ϕ_t^H so that

$$\phi_t^H\big(q, -S_q(Q,t;q)\big) = \big(Q, S_Q(Q,t;q)\big), \quad q(0) = q, \quad q(t) = Q, \quad p(t) = S_Q(Q,t;q),$$

then

$$S(Q,t;q) = S(q(t),t;q) = \Lambda(q(0),p(0),t) = \int_0^t \big[p(s)\cdot\dot q(s) - H(q(s),p(s),s)\big]\ ds.$$

Differentiating both sides with respect to t yields

$$S_t(Q,t;q) + S_Q(Q,t;q)\cdot\dot q = p(t)\cdot\dot q(t) - H(q(t),p(t),t).$$

As a result,

$$S_t(q,Q,t) + H\left(Q, S_Q(q,Q,t),t\right) = 0, \tag{1.15}$$

for $t \in (a,b)$. Similarly if we set $W = \Lambda + q\cdot p$, and regard $W(Q,t;p)$ as a function of (Q,p), then

$$W(q(t),t;p(0)) = p(0)\cdot q(0) + \int_0^t \big[p(s)\cdot\dot q(s) - H(q(s),p(s),s)\big]\ ds.$$

Differentiating both sides with respect to t yields

$$W_t(q(t), p(0), t) + W_Q(q(t), p(0), t) \cdot \dot{q}(t) = p(t) \cdot \dot{q}(t) - H(q(t), p(t), t).$$

This yields

$$W_t(Q, p, t) + H\left(Q, W_Q(Q, p, t), t\right) = 0, \tag{1.16}$$

because $W_Q(q(t), p(0), t) = p(t)$. In summary

- Critical points of $\mathcal{A}$ are orbits of X_H.
- Critical values of $\mathcal{A}$ are solutions of HJE.

Remark 1.1 In particular, if H is 1-periodic in t, $T = 1$, and we define $\mathcal{A}$ on the space of 1-peridic paths (loops), then the critical points of $\mathcal{A}$ correspond to the periodic orbits of X_H. Floer uses the gradient equation

$$w_s = -\partial\mathcal{A}(w), \tag{1.17}$$

to prove the existence of periodic orbits by showing that

$$\lim_{s\to\infty} w(\cdot, s),$$

exists. In fact (1.18) is an elliptic (or rather Cauchy-Riemann type) PDE, and one may hope to use elliptic regularity of the solutions to obtain the compactness of path w in a suitable Sobolev space. □

The action functional simplifies when H is convex in the momentum variable. To explain this, let us assume that there exists a C^2 function $L : TM \to \mathbb{R}$, $L = L(q, v)$, that is convex in the velocity v. Moreover when H is a Tonelli Hamiltonian, the transformation $\mathbb{L} : TM \to T^*M$,

$$\mathbb{L}(q, v) = \big(q, L_v(q, v)\big), \tag{1.18}$$

is a C^1 diffeomorphism with

$$p = L_v(q, v) \quad \text{iff} \quad v = H_p(q, p).$$

(Here we identify $\big(T_qM\big)^{**}$ with T_qM.) The *Lagrangian* function L and the Hamiltonian function H are related to each other by Legendre Transform

$$L(q, v) = \sup_{p\in T_q^*M} \big(p_q(v) - H(q, p)\big), \qquad H(q, p) = \sup_{v\in T_qM} \big(p_q(v) - L(q, v)\big).$$

Moreover

$$H \circ \mathbb{L}(q,v) = L_v(q,v)(v) - L(q,v).$$

Note that if $x(t) = \phi_t^H(a)$ is a solution of (6.1), then

$$\lambda_x(\dot{x}) - H(x) = p_q\big((d\pi)_x(\dot{x})\big) - H(q,p) = p_q(\dot{q}) - H(q,p) = L(q,\dot{q}).$$

Hence

$$\mathcal{A}\big(x(\cdot)\big) = \int_0^T \big(\lambda_x(\dot{x}) - H(x)\big)\ dt = \int_0^T L(q,\dot{q})\ dt =: \mathcal{L}\big(q(\cdot)\big).$$

Since L is convex in v, we may find solutions to (6.1) by finding minimizers of $\mathcal{L}$ that is defined for paths $q : [0,T] \to M$ with specified endpoints. By a classical result of Tonelli, the action functional $\mathcal{L}$ has a minimizer that satisfies the Euler-Lagrange equation

$$\frac{d}{dt} L_v(q,\dot{q}) = L_q(q,\dot{q}). \tag{1.19}$$

We now argue that we can use the action functional to construct a generating function for ϕ_T^H. To explain this, let us define $\Lambda_H^T : T^*M \to \mathbb{R}$, by

$$\Lambda_H^T(x) = \mathcal{A}_H\left(\eta_T^x\right), \qquad \text{where} \quad \eta_T^x(t) = \phi_t^H(x) \ \text{ for } \ t \in [0,T]. \tag{1.20}$$

We now claim that Λ is a generating function for ϕ^H.

Proposition 1.2 *For every $T \geq 0$ and any Hamiltonian H, we have*

$$d\Lambda_H^T = \big(\phi_T^H\big)^* \lambda - \lambda. \tag{1.21}$$

Proof Set

$$A(x) = \int_{\eta_T^x} \lambda, \qquad B(x) = \int_0^T H\big(\eta_T^x(t),t\big)\ dt.$$

Take any $\big(\tau(\theta) :\ 0 \leq \theta \leq h\big)$ with $\tau(0) = x$ and $\dot{\tau}(0) = v \in T_xM$. Set $y(t,\theta) = \phi^H - t\big(\tau(\theta)\big)$,

$$\Theta = \{y(t,\theta) :\ 0 \leq t \leq T,\ 0 \leq \theta \leq h\},$$

and use Stokes' theorem to assert

$$h^{-1}\int_0^h \int_0^T \omega_y(y_t,y_\theta)\ dtd\theta = h^{-1}\int_\Theta d\lambda = h^{-1}\left[\int_{\eta^{\tau(0)}} \lambda - \int_{\eta^{\tau(h)}} \lambda + \int_{\varphi\circ\tau(\cdot)} \lambda - \int_{\tau(\cdot)} \lambda\right],$$

$$h^{-1}\int_0^h \int_0^T \left(i_{X_H}\omega\right)_y(y_\theta)\ dtd\theta = h^{-1}\left[\int_{\eta^{\tau(0)}} \lambda - \int_{\eta^{\tau(h)}} \lambda + \int_{\tau(\cdot)} (\varphi^*\lambda - \lambda)\right],$$

where $\varphi = \phi_T^H$. Sending $h \to 0$ yields

$$-(dB)_x(v) = -(dA)_x(v) + \big(\varphi^*\lambda - \lambda\big)_x(v).$$

This is exactly (1.21). □

1.9 Discrete Models

Any symplectic map ψ from a symplectic manifold to itself serves as an example of a discrete analog of a Hamiltonian flow. We will be mainly interested in those symplectic diffeomorphism for which a global generating function exist. If the generating function is of the first kind, i.e., (1.7) holds for some $S(q,Q)$. In the Euclidean setting, we may write $S(Q,q) = L(q, Q-q)$, and if $L(q,v)$ is bounded below and has a superlinear growth at infinity in the *velocity* variable v, we call the corresponding map ψ a twist map and the corresponding dynamical model is a generalization of the Frenkel-Kontorova Model. Given a sequence $\mathbf{q} = (q_0, q_1, \ldots, q_n)$, we define its *action* by

$$\mathcal{A}(\mathbf{q}) = \sum_{i=1}^{n} S(q_{i-1}, q_i) = \sum_{i=1}^{n} L(q_i - q_{i-1}, q_{i-1}).$$

The critical points of $\mathcal{A}$ correspond to the orbits of ψ. Because of our assumption on L, we may use the minimizers of $\mathcal{A}$ to construct interesting orbits of ψ.

Example 1.1 *(Standard Map)* When $L(q,v) = \frac{1}{2}|v|^2 - V(q)$, then

$$Q = q + P, \qquad P = p - \nabla V(q).$$

□

We may also consider a generating function $V(Q,p) = Q \cdot p - v(Q,p)$ of Type III so that

$$\psi\big(Q - v_p(Q,p), p\big) = \big(Q, p - v_Q(Q,p)\big).$$

In other words,

$$Q = q + v_p(Q,p), \qquad P = p - v_Q(Q,p).$$

2 Twist Maps and Their Generalizations

The origin of the twist maps goes back to Poincaré's work on area-preserving maps on annulus that he encountered in his work on 3-body problem of celestial mechanics. Before embarking on studying twist maps, we give an overview of circle diffeomorphisms and their rotation numbers.

Definition 2.1(i) Regarding $\mathbb{S}^1$ as the interval $[0,1]$ with $0 = 1$, let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be an orientation preserving homeomorphism. Its *lift* $F = \ell(f)$ is an increasing map $F : \mathbb{R} \to \mathbb{R}$ such that $f(x) = F(x) \pmod 1$, and F can be written as $F(x) = x + G(x)$, for a 1-periodic function $G : \mathbb{R} \to \mathbb{R}$. We may also regard G as a map on the circle: $g : \mathbb{S}^1 \to \mathbb{R}$, $g(x) = G(x)$ for $x \in [0,1)$.

(ii) We define $\pi : \mathbb{R} \to \mathbb{S}^1$ by $\pi(x) = e^{2\pi x}$. For f and F as in **(i)**, we define its rotation number

$$\rho(F) = \lim_{n\to\infty} n^{-1}F^n(x), \qquad \rho(f) = \pi\big(\rho(F)\big). \tag{2.1}$$

(iii) Given $\rho \in [0,1)$, we write r_ρ for a rotation of the circle through the angle ρ. Its lift R_ρ is given by $R_\rho(x) = x + \rho$. □

Theorem 2.1 *(Poincaré) Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be an orientation preserving homeomorphism and write F for its lift. Then the following statements are true:*

(i) *The rotation number always exists and is independent of x.*

(ii) *f has a fixed point iff $\rho(f) = 0$.*

(iii) *$\pm\rho(F) > 0$ iff $\pm(F(x) - x) > 0$.*

(iv) *Let (r,s) be a pair of coprime positive integers. Then f has a (r,s)-periodic orbit (this means that $F^s(x) = F(x) + r$ for $F = \ell(f)$), iff $\rho(f) = r/s$.*

(v) *If $\rho(f) \notin \mathbb{Q}$, then the set $\Omega_\infty(x)$ of the limit points of the sequence $\big\{f^n(x) : \ n \in \mathbb{N}\big\}$ is independent of x, and is either $\mathbb{S}^1$ or nowhere dense.*

Proof We only prove **(i)**. By induction, we can readily show that if $F(x) = x + g(x)$ for a periodic function g, then $F^n(x) = x + G_n(x)$ for a periodic function G_n that is simply given by

$$G_n(x) = \sum_{i=0}^{n-1} G\big(F^i(x)\big) = \sum_{i=0}^{n-1} g\big(f^i(x)\big). \tag{2.2}$$

Observe that since F^n is increasing, we have

$$0 \le x \le y < 1 \quad \Longrightarrow \quad G^n(y) - G^n(x) \le y - x < 1.$$

Hence

$$G_{m+n} = G_m + G_n \circ F^m \le G_m + G_n + 1.$$

From this we deduce

$$\rho(x) = \lim_{n\to\infty} n^{-1} G_n(x) = \lim_{n\to\infty} n^{-1}\big(F_n(x) - x\big) = \lim_{n\to\infty} n^{-1} F_n(x),$$

exists and is a periodic non-decreasing function of x. Hence $\rho(x)$ must be a constant. □

Theorem 2.2 *Let f and F be as in Proposition 1.1.*

(i) *(Denjoy) If $f \in C^1$ with f' a function of bounded variation, and $\rho = \rho(f) \notin \mathbb{Q}$, then there exists a homeomorphism h such that $f = h^{-1} \circ r_\rho \circ h$.*

(ii) *(Herman) If $f \in C^{2+\alpha}$ with $\alpha \in [0, 1)$, and $\rho(F) \in D(\tau)$ satisfies a Diophantine condition for some $\tau > 2$, then h is Part* **(i)** *is in $C^{1+\alpha}$.*

Remark 2.1(i) Since the Lebesgue measure is invariant for r_ρ, and $h \circ f \circ h^{-1} = r_\rho$, we learn

$$\int (h \circ J)(f(x))\ dh(x) = \int J\ dx, \quad \text{or} \quad \int J \circ f\ dh = \int J\ dh.$$

In other words, the measure μ with $\mu[0, x] = h(x)$ is invariant for f. Hence Part **(ii)** is equivalent to the statement that if $f \in C^{2+\alpha}$, then the invariant measure had a density in C^α.

(ii) In terms of the invariant measure, the rotation number can be express as

$$\rho(f) = \int g\ d\mu,$$

by (6.1).

(iii) Define $\mathcal{F}$ to be the set of continuous increasing functions $F : \mathbb{R} \to \mathbb{R}$ such that

$$\sup_x |F(x) - x| < \infty.$$

Writing $F(x) = x + G(x)$, we define a translation operator that translates G:

$$(\tau_a F)(x) = F(x + a) - a = x + G(x + a).$$

Let $\mathbb{P}$ be an ergodic probability measure on $\mathcal{F}$. Then on can show that there exists a constant $\rho(\mathbb{P})$ such that

$$\lim_{n\to\infty} n^{-1}F^n(x) = \rho(\mathbb{P}),$$

for $\mathbb{P}$-almost all choices of F. □

Definition 2.2(i) Let $\varphi : \mathbb{S}^1 \times [-1,1] \to \mathbb{S}^1 \times [-1,1]$, be an orientation preserving homeomorphism. Its *lift* is a homeomorphism $\Phi : \mathbb{R} \times [-1,1] \to \mathbb{R} \times [-1,1]$ such that

$$\varphi(x) = \Phi(x) \pmod 1,$$

and $\Phi = \ell(\varphi)$ can be written as $\Phi(q,p) = (q,0) + \Psi(q,p)$, for a continuous $\Psi : \mathbb{R} \times [-1,1] \to \mathbb{R} \times [-1,1]$, that is 1-periodic function in q-variable.

(ii) An orientation-preserving diffeomorphism $\varphi : \mathbb{S}^1 \times [-1,1] \to \mathbb{S}^1 \times [-1,1]$ is called a *twist map* if the following conditions are met:

(i) φ (or equivalently its lift Φ) is area-preserving.

(ii) If we define $\Phi^{\pm}$ by $(\Phi^{\pm}(q), \pm 1) = \Phi(q, \pm 1)$, then $\pm\big(\Phi^{\pm}(x) - x\big) > 0$.

□

Our main result about twist maps is the following:

Theorem 2.3 *(Poincaré and Birkhoff) Any twist map has at least two fixed points.*

To see Poincaré-Birkhoff's theorem within a larger context, we interpret it in the following way: since $0 \in \big(\rho(\Phi^-), \rho(\Phi^+)\big)$, then φ has at least two orbits in the interior of the cylinder that are associated with 0 rotation number, namely fixed points. In fact an analogous result is true for periodic orbits which may be regarded as a variant of Theorem 1.1**(ii)** for the twist maps.

Theorem 2.4 *(Birkhoff) Let $\varphi : \mathbb{S}^1 \times [-1,1] \to \mathbb{S}^1 \times [-1,1]$, be an area and orientation preserving C^1-diffeomorphism. If $r/s \in \big(\rho(\Phi^-), \rho(\Phi^+)\big)$ is a rational number with r and s coprime, then φ has at least two (r,s)-periodic orbits in the interior of $\mathbb{S}^1 \times [-1,1]$.*

Naturally we are led to the following question: How about an irrational $\rho \in \big(\rho(\Phi^-), \rho(\Phi^+)\big)$? Can we find an orbit of φ associated with such ρ? The answer to this question is affirmative and this is the subject of the *Aubry-Mather Theorem.* For any irrational $\rho \in \big(\rho(\Phi^-), \rho(\Phi^+)\big)$, there exists an invariant set on the cylinder that in some sense has the rotation number ρ. This invariant set q-projects onto either a Cantor-like subset of S^1 or the whole S^1. The invariant set lies on a graph of a Lipschitz function defined on S^1. These invariant sets are known as *Aubry–Mather sets.*

Poincaré established Theorem 2.3 provided that φ has a global generating function. Such a generating function exists if φ is a *monotone twist map.* To explain Poincare's argument, let us formulate a condition on $\Phi = \ell(\varphi)$ that would guarantee the existence of a global generating function $S(q, Q)$ for Φ.

Definition 2.3 An area-preserving map φ or its lift $\Phi(q, p) = (Q(q, p), P(q, p))$ is called *positive (monotone)* if $Q(q, p)$ is increasing in p for every $q \in \mathbb{R}$. We say φ is *negative (monotone)* twist if φ^{-1} is a positive twist. □

Proposition 2.1 *Let Φ be a C^1 monotone twist map. Then there exists a C^2 function $S : U \to \mathbb{R}$ with*

$$U = \{(q, Q) : Q(q, -1) \leq Q \leq Q(q, +1)\}$$

such that

$$\Phi(x, -S_q(q, Q)) = (Q, S_Q(q, Q)).$$

Moreover

$$S(q + 1, Q + 1) = S(q, Q), \qquad S_{qQ} < 0. \tag{2.3}$$

Proof The image of the line segment $\{q\} \times [-1, 1]$ under Φ is a curve γ with parametrization $\gamma(p) = (Q(q, p), P(q, p))$. By the monotonicity, the relation $Q(q, p) = Q$ can be inverted to yield $p = p(q, Q)$ which is increasing in Q. The set $\gamma[-1, 1]$ can be viewed as a graph of the function

$$Q \mapsto P(q, p(q, Q))$$

with $Q \in [Q(q, -1), Q(q, +1)]$. The anti-derivative of this function yields S. This can be geometrically described as the area of the region Δ between the curve $\gamma([-1, 1])$, the line $P = -1$ and the vertical line $\{q\} \times [-1, 1]$. We now apply Φ^{-1} on this region. The line segment $\{Q\} \times [-1, 1]$ is mapped to a curve $\hat{\gamma}([-1, 1])$ which coincides with a graph of a function $q \mapsto p$. Since Φ is area preserving the area of $\Phi^{-1}(\Delta)$ is $S(q, Q)$. From this we deduce that $S_Q = -p$. Here we have used the fact that Φ^{-1} is a (negative) twist map. This is because if we write $\Phi^{-1}(Q, P) = (q(Q, P), p(Q, P))$, then

$$(\Phi^{-1})' = \begin{bmatrix} q_Q & q_P \\ p_Q & p_P \end{bmatrix} = \begin{bmatrix} Q_q & Q_p \\ P_q & P_p \end{bmatrix}^{-1} = \begin{bmatrix} P_p & -Q_p \\ -P_q & Q_q \end{bmatrix}$$

which implies that $q_P = -Q_p < 0$.

The periodicity (2.3) is an immediate consequence of $\Phi(q + 1, p) = \Phi(q, p) + (1, 0)$;

$$\Phi(\{q + 1\} \times [-1, 1]) = \Phi(\{q\} \times [-1, 1]) + (1, 0).$$

As for the second assertion in (2.3), recall that $p(q, Q)$ is increasing in Q. Hence

$$S_{qQ} = -p_Q < 0.$$

□

A partial converse to Proposition 2.1 is true, namely if a function S satisfies (**??**), then it generates a map Φ which is area preserving. We don't address the behavior of Φ on the boundary lines and for simplicity assume that S is defined on $\mathbb{R}^2$.

Proposition 2.2 *Let S be a C^2 function satisfying* (2.3)*. Then there exists a C^1-function Ψ such that*

(i) $\Phi(q+1,p) = \Phi(q,p) + (1,0)$

(ii) $\Psi(q, -S_q(q,Q)) = (Q, S_Q(q,Q))$

(iii) $\det \Phi' \equiv 1.$

Proof Since $S_{qQ} < 0$, the function $Q \mapsto -S_q(q,Q)$ is increasing. As a result, $p = -S_q(q,Q)$ can be inverted to yield $Q = Q(q,p)$. We then set

$$P(q,p) = S_Q(q, Q(q,p)) \quad \text{and} \quad \Phi(q,p) = (Q(q,p), P(q,p)).$$

Evidently (ii) is true and (i) follows from (ii) and (**??**) because $S_q(q+1, Q+1) = S_q(q,Q)$, and $S_Q(q+1, Q+1) = S_Q(q,Q)$. It remains to verify (iii). For this, set $\hat{S}(q,p) = S(q, Q(q,p))$. We have

$$\begin{aligned} \hat{S}_q &= S_q + S_Q Q_q = -p + PQ_q, \\ \hat{S}_p &= S_Q Q_p = PQ_p. \end{aligned}$$

Differentiating again yields

$$\begin{aligned} \hat{S}_{qp} &= -1 + P_p Q_q + PQ_{qp}, \\ \hat{S}_{pq} &= P_q Q_p + PQ_{pq}. \end{aligned}$$

Since $S \in C^2$, we must have $\hat{S}_{qp} = \hat{S}_{pq}$, which yields $P_p Q_q - P_q Q_p = 1$, as desired. □

We now show how the existence of a generating function can be used to prove the existence of fixed points.

Proof of Theorem 2.3 for a monotone twist map Define $L(q) = S(q,q)$. We first argue that a critical point of L corresponds to a fixed point of Φ. Indeed, if $L'(q^0) = 0$, then $S_q(q^0, q^0) + S_Q(q^0, q^0) = 0$. Since $\Phi(q^0, -S_q(q^0, q^0)) = (q^0, S_Q(q^0, q^0))$, we deduce that $\Phi(q^0, y^0) = (q^0, y^0)$ for $y^0 = -S_q(q^0, q^0) = S_Q(q^0, q^0)$. On the other hand, by (**??**), we have that $L(q+1) = L(q)$. Either L is identically constant which yields a continuum of fixed points for Ψ, or L is not constant. In the latter case, L has at least two distinct critical points, namely a maximizer and minimizer. These yield two distinct critical points of Φ.

□

We may wonder whether a similar strategy as in the above proof can be used to Prove Theorem 9.2 when φ is a monotone area-preserving map. Indeed if Φ is a monotone twist map, then we can associated with it a variational principle which is the discrete analog of the Lagrange Variational Principle, as can be seen in the following proposition.

Proposition 2.3 *Let Φ be a monotone twist map with generating function S. Given q and $Q \in \mathbb{R}$, define*

$$L(q, Q; q_1, q_2, \ldots, q_{k-1}) = \sum_{j=0}^{n-1} S(q_j, q_{j+1}),$$

with $q_0 = q$, and $q_n = Q$. Then the following statements are true.

(i) *The point $(q_1, q_2, \ldots, q_{k-1})$ is a critical point of $L(\cdot; q, Q)$ iff there exist $p_0, p_1, \ldots, p_k$ such that $\Phi^j(q_j, p_j) = (q_{j+1}, p_{j+1})$ for $j = 1, 2, \ldots, k-1$.*

(ii) *The point $(q_0, q_1, q_2, \ldots, q_{s-1})$ is a critical point of*

$$K(q_1, q_2, \ldots, q_s) = S(q_{s-1}, q_0 + r) + \sum_{j=0}^{s-2} S(q_j, q_{j+1})$$

if and only if there exist $p_0, p_1, p_2, \ldots, p_{s-1}$ such $\Phi^j(q_j, p_j) = (q_{j+1}, p_{j+1})$ for $j = 0, \ldots, s-1$, with $q_s = q_0 + r$.

Proof We only prove **(ii)** because **(i)** can be proved by a verbatim argument. Let $(q_0, \ldots, q_{s-1})$ be a critical point and set $q_s = q_0 + r$. We also set $p_j = -S_q(q_j, q_{j+1})$. The result follows because if $P_j = S_Q(q_j, q_{j+1})$, then

$$K_{q_j} = p_j - P_{j-1}$$

for $j = 0, 1, 2, \ldots, s-1$ and $\Psi(q_j, p_j) = (q_{j+1}, P_j)$. □

Given a Hamiltonian function $H : M \times \mathbb{R} \to \mathbb{R}$ on a symplectic manifold (M, ω), we may wonder whether or not the corresponding Hamiltonian vector field $X_H = X_H^\omega$ has T-periodic orbits for a given period T. *Arnold's Conjecture* offers a non-trivial lower bounds on the number of such periodic orbits. To convince that such a question is natural and important, let us examine this question when the Hamiltonian function is time-independent first. We note that for the autonomous X_H we can even find rest points (or constant orbits) and there is a one-one correspondence between the constant orbits of X_H and the critical points of H. We can appeal to the following classical theories in Algebraic Topology to obtain sharp universal lower bounds on the number of critical points of a smooth function on M where M is a smooth closed manifold. Let us write $Crit(H)$ for the set of critical points of $H : M \to \mathbb{R}$.

(i) According to Lusternik-Schnirelmann (LS) Theorem,

$$\sharp Crit(H) \geq c\ell(M), \tag{2.4}$$

where $c\ell(M)$ denotes the *cuplength* of M.

(ii) According to Morse Theory, for a Morse function H,

$$\sharp Crit(H) \geq \sum_k \beta_k(M), \tag{2.5}$$

where $\beta_k(M)$ denotes the k-th *Betti's number* of M.

According to Arnold's conjecture, the analogs of (2.4) and (2.5) are true for the non-autonomous Hamiltonian functions provided that we count 1-periodic orbits of X_H instead of constant orbits. For the sake of comparison, we may regard (2.4) and (2.5) as a lower bound on the number of 0-periodics orbit when H is 0-periodic in t. In Arnold's conjecture, we replace 0-periodicity with 1-periodicity. Note that if H is 1-periodic in time, then $\phi_{t+1}^H(x) = \phi_t^H(x)$ for all t iff $\phi_1^H(x) = x$. To this end, define

$$Per(H) := \big\{x \in M : \ \phi_1^H(x) = x\big\} = Fix\big(\phi_1^H\big). \tag{2.6}$$

Arnold's Conjecture: Let (M, ω) be a closed symplectic manifold and let $H : M \times [0, \infty) \to \mathbb{R}$ be a smooth Hamiltonian function that is 1-periodic in the time variable. Then

$$\sharp Per(H) \geq c\ell(M). \tag{2.7}$$

Moreover, if $\varphi := \phi_1^H$ is non-degenerate in the sense that $\det(d\varphi - id)_x \neq 0$ for every $x \in Fix(\varphi)$, then

$$\sharp Per(H) \geq \sum_k \beta_k(M). \tag{2.8}$$

□

We now describe our strategies for establishing Arnold's conjecture under some additional conditions on M: A natural way to tackle Arnold's conjecture is to study the set of critical points of $\mathcal{A}_H : \Gamma \to \mathbb{R}$, where Γ is the space of 1-periodic $x : \mathbb{R} \to M$ and

$$\mathcal{A}_H(x(\cdot)) = \int_w \omega - \int_0^1 H(x(t), t)\ dt, \tag{2.9}$$

where $w : \mathbb{D} \to M$ is any extension of $x\cdot : \mathbb{S}^1 \to M$ to the unit disc $\mathbb{D}$. (Note that since ω is closed, the right-hand side of (2.9) is independent of the extension.) We may try to apply LS and Morse Theory to the functional $\mathcal{A}_H$ in order to get lower bounds on $\sharp Per(H)$. Of course we cannot apply either Morse Theorem (2.5) or LS Theorem (2.4) to $\mathcal{A}_H$ directly because Γ is neither compact nor finite-dimensional. However in the case of the torus or when M is a cotangent bundle, we may reduce the problem by using *generalized generating functions*. In fact, one can show that ϕ_t^H has a type II or III generating functions (as we discussed in Subsections 1.8 and 1.9) provided that t is sufficiently small. We then use the group property of the flow to write

$$\varphi = \phi_1^H = \psi_1 \circ \cdots \circ \psi_N,$$

where each ψ_i has a generating function. This can be used to build a generalized generating function for φ a la Chaperon. We may establish Arnold's conjecture with the aid of generalized generating functions in some cases. We note that when $M = \mathbb{T}^{2d}$, then the symplectic map $\varphi = \Phi_1^H : \mathbb{T}^{2d} \to \mathbb{T}^{2d}$ has a lift $\Phi : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ such that $\Phi - id$ is periodic. Motivated by Arnold's conjecture, we may wonder where or not any symplectic diffeomorphism of $\mathbb{T}^{2d}$ possesses fixed points and a non-trivial lower bound can be given for the number of its fixed points. This is not the case in general as a non-zero translation on a torus has no fixed point. However note that there is an additional feature of such $\varphi = \Phi_1^H$ that we have not discussed and will play an essential role for our purposes, namely since

$$\Phi(x) - x = \int_0^1 J\nabla H\big(\phi_t^H(x), t\big)\ dt,$$

and regarding $\mathbb{T}^{2d}$ as $[0,1)^{2d}$, we have

$$\int_{\mathbb{T}^{2d}} \big(\Phi(x) - x\big)\ dx = \int_0^1 J \int_{\mathbb{T}^{2d}} \nabla H\big(\phi_t^H(x), t\big)\ dx\ dt = \int_0^1 J \int_{\mathbb{T}^{2d}} \nabla H(x,t)\ dx\ dt = 0. \tag{2.10}$$

Arnold's conjecture was established by Conley and Zehnder when $M = \mathbb{T}^{2d}$. In fact an equivalent formulation goes as follows.

Theorem 2.5 *Let $\varphi : \mathbb{T}^{2d} \to \mathbb{T}^{2d}$ be a symplectic diffeomorphism such that its lift Φ satisfies*

$$\int_{\mathbb{T}^{2d}} \big(\Phi(x) - x\big)\ dx = 0 \tag{2.11}$$

Then φ has at least $2d + 1$ fixed points.

A variant of Theorem 2.5 can be proved when the periodicity of $\Phi - id$ is replaced with almost periodicity, or even when $\Phi - id$ is selected randomly according to a translation invariant probability measure.

Definition 2.4(i) Let us write $\mathcal{H} = \mathcal{H}(\mathbb{R}^{2d})$ for the space of C^2 Hamiltonian functions $H : \mathbb{R}^{2d} \times \mathbb{R} \to \mathbb{R}$. For each $a = (b, c) \in \mathbb{R}^d \times \mathbb{R}^d$, we define

$$(\tau_b H)(q, p, t) = H(q+b, p), \quad (\eta_c H)(q, p, t) = H(q, p+c, t), \quad (\theta_a H)(q, p, t) = H(q+b, p+c, t).$$

(ii) We write $\mathcal{C}^1$ for the set of C^1 maps $\Phi : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$. We set $\mathcal{F}(\Phi) = \Phi - id$, where id denotes the identity map. We write $\mathcal{S}$ for the set of symplectic diffeomorphism $\Phi : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ and set $\tilde{\mathcal{S}} = \mathcal{F}(\mathcal{S})$. For $a \in \mathbb{R}^{2d}$, the translation operators $\theta_a : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ and $\theta_a, \theta'_a : \mathcal{C}^1 \to \mathcal{C}^1$ are defined by

$$\theta_a(x) = x + a, \quad \theta_a \omega = \omega \circ \theta_a, \quad \theta'_a = \mathcal{F}^{-1} \circ \theta_a \circ \mathcal{F},$$

for $x \in \mathbb{R}^{2d}$ and $\omega \in \mathcal{C}^1$. Note that for $\Phi \in \mathcal{C}^1$,

$$\big(\theta'_a \Phi\big)(x) = \theta_{-a} \circ \Phi \circ \theta_a = \Phi(x + a) - a.$$

(iii) Let Φ be a symplectic diffeomorphsim with

$$\Phi(q, p) = \big(Q(q, p), P(q, p)\big).$$

We say that Φ is *exact* if for every $p \in \mathbb{R}^d$, the map $q \mapsto Q(q, p)$ is a diffeomorphism of $\mathbb{R}^d$. We write $\hat{q}(Q, p)$ for the inverse:

$$Q(q, p) = Q \quad \Leftrightarrow \quad q = \hat{q}(Q, p).$$

We also set $\hat{P}(Q, p) = P\big(\hat{q}(Q, p), p\big)$, and

$$\widehat{\Phi}(Q, p) = \big(\hat{q}(Q, p), \hat{P}(Q, p)\big), \qquad \widetilde{\Phi}(Q, p) = \big(\hat{P}(Q, p), \hat{q}(Q, p)\big).$$

□

Proposition 2.4 **(i)** *$\phi^{\theta_a H} = \theta_{-a} \circ \phi^H \circ \theta_a = \theta'_a \phi^H$. In particular, if H is 1-periodic, i.e., $\theta_n H = H$, for all $n \in \mathbb{Z}^{2d}$, and $\Phi = \phi_1^H$, then $\mathcal{F}\big(\Phi\big)$ is also 1-periodic.*

(ii) *For every exact Φ, and $a \in \mathbb{R}^d$, we have*

$$\widehat{\theta'_a \Phi} = \theta'_a \widehat{\Phi}.$$

In particular, if $\mathcal{F}(\Phi)$ is 1-periodic, then so is $\mathcal{F}\big(\widehat{\Phi}\big)$.

(iii) *Assume that $\Phi \in \mathcal{S}$ is exact. Then there exists a C^2 function $W : \mathbb{R}^{2d} \to \mathbb{R}$ such that $\widetilde{\Phi} = \nabla W$.*

(iv) *If $\omega = \mathcal{F}(\Phi)$ is 1-periodic, with*

$$\int_{\mathbb{T}^{2d}} \omega(x) \; dx = 0,$$

then

$$W(Q, p) = Q \cdot p - w(Q, p),$$

for a function w that is 1-periodic.

Proof(i) This is an immediate consequence of the fact that if $y(\cdot)$ is an orbit of $X_{\theta_a H}$, then $x(\cdot) = \theta_{-a} y(\cdot) = y(\cdot) - a$ is an orbit of X_H.

(ii) Let use write

$$\Phi'(q,p) := (\theta'_a \Phi)(q,p) = \big(Q'(q,p), P'(q,p)\big), \qquad \widehat{\Phi}'(Q,p) = \big(\hat{q}'(Q,p), \hat{P}'(Q,p)\big).$$

This implies

$$\begin{aligned} Q(q+b, p+c) - b = Q \quad &\Leftrightarrow \quad \hat{q}'(Q,p) = q, \\ Q(q+b, p+c) = Q + b \quad &\Leftrightarrow \quad \hat{q}(Q+b, p+c) = q + b. \end{aligned}$$

Hence $\hat{q}'(Q,p) = \hat{q}(Q+b, p+c) - b$. On the other hand

$$\begin{aligned} \hat{P}'(Q,p) &= P'\big(\hat{q}'(Q,p), p\big) = P\big(\hat{q}'(Q,p) + b, p + c\big) - c \\ &= P\big(\hat{q}(Q+b, p+c), p+c\big) - c = \hat{P}\big(Q+b, p+c\big) - c, \end{aligned}$$

as desired.

(iii) Since Φ is symplectic, we have

$$d\big(\hat{P} \cdot dQ + \hat{q} \cdot dp\big) = d\big(\hat{P} \cdot dQ - dp \cdot \hat{q}\big) = d\big(P \cdot dQ - dp \cdot q\big) = 0.$$

Hence, there exists a function $W = W(Q,p)$ such that

$$dW = \hat{P} \cdot dQ + \hat{q} \cdot dp.$$

As a result, $\nabla W = \widetilde{\Phi}$.

(iv) We write $\hat{\omega} = \mathcal{F}\big(\widehat{\Phi}\big)$, and $\widehat{\nabla} w = (w_p, w_Q)$, so that

$$\begin{aligned} \big(W_p(Q,p), W_Q(Q,p)\big) &= \big(Q - w_p(Q,p), p - w_Q(Q,p)\big) \\ &= (Q,p) - \widehat{\nabla} w(Q,p) = (Q,p) + \hat{\omega}(Q,p). \end{aligned}$$

By **(ii)** we know that $\hat{\nabla} w = -\omega$ is a periodic function. We wish to show that w is also a periodic function. The periodicity of w is equivalent to

$$\int_{[0,1]^{2d}} \hat{\omega}(x)\, dx = -\int_{[0,1]^{2d}} \hat{\nabla} w(x)\, dx = 0.$$

To verify this, observe that if

$$A = (B, C) = \int_{[0,1]^{2d}} \hat{\omega}(x)\, dx,$$

then there exits a C^2 periodic function $v(Q,p)$ such that $\hat{\omega} - A = -\hat{\nabla} v$, or

$$\hat{P}(Q,p) = C + p - v_Q(Q,p), \qquad \hat{q} = B + Q - v_p(Q,p).$$

On the other hand, by assumption,

$$\begin{aligned}
0 &= \int_{[0,1]^{2d}} \omega(q,p)\, dqdp = \int_{[0,1]^{2d}} \big(Q(q,p) - q, P(q,p) - p\big)\, dqdp \\
&= \int_{[0,1]^{2d}} \big(Q - \hat{q}(Q,p), \hat{P}(Q,p) - p\big)\, dqdp \\
&= \int_{[0,1]^{2d}} \big(Q - \hat{q}(Q,p), \hat{P}(Q,p) - p\big)\ \det\big(\hat{q}_Q(Q,p)\big)\, dQdp \\
&= \int_{[0,1]^{2d}} \big(v_p(Q,p) - B, C - v_Q(Q,p)\big)\ \det\big(I - v_{Qp}(Q,p)\big)\, dQdp \\
&= (-B, C) + \int_{[0,1]^{2d}} J\nabla v(Q,p)\ \det\big(I + v_{Qp}(Q,p)\big)\, dQdp.
\end{aligned}$$

We are done if we can show

$$\int_{[0,1]^{2d}} \nabla v(Q,p)\ \det\big(I + v_{Qp}(Q,p)\big)\, dQdp = 0. \tag{2.12}$$

□

Proposition 2.5 *Let $\Phi^i, i = 1, \ldots, k$, be k exact symplectic diffeomorphisms with generating functions $W^i(Q,p) = Q \cdot p - w^i(Q,p), i = 1, \ldots, k$, respectively. Let $\Phi = \Phi^k \circ \cdots \circ \Phi^1$.*

(i) *Define*

$$\begin{aligned}
W(Q,p;\xi) &= \sum_{i=1}^{k} W^i(q_i, p_{i-1}) - \sum_{i=1}^{k-1} q_i \cdot p_i \\
&= q_1 \cdot p_0 + \sum_{i=2}^{k} p_{i-1} \cdot (q_i - q_{i-1}) - \sum_{i=1}^{k} w^i(q_i, p_{i-1}) \\
&= Q \cdot p + \sum_{i=2}^{k} (p_{i-1} - p_0) \cdot (q_i - q_{i-1}) - \sum_{i=1}^{k} w^i(q_i, p_{i-1}) \\
&=: Q \cdot p + w(Q,p;\xi).
\end{aligned}$$

where $p_0 = p, q_k = Q$, and $\xi = \big(q_1, p_1, \ldots, q_{k-1}, p_{k-1}\big)$. Then

$$W_\xi(Q,p;\xi) = 0 \quad \Longrightarrow \quad \Phi\big(W_p(Q,p;\xi), p\big) = \big(Q, W_Q(Q,p;\xi)\big). \tag{2.13}$$

Moreover (Q, p) is a fixed point of Φ iff for some ξ, we have $\nabla w(Q, q; \xi) = 0$.

(ii) *Given $\mathbf{x} = (x_0, \ldots, x_{k-1})$, $x_0 = (q_0, p_0), \ldots, x_{k-1} = (q_{k-1}, p_{k-1})$, define*

$$\begin{aligned}\mathcal{A}^k(\mathbf{x}) &= \sum_{i=1}^k W^i(q_i, p_{i-1}) - \sum_{i=1}^k q_i \cdot p_i \\ &= \sum_{i=1}^k \big(p_{i-1} \cdot (q_i - q_{i-1}) - w^i(q_i, p_{i-1})\big),\end{aligned}$$

with $x_0 = x_k = (q_k, p_k)$. (In other words, $\mathcal{A}^k$ is defined for k-periodic sequences.) Then any critical point $\mathbf{x}$ of $\mathcal{A}^k$ yields an orbit $\Phi_i(x_{i-1}) = x_i, i = 1, \ldots, k$. In particular $x_0 = x_k$ is a fixed point of Φ.

Proof(i) If we write $\hat{q}_{i-1} = W_p\big(q_i, p_{i-1}\big)$, and $\hat{p}_i = W_Q\big(q_i, p_{i-1}\big)$, then $\Phi^i\big(\hat{q}_{i-1}, p_{i-1}\big) = \big(q_i, \hat{p}_i\big)$. On the other hand, for $i = 1, \ldots, k-1$,

$$\begin{aligned}W_{q_i}(Q, p; \xi) &= \hat{p}_i - p_i, \quad W_{p_i}(Q, p; \xi) = \hat{q}_i - q_i, \\ W_p(Q, p; \xi) &= W_p^1(q_1, p), \quad W_Q(Q, p; \xi) = W_Q^k(Q, p_k).\end{aligned}$$

From this, we can readily deduce (2.13).

(ii) Observe that if we set $\hat{q}_{i-1} = W_p\big(q_i, p_{i-1}\big)$, and $\hat{p}_i = W_Q\big(q_i, p_{i-1}\big)$, then

$$\begin{aligned}\mathcal{A}^k_{q_i}(\mathbf{x}) &= \hat{p}_i - p_i, \quad \mathcal{A}^k_{p_i}(\mathbf{x}) = \hat{q}_i - q_i, \\ \mathcal{A}^k_{q_k}(\mathbf{x}) &= \hat{p}_k - p_k, \quad \mathcal{A}^k_{p_0}(\mathbf{x}) = \hat{q}_0 - q_0\end{aligned}$$

for $i = 1, \ldots, k-1$. Hence at a critical point we have $\Phi_i(x_{i-1}) = x_i$ for $i = 1, \ldots, k$. This completes the proof. □

Proof of Theorem 2.5 *(Sketch)* For some sufficiently large k, we can find exact symplectic diffeomorphisms $\Phi^i, i = 1, \ldots, k$, such that $\Phi = \Phi^k \circ \cdots \circ \Phi^1$. By Proposition 2.5**(ii)**, there is a one-to-one correspondence between Φ fixed points x_0 and critical points $\mathbf{x} = (x_0, \ldots, x_{k-1})$. Observe that when $\mathcal{F}(\Phi)$ is periodic of 0 average, then $w^1, \ldots, w^k$ are periodic. On he other hand, since $x_k = x_0$ in the definition of $\mathcal{A}^k$, we may write

$$\mathcal{A}^k(\mathbf{x}) = \sum_{i=1}^k \big[(p_{i-1} - p_0) \cdot (q_i - q_{i-1}) - w^i(q_i, p_{i-1})\big].$$

This implies that if we set $z_i = x_i - x_{i-1} = (q_i', p_i')$, and $\mathbf{z} = (z_1, \dots, z_{k-1})$, then we can write

$$\begin{aligned}\frac{1}{2} B\mathbf{z} \cdot \mathbf{z} &:= \sum_{i=1}^{k} p_{i-1} \cdot (q_i - q_{i-1}) = -\sum_{i=1}^{k} (p_i - p_{i-1}) \cdot q_i \\ &= \sum_{i=1}^{k} (p_{i-1} - p_0) \cdot (q_i - q_{i-1}) = -\sum_{i=1}^{k} (p_i - p_{i-1}) \cdot (q_i - q_0) \\ &= \sum_{i=1}^{k} (p_{i-1}' + \dots + p_1') \cdot q_i' = -\sum_{i=1}^{k} (q_i' + \dots + q_1') \cdot p_i',\end{aligned}$$

where $B = [B_{ij}]_{i,j=1}^{k-1}$, with each B_{ij} a $(2d) \times (2d)$ matrix. We may express B as

$$B = \begin{bmatrix} 0 & C \\ -D & 0 \end{bmatrix},$$

with both C and D invertible. Hence B is non-singular. Moreover, since for each $m \in \mathbb{Z}^{2d}$,

$$\mathcal{A}^k(x_0 + m, \dots, x_{k-1} + m) = \mathcal{A}^k(x_0, \dots, x_{k-1}),$$

we can write

$$\mathcal{A}^k(\mathbf{x}) = \frac{1}{2} B\mathbf{z} \cdot \mathbf{z} + \hat{w}(x_0, \mathbf{z}),$$

for a bounded C^2 function $\hat{w}(x_0, \mathbf{z})$ that is periodic in x_0. writing $\mathbf{y} = (x_0, \mathbf{z})$, and $\mathcal{B}(\mathbf{y})$ for $\mathcal{A}^k$, we may regard $\mathcal{B}$ as a function on $\mathbb{T}^{2d} \times \mathbb{R}^{2d(k-1)}$. We may study the set of critical points of $\mathcal{B}$ by analyzing the corresponding gradient flow $\dot{\mathbf{y}} = -\nabla \mathcal{B}(\mathbf{y})$. Equivalently,

$$\dot{\mathbf{z}} = B\mathbf{z} + \hat{w}_{\mathbf{z}}(x_0, \mathbf{z}), \qquad \dot{x}_0 = \hat{w}_{x_0}(x_0, \mathbf{z}). \tag{2.14}$$

Note that if $\hat{w} = 0$, then $\mathbb{T}^{2d} \times \{0\}$ is the the set rest points for the flow associated with (2.14). In fact 0 is a hyperbolic (saddle-like) critical point for $\dot{\mathbf{z}} = B\mathbf{z}$. In (2.14) we have a rather *compact* perturbation of $\dot{\mathbf{z}} = B\mathbf{z}$.

Writing ψ_t for the flow of (2.14), we set

$$\Gamma = \left\{ \mathbf{y} : \sup_t \left| \psi_t(\mathbf{y}) \right| < \infty \right\}.$$

□

Let us study an example of a map which is not quite a twist map but still possesses a global generating function and Theorem 2.4 may be applied to guarantee the existence of its periodic orbits.

Example 2.1 (Billiard map in a convex domain). Let C be a strictly bounded convex domain in $\mathbb{R}^2$ and denote its boundary by Γ. Without loss of generality, we assume that the total length of Γ is 1. First we describe the billiard flow in C. This is the flow associated with the Hamiltonian function $H(q,p) = \frac{1}{2}|p|^2 + V(q)$ where

$$V(q) = \begin{cases} 0 & \text{if } q \in C \\ \infty & \text{if } q \notin C. \end{cases}$$

Here is the interpretation of the corresponding flow: A ball of velocity x starts from a point $a \in C$ and is bounced off the boundary Γ by the law of reflection. This induces a transformation for the hitting location and reflection angle. More precisely, if a trajectory $a+tv$, $t > 0$ hits the boundary at a point $\gamma(q)$ and a post-reflection angle θ, then we write $\gamma(Q)$ and Θ for the location and post-reflection angle of the next reflection. Here q is the length of arc between a reference point $A \in \Gamma$ and $\gamma(q)$ on Γ in positive direction, and θ measures the angle between the tangent at $\gamma(q)$ and the post-reflection velocity vector. We write ψ for the map $(q,\theta) \mapsto (Q,\Theta)$ with $q, Q \in S^1$ and $\theta, \Theta \in [0,\pi]$. It is more convenient to define $p = -\cos\theta$ so that in the (q,p) coordinates, we have a map $\varphi : \mathbb{S}^1 \times [-1,1] \to \mathbb{S}^1 \times [-1,1]$. As before, we write Φ for its lift. We claim that Φ is a monotone area-preserving map; it is not a twist map because the twist conditions on the boundary lines $p = \pm 1$ are violated. We show this by applying Proposition 9.4. In fact the generating function is simply given by

$$S(q,Q) = -|\gamma(q) - \gamma(Q)|,$$

because

$$\begin{aligned} -S_q(q,Q) &= -\frac{(\gamma(Q) - \gamma(q))}{|\gamma(Q) - \gamma(q)|} \cdot \dot\gamma(q) = -\cos\theta, \\ S_Q(q,Q) &= -\frac{(\gamma(Q) - \gamma(q))}{|\gamma(Q) - \gamma(q)|} \cdot \dot\gamma(Q) = \cos\Theta, \\ S_{Qq}(q,Q) &= \Theta_q \sin\Theta. \end{aligned}$$

Note that if $\Theta \in (0,\pi)$, then $\sin\Theta > 0$, and Θ is decreasing in q which means that $S_{Qq} < 0$. Here of course we are using the strict convexity. As for the boundary lines, we have $\Phi(q,-1) = (q,-1)$, $\Phi(q,1) = (q+1,1)$. Note that $S(q,Q)$ is defined for (q,Q) satisfying $Q \in [q, q+1]$. Also note that Φ has no fixed point inside $\mathbb{R} \times (-1,1)$. It is not hard to show that $\rho^- = \rho(\varphi^-) = 0$ and $\rho^+ = \rho(\varphi^+) = 1$. □

Exercises(i) Show that if f is an orientation preserving homeomorphism with $\rho(f) = 0$, then f has a fixed point.

(ii) Let $b : \mathbb{R} \to \mathbb{R}$ be a positive 1-periodic function and write ϕ_t for the flow of the ODE $\dot x = b(x)$. Find the rotation number of this ODE by evaluating the following limit:

$$\lim_{t\to\infty} t^{-1}\big(\phi_t(x) - x\big).$$

Also, find a strictly increasing function $K : \mathbb{R} \to \mathbb{R}$ such that

$$K \circ \phi_t \circ K^{-1},$$

is a free motion on $\mathbb{R}$.

(iii) Define $\tau_a b(x) = b(x+a)$, and write $\mathcal{B}$ for the set of uniformly positive Lipshitz function $b : \mathbb{R} \to \mathbb{R}$. Let $\mathbb{P}$ be a τ-invariant ergodic probability measure on $\mathcal{B}$. For each b, write $\phi_t(x; b)$ for the flow of the ODE $\dot{x} = b(x)$. Show that

$$\lim_{t\to\infty} t^{-1}\big(\phi_t(x; b) - x\big),$$

exists $\mathbb{P}$-almost surely, and evaluate the limit.

(iv) Verify (2.12). □

3 Hamilton-Jacobi Equation and Its Discrete Variant

We have discussed two types of generating functions. They have led to two types of action functionals. In Chapter 2 we learned how the critical points of the action functional yield the orbits of the corresponding dynamical system. In this chapter we focus on the critical values of the action functional. We also examine how the stochasticity can play a role. We may choose the generating function randomly according to a probability law, or add some noise to the dynamics. We first focus on Type I generating functions:

3.1 Frenkel-Kontorova Model

Imagine that we have a sequence of symplectic maps $\big(\Phi_i:\ i \in \mathbb{N}\big)$ such that each Φ_i has a Type I generating function $S^i(q,Q)$, so that

$$\Phi_i\big(q, S^i_q(q,Q), q\big) = \big(Q, S^i_Q(q,Q)\big).$$

We may define a dynamical system with orbits $(x_0, x_1, \dots, x_n, \dots)$ with the rule

$$x_n = \Phi_i(x_0), \qquad \text{or} \qquad x_n = \Phi_n \circ \cdots \circ \Phi_1(x_0).$$

If $\Phi_i = \Phi$ is independent of i, then we have an autonomous dynamical system with $x_n = \Phi^n(x_0)$. This dynamical system is equivalent to a second order dynamical system in q components. By this we mean that if $(x_n:\ n = 0, 1, \dots)$ is an orbit with $x_i = (q_i, p_i)$, then $(q_n:\ n = 0, 1, \dots)$ is an orbit of the dynamical system with the rule $q_n = F_n\big(q_{n-2}, q_{n-1}\big)$, where F_n is defined implicitly from

$$S^{n-1}_q\big(q_{n-2}, q_{n-1}\big) + S^n_Q\big(q_{n-1}, q_n\big) = 0. \tag{3.1}$$

Moreover, given q and Q, we can find an orbit $(q_0, \dots, q_n)$, with $q_0 = q, q_n = Q$, iff $\big(q_1, \dots, q_{n-1}\big)$ is a critical point of

$$\mathcal{S}^n\big(q_1, \dots, q_{n-1}; q, Q\big) = \sum_{i=1}^{n} S^i\big(q_{i-1}, q_i\big).$$

For the construction of invariant measures, we may consider the following variation: given a continuous function $g: \mathbb{R}^d \to \mathbb{R}$, consider

$$\mathcal{S}^n\big(q_0, q_1, \dots, q_{n-1}; g; Q\big) = g(q_0) + \mathcal{S}^n\big(q_1, \dots, q_{n-1}; q_0, Q\big).$$

Given q and Q, a critical point of $\mathcal{S}^n\big(q_0, q_1, \dots, q_{n-1}; g; Q\big)$ yields an orbit $(x_0, \dots, x_n)$ of our dynamical system with properties

$$p_0 = -S^1(q_0, q_1) = \nabla q(q_0), \qquad p_n = S^n(q_{n-1}, Q).$$

As we mentioned in Chapter 2, it is more convenient to write $S(q,Q) = L(q, Q-q)$, and in the case of q-periodic Hamiltonian, the function $L(q,v)$ is periodic in q. Because of examples we have in mind, it is quite natural to assume that

$$\liminf_{|v|\to\infty} \inf_q |v|^{-1} L(q,v) = \infty. \tag{3.2}$$

Note that this condition is satisfied for a standard map associated with $L(q,v) = |v|^2/2 - V(q)$, for a bounded C^1 function V. Assuming (3.2) is valid for each S^i, we define two operators

$$(\mathcal{T}_i g)(Q) = \inf_q \big(g(q) + S^i(q,Q)\big), \quad (\widehat{\mathcal{T}}_i g)(q) = \inf_Q \big(g(Q) - S^i(q,Q)\big), \tag{3.3}$$

on the space Λ of Lipschitz functions $g : \mathbb{R}^d \to \mathbb{R}$. Note that if $S(q,Q)$ is a generating function for Φ, then $S'(q,Q) = -S(Q,q)$ is a generating function for Φ^{-1}. More or less going from $\mathcal{T}$ to $\widehat{\mathcal{T}}$ is a matter of reversing the direction of time. (With some modifications, we can replace $\mathbb{R}^d$ with a Riemannian manifold M for what follows.) We will see later that $\mathcal{T}_i g \in \Lambda$ when $g \in \Lambda$. Observe

$$u_n(Q) := \big(\mathcal{T}_n \circ \cdots \circ \mathcal{T}_1\big)(g)(Q) = \inf_{q_0,\dots,q_{n-1}} \Big(g(q_0) + \mathcal{S}^n\big(q_1,\dots,q_{n-1};q_0,Q\big)\Big).$$

We regard

$$u_n = \mathcal{T}_n\big(u_{n-1}\big), \quad u_0 = g,$$

as a discrete variant of the (time inhomogeneous) HJE, where g is the initial data. Similarly,

$$u_{-n} = \widehat{\mathcal{T}}_n\big(u_{1-n}\big), \quad \hat{u}_0 = g,$$

is a discrete HJE with final condition $u_0 = g$. In particular, when $S^i = S$ is independent of i, we simply have $u_n = \mathcal{T}^n(g)$, and $u_n = \widehat{\mathcal{T}}^n(g)$, where

$$u(Q) := (\mathcal{T}g)(Q) = \inf_q \big(g(q) + S(q,Q)\big), \quad \hat{u}(q) := (\widehat{\mathcal{T}}g)(q) = \sup_Q \big(g(Q) - S(q,Q)\big). \tag{3.4}$$

Given Q, let us write $q = q(Q)$ for a minimizer in (3.4). If g is differentiable at q and u is differentiable at Q, then we have $\nabla g(q) + S_q(q,Q) = 0$, and if we write $A(q;Q) = g(q) + S(q,Q)$, then

$$\nabla u(Q) = A_q(q,Q)\ Dq(Q) + A_Q(q,Q) = S_Q(q,Q).$$

As a result,

$$\Phi\big(q, \nabla g(q)\big) = \big(Q, \nabla u(Q)\big).$$

In particular, if $\mathcal{T}(U) = U + c$ for a constant c, then $\nabla\mathcal{T}(U) = \nabla U$ and we learn that if

$$Gr(U) = \{(q, \nabla U(q)) : \ U \ \text{ differentiable at } \ q\},$$

then

$$\Phi^{-1}\big(Gr(U)\big) \subseteq Gr(U). \tag{3.5}$$

Similarly, given q, if we write $Q = Q(q)$ for a maximizer in (3.4), then

$$\Phi\big(q, \nabla\hat{u}(q)\big) = \big(Q, \nabla g(q)\big),$$

provided that g is differentiable at Q and $\hat{u}$ is differentiable at q. In particular, if we find $\hat{U}$ such that $\hat{\mathcal{T}}(\hat{U}) = \hat{U} + c'$, for a constant c', then $\nabla\hat{\mathcal{T}}(\hat{U}) = \nabla\hat{U}$, and

$$\Phi\big(Gr(\hat{U})\big) \subseteq Gr(\hat{U}). \tag{3.6}$$

This and (3.5) suggests that we should look for the HJE of the form

$$\mathcal{T}(U) = U + c, \qquad \hat{\mathcal{T}}(\hat{U}) = \hat{U} + c', \tag{3.7}$$

for suitable constants c and c', and use the solutions to construct invariant sets for Φ.

To have some concrete regularity estimate let us assume that $L(q, v) = S(q, q + v)$ has a super linear growth at infinity.

Assumption 3.1 There exists constants c_0, c_1 and $\delta > 0, \alpha > 1$ such that

$$\begin{aligned} &\inf_q L(q, v) \geq \delta|v|^\alpha - c_0, \qquad \sup_q L(q, 0) \leq c_1, \\ &\sup_q \sup_{|v| \leq \ell} |L(q + z, v) - L(q, v)| \leq c_2(\ell)|z|. \end{aligned} \tag{3.8}$$

Proposition 3.1 *Assume that* (3.8) *holds and that* $|g(q') - g(q)| \leq \ell|q' - q|$ *for all* q, q'. *Then*

$$(\mathcal{T}q)(Q) = \inf_{q : |Q-q| \leq \ell'} \big(g(q) + S(q, Q)\big), \qquad |u(Q') - u(Q)| \leq \ell''|Q' - Q|, \tag{3.9}$$

for $\ell' = c_0 + c_1 + \big(\delta^{-1}(\ell + 1)\big)^{\frac{1}{\alpha - 1}}$, *and* $\ell'' = \ell + c_2(\ell')$.

Proof Observe

$$g(q) + S(q, Q) \geq g(Q) - \ell|Q - q| + \delta|Q - q|^\alpha - c_0.$$

Hence

$$g(Q) + S(Q, Q) \leq g(q) + S(q, Q),$$

if $c_0 + c_1 \le \delta|v|^\alpha - \ell|v|$, for $v = Q - q$. Note $\delta|v|^\alpha - \ell|v| \ge |v|$ if $|v| \ge \big(\delta^{-1}(\ell+1)\big)^{\frac{1}{\alpha-1}}$. This implies the identity in (3.9).

If $u(Q) = g(q) + L(q, Q-q)$ for some Q with $|Q - q| \le \ell'$, then for $q' = q + Q' - Q$,

$$\begin{aligned} u(Q') &\le g(q') + L(q', Q-q) \le g(q) + L(q, Q-q) + \ell|Q' - Q| + c_2(\ell')|Q' - Q| \\ &= u(Q) + \big(\ell + c_2(\ell')\big)|Q' - Q|, \end{aligned}$$

as desired. □

3.2 Type II and III Generating Functions

If we consider a symplectic map with generating function $W(Q,p) = Q \cdot p - w(Q,p)$, then a candidate for the action is

$$A(q,p;Q) = A(x;Q) = g(q) + W(Q,p) - q \cdot p = g(q) + (Q-q)\cdot p - w(Q,p).$$

Given Q, at any critical point $x = x(Q) = (q,p)$ of A we have

$$\begin{aligned} &0 = A_q(q,p;Q) = \nabla g(q) - p, \qquad 0 = A_p(q,p;Q) = W_p(Q,p) - q, \\ &A_Q(q,p;Q) = A_x(x;Q)(Dx)(Q) + W_Q(Q,p) = W_Q(Q,p). \end{aligned}$$

This means that for $u(Q) = A(x;Q)$ at the critical point,

$$\Phi\big(q, \nabla g(q)\big) = \big(Q, \nabla u(Q)\big),$$

provided that g is differentiable at q and u is differentiable at Q. In the case of Type I generating function, we simply take the minimum of the action because the action is bounded blow. This is no longer the case for Type II generating function. For example if Φ is a symplectic map, then w is periodic, and if g is also periodic, then A is a periodic perturbation of the quadratic function $A^0(x;Q); = (Q-q)\cdot p$. Hence A is neither is bounded from below nor above. The best we can hope for that given Q, the function $A(x;Q)$ has a critical point which is of the same type as the type 0 is for $A^0(x;Q)$. Now imagin that we come up with a universal way of selecting a critical value of A no matter what g is. This critical value yields an operator

$$\mathcal{V}(g)(Q) = A(x;Q) = A\big(x(Q);Q\big),$$

where $x(Q)$ is our selected critical point. If we can find a function U such that $\mathcal{V}(U) = U + c$, for a constant c, then $\Phi^{-1}\big(Gr(U)\big) \subseteq Gr(U)$.

More generally, assume that $\Phi = \Phi_k \circ \cdots \circ \Phi_1$ and each Φ^i has a generating function $W^i(q_i, p_{i-1}) = q_i \cdot p_{i-1} - w^i(q_i, p_{i-1})$. Then Φ has a generalized generating function of the form

$$\begin{aligned} W(q_k, p_0; \xi) &= W\big(q_k, p_0; q_1, p_1, \dots, q_{k-1}, p_{k-1}\big) \\ &= q_1 \cdot p_0 + \sum_{i=2}^{k} p_{i-1} \cdot (q_i - q_{i-1}) - \sum_{i=1}^{k} w^i(p_{i-1}, q_i). \end{aligned}$$

Recall

$$W_\xi(q_k, p_0; \xi) = 0 \quad \Longrightarrow \quad \Phi\big(W_{p_0}(q_k, p_0; \xi), p_0\big) = \big(q_k, W_{q_k}(q_k, p_0; \xi), p_0\big)$$

Given an initial data g, we set

$$\begin{aligned} A(\xi'; q_k) = A\big(q_1, p_1, \dots, q_{k-1}, p_{k-1}; q_k\big) &= g(q_0) - p_0 \cdot q_0 + W(q_k, p_0; \xi) \\ &= g(q_0) + \sum_{i=1}^{k} \big(p_{i-1} \cdot (q_i - q_{i-1}) - w^i(p_{i-1}, q_i)\big). \end{aligned}$$

We then have

$$A_{\xi'}(q_k; \xi') = 0 \quad \Longrightarrow \quad p_0 = \nabla g(q_0), \quad \Phi\big(q_0, p_0\big) = \big(q_k, \nabla u_k(q_k)\big),$$

where $u_k(q_k) = A(q_k; \xi'(q_k))$, is the value of the action at the critical point $\xi'(q_k)$. If we set $\mathcal{V}_k(g)(q_k)$ for this critical value, and U is chosen so that $\mathcal{V}_k(U) = U + c_k$, for a constant c_k, then we have $\Phi^{-1}\big(Gr(U)\big) \subseteq Gr(U)$.

Observe that if $W'(q, P)$ is a Type II generating function for Φ^{-1}, then it is a Type III generating function for Φ. Motivated by this, let us choose a generating function $V(q, P) = q \cdot P - v(q, P)$ of type III, so that

$$\Phi\big(q, V_q(q, P)\big) = \big(V_P(q, P), P\big).$$

Again

$$A(Q, P; q) = A(X; q) = g(Q) - Q \cdot P + V(q, P) = g(Q) + (q - Q) \cdot P - v(q, P),$$

is the action, and at a critical point $A_X(X; q) = 0$, we have

$$P = \nabla g(Q), \qquad Q = V_P(q, P), \qquad \nabla \hat{u}(q) = p,$$

where

$$\hat{u}(q) = \widehat{\mathcal{V}}(g)(q) = A(X(q); q),$$

is the corresponding critical value. Again if for some $\hat{U}$, we have that $\widehat{\mathcal{V}}(\hat{U}) = \hat{U} + c$, for a constant c, we learn that $\Phi\big(Gr(\hat{U})\big) \subseteq Gr(\hat{U})$.

3.3 Gibbs Measures

There is a viscous variant of the discrete HJE that is related to orbits (or rather realizations) of a Markov chain. Instead of minimizing $\mathcal{S}^n$, we define a probability measure on M^{n-1} that favors states $\mathbf{q}^n = (q_1, \dots, q_{n-1})$ of lower *energy* $\mathcal{S}^n$. More precisely, we define a *Gibbs measure* $\mathbb{P}_n(\cdot; q, Q)$ on M^{n-1} as

$$\mathbb{P}(d\mathbf{q}^n) = Z_n(q, Q)^{-1} \exp\big(- \beta \mathcal{S}^n(\mathbf{q}^n; q, Q)\big) \prod_{i=1}^{n-1} \nu(dq_i),$$

where β is a positive scalar, $\nu(dq)$ is a *reference measure* (for example a volume form associated with a metric when M is a Riemannian manifold), and Z is the normalizing constant:

$$Z_n(q, Q) = \int_{M^{n-1}} \exp\big(- \beta \mathcal{S}^n(\mathbf{q}^n; q, Q)\big) \prod_{i=1}^{n-1} \nu(dq_i).$$

For simplicity, let us assume that $S^i = S$ for all i. Now, if we attempt to normalize our measure inductively, we need to evaluate $n = 2$, we need to calculate

$$Z(q_{n-2}, Q) := \int_M \exp\big(- \beta S(q_{n-2}, q_{n-1}) - \beta S(q_{n-1}, Q)\big)\ \nu(dq_{n-1}),$$

which depends on q_{n_2}. Dividing the integrand by $Z(q_{n-2}, Q)$ would alter S. To avoid this, observe that if we replace $S(q, Q)$ with $S(q, Q) + u(Q) - u(q)$, then the corresponding Gibbs measure would not be affected (it only changes the normalizing constant). Motivated by this, we define

$$\mathcal{R}_\beta(h)(g)(Q) = \int_M e^{-\beta S(q,Q)} h(Q)\ \nu(dQ), \qquad \mathcal{R}^*_\beta(h)(g)(Q) = \int_M e^{-\beta S(q,Q)} h(q)\ \nu(dq).$$

Note that with respect to the inner product

$$\langle h, k\rangle = \int_M hk\ d\nu,$$

we can readily show that $\mathcal{R}^*_\beta$ is the adjoint of $\mathcal{R}_\beta$. A generalization of Perron-Frobenius Theorem offers a way of modifying S so that we can normalize our measure inductively: For simplicity, let us assume that $M = \mathbb{R}^d$, ν is the Lebesgue measure and that (2.3) holds.

Theorem 3.1 *The largest eigenvalue $\lambda'_\beta = e^{\beta\lambda_\beta}$ of $\mathcal{R}_\beta$ is positive and λ'_β satisfies $\lambda'_\beta \geq |\lambda'$ for any other eigenvalue λ'. Moreover λ'_β is simple, and there exist functions $u_\beta, u^*_\beta : M \to \mathbb{R}$ such that*

$$\mathcal{R}_\beta\big(e^{\beta u_\beta}\big) = e^{\beta\lambda_\beta} e^{\beta u_\beta}, \qquad \mathcal{R}^*_\beta\big(e^{-\beta u^*_\beta}\big) = e^{\beta\lambda_\beta} e^{-\beta u^*_\beta}.$$

Motivated by Theorem 3.1, we set

$$\hat{S}(q,Q) := S(q,Q) - (u_\beta(Q) - u_\beta(q)) + \lambda_\beta, \qquad p(q,dQ) = \exp\big(-\beta\hat{S}(q,Q)\big)\ \nu(dQ).$$

By Theorem 3.1, the *kernel* $p(q,dQ)$ is a probability measure for each q. Using this kernel, we may define a Markov chain $\mathbf{q} = (q_0, q_1, \dots, q_n, \dots)$ such that

$$\mathbb{P}^q\big(q_n \in A \mid q_0, \dots, q_{n-1}\big) = \int_A p(q_{n-1}, dq_n), \qquad q_0 = q,$$

for every measurable set $A \subseteq M$. Here $\mathbb{P}^q$ is a probability measure on the set of sequences $\mathbf{q}$ with $q_0 = q$. Hence

$$\begin{aligned}\mathbb{P}^q\big(q_1 \in A_1, \dots, q_n \in A_n\big) &= \int_{A_1} \cdots \int_{A_n} \prod_{i=1}^n p(q_{i-1}, dq_i)\\ &= \int_{A_1} \cdots \int_{A_n} \exp\left(-\sum_{i=1}^n \beta\hat{S}(q_{i-1}, q_i)\right)\ \prod_{i=1}^n \nu(dq_i).\end{aligned}$$

Writing $\mathbb{P}_n^q(dq_1, \dots, dq_n)$ for the n-dimensional marginal of $\mathbb{P}^q$, we deduce

$$\mathbb{P}_n\big(dq_1, \dots, dq_{n-1}; q, Q\big) = \mathbb{P}_n^q\big(dq_1, \dots, dq_n \mid q_n = Q\big).$$

Also, if we define

$$\hat{\mathcal{T}}_\beta(g) = \beta^{-1}\log \mathcal{R}_\beta\big(e^{\beta g}\big),$$

then

$$u_n = \hat{\mathcal{T}}_\beta(u_{n-1}),$$

is a discrete analog of viscous HJE. Note that

$$\lim_{\beta\to\infty} \hat{\mathcal{T}}_\beta(g) = \hat{\mathcal{T}}(g).$$

In the same vein, we set

$$\mathcal{T}_\beta(g) = -\beta^{-1}\log \mathcal{R}_\beta^*\big(e^{-\beta g}\big),$$

then

$$u_n = \mathcal{T}_\beta(u_{n-1}),$$

is a discrete analog of viscous backward HJE. Note that

$$\lim_{\beta\to\infty} \mathcal{T}_\beta(g) = \mathcal{T}(g).$$

Moreover, the eigenfunctions $e^{\beta u_\beta}$, and $e^{-\beta u^*_\beta}$, can be used to find an invariant measure for our Markov Chain. For this, observe that if we have an invariant measure of the form $\mu(dq) = Z^{-1}e^h\ dq$, then we must have

$$e^{h(Q)} = \int e^{h(q)}p(q, dQ) = e^{\beta(u_\beta(Q)-\lambda_\beta)}R^*_\beta\left(e^{h-\beta u_\beta}\right)(Q)$$

This means $e^{h-\beta u_\beta} = e^{-u^*_\beta}$. Hence for an invariant measure, we may choose a measure of the form

$$\mu(dq) = Z^{-1}e^{\beta(u_\beta - u^*_\beta)(q)}\ dq.$$

We note

$$\hat{\mathcal{T}}(u_\beta) = u_\beta + \lambda_\beta, \quad \mathcal{T}(u^*_\beta) = u^*_\beta - \lambda_\beta, \tag{3.10}$$

which is in line with (3.5) and (3.4) as we solved the corresponding HJE (3.7).

4 Homogenization

Let us write $\mathcal{L}$ for the set of maps $S : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that the map $L(q,v) = S(q, q+v)$ satisfies Assumption 3.1. We also write Ω for the set of C^1 functions $F : \mathbb{R}^d \to \mathbb{R}^d$ such that $G(q) = F(q) - q$ is bounded. For the question of homogenization, we define an operator that turns a microscopic height function $g : \mathbb{R}^d \to \mathbb{R}$ to a macroscopic height function. Its inverse does the opposite:

$$(\Gamma_n g)(q) = n^{-1} g(nq), \qquad \big(\Gamma_n^{-1} g)(q) = n g\big(n^{-1} q\big).$$

We think of g as an initial macroscopic height function. Its growth is governed microscopically by the operator $\mathcal{T}$ or $\widehat{\mathcal{T}}$. The macroscopic height function after one macroscopic time step is given by

$$u_n = \widehat{\mathcal{T}}^n_{\Gamma_n(S)}(g) = \left(\mathcal{G}_n \circ \widehat{\mathcal{T}}^n_S \circ \Gamma_n^{-1}\right)(g).$$

A homogenization occurs if the limit

$$\lim_{n\to\infty} u_n(q),$$

exists for every Lipschitz function g. We may write

$$\begin{aligned} u_n &= \sup_{q_1,\dots,q_n} \left[g\big(n^{-1}q_n\big) - n^{-1}\big(S(nq,q_1) + S(q_1,q_2) + \cdots + S(q_{n-1},q_n)\big)\right] \\ &= \sup_Q \left[g(Q) - S_n(q,Q)\right], \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} S_n(q,Q) &= \inf_{q_1,\dots,q_{n-1}} n^{-1}\big(S(nq,q_1) + S(q_1,q_2) + \cdots + S(q_{n-1},nQ)\big) \\ &= \inf_{q_1,\dots,q_{n-1}} \big((\Gamma_n S)(q,q_1) + (\Gamma_n S)(q_1,q_2) + \cdots + (\Gamma_n S)(q_{n-1},Q)\big). \end{aligned}$$

One approach for establishing the homogenization is based on the following intuition that we partially discussed in Chapter 3: If for some C^1 $U \in \Lambda$, we have $\widehat{\mathcal{T}}(\hat{U}) = \hat{U} + c$, then $\Phi\big(q, \nabla\hat{U}(q)\big) = \big(Q, \nabla\hat{U}(Q)\big)$. The relationship between q and $Q = F(q)$ is that Q is a critical point of $A(Q;q) = \hat{U}(Q) - S(q,Q)$. So, $F(q)$ is implicitly given by

$$\nabla\hat{U}(F(q)) = S_Q(q, F(q)). \tag{4.2}$$

Hence for such U, the set $Gr(\hat{U})$ is invariant for Φ. Moreover, the q-component of the flow associated with the restriction of Φ to the set $Gr(\hat{U})$ is given by $F : \mathbb{R}^d \to \mathbb{R}^d$.

The homogenization maybe achieved in three steps that we now sketch:

Step 1 *(Lower Bound)* Motived by the flow F of (4.2), we pick any $F \in \Omega$ with $F(q) = q + G(q)$. We select $q_i = F^i(q_0)$ with $q_0 = nq$ in (4.1). Note

$$n^{-1}q_n = q + n^{-1}\sum_{i=0}^{n-1} G\big(F^i(q_0)\big), \qquad \sum_{i=0}^{n-1} S(q_i, q_{i+1}) = \sum_{i=0}^{n-1} S^F\big(F^i(q_0)\big),$$

where $S^F(q) = S(q, F(q)) = L(q, G(q))$. We certainly have

$$u_n(q) \geq g\left(q + n^{-1}\sum_{i=0}^{n-1} G\big(F^i(q_0)\big)\right) - n^{-1}\sum_{i=0}^{n-1} S^G\big(F^i(q_0)\big). \tag{4.3}$$

We wish to find the limit of the right-hand side of (4.3). For example, when $L(q, v)$ is periodic in q, we choose F to be a lift of a map $f : \mathbb{T}^d \to \mathbb{T}^d$. Then G is also periodic, which implies that S^F is periodic. Now if we pick any ergodic invariant measure for F, then we have

$$\lim_{n\to\infty} n^{-1}\sum_{i=0}^{n-1} G\big(F^i(q_0)\big) = \int G\ d\mu, \qquad \lim_{n\to\infty} n^{-1}\sum_{i=0}^{n-1} S^F\big(F^i(q_0)\big) = \int L^G\ d\mu, \tag{4.4}$$

almost surely for μ almost choices of q_0. From this we obtain

$$\liminf_{n\to\infty} u_n \geq g\left(q + \int G\ d\mu\right) - \int L^G\ d\mu.$$

This being true for any such pair (F, μ), we deduce

$$\liminf_{n\to\infty} u_n \geq \sup_{(F,\mu)} \left[g\left(q + \int G\ d\mu\right) - \int S^F\ d\mu\right] = \sup_v \big[g(q+v) - \hat{L}(v)\big], \tag{4.5}$$

where the first supremum is over the pair (F, μ) such that μ is an ergodic invariant measure for that map F, and

$$\hat{L}(v) = \inf_{(F,\mu)} \left\{\int S(q, F(q))\ \mu(dq) :\ \int (F(q) - q)\ \mu(dq) = v\right\}. \tag{4.6}$$

Step 2 *(Upper Bound)* Given any $p \in \mathbb{R}^d$ and any continuous function $u : \mathbb{R}^d \to \mathbb{R}$, we define

$$\bar{H}(p; u) = \sup_{q,Q} \big(u(Q) - u(q) + p \cdot (Q - q) - S(q, Q)\big).$$

We write $\mathcal{U}$ for the set continuous $u : \mathbb{R}^d \to \mathbb{R}$, such that

$$\lim_{|q|\to\infty} |u|^{-1}u(q) = 0.$$

We then use any $u \in \mathcal{U}$ to produce an upper bound for u_n:

$$
\begin{aligned}
u_n(q) &\le \sup_{q_1,\dots,q_n} \big[g\big(n^{-1}q_n\big) - \big(n^{-1}q_n - q\big)\cdot p - n^{-1}\big(u(q_n) - u(nq)\big)\big] + H(p;u) \\
&= \sup_Q \big(g(Q) - (Q-q)\cdot p - n^{-1}\big(u(nQ) - u(nq)\big)\big) + H(p;u).
\end{aligned}
$$

As a result,

$$
\limsup_{n\to\infty} u_n \le \inf_p \inf_{u\in\mathcal{U}} \left[\sup_Q \big(g(Q) - (Q-q)\cdot p - n^{-1}\big(u(nQ) - u(nq)\big)\big) + \bar{H}(p;u)\right]. \tag{4.7}
$$

If we can interchange inf with sup, we obtain

$$
\begin{aligned}
\limsup_{n\to\infty} u_n &\le \sup_Q \left[g(Q) - \inf_p \left((Q-q)\cdot p - \inf_{u\in\mathcal{U}} \bar{H}(p;u)\right)\right] \\
&= \sup_Q \big[g(Q) - \bar{L}(Q-q)\big),
\end{aligned} \tag{4.8}
$$

where

$$
\bar{L}(v) = \sup_p \big(p\cdot v - \bar{H}(p)\big), \qquad \bar{H}(p) = \inf_{u\in\mathcal{U}} \bar{H}(p;u).
$$

Step 3 ($\hat{L} = \bar{L}$) To establish homogenization, it remains to show that the upper and lower limits of Steps 1 and 2 coincide. This may be achieved by an introduction of a *Lagrange multiplier*, and an application of *Minimax Principle*. Indeed, if we write $\hat{H}$ for the Legendre Transform of $\hat{L}$:

$$
\hat{H}(p) := \sup_v \big(p\cdot v - \hat{L}(v)\big),
$$

then we can write

$$
\begin{aligned}
\hat{H}(p) &= \sup_{(F,\mu)} \left(\int \big((F(q)-q)\cdot p - S(q,F(q))\big)\mu(dq)\right) \\
&= \sup_F \sup_\mu \inf_{u\in C_b} \left(\int \big((F(q)-q)\cdot p - S(q,F(q))\big)\mu(dq) + \int \big(u(F(q)) - u(q)\big)\ \mu(dq)\right) \\
&= \inf_{u\in C_b} \sup_F \sup_\mu \left(\int \big((F(q)-q)\cdot p - S(q,F(q))\big)\mu(dq) + \int \big(u(F(q)) - u(q)\big)\ \mu(dq)\right) \\
&= \inf_{u\in C_b} \sup_F \sup_q \big((F(q)-q)\cdot p - S(q,F(q)) + u(F(q)) - u(q)\ \big) \\
&= \inf_{u\in C_b} \sup_Q \sup_q \big((Q-q)\cdot p - S(q,Q) + u(Q) - u(q)\big) = \bar{H}(p).
\end{aligned}
$$

Here $C_b = C_b(\mathbb{R}^d)$, is the space of bounded continuous functions. $\square$

Motivated by the above discussion, we now state our homogenization result. We start with the periodic case.

Theorem 4.1 *Let $S(q,Q) = L(q, Q-q)$ with $L(q,v)$ periodic in q and satisfying (**??**). Then for every Lipschitz function g, we have*

$$\lim_{n\to\infty} \Gamma_n \circ \widehat{\mathcal{T}}^n \circ \Gamma_n^{-1}(g) = \sup_Q \big(g(Q) - \bar{L}(Q-q)\big). \tag{4.9}$$

Here $\bar{L}$ the Legendre transform of $\bar{H}$, given by

$$\bar{H}(p) = \inf_u \sup_{q,Q} \big(p\cdot(Q-q) + u(Q) - u(q) - S(q,Q)\big), \tag{4.10}$$

with the infimum over periodic continuous functions $u : \mathbb{R}^d \to \mathbb{R}$.

We may also establish a homogenization when S is selected randomly according to an ergodic τ-invariant measure. Before we state the main result, we remark that in the formula (4.1), we compared $S(q,Q)$ with

$$w(q,Q) = p\cdot(Q-q) + u(Q) - u(q),$$

which should be regarded as a discrete analog of a 1-form. Think of $w(q,Q) = w'(q, Q-q)$ as a function that acts on velocities $Q-q$ at the base point q. Note that our $w'(q,v)$ is periodic in q, but not linear in $Q-q$ because it is not defined on the tangent fiber at q as in the continuous setting. Though it is a discrete 1-form because we can "integrate" it over $\mathbb{T}^d$ sequences $\mathbf{q} = (q_0, \dots, q_n)$:

$$w(\mathbf{q}) := \sum_{i=0}^{n-1} w(q_i, q_{i+1}).$$

Now if $u : \mathbb{R}^d \to \mathbb{R}$ is periodic, then the form $u(q,Q) := u(Q) - u(q)$ is an example of an exact form because $u(\mathbf{q}) = 0$ for any periodic sequence (whenever $q_n - q_0 \in \mathbb{Z}^d$.) However the form $\bar{p}(q,Q) = p\cdot(Q-q)$ is not exact but closed because $\bar{p}(\mathbf{q})$ depends only on the end points q_0 and q_n.

In the random case, the torus $\mathbb{T}^d$ is replaced with the space $\mathcal{S}$.

Definition 4.1(i) Given $\delta, c_0, c_1, c_2, c_3 > 0$, and $\alpha, \beta > 1$, we write $\mathcal{S}$ for the set of continuous $S : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that

$$\begin{aligned}
&\inf_q S(q, q+v) \ge \delta|v|^\alpha - c_0, \qquad \sup_q S(q,q) \le c_1,\\
&\sup_{q,Q} |S(q+z, Q+z) - S(q,Q)| \le c_2|z|,\\
&\sup_{q,Q} |S(q, Q+z) - S(q,Q)| \le c_3|z||Q-q+z|^{\beta-1}.
\end{aligned}$$

(ii) For measurable functions $u : \mathbb{R}^d \to \mathbb{R}$, $L : \mathcal{S} \times \mathbb{R}^d \to \mathbb{R}$, $S : \mathbb{R}^d \times \mathbb{R}^d$ and $a \in \mathbb{R}^d$, we set

$$\tau_a u(q) = u(q+a), \qquad \tau_a L(q,v) = L(q+a,v), \qquad (\tau_a' S)(q,Q) = S(q+a,Q+a).$$

□

We have a τ'-invariant probability measure on $\mathcal{S}$. If for example $S(q,Q) = L(q,Q-q)$, with $L(q,v)$ periodic in q, then we take a probability measure that is concentrated on $\{\tau_q S : \ q \in \mathbb{R}^d\}$. This set is closed with respect to the uniform topology, and topologically homeomorphic to $\mathbb{T}^d$. In fact the only τ'-invariant measure on this set is isomorphic to the Lebesgue measure on $\mathbb{T}^d$. Though if we take a quasi-periodic S, and consider the set $\{\tau_q' S : \ q \in \mathbb{R}^d\}$, it is no longer closed and its closure would be homeomorphic to $\mathbb{T}^{dN}$ for some $N \in \mathbb{N}$. Given $S \in \mathcal{S}$, we may define the operator

$$\widehat{\mathcal{T}}(g) = \widehat{\mathcal{T}}(g;S) = \sup_Q \big(g(Q) - S(q,Q)\big),$$

as before. Recall that our homogenization proof was based on the existence of a function $u : \mathbb{R}^d \to \mathbb{R}$, and $F : \mathbb{R}^d \to \mathbb{R}^d$, with $F(q) = G(q) + q$, such that

$$\sup_Q \big(p \cdot (Q-q) + u(Q) - u(q) - S(q,Q)\big) = \bar{H}(p),$$

$$S_q(q,F(q)) + \nabla u(q) + p = S_q\big(q, G(q)+q\big) + \nabla u(q) + p = 0. \tag{4.11}$$

We note that if we can replace S with $\tau_a' S$, u with $\tau_a u$, and G with $\tau_a G$, our equations in(4.11) are still valid. This suggests finding $u(q) = u(q;S)$ and $G(q) = G(q;S)$, such that

$$u(q;\tau_a S) = u(q+a;S), \qquad G(q;\tau_a S) = G(q+a;S).$$

In fact, if we define $\hat{u}(S) = u(0;S)$, then $u(q,S) = \hat{u}\big(\tau_q' S\big)$. Equivalently, we may look for functions

$$\hat{u} : \mathcal{S} \to \mathbb{R}, \qquad \hat{G} : \mathcal{S} \to \mathbb{R}^d,$$

such that $u(q) = \hat{u}\big(\tau_q S\big)$ and $G(q) = \hat{G}\big(\tau_q S\big)$ satisfy (4.11). Given $p \in \mathbb{R}^d$, we may wonder whether or not there exists a continuous $\hat{u} : \mathcal{S} \to \mathbb{R}$, and a constant $\bar{H}(p)$ such that

$$\sup_Q \big(p \cdot (Q-q) + \hat{u}\big(\tau_Q' S\big) - \hat{u}\big(\tau_q' S\big) - S(q,Q)\big) = \bar{H}(p). \tag{4.12}$$

It turns out that (4.12) does not have a solution if we go beyond the periodic case. Instead, we need to consider functions of the form $u(q;S)$ that are in some sense acts like an *exact form* in the discrete setting.

Definition 4.2 A measurable function $u : \mathbb{R}^d \times \mathcal{S} \to \mathbb{R}$ is an *exact form* if the following conditions hold:

(i) $u\big(Q-q;\tau'_q S\big) = u(Q;S) - u(q;S)$, for all $(q,Q,S) \in \mathbb{R}^d \times \mathbb{R}^\times \mathcal{S}$.

(ii) $u(q;S)$ is Lipschitz continuous in q.

(iii) For some $r > d$, and every $q \in \mathbb{R}^d$, $\int |u(q;S)|^r \ \mathbb{P}(dS) < \infty$.

(iv) $u(0;S) = 0$, and for every q, we have $\int u(q;S) \ \mathbb{P}(dS) = 0$.

We write $\mathcal{U}$ for the set of exact forms. □

Remark 4.1 Note that the first and the last properties are satisfied if $u(q,S) = h\big(\tau'_q S\big) - h(S)$ for some function $h : \mathbb{S} \to \mathbb{R}$. We also note that (i) is equivalent to the following property: If $q_0, \dots, q_k$ is any sequence with $q_0 = q_k$, then

$$\sum_{i=0}^{k} u\big(q_{i+1} - q_i; \tau'_{q_i} S\big) = 0.$$

□

Theorem 4.2 *Let $\mathbb{P}$ be an ergodic τ'-invariant measure on the set $\mathcal{S}$. Then for every Lipschitz function g,* (4.9) *holds for $\bar{L}$, the Legendre transform of $\bar{H}$. The function $\bar{H}$ is given by*

$$\bar{H}(p) = \inf_{u \in \mathcal{U}} \sup_q ess \sup_S \big(p \cdot q + u(q;S) - S(0,q)\big). \tag{4.13}$$

As a preparation, we establish a variant of Proposition 3.1 for u_n.

Proposition 4.1 *Assume that g is Lipschitz with Lipschitz constant ℓ. Then we can restrict the supremum in* (4.1) *to those Q such that*

$$|Q-q| \le \ell', \qquad 0 \le S_n(q,Q) + c_0 \le c'_0, \tag{4.14}$$

with ℓ' is as in Proposition 3.1 and $c'_0 = \delta^{1/(1-\alpha)} + (1-\alpha^{-1})\, c_0$.

Proof Note that by (3.8),

$$\begin{aligned} S_n(q,Q) &\ge \delta n^{-1} \left(|nq - q_1|^\alpha + \cdots + |q_{n-1} - nQ|^\alpha\right) - c_0 \\ &\ge \delta \left| n^{-1}\big((nq - q_1) + \cdots + (q_{n-1} - nQ)\big)\right|^\alpha - c_0 \\ &= \delta |q - Q|^\alpha - c_0. \end{aligned}$$

This allows to repeat the proof of Proposition 3.1 and deduce the first inequality in (**??**). On the other hand, if the supremum in (4.1) is attained in Q, then

$$\begin{aligned} 0 &\le g(Q) - g(q) - S_n(q,Q) \le \ell|Q-q|^\alpha - S_n(q,Q) \\ &\le \delta^{-1/\alpha}\big(S_n(q,Q)+c_0\big)^{1/\alpha} - S_n(q,Q) \\ &\le \big(1-\alpha^{-1}\big)\,\delta^{1/(1-\alpha)} + \alpha^{-1}\big(S_n(q,Q)+c_0\big) - S_n(q,Q) \end{aligned}$$

This implies the second bound in (4.14). □

The main ingredients for the proof of Theorem 4.1 are the following existence and regularity of exact forms and an application of Subadditive Ergodic Theorem.

Theorem 4.3 *For every $u \in \mathcal{U}$,*

$$\lim_{|q|\to\infty} |q|^{-1} u(q;S) = 0, \tag{4.15}$$

$\mathbb{P}$-almost surely.

Theorem 4.4 *For every p, there exist a constant $\hat{H}(p)$, and $u \in \mathcal{U}$ such that*

$$\sup_q \big(p\cdot q + u(q,S) - S(0,q)\big) = \hat{H}(p). \tag{4.16}$$

Theorem 4.5 *For each v, the limit*

$$\hat{L}(v,S) = \lim_{n\to\infty} n^{-1} S_n(0,v), \tag{4.17}$$

exists $\mathbb{P}$-almost surely. Moreover $\hat{L}(v) = \hat{L}(v,S)$ is independent of S, and convex in v.

Proof Observe that if

$$S_n'(q,Q) = \inf_{q_1,\dots,q_{n-1}} \sum_{i=0}^{n-1} S(q_i,q_{i+1}),$$

with $q_0 = q, q_n = Q$, then we have the following subadditivity:

$$S_{m+n}'(a,c) \le S_m'(a,b) + S_n'(b,c).$$

As a result, if we pick $v \in \mathbb{R}^d$, and set

$$T = \tau_v', \qquad F_n(S) = S_n'(0,nv),$$

then

$$F_{m+n}(S) \le F_m(S) + F_n\big(T^m S\big).$$

Note that $\mathbb{P}$ is T-invariant but may not be ergodic. Nonetheless we may apply *Kingman Subadditive Ergodic Theorem* to assert that the limit

$$\hat{L}(v,S) := \lim_{n\to\infty} n^{-1} S_n'(0,nv), \tag{4.18}$$

exists $\mathbb{P}$-almost surely, and that

$$\int \hat{L}(v,S)\ \mathbb{P}(dS) = \inf_n \int n^{-1} S_n'(0,nv)\ d\mathbb{P}.$$

Moreover, using the ergodicity of $\mathbb{P}$ and the subadditivity of S', we can show that $\bar{L}$ is independent of S and convex in v. □

Proof of Theorem 4.3 *(Step 1)* To ease the notation, let us write

$$S^p(q,Q) = S(q,Q) - p\cdot(Q-q).$$

We note that if $S \in \mathcal{S}$, with constants $\delta, c_0, c_1, c_2, c_3$, then then $S^p \in \mathcal{S}$ for constants $\delta', c_0', c_1', c_2', c_3'$. ($\alpha, \beta$ will not change.) For example

$$S^p(q,Q) \geq p\cdot(Q-q) + \delta|Q-q|^\alpha - c_0 \geq -|Q-q||p| + \delta'|Q-q|^\alpha - c_0' \geq -c_1 - c_0,$$

where δ' can be chosen to be $\delta/2$, and $c_0' = c_0 + c_0''|p|^{\frac{\alpha}{\alpha-1}}$ for a constant $c_0'' = c_0''(\alpha,\delta)$. From now on we assume that $p = 0$.

Pick $\lambda \in (0,1)$, and define $h^\lambda : \mathcal{S} \to \mathbb{R}$, by

$$h^\lambda(S) = -\inf_{q_1,q_2,\dots} \sum_{n=0}^{\infty} S(q_n, q_{n+1})\lambda^n,$$

with $q_0 = 0$. We then have

$$h^\lambda\big(\tau_q' S\big) = -\inf_{q_1,q_2,\dots} \sum_{n=0}^{\infty} S(q_n, q_{n+1})\lambda^n,$$

with $q_0 = q$. Moreover,

$$\sup_q \big(\lambda h^\lambda(\tau_q' S) - S(0,q)\big) = h^\lambda(S).$$

Equivalently,

$$\sup_q \big(\lambda h^\lambda(\tau_q' S) - \lambda h^\lambda(S) - S(0,q)\big) = (1-\lambda) h^\lambda(S).$$

Hence if $u^\lambda(q;S) = h^\lambda(\tau_q' S) - h^\lambda(S)$, then

$$\sup_Q \big(\lambda u^\lambda(q;S) - S(0,q)\big) = (1-\lambda) h^\lambda(S). \tag{4.19}$$

We claim that for a subsequence, the limits

$$u(q;S)=\lim_{\lambda\to 1}u^{\lambda}(q;S),\qquad \hat{H}(p;S)=\lim_{\lambda\to 1}(1-\lambda)h^{\lambda}(S), \tag{4.20}$$

exist and they are the desired u and $\hat{H}$ we are searching for. We note that if the limits in (4.28) exit, then

$$\begin{aligned} &u\big(Q-q;\tau_q'S\big)=u(Q;S)-u(q;S),\\ &\hat{H}\big(p;\tau_q'S\big)-\hat{H}(p;S)=\lim_{\lambda\to 1}(1-\lambda)u^{\lambda}(q;S)=0. \end{aligned} \tag{4.21}$$

From the latter and the ergodicity of $\mathbb{P}$, we deduce that $\hat{H}$ is independent of S. The former implies that Property (i) of an exact form is satisfied.

(Step 2) Evidently $(1-\lambda)h^{\lambda}(S)\ge -S(0,0)\ge -c_1$, by choosing $0=q_0=q_1=\dots$ in the definition of h^{λ}. From this and $S(q,Q)\ge -c_0$ we deduce

$$-c_1\le(1-\lambda)h^{\lambda}(S)\le c_0. \tag{4.22}$$

We now examine u^{λ}. From (4.27) and (4.29) we learn

$$\lambda u^{\lambda}(q;S)\le S(0,q)+c_0.$$

Hence

$$\lambda u^{\lambda}(q;S)=-\lambda u^{\lambda}\big(-q;\tau_q'S)\ge -S(q,0)-c_0.$$

Proof of Theorem 4.2 *(Step1)* Observe that if $\mathcal{V}_n^S=\Gamma_n\circ\widehat{\mathcal{T}}_S^n\circ\Gamma_n^{-1}=\widehat{\mathcal{T}}^n_{\Gamma_n(S)}$, then

$$\mathcal{V}_n^{\tau_{na}'S}=\tau_a\circ\mathcal{V}_n^S\circ\tau_{-a}\quad\text{ or }\quad \mathcal{V}_n^{\tau_{na}'S}\circ\tau_a=\tau_a\circ\mathcal{V}_n^S. \tag{4.23}$$

This means

$$u_n(q)=u_n^S(q)=\big(\mathcal{V}_n^Sg\big)(q)=\left(\mathcal{V}_n^{\tau_{nq}'S}(\tau_qg)\right)(0) \tag{4.24}$$

From this, we learn that for any $\bar{u}:\mathbb{R}^d\to\mathbb{R}$,

$$\begin{aligned}\int\left[\int_{[-\ell,\ell]^d}\big|\big(\mathcal{V}_n^Sg\big)(q)-u(q)\big|\ dq\right]\mathbb{P}(dS)&=\int\left[\int_{[-\ell,\ell]^d}\left|\left(\mathcal{V}_n^{\tau_{nq}'S}(\tau_qg)\right)(0)-u(q)\right|\ dq\right]\mathbb{P}(dS)\\ &=\int\left[\int_{[-\ell,\ell]^d}\big|\big(\mathcal{V}_n^S(\tau_qg)\big)(0)-\bar{u}(q)\big|\ dq\right]\mathbb{P}(dS).\end{aligned}$$

This means that for a local L^1 convergence of $u_n(q)=\big(\mathcal{V}_n^Sg\big)(q)$ to a deterministic function u, we only need to show the existence of the limit for $q=0$.

(Step 2) Let us simply write $u_n(S)$ for $\big(\mathcal{V}_n^S g\big)(0)$. Set

$$\underline{u}(S) = \liminf_{n\to\infty} u_n(S).$$

We claim that $\mathbb{P}$-almost surely, the function $\underline{u}(S)$ is constant. By the ergodicity of $\mathbb{P}$, it suffices to show that $\underline{u}(\tau'_a S) = \underline{u}(S)$, for every $a \in \mathbb{R}^d$. Indeed by (4.24),

$$u_n(\tau'_a S) = \sup_Q \big(g\big(Q - n^{-1}a\big) - S_n\big(n^{-1}a, Q\big)\big) = \sup_Q \big(g\big(Q\big) - S_n\big(n^{-1}a, Q\big)\big) + O\big(n^{-1}\big),$$

where

$$S_n\big(n^{-1}a, Q\big) = \inf_{q_1,\dots,q_{n-1}} n^{-1}\left(S(a, q_1) + \dots + S(q_{n-2}, q_{n-1}) + S(q_{n-1}, nQ)\right),$$

and $S_n\big(n^{-1}a, Q\big) \le c_3$ by Proposition 4.1. In particular

$$c_3 \ge n^{-1}S(a, q_1) \ge \delta n^{-1}|a - q_1|^\alpha - n^{-1}c_0.$$

This leads to

$$|a - q_1| \le \delta^{-1/\alpha}(nc_3 + c_0)^{1/\alpha}.$$

This implies

$$|S(a, q_1) - S(0, q_1)| \le c|a|(|q_1| + 1) \le c' n^{1/\alpha},$$

which in turn implies

$$\big|S_n\big(n^{-1}a, Q\big) - S_n(0, Q)\big| \le c' n^{1/\alpha - 1}.$$

From this we deduce that $\underline{u}(\tau'_a S) = \underline{u}(S)$, as desired.

(Step 3) Given $\hat{G} : \mathcal{S} \to \mathbb{R}$, we may set $F(q) = q + \hat{G}(\tau_q S)$, and choose a sequence of the form $q_n = F^n(0)$. Observe that if $f(S) = f_G(S) = \tau'_{G(S)} S$, $L(S) = L^G(S) = S(0, G(S))$, then

$$\begin{aligned} &q_n = S + G(S) + \dots + G\big(f^{n-1}(S)\big), \qquad \tau'_{q_n} S = f^n(S), \\ &S(0, q_1) + \dots + S(q_{n-1}, q_n) = L^G(S) + \dots + L^G\big(f^{n-1}(S)\big). \end{aligned} \tag{4.25}$$

These identities can be readily verified by induction on n. We now set Θ to be the set of measurable pairs $\big(\hat{G}, \rho\big)$, $\hat{G}, \rho : \mathcal{S} \to \mathbb{R}$ such that the probability measure $\mathbb{Q}(dS) = \rho(dS)\ \mathbb{P}(dS)$ is an invariant measure for $f = f_G$. We write Θ_{er} for the set of measurable pairs $\big(\hat{G}, \rho\big) \in \Theta$ such that the measure $\mathbb{Q}$ is also ergodic. By Ergodic Theorem, the limits

$$\bar{q}^G(S) := \lim_{n\to\infty} n^{-1} q_n, \quad \bar{L}^G(S) = \lim_{n\to\infty} n^{-1}\left(L^G(S) + \dots + L^G\big(f^{n-1}(S)\big)\right),$$

exists $\mathbb{Q}$-almost surely, and

$$\bar{q}^G = \int G\ d\mathbb{Q}, \quad \bar{L}^G = \int L^G\ d\mathbb{Q}. \tag{4.26}$$

As a result,

$$\underline{u}(S) \geq g\big(\bar{q}^G\big) - \bar{L}^G, \tag{4.27}$$

$\mathbb{Q}$-almost surely. Since $\mathbb{P} << \mathbb{Q}$, we learn that (4.27) is also true $\mathbb{P}$-almost surely.

$$\underline{u}(S) \geq \sup_{(\hat{G},\rho)\in\Theta_{er}} \big(g\big(\bar{q}^G\big) - \bar{L}^G\big) = \sup_v \big(g(v) - \hat{L}(v)\big), \tag{4.28}$$

$\mathbb{P}$-almost surely. Here

$$\hat{L}(v) = \inf\left\{\int L^G\rho\ d\mathbb{P}:\ (\hat{G},\rho)\in\Theta_{er},\ \int G\rho\ d\mathbb{P} = v\right\}.$$

(Step 4) If we choose $g(Q) = p\cdot Q$, and write $\underline{u}^p$ for the corresponding $\underline{u}$, then for every pair $(\hat{G},\rho)\in\Theta_{er}$, we have

$$\underline{u}^p(S) \geq p\cdot\bar{q}^G(S) - \bar{L}^G(S).$$

Since $\underline{u}^p(S)$ is $\mathbb{P}$-almost surely constant, we learn,

$$\int \underline{u}^p\ d\mathbb{P} = \int \underline{u}^p\ d\mathbb{Q} \geq \int \big[p\cdot\bar{q}^G(S) - \bar{L}^G(S)\big]\ \mathbb{Q}(dS) = \int G\ d\mathbb{Q} - \int L^G\ d\mathbb{Q}.$$

As a result,

$$\int \underline{u}^p\ d\mathbb{P} \geq \sup_{(\hat{G},\rho)\in\Theta}\left(\int G\ d\mathbb{Q} - \int L^G\ d\mathbb{Q}\right).$$

Hence, $\mathbb{P}$-almost surely,

$$u^p(S) \geq \sup_{(\hat{G},\rho)\in\Theta}\left(\int G\ d\mathbb{Q} - \int L^G\ d\mathbb{Q}\right). \tag{4.29}$$

Exercise(i) Show that $\hat{L}$ defined by (4.26) is independent of S and convex in v.

5 Viscosity Solution verses Variational Solution

Let $\Phi : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ be a symplectic map with generating function $W(Q,p) = Q \cdot p - w(Q,p)$. In Chapter 3 we learned that if g is a C^1 function, and

$$A\big(q_0, p_0, \dots, q_{n-1}, p_{n-1}; q_n; g\big) = g(q_0) + \sum_{i=1}^{k} \big(p_{i-1} \cdot (q_i - q_{i-1}) - w(p_{i-1}, q_i)\big),$$

then a critical point of A yields an orbit $x_i = (q_i, p_i) = \Phi^i(x_0), i = 1, \dots, n$, with $p_0 = \nabla g(q_0)$. Motivated by this, let us define

$$\mathcal{W}_n(x_0) = \sum_{i=1}^{n} \big(p_{i-1} \cdot (q_i - q_{i-1}) - w(p_{i-1}, q_i)\big),$$

where $x_i(q_i, p_i) = \Phi^i(x_0)$ for $i = 1, \dots, n$. In other words, $\mathcal{W}_n(x_0)$ denotes the action at time n of an orbit that starts from x_0. We then set

$$\mathcal{F}_n(g) = \left\{\big(Q, g(q) + \mathcal{W}_n\big(q, \nabla g(q)\big)\big) : \ q \in \mathbb{R}^d, \ \Phi^n(q, \nabla g(q)) = (Q, P)\right\}.$$

We may extend the definition of $\mathcal{F}_n$ to Lipschitz g.

Definition 5.1(i) Given a Lipschitz function $g : \mathbb{R}^d \to \mathbb{R}$, we write $\hat{\partial} g(q)$ for the set of vectors p such that there exists a sequence q_k such that $\nabla g(q_k)$ exists, and

$$q = \lim_{k \to \infty} q_k, \qquad p = \lim_{k \to \infty} \nabla g(q_k).$$

The convex hull of the set $\hat{\partial} g(q)$ is denoted by $\partial g(q)$.

(ii) Given a Lipschitz function g, we set

$$\mathcal{F}_n(g) = \left\{(q_n, g(q_0) + \mathcal{W}_n(q_0, p_0)) : \ q_0 \in \mathbb{R}^d, p_0 \in \partial g(q_0), \ \Phi^n(q_0, p_0) = (q_n, p_n)\right\}.$$

(iii) By a *variational solution* associated with Φ, we mean a collection of operators $\widehat{\mathcal{V}}_n = \widehat{\mathcal{V}}_n^S : \Lambda \to \Lambda, \ n \in \mathbb{N}$ with the following properties:

- $\widehat{\mathcal{V}}_n(g + c) = \widehat{\mathcal{V}}_n(g) + c$ for each n and every constant $c \in \mathbb{R}$.
- For $g, g' \in \Lambda$ with $g \le g'$, we have $\widehat{\mathcal{V}}_n(g) \le \widehat{\mathcal{V}}_n(g')$.
- For every $g \in \Lambda$, and $n \in \mathbb{N}$,

$$\left\{\big(q, \widehat{\mathcal{V}}_n(g)(q)\big) : \ q \in \mathbb{R}^d\right\} \subseteq \mathcal{F}_n(g).$$

□

Likewise, we may define a variational solution of the HJE (1.10). Recall that for $\gamma : [0,t] \to \mathbb{R}^{2d}$, with $\gamma(s) = (q(s), p(s))$, and a C^1 Hamiltonian function $H : \mathbb{R}^{2d} \to \mathbb{R}$, the action is defined by

$$\mathcal{A}_t(\gamma) = \mathcal{A}_t^H(\gamma) = \int_0^t \big[p \cdot \dot{q} - H(x)\big]\ ds.$$

Definition 5.2(i) We set $\phi^H_{[0,t]}(a)$ for the restriction of the flow $\phi^H_s(a)$ to the interval $[0,t]$. Given $a \in \mathbb{R}^{2d}$, we define

$$A_t^H(a) = \mathcal{A}_t^H\big(\phi^H_{[0,t]}(a)\big).$$

(ii) Given a Lipschitz function g, we set

$$\mathcal{F}_t(g) = \big\{(q(t), g(q_0) + \mathcal{A}_n(q_0,p_0)) :\ q_0 \in \mathbb{R}^d, p_0 \in \partial g(q_0),\ \phi_t^H(q_0,p_0) = (q(t),p(t))\big\}\,.$$

(iii) By a *variational solution* associated with Φ, we mean a collection of operators $\widehat{\mathcal{V}}_t : \Lambda \to \Lambda,\ t \in [0,\infty)$ with the following properties:

- $\widehat{\mathcal{V}}_0$ is identity, and $\widehat{\mathcal{V}}_t(g+c) = \widehat{\mathcal{V}}_t(g) + c$ for each t and every constant $c \in \mathbb{R}$.
- For $g, g' \in \Lambda$ with $g \le g'$, we have $\widehat{\mathcal{V}}_t(g) \le \widehat{\mathcal{V}}_t(g')$.
- For every $g \in \Lambda$, and $t \in [0,\infty)$,

$$\Big\{\big(q, \widehat{\mathcal{V}}_t(g)(q)\big) :\ q \in \mathbb{R}^d\Big\} \subseteq \mathcal{F}_t(g).$$

□

When H is independent of q, then $\mathcal{F}_t$ can simply be described as

$$\begin{aligned}\mathcal{F}_t(g) &= \big\{\big(q + t\nabla H(p), g(q) + t(p\cdot\nabla H(p) - H(p))\big) :\ q \in \mathbb{R}^d,\ p \in \partial g(q)\big\}\\ &= \big\{\big(Q, g(q) + p\cdot(Q-q) - tH(p)\big) :\ Q \in \mathbb{R}^d,\ Q - q = t\nabla H(p),\quad p \in \partial g(q)\big\}\\ &= \big\{\big(Q, A^t(x;Q;g)\big) :\ Q \in \mathbb{R}^d,\ 0 \in \partial_x A(x;Q;g)\big\}\,,\end{aligned} \tag{5.1}$$

where $A^t(q,p;Q;g) = A^t(x;Q;g) = g(q) + p\cdot(Q-q) - tH(p)$.

Before examining some examples in dimension one, we define a type of discontinuity of u_q is will play an essential role as we compare variational solutions with *viscosity solutions*.

Definition 5.3 We say that a pair of momenta (p^-, p^+) satisfies the *Oleinik Condition* if either $p^- > p^+$, and the graph of the restriction of H to $[p^-, p^+]$ is above the chord connecting $(p^-, H(p^-))$ to $(p^+, H(p^+))$, or $p^- < p^+$, and the graph of the restriction of H to $[p^-, p^+]$ is below the chord connecting $(p^-, H(p^-))$ to $(p^+, H(p^+))$. □

Example 5.1 Assume that $d = 1$ and that H is independent of q. Set $K(p) = pH'(p) - H(p)$. Then

$$\mathcal{F}_t(g) = \left\{ \left(q + tH'(p), g(q) + tK(p)\right) : \ q \in \mathbb{R}, \ p \in \partial g(q) \right\}.$$

For example, if $g(q) = p^- q 1\!\!1(q \le 0) + p^+ q 1\!\!1(q \ge 0)$, with $p^- > p^+$, then $\mathcal{F}_t(g) = \mathcal{F}_t^- \cup \mathcal{F}_t^0 \cup \mathcal{F}_t^+$, where

$$\begin{aligned} \mathcal{F}_t^- &= \left\{ \left(q + tH'(p^-), p^- q + tK(p^-)\right) : \ q \le 0 \right\} = \left\{ \left(q, p^- q - tH(p^-)\right) : \ q \le tv_- \right\}, \\ \mathcal{F}_t^+ &= \left\{ \left(q + tH'(p^+), p^+ q + tK(p^+)\right) : \ q \ge 0 \right\} = \left\{ \left(q, p^+ q - tH(p^+)\right) : \ q \ge tv_+ \right\}, \\ \mathcal{F}_t^0 &= \left\{ \left(tH'(p), tK(p)\right) : \ p \in [p^+, p^-] \right\}, \end{aligned}$$

with $v^{\pm} = H'(p^{\pm})$. Note

$$\mathcal{F}_t^{\pm} = t\mathcal{F}_1^{\pm} =: t\mathcal{F}^{\pm}, \qquad \mathcal{F}_t^0 = t\mathcal{F}_1^0 =: t\mathcal{F}^0.$$

Hence we only need to determine $\mathcal{F} = \mathcal{F}_1$. To analyze $\mathcal{F}$ further, we examine several cases:

(i) If H is strictly convex then H' is increasing. We then set $L = K \circ (H')^{-1}$, which is simply the Legendre transform of H. Moreover $v^- > v^+$, and

$$\mathcal{F}^0 = \left\{ \left(v, L(v)\right) : \ v \in [v^+, v^-] \right\}.$$

Note that $\mathcal{F}^{\pm}$ are lines that intersect at the point $(\bar{q}, \bar{u})$ where $\bar{q} = H[p^-, p^+]$, and $\bar{u} = p^{\pm}\bar{q} - H(p^{\pm})$, with $\bar{v}$ given by *Rankine-Hugoniot Formula*

$$H[p^-, p^+] = \frac{H(p^+) - H(p^-)}{p^+ - p^-}.$$

In fact the only continuous function $\hat{u}(\cdot)$ such that the graph of $\hat{u}$ is a subset of $\mathcal{F}(g)$ is

$$\hat{u}(q) = \left(p^- q - H(p^-)\right) 1\!\!1\left(q \le \bar{v}\right) + \left(p^+ q - H(p^+)\right) 1\!\!1\left(q \ge \bar{v}\right). \tag{5.2}$$

This yields the solution $\hat{u}(q, 1) = \hat{u}(q)$ when $t = 1$. The general t follows from multiplying the graph of u by t. The solution (5.2) is an example of a *shock wave*.

Observe that $g = \min\{g^-, g^+\}$, with $\gamma^{\pm}(q) = qp^{\pm}$, and $\widehat{\mathcal{V}}_t(g) = \min\left\{\widehat{\mathcal{V}}_t(g^-), \widehat{\mathcal{V}}_t(g^+)\right\}$. This strong form of monotonicity is true for any pair of initial data $g^{\pm}$, and is a consequence of the convexity of H.

(ii) If H is strictly concave, then H' is decreasing. As before, we set $L = K \circ (H')^{-1}$, which is now concave . It may be defined by

$$L(v) = \min_{p \in [p^+, p^-]} (vp - H(p)).$$

Moreover, if $v^{\pm} = H'(p^{\pm})$, then $v^- < v^+$, and

$$\mathcal{F}^0 = \big\{\big(v, L(v)\big) : \ v \in [v^-, v^+]\big\}.$$

In fact $\mathcal{F}(g)$ is the graph of a function $u(\cdot)$ that is given by

$$\hat{u}(q) = \big(p^- q - H(p^-)\big)\mathbb{1}\big(q \le v^-\big) + \big(p^+ q - H(p^+)\big)\mathbb{1}\big(q \ge v^+\big) + L(q)\mathbb{1}\big(v^- \le q \le v^+\big).$$

What we have is an example of a *rarefaction wave.*

(iii) We now relax the convexity assumption of part **(i)** to the Oleinik Condition. More precisely, we assume that the graph of $H : [p^+, p^-] \to \mathbb{R}$ lies below the chord connecting $(p^+, H(p^+))$ to $(p^-, H(p^-))$. We claim that under Oleinik condition, the only possible u with its graph subset of $\mathcal{F}_1(g) = \mathcal{F}(g)$, is given by (5.2). For this, it suffices to show that no point of $\mathcal{F}^0$ can reach the set below the graph of u. Indeed by Oleinik Condition

$$\frac{H(p) - H(p^+)}{p - p^+} \le \bar{v} = \frac{H(p^+) - H(p^-)}{p^+ - p^-} \le \frac{H(p^-) - H(p)}{p^- - p},$$

for every $p \in [p^+, p^-]$. Hence

$$\begin{aligned}
\bar{v} \le q \quad &\Longrightarrow \quad \frac{H(p) - H(p^+)}{p - p^+} \le q \quad \Longrightarrow \quad p^+ q - H(p^+) \le pq - H(p),\\
\bar{v} \ge q \quad &\Longrightarrow \quad \frac{H(p^-) - H(p)}{p^- - p} \ge q \quad \Longrightarrow \quad p^- q - H(p^-) \le pq - H(p).
\end{aligned}$$

Hence

$$\hat{u}(q) \le \min_{p \in [p^+, p^-]} (pq - H(p)),$$

for every q. This means that the set $\mathcal{F}^0$ lies above the graph of $\hat{u}$. On the other hand, if for some point $(H'(p), pH'(p) - H(p))$ lies on the graph of $\hat{u}$ for some $p \in [p^+, p^-]$, then

$$\text{either} \quad \bar{v} \le q = H'(p) = \frac{H(p) - H(p^+)}{p - p^+} \qquad \text{or} \quad \bar{v} \ge q = H'(p) = \frac{H(p^-) - H(p)}{p^- - p}.$$

By Oleinik Condition, we must have $\bar{v} = q$, which implies that the only possible intersection point between the graph of $\hat{u}$ and $\mathcal{F}^0$ is the corner point of the graph of $\hat{u}$. This completes the proof of our claim.

(iv) Assume that $H(p^+) = H(p^-) = H'(p^-) = 0, H'(p^+) < 0$, and $H(p) < 0$ for every $p \in (p^+, p^-)$. Then the Oleinik Condition is satisfied. We note that $\mathcal{F}^-$ ends at the origin, $\mathcal{F}^+$ passes through the origin, and $\mathcal{F}^0$ has two concave and convex pieces that are tangent to $\mathcal{F}^-$ and $\mathcal{F}^+$ respectively. The shock location is the origin, and $\hat{u}(q,t) = g(q)$ for all $t \geq 0$. □

As Example 5.1 indicates, we may have a simple formula for the variational solution when H is convex in momentum variable. Note that the action can be expressed in terms of the Lagrangian because when $\dot{x} = J\nabla H(x)$ for $x = (q,p)$, then

$$p \cdot \dot{q} - H(q,p) = L(q,\dot{q}).$$

In fact in this case the variational solution is given by Lax-Oleinik Formula.

Theorem 5.1 *For a Tonelli Hamiltonian function H, we have*

$$\widehat{\mathcal{V}}_t^H(g)(Q) = \inf \left\{ g(q(0)) + \int_0^t L(q,\dot{q})\ ds :\ q(\cdot \in C^1[0,t],\ q(t) = Q \right\}. \tag{5.3}$$

In particular if H is convex and independent of q, we may use (5.3) and (5.1) to write

$$\begin{aligned} \widehat{\mathcal{V}}_t^H(g)(Q) &= \inf_q \left(g(q) - tL\left(\frac{Q-q}{t}\right)\right) \\ &= \inf_q \sup_p \big(g(q) + p\cdot(Q-q) - tH(p)\big) = \inf_q \sup_p A^t(q,p;Q;g). \end{aligned} \tag{5.4}$$

This formula is not surprising; after all we are looking for a critical value of $A^t(\cdot;Q;g)$ that is concave in p. So it is natural to try a simplex minimax critical value that happens to be finite when H is convex.

In fact if we set $t = 1$, then the role of q and p are of the same flavor. Because of this, we may wonder whether or not we have a simple formula for a variational solution when, for example g is concave. This is indeed the case as the following result confirms.

Theorem 5.2 *Assume that H is Tonelli and independent of q, and g is Lipschitz and concave. Then*

$$\widehat{\mathcal{V}}_t^H(g)(Q) = \inf_p \sup_q \big(g(q) + p\cdot(Q-q) - tH(p)\big). \tag{5.5}$$

The identity (5.5) is known as *Hopf's formula* and can be rewritten as

$$\widehat{\mathcal{V}}_t^H(g)(Q) = \inf_p \big(p\cdot Q - g^\dagger(p) - tH(p)\big) = \big(g^\dagger + tH\big)^\dagger(Q), \tag{5.6}$$

where we have used † for the Legendre Transform:

$$g^\dagger(p) = \inf_q \big(p\cdot q - g(q)\big).$$

Note that $(g+tH)^\dagger$ is always well-defined and a concave, even when H is not concave. If g is convex instead, then (5.5) and (5.6) change to

$$\widehat{\mathcal{V}}_t^H(g)(Q) = \sup_p \inf_q \big(g(q) + p\cdot(Q-q) - tH(p)\big) = \big(g^* + tH\big)^*(Q), \tag{5.7}$$

where we have used $*$ for the Legendre Transform:

$$g^*(p) = \sup_q \big(p\cdot q - g(q)\big).$$

Example 5.2(i) If the restriction of H to $[p^+,p^-]$ consists of a collection of concave and convex pieces, then the set $\mathcal{F}^0$ is a union of the graphs of the Legendre transforms of such pieces. However, when $g(q) = \min\{p^-q, p^+q\}$ with $p^+ < p^-$, then g is concave, and the corresponding function u depends only the the concave hull of the restriction of H to $[p^+,p^-]$. Indeed from (5.6), and the elementary fact that $g^\dagger(p) = -\infty 1\!\!1\big(p \notin [p^+,p^-]\big)$, we deduce

$$\hat{u}(q,1) = \hat{u}(q) = \min_{p\in[p^-,p^+]} \big(pq - H(p)\big) = \min_{p\in[p^+,p^-]} \big(pq - \hat{H}(p)\big),$$

where $\hat{H}$ denotes the concave hull of the restriction of H to $[p^+,p^-]$. Note that the graph of H is below the chord connecting $\big(p^+,H(p^+)\big)$ to $\big(p^-,H(p^-)\big)$, iff the concave hull of the restriction of H to $[p^+,p^-]$ is this cord. If this is the case, then the Oleinik Condition is satisfied, and we have a shock.The solution is simply given by

$$\hat{u}(q) = \min_{p\in[p^+,p^-]} \big(pq - H(p)\big) = \min\big\{p^-q - H(p^-), p^+q - H(p^+)\big\},$$

as in part **(i)**. Note that the graph of u now can have pieces that lie on $\mathcal{F}^0$. In order to have a feel for complex u could be, imagine that there are points p_1,p_2,p_3 with $p^+ < p_1 < p_2 < p_3 < p^-$ such that $\hat{H} = H$ in the set $[p_1,p_2]\cup[p_3,p^-]$, and $\hat{H} \neq H$ in its complement. Then the graph of u would have two pieces of $\mathcal{F}^0$ associated with the intervals $[p_1,p_2]$ and $[p_3,p^-]$. More precisely we may express the graph of u as $F_1\cup F_2\cup F_3\cup F_4$, where $F_1 = \mathcal{F}^-$,

$$F_2 = \big\{(H'(p),K(p)):\ p\in[p_3,p^-]\big\},\qquad F_3 = \big\{(H'(p),K(p)):\ p\in[p_1,p_2]\big\},$$

and $F_4 \subset \mathcal{F}^+$. The momentum u' consists of two rarefaction waves associated with F_2 and F_3 that are separated by a discontinuity. The rarefaction F_3 is separated from F_4 by a shock discontinuity.

(ii) Let us now assume that $p^- < p^+$. Then g is convex and we may apply (5.7) to assert

$$\hat{u}(q,1) = \hat{u}(q) = \max_{p\in[p^-,p^+]} \big(pq - H(p)\big) = \max_{p\in[p^-,p^+]} \big(pq - \tilde{H}(p)\big),$$

where $\tilde{H}$ denotes the convex hull of H. In particular if the graph of the restriction of H to $[p^-, p^+]$ is above the chord connecting $(p^-, H(p^-))$ to $(p^+, H(p^+))$, then $H(p^\pm) = \tilde{H}(p^\pm)$, and

$$\hat{u}(q,t) = \max\left\{qp^+ - H(p^+), qp^- - H(p^-)\right\}.$$

In other words, the *Oleinik Condition* is satisfied and we have a shock discontinuity. □

We now turn to *viscosity solutions*.

Definition 5.3(i) Given a function $u : \mathbb{R}^k \to \mathbb{R}$, we write $\bar{\partial} u(z)$ for the set of vectors $a \in \mathbb{R}^k$ such that

$$\limsup_{h \to 0} |h|^{-1}\big(u(z+h) - u(z) - a \cdot h\big) \le 0.$$

Equivalently, $a \in \bar{\partial} u(z)$ iff there exists a C^1 function $\varphi : \mathbb{R}^k \to \mathbb{R}$ such that $\varphi(z) = u(z)$, $\nabla \varphi(z) = a$, and $u \le \varphi$. Similarly, $a \in \underline{\partial} u(z)$ iff

$$\liminf_{h \to 0} |h|^{-1}\big(u(z+h) - u(z) - a \cdot h\big) \ge 0.$$

Equivalently, $a \in \underline{\partial} u(z)$ iff there exists a C^1 function $\varphi : \mathbb{R}^k \to \mathbb{R}$ such that $\varphi(z) = u(z)$, $\nabla \varphi(z) = a$, and $u \ge \varphi$.

(ii) We say a uniformly continuous function $u : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$ is a solution of (1.10) if for every $(p, r) \in \bar{\partial} u(q, t), t > 0$ satisfies $r + H(q, p) \le 0$, and for every $(p, r) \in \underline{\partial} u(q, t), t > 0$ satisfies $r + H(q, p) \ge 0$. □

Example 5.3 Assume that $u : \mathbb{R}^k \to \mathbb{R}$ is continuous and there exists a C^1 surface Γ of codimension one such that u is C^1 on $\mathbb{R}^k \setminus \Gamma$. Write $u^\pm$ for the restriction of u on each side of Γ. (This is well-defined for points near Γ.) We assume that $u^\pm$ are C^1 functions up to the boundary points on Γ. Pick a point on Γ. We wish to determine $\bar{\partial} u(a)$ in terms of $\nabla u^\pm(a)$. Assume that $v \in \bar{\partial} u(a) \neq \emptyset$. Let us write $T_a\Gamma$ for the tangent fiber at a to Γ, P_a for the orthogonal projection onto $T_a\Gamma$, and ν_a for the unit normal vector at a that points from $-$-side (on which u^- is defined) to the $+$-side (on which u^+ is defined). First take a smooth path $\gamma : (-\delta, \delta) \to \Gamma$ with $\gamma(0) = a$, $\dot{\gamma}(0) = \tau$. Using $v \in \bar{\partial} u(a)$, and

$$\left(\frac{d}{dt} u \circ \gamma\right)(0) = \nabla u^\pm(a) \cdot \tau,$$

we deduce that $\nabla u^\pm(a) \cdot \tau \le v \cdot \tau$. This also being also true for $-\tau \in T_a\Gamma$ implies that $\nabla u^\pm(a) \cdot \tau = v \cdot \tau$. Hence $\nabla u^+(a) - \nabla u^-(a)$ is orthogonal to $T_a\Gamma$. This is not surprising and follows from the continuity of u; since $u^+ = u^-$ on Γ, we have that the τ-directional derivative of u^+ and u^- coincide whenever $\tau \in T_a\Gamma$. Now if we vary a in the direction of ν_a or $-\nu_a$, we deduce

$$\nabla u^+(a) \cdot \nu_a \le v \cdot \nu_a, \qquad \nabla u^-(a) \cdot (-\nu_a) \le v \cdot (-\nu_a).$$

Equivalently,

$$\nabla u^+(a)\cdot \nu_a \le v\cdot \nu_a \le \nabla u^-(a)\cdot \nu_a.$$

Hence, if $\bar\partial u(a)\neq \emptyset$, then $P_a\nabla u^+(a)=P_a\nabla u^-(a)$, $\nabla u^+(a)\cdot\nu_a\le \nabla u^-(a)\cdot \nu_a$, and

$$\bar\partial u(a)=\left\{P_a\nabla u^\pm(a)+r\nu_a:\ a\in\left[\nabla u^+(a)\cdot\nu_a,\nabla u^-(a)\cdot\nu_a\right]\right\}.$$

Likewise, if $\underline{\partial} u(a)\neq\emptyset$, then $P_a\nabla u^+(a)=P_a\nabla u^-(a)$, $\nabla u^+(a)\cdot\nu_a\ge \nabla u^-(a)\cdot\nu_a$, and

$$\underline{\partial} u(a)=\left\{P_a\nabla u^\pm(a)+r\nu_a:\ a\in\left[\nabla u^-(a)\cdot\nu_a,\nabla u^+(a)\cdot\nu_a\right]\right\}.$$

In summary, we always have $P_a\nabla u^+(a)=P_a\nabla u^-(a)$, and there are three possibilities:

$$\begin{aligned}
\nabla u^+(a)\cdot\nu=\nabla u^-(a)\cdot\nu \quad &\Longrightarrow \quad \bar\partial u(a)=\underline{\partial} u(a)=\{\nabla u^\pm(a)\},\\
\nabla u^+(a)\cdot\nu<\nabla u^-(a)\cdot\nu \quad &\Longrightarrow \quad \bar\partial u(a)\neq\emptyset,\quad \underline{\partial} u(a)=\emptyset,\\
\nabla u^+(a)\cdot\nu>\nabla u^-(a)\cdot\nu \quad &\Longrightarrow \quad \bar\partial u(a)=\emptyset,\quad \underline{\partial} u(a)\neq\emptyset.
\end{aligned}$$

□

In Example 5.1**(i)**, **(iii)**, **(iv)**, and Example 5.2, we have variational solutions for which $\hat u_q$ has shock discontinuities. In all these examples, the jump discontinuity of $\hat u_q$ satisfies an Oleinik Condition. However it is known that in general Oleinik condition may be violated for a variational solution. Several explicit examples have been discovered for such a violation. The following recent example is due V. Roos (2017). The idea is that in Example 5.1**(iv)** Oleinik Condition is satisfied but a small perturbation of $p^\pm$ may lead to a violation of Oleinik Condition. Indeed if we change p^- to $p^- - \eta$ for some small $\eta>0$, then we can find small $\eta'>0$ such that the Oleinik Condition is violated for the left and right limits $p^+ + \eta'$ and $p^- - \eta$. This may be achieved by perturbing the right arm of the graph of g by a convex function.

Theorem 5.3 *Assume $d=1$, and $H\in C^2$ is independent of q. Assume that $p^+<p^-$, $H(p^+)=H(p^-)=H'(p^-)=0>H'(p^-)$, and $H(p)<0$ for every $p\in(p^+,p^-)$. Let $f\in C^2$ be a strictly Lipschitz convex function with $f(0)=f'(0)=0$ and set*

$$g(q)=p^-q\mathbb{1}(q\le 0)+\big(p^+q+f(q)\big)\mathbb{1}(q\ge 0).$$

Then there exists $t_0>0$ such that for every $t\in(0,t_0)$, there exists a point $q(t)>0$ such that for every variational solution $\hat u_q(q,t)$ is discontinuous at $q(t)$. Moreover the momenta $\hat u_q(q(t)\pm,t)$ violate the Oleinik Condition.

Proof *(Step 1)* As before, $\mathcal{F}_t(q)=\mathcal{F}_t^+\cup\mathcal{F}_t^0\cup\mathcal{F}_t^-$, where

$$\begin{aligned}
\mathcal{F}_t^+&=t\hat{\mathcal{F}}_t^+=\left\{t\big(q+H'(g'(tq)),t^{-1}g(tq)+K(g'(tq))\big):\ q>0\right\},\\
\mathcal{F}_t^-&=\mathcal{F}^-=\left\{\big(q,qp^-\big):\ q<0\right\},\\
\mathcal{F}_t^0&=t\mathcal{F}^0=\left\{t\big(H'(p),K(p)\big):\ p\in[p^+,p^-]\right\}.
\end{aligned}$$

Note that the sets $\mathcal{F}^-$ and $\mathcal{F}^0$ are independent of f and the same as what we had in Example 5.1**(iv)**. Let us write

$$\mathcal{F}^+ = \big\{\big(q, qp^+ + H(p^+)\big) :\ q > H'(p^+)\big\},$$

which is what we get when $f = 0$ and $t = 1$.

We now examine the set $\mathcal{F}_t^+$. We claim that for $t \in (0, t_0)$, with

$$t_0 = \big[\sup_q |H''(q)| \sup_q |f''(q)|\big]^{-1},$$

the set $\mathcal{F}_t^+$ is a graph of a convex function that is above $t\mathcal{F}^+$, and is tangent to $t\mathcal{F}^+$ at its end point. For convexity, observe that if

$$a(q) = q + H'(g'(tq)), \qquad b(q) = t^{-1}g(tq) + K(g'(tq)),$$

then $a'(q) = 1 + tH''(g'(tq))g''(tq) = 1 + tH''(g'(tq))f''(tq) > 0$, and

$$b'(q) = g'(tq) + tg'(tq)H''(g'(tq))g''(tq) = g'(tq)a'(tq).$$

Hence the slope of $\mathcal{F}_t^+$ at the point $t(a(q), b(q))$ is $g'(tq)$. Since both a' and g' are increasing, $\mathcal{F}_t^+$ is convex. At $q = 0$ this slope is p^+, which means that the line $\mathcal{F}^+$ is tangent to $\hat{\mathcal{F}}_t^+$ at its end point $(a(0), b(0))$, hence it lies above this line.

(Step 2) For small $\delta > 0$, the set

$$\hat{\mathcal{F}}_t^0 = t\hat{\mathcal{F}}^0 = \big\{t\big(H'(p), K(p)\big) :\ p \in [p^- - \delta, p^-]\big\},$$

is a graph of concave function that starts from the origin and lies below a line of slope p^- that passes through the origin. As a result, the set $\mathcal{F}_t^+$ will intersect $\hat{\mathcal{F}}_t^0$ at some point $t(a(q^t), b(q^t)), q^t > 0$, for small and positive t. One can see this by comparing the set $\hat{\mathcal{F}}_t^+$ with $\hat{\mathcal{F}}^0$ and find an intersection point for these two sets. Observe that the set $\hat{\mathcal{F}}_t^+$ is above $\mathcal{F}^+$ and tangent to $\mathcal{F}^+$ at its end point. Moreover, since

$$g'(tq) = p^+ + f'(tq) = p^+ + o(1), \qquad t^{-1}g(tq) = qp^+ + t^{-1}f(tq),$$

we have that $\hat{\mathcal{F}}_t^+ \to \mathcal{F}^+$ as $t \to 0$. This means that the sets $\hat{\mathcal{F}}_t^+$ and $\hat{\mathcal{F}}^0$ intersect at a some point $(a(q^t), b(q^t)$ near the origin for small t. This means that the variational solution $u(q,t)$ has a corner at tq^t. The left and right derivatives of $u(\cdot, t)$ at tq^t, are given by the slope of $\mathcal{F}_t^0$ and $\mathcal{F}_t^+$ at the point $t(a(q^t), b(q^t))$. The right derivative is given by $g'(q^t)$ as we showed in *Step 1*. For the right derivative, if for some $\hat{p}- \in [p^- - \delta, p^-]$, we have $H'(\hat{p}) = a(q^t)$, then

$$b(q^t) = \hat{p}H'(\hat{p}^-) - H(\hat{p}^-),$$

and the tangent vector to $\hat{\mathcal{F}}_t^0$ at $(a(q^t), b(q^t))$ is $\big(H''(\hat{p}^-), \hat{p}^- H''(\hat{p}^-)\big)$, which has a slope $\hat{p}^-$. It remains to show that the Oleinik Condition is violated for the left and right momenta $\hat{p}^-$ and $\hat{p}^+ := g'(q^t)$.

(Final Step) For small t, we have $\hat{p}^- = p^- + o(1)$, $\hat{p}^+ = p^+ + o(1)$. So $\hat{p}^- < \hat{p}^+$. By $H'(\hat{p}^-) = a(q^t)$, we know that $H'(\hat{p}^+) = H'(\hat{p}^-) - q^t$. Hence,

$$\begin{aligned}\hat{p}^- H'(\hat{p}^-) - H(\hat{p}^-) &= b(q^t) = t^{-1} g(tq^t) + \hat{p}^+ H'(\hat{p}^+) - H(\hat{p}^+) \\ &= t^{-1} g(tq^t) - \hat{p}^+ q^t + \hat{p}^+ H'(\hat{p}^-) - H(\hat{p}^+).\end{aligned}$$

Equivalently,

$$(\hat{p}^- - \hat{p}^+) H'(\hat{p}^-) + H(\hat{p}^+) - H(\hat{p}^-) = t^{-1}\big(g(tq^t) - g'(q^t)tq^t\big) = t^{-1}\big(f(tq^t) - f'(tq^t)tq^t\big) =: \varphi(q^t).$$

We note that $\varphi(0) = 0$ and $\varphi'(q) < 0$ for $q > 0$ by convexity of f. As a result,

$$(\hat{p}^- - \hat{p}^+) H'(\hat{p}^-) < H(\hat{p}^-) - H(\hat{p}^+).$$

This violates the Oleinik Condition because $\hat{p}^+ < \hat{p}^-$. □

As we will see in Exercise**(i)** below, the Oleinik Condition is always satisfied by the pair $(u_q(q-,t), u_q(q+,t)$ at every discontinuity point (q,t) of u_q, where u is a viscosity solution. Hence the variational solution of Theorem 5.3 is not a viscosity solution. We now explore the viscosity solution for H and g as in Theorem 5.3.

Example 5.4 Let H and g be as in Theorem 5.3. Assume that H is concave near p^-, and for some $\delta, \delta_1, \delta_2 > 0$,

$$\big\{p \in [p^+, p^-] :\ H(p) \in [-\delta, 0]\big\} = [p^+, p^+ + \delta_1] \cup [p^- - \delta_2, p^-].$$

Choose $\delta^- < \delta_2, \delta^+ < \delta_1$ such that for each $p \in [p^+, p^+ + \delta^+]$, there is unique $\psi(p) \in [p^- - \delta^-, p^-]$ such that $\psi(p^+) = p^-$, and

$$H\big(\psi(p)\big) - H(p) = H'\big(\psi(p)\big)\big(\psi(p) - p\big). \tag{5.8}$$

We claim that the viscosity solution u as a corner at $q(t)$ such that $q(0) = 0$, and for small t,

$$\dot{q}(t) = H'\big(p^-(t)\big), \quad p^-(t) = \psi\big(p^+(t)\big), \tag{5.9}$$

where $p^\pm(t) = u_q\big(q(t)\pm, t\big)$ represent the left and right values of u_q at $q(t)$. We now express $p^+(t)$ in terms of $q(t)$, so that the ODE (5.9) can be solved uniquely for the initial condition $q(0) = 0$. For this, let us write $h : [p^+, \infty) \to [0, \infty)$ for the Legendre transform of $g : [0, \infty) \to (-\infty, 0]$, so that $h'(p^+) = 0$, and $g'(q) = \rho$ iff $h'(\rho) = q$. Note if for q, we have $q(t) = q + tH'(g'(q))$, then $p^+(t) = g'(q)$. Equivalently,

$$q(t) = h'(\rho) + tH'(\rho), \quad p^+(t) = \rho.$$

Let us write $\ell(q,t)$ for the inverse of $\rho \mapsto h'(\rho) + tH'(\rho)$, that is increasing and well-defined for small t. This gives us the formula

$$p^+(t) = \ell\big(q(t), t\big),$$

which allows us to express $p^-(t)$ as a function of $q(t)$. The function $\ell(q,t)$ can be expressed as $\ell = w_q$, where w solves the HJE with initial condition $g(q),\ q \geq 0$, and our formula for ℓ is compatible with (5.7). In particular

$$\ell_t + H'(\ell)\ell_q = 0.$$

We note that $\dot{q}(0) = 0$ but $\dot{q}(t) > 0$ for $t > 0$ and small because $H'\big(p^-(t)\big) > 0$. On the other hand,

$$\dot{p}^+(t) = \ell_t\big(q(t), t\big) + \ell_q\big(q(t), t\big)\dot{q}(t) = \ell_q\big(q(t), t\big)\big(H'(p^-(t)) - H'(p^+(t))\big).$$

Since $\ell_q > 0, H'(p^-(t)) > 0, H'(p^+(t)) < 0$, we deduce that $p^+(t)$ is increasing as a function of t. Since ψ is decreasing, we learn that $p^-(t)$ is decreasing. On the other hand,

$$\ddot{q}(t) = H''\big(p^-(t)\big)\dot{p}^-(t) > 0,$$

for small t. This means that $q(\cdot)$ as a function of t is convex. Here how the viscosity solution for short times look like:

- For $Q > q(t)$ we have $u(Q,t) = g(q) + tK(\rho)$, where $\rho = \ell(Q,t)$, and $g'(q) = \rho$.
- For $Q \leq 0$, we have $u(Q,t) = p^- Q$.
- For $Q \in [0, q(t)]$, we first set $Q(s,t) = q(s) + (t-s)H'(p^-(s))$, for $s \geq t$. We note that $Q_s = (t-s)H''(p^-(s))\dot{p}^-(s) > 0$, so that $s \mapsto Q(s,t)$ is increasing with $Q(0,t) = 0, Q(t,t) = q(t)$. Its inverse is denoted by $s(Q,t)$, and $u(Q,t) = u(q(s)) + (t-s)H'(p^-(s))$, for $s = s(Q,t)$.

What we have constructed is a viscosity solution because it solves HJE outside the set $\big\{(q(t),t) :\ t \in [0,\delta)\big\}$ for small δ, and on this set the Oleinik Condition is satisfied. It also coincides with g initially. So u must be the unique viscosity solution.

For comparison, let us write $\hat{u}$ for the variational solution which has a corner at $\hat{q}(t)$ with the left and right momenta at $\hat{q}(t)$ given by $\hat{p}^{\pm}(t)$. Indeed

$$\begin{aligned}
&H\big(p^-(t)\big) - H\big(p^+(t)\big) - H'\big(p^-(t)\big)\big(p^-(t) - p^+(t)\big) = 0,\\
&H\big(\hat{p}^-(t)\big) - H\big(\hat{p}^+(t)\big) - H'\big(\hat{p}^-(t)\big)\big(\hat{p}^-(t) - \hat{p}^+(t)\big) = t^{-1}\big(\hat{q}(t)g'(\hat{q}(t)) - g(\hat{q}(t))\big) > 0.
\end{aligned}$$

Hence $p^-(t) = \psi\big(p^+(t)\big)$, but $\hat{p}^-(t) > \psi\big(\hat{p}^+(t)\big)$. From this we can deduce

$$\dot{q}(t) = H'\big(p^-(t)\big) = H'\big(\psi(p^+(t))\big), \quad \text{but} \quad \frac{d\hat{q}}{dt}(t) = H\big[\hat{p}^+(t), \hat{p}^-(t)\big] < H'\big(\psi(\hat{p}^+(t))\big).$$

Since $q(0) = \hat{q}(0) = 0$, and

$$p^+(t) = \ell(q(t), t), \quad \hat{p}^+(t) = \ell\big(\hat{q}(t), t\big),$$

we deduce that $\hat{q}(t) \leq q(t)$ for $t > 0$. Note that $u(q,t) = \hat{u}(q,t)$ for $q \notin (0, q(t))$. We now that $u(q,t) < \hat{u}(q,t)$ if $q \in (0, q(t))$, and t is small. For this it suffices to show that if $\rho = u_q$ and $\hat{\rho} = \hat{u}_q$, then $\hat{\rho}(q,t) < \rho u(q,t)$ for $q \in (0, q(t))$. Simply because

$$\begin{aligned} u(q,t) &= u(q(t),t) - \int_q^{q(t)} \rho(a,t)\ da = \hat{u}(q(t),t) - \int_q^{q(t)} \rho(a,t)\ da \\ &< \hat{u}(q(t),t) - \int_q^{q(t)} \hat{\rho}(a,t)\ da = \hat{u}(q,t). \end{aligned}$$

We first consider the case $q \in (\hat{q}(t), q(t))$. For small t, $\hat{\rho}(q,t) = \hat{\rho}(q_0, 0) = g'(q_0)$ for some q_0 that is close to 0. Hence $\hat{\rho}(q,t)$ is close to p^+. However, on the other side of the jump discontinuity, we have $\rho(q,t)$ that is close to ρ^-. Hence larger than $\hat{\rho}$. In the same fashion we can show that $\rho > \hat{\rho}$ for $q \in (0, \hat{q}(t))$ and small t. □

As we have seen in the proof of Theorem 5.3, we can easily calculate solution for small times if the second derivative of the initial data is uniformly bounded.

Proposition 5.1 *Assume that D^2H and D^2g are uniformly bounded and g is C^1 and Lipschitz. Write u and $\hat{u}$ for viscosity and variational solution with initial condition g. Then for $t \geq t_0$ (with t_0 depending on the bounds on D^2H and ∇g only), we have*

$$u(Q,t) = \hat{u}(Q,t) = g(q(0)) + \int_0^t [p \cdot \dot{q} - H(q,p)]\ ds,$$

where $(q(s), p(s)) = \phi_s\big(q(0), \nabla g(q(0))\big)$ is the unique Hamiltonian orbit such that $q(t) = Q$.

Proof We can readily show that the map $F(a) = q(t)$ where $(q(s), p(s)) = \phi_s\big(a, \nabla g(a)\big)$, is a homeomorphism. □

Theorem 5.4 *(Bernard) Assume that D^2H is uniformly bounded and g is Lipschitz. We also assume that D^2g is either uniformly bounded above, or uniformly bounded from below. Write u and $\hat{u}$ for viscosity and variational solution with initial condition g. Then there exists $t_0 > 0$ that depends only on the bounds on D^2H and $\Delta^2 g$ such that the following are true for $t \geq t_0$:*

(i) $u(q,t) \leq \hat{u}(q,t)$.

- *If D^2g is bounded from above, then*

$$\hat{u}(q,t) = \inf\big\{u :\ (q,u) \in \mathcal{F}(q)\big\}. \tag{5.10}$$

- *If $D^2 g$ is bounded from below, then*

$$\hat{u}(q,t) = \sup\big\{u: \ (q,u) \in \mathcal{F}(q)\big\}. \tag{5.11}$$

Proof Assume that $D^2 g$ is bounded from above by c_0. Then we can find a family $\mathcal{G}$ of C^2 Lipschitz functions such that

$$g = \inf_{\hat{g} \in \mathcal{G}} \hat{g}, \qquad \sup_{\hat{g} \in \mathcal{G}} \sup_{q} |D^2 \hat{g}(q)| \leq c_0,$$

and for every $a \in \mathbb{R}^d$, and $p \in \partial g(a)$, there exists $\hat{g} \in \mathcal{G}$ such that $g(a) = \hat{g}(q)$, and $\nabla \hat{g}(a) = p$. Now given Q, pick (a,p) such $p \in \partial g(a)$, and if $(q(s), p(s)) = \phi_s(a,p)$, then $q(t) = q$. Pick $\hat{g} \in \mathcal{G}$ such that $g(a) = \hat{g}(q)$, and $\nabla \hat{g}(a) = p$. We then have

$$\mathcal{V}_t g(Q) \leq \mathcal{V}_t \hat{g}(Q) = \hat{\mathcal{V}}_t \hat{g}(Q) = g(a) + \int_0^t [p \cdot \dot{q} - H(q,p)] \ ds$$
$$\hat{\mathcal{V}}_t g(Q) \leq \hat{\mathcal{V}}_t \hat{g}(Q) = g(a) + \int_0^t [p \cdot \dot{q} - H(q,p)] \ ds.$$

As a result,

$$\mathcal{V}_t g(Q) \leq \inf\big\{u: \ (q,u) \in \mathcal{F}(q)\big\} \leq \hat{\mathcal{V}}_t g(Q)$$
$$\hat{\mathcal{V}}_t g(Q) \leq \inf\big\{u: \ (q,u) \in \mathcal{F}(q)\big\}.$$

□

5.1 Variational selectors

We now give a recipe for the construction of variational solutions in discrete setting. Recall that we write Λ for the set of Lipschitz functions, and Λ_r for the set of $g \in \Lambda$ such that $|g(q) - g(q')| \leq r|q - q'|$. Recall that a variational solution $u_n(Q)$ is a critical value of

$$A(\mathbf{x}_n; Q; g) = g(q_0) + \sum_{i=1}^{n} \big[p_{i-1} \cdot (q_i - q_{i-1}) - w(p_{i-1}, q_i)\big],$$

where $q_n = Q$, and $\mathbf{x}_n = (x_0, \dots, x_{n-1})$, with $x_i \in (q_i, p_i) \in \mathbb{R}^{2d}$. We assume that $w : \mathbb{R}^{2d} \to \mathbb{R}$ is a C^1 and Lipschitz function. We may write $A = \ell + f$, where ℓ is a quadratic function and f is a Lipschitz function. Writing $\mathbf{x}_n = x = (q,p) \in \mathbb{R}^k$ for $k = 2nd$, then

$$\ell(x) = \frac{1}{2} Bx \cdot x = \sum_{1}^{n-1} p_{i-1} \cdot (q_i - q_{i-1}) - p_{n-1} \cdot q_{n-1},$$

where B is a matrix of the form

$$B = \begin{bmatrix} 0 & D \\ D^t & 0 \end{bmatrix},$$

where D is a matrix which has -1 on its main diagonal, 1 right above the main diagonal, and 0 anywhere else. As a result, ℓ is a non-degenerate quadratic form. Because of the very form of A, we make the following definition.

Definition 5.4(i) We write $\mathcal{Q}_k$ for the set of non-degenerate quadratic functions $\ell : \mathbb{R}^k \to \mathbb{R}$. In other words, $\ell(x) = \frac{1}{2} Bx \cdot x$ for a nonsingular symmetric matrix B. We write $\Omega_k(\ell; r)$ for the set of functions $F : \mathbb{R}^k \to \mathbb{R}$ such that $F = \ell + f$ for some $f \in \Lambda_r$. We write

$$\mathcal{Q} = \cup_{k=1}^{\infty} \mathcal{Q}_k, \qquad \Omega_k = \cup_{r=1}^{\infty} \cup_{\ell \in \mathcal{Q}_k} \Omega_k(\ell; r), \qquad \Omega = \cup_{k=1}^{\infty} \Omega_k.$$

(ii) We call $\mathcal{C} : \Omega \to \mathbb{R}$ a em variational selector if it satisfies the following conditions:

(1) If $F \in \Omega$ and $F \in C^1$, then $\mathcal{C}(F) = F(\bar{x})$, for some $\bar{x}$ with $\nabla F(\bar{x}) = 0$.

(2) If $f_1, f_2 \in \Lambda$, with $f_1 \leq f_2$, and $\ell \in \mathcal{Q}$, then $\mathcal{C}(\ell + f_1) \leq \mathcal{C}(\ell + f_2)$.

(3) $\mathcal{C}(F + c) = \mathcal{C}(F) + c$, for every $F \in \Omega$ and $c \in \mathbb{R}$.

(4) If $F \in \Omega$ is bounded below, then $\mathcal{C}(F) = \min F$.

(5) If $\psi : \mathbb{R}^d \to \mathbb{R}^d$ is a Lipschitz smooth diffeomorphism, and $F \in \Omega_k$, then $\mathcal{C}(F) = \mathcal{C}(F \circ \psi)$.

(6) If $F \in \Omega_k$, $\ell' \in \mathcal{Q}_{k'}$, and $F'(x, y) = F(x) + \ell'(y)$, then $\mathcal{C}(F') = \mathcal{C}(F)$.

□

Once a variational selector is known, then we can use it to construct a variational solution by setting

$$\widehat{\mathcal{V}}_n(g)(Q) = \mathcal{C}\big(A(\cdot; Q; g)\big). \tag{5.12}$$

As we mentioned before we use Lusternik-Schnirleman (LS) Theory to construct a selector. Before we give a precise recipe for $\mathcal{C}$, we make some remarks:

Proposition 5.2 **(i)** *If $F \in \Omega_k(\ell; r)$, with $F = \ell + f$, $\ell \in \mathcal{Q}_k$, and $\nabla F(\bar{x}) = 0$, then*

$$|\bar{x}| \leq r\delta(B)^{-1}, \qquad \text{where} \qquad \delta(\ell) = \inf_{|x|=1} |Bx|.$$

(ii) *If $\ell + f = \ell' + f'$, for $f, f' \in \Lambda$, $\ell, \ell' \in \mathcal{Q}_k$, then $\ell = \ell'$, and $f = f'$.*

Proof(i) At a critical point $\bar{x}$ we have $B\bar{x} = -\nabla f(\bar{x})$, which implies

$$\delta(B)|\bar{x}| \le |B\bar{x}| = \big|\nabla f(\bar{x})\big| \le r,$$

as desired.

(ii) If $\ell + f = \ell' + f'$, then $\ell'' = f''$, where $\ell'' = \ell' - \ell$, $f'' = f - f'$. Since f'' is Lipschitz, then $\ell'' = 0$. In fact if $\ell''(x) = B''x \cdot x$, and v is an eigenvector of B'' associated with eigenvalue λ, then $\varphi(t) = \lambda |v|^2 t^2$ must be Lipschitz in t, which is impossible unless $\lambda |v|^2 = 0$. □

LS Theory is normally applied to continuous maps $F : M \to \mathbb{R}$, for a compact manifold M. In our case the non-degenerate quadratic function ℓ make up for the lack of compactness. A standard way to find a critical value of F is by designing a collection $\mathcal{F}$ of subsets of $\mathbb{R}^k$ such that

$$c(F, \mathcal{F}) = \inf_{A \in \mathcal{F}} \sup_A F,$$

is a critical value of F. This is guaranteed if the collection $\mathcal{F}$ satisfies the following property:

$$A \in \mathcal{F}, t > 0 \quad \implies \quad \varphi\phi_t^F(A) \in \mathcal{F},$$

where $\varphi\phi_t$ denotes the flow of the vector field $-\nabla F$. To have a universal collection $\mathcal{F}$ that words for all F, we assume two properties for $\mathcal{F}$:

(1) If $A \in \mathcal{F}$, and φ is a homeomorphism, then $\varphi(A) \in \mathcal{F}$.

(2) If $A \in \mathcal{F}$, and $A \subset B$, then $B \in \mathcal{F}$.

Note that the second property is harmless can always be assumed because we take an infimum over subsets of $A \in \mathcal{F}$. Especially this property implies

$$c(F, \mathcal{F}) = \inf_{A \in \mathcal{F}} \sup_A F = \inf_{r \in \mathbb{R}} \big\{ r : \ M_r(F) \in \mathcal{F} \big\}, \tag{5.13}$$

where

$$M_r(F) = \big\{ x : \ F(x) < r \big\}.$$

Indeed if we write c and $\bar{c}$ for the left and right-hand sides of the second equality in (5.12), then for any $a > c$, we can find $A \in \mathcal{F}$ such that $\sup_A F \le a$, which means that $A \subseteq M_F(a)$. This in turn implies that $M_a(F) \in \mathcal{F}$, which leads to $\hat{c} \le c$. In the same fashion, we can verify $c \le \hat{c}$.

It remains to design a family $\mathcal{F}$ such that **(1)** and **(2)** hold, and $c(F, \mathcal{F})$ is finite. Once such a family is found, we set $\mathcal{F}(F) = c(F, \mathcal{F})$. In view of (5.13), and property **(1)**, we my choose $\mathcal{F}$ the collection of sets with certain degree of topological complexity, so that $c(F, \mathcal{F})$ is the first r for which the sublevel set $M_r(F)$ reaches such complexity. We now describe the

LS strategy. Write $\Omega_k^0(\ell, r_0)$ for the set of $F \in \Omega_k(\ell, r_0)$ such that $F(0) = 0$. Let us consider $F \in \Omega_k^0(\ell, r_0)$, with $\ell(x) = \frac{1}{2}Bx \cdot x$. Set $c_0 = r_0\delta(B)^{-1}$, and $c_1 = r_0 c_0$,so that

$$\nabla F(\bar{x}) = 0 \quad \implies \quad |\bar{x}| \le c_0 \quad \implies \quad |F(\bar{x})| \le c_1.$$

Note that ℓ has a single critical point at the origin. Hence for $a < 0 < b$, the sets $M_b(\ell)$ is topologically more complex than $M_a(\ell)$. Since F is a Lipschitz perturbation of ℓ, and all critical values of F are in the interval $[-c_1, c_1]$, we expect $M_{c_1}(F)$ to be topologically more complex than $M_{-c_1}(F)$. We wish to design a collection $\mathcal{F}$ that captures such complexity. *Relative Cohomology Classes* allow us to measure such complexity.

Definition 5.5 Given two open sets $A \subset B$, we write $\Lambda^j(B, A)$ for the set of closed j forms α in B such that the restriction of α to the set A is exact. We write $\alpha \sim \beta$ for two forms in $\Lambda^j(B, A)$ such that $\beta - \alpha$ is exact in B. We write $H^j(B, A)$ for the set of equivalent classes and $H^*(B, A)$ for the union of $H^j(B, A), j = 0, 1, \dots$. □

For example, for $a < 0 < b$, one can show that $H^*\big(M_b(\ell), M_a(\ell)\big)$ is the same as $H^*(D, \partial D)$, where D is a disc in $\mathbb{R}^{r^-}$, with r^- denoting the number of the negative eigenvalues of B. In fact the set $M_{-c_1}(F)$ is homeomorphic to $M_{-c_1}(\ell)$, and we may define

$$\mathcal{C}(F) = \inf\big\{r : \ H^*\big(M_r(F), M_{-c_1}(F)\big) \neq 0\big\} = \sup\big\{r : \ H^*\big(M_r(F), M_{-c_1}(F)\big) = 0\big\}.$$

More generally, we may take any $\alpha \in H^*\big(M_b(\ell), M_a(\ell)\big)$, and set

$$\begin{aligned}\mathcal{C}(F; \alpha) &= \inf\big\{r : \ \text{ the restriction of } \alpha \text{ to } M_r(F) \text{ is not exact}\big\}\\ &= \sup\big\{r : \ \text{ the restriction of } \alpha \text{ to } M_r(F) \text{ is exact}\big\}.\end{aligned}$$

5.2 Game Theory

We now offer a way of constructing viscosity solutions. For our purposes, it is more convenient to solve the final value problem

$$(5.14) \qquad \begin{cases} u_t + H(q, u_q) = 0, & t < T, \\ u(q, T) = g(q). \end{cases}$$

We assume that H is of the following form

$$H(q, p) = \inf_{z \in Z} \hat{H}(q, p; z) = \inf_{z \in Z} \sup_{v} \big(p \cdot v - \hat{L}(q, v; z)\big),$$

where Z is some measure space, $H(q, p; z)$ is convex in p for each $z \in Z$, and we writing $\hat{L}(q, v; z)$ for its Legendre transform in the p-variable. We assume that the family $\big\{\hat{L}(\cdot, \cdot; z) :$

$z \in Z\big\}$ is Tonelli, uniformly in z: There exist constants $\eta_0 > 1, \delta_0 > 0$, and a_0 such that

$$\hat{L}(q, v; z) \geq L_0(v) := \delta_0 |v|^{\eta_0} - a_0, \qquad \sup_{|v'| \leq 1} \hat{L}(q, v'; z) \leq a_0, \tag{5.15}$$
$$\lim_{\delta \to 0} \sup_{z' \in Z} \sup_{|x| \leq 1} \sup_{|x - x'| \leq \delta} \big| \hat{H}(x'; z') - \hat{H}(x; z') \big| = 0,$$

for all $q, v \in \mathbb{R}^d$ and $z \in Z$.

Definition 5.6 We write $V(t)$ for the set of bounded measurable maps $v : [t, T] \to \mathbb{R}^d$, and $Z(t, T)$ for the set of measurable maps $z : [t, T] \to Z$. We write $\Delta(t, T)$ for the set of *strategies*. By a strategy, we mean a map $\alpha : Z(t, T) \to V(t, T)$ such that if $t < s \leq T$, and $z = z'$ on $[t, s]$, then $\alpha[z] = \alpha[z']$ on $[t, s]$. □

We are now ready to offer a solution to (5.14). For $t \leq T$, set

$$u(q, t) = \mathcal{V}_t^T(g)(q) = \sup_{\alpha \in \Delta(t,T)} \inf_{z \in Z(t,T)} \left[g(q(T)) - \int_t^T \hat{L}\big(q(\theta), \dot{q}(\theta); z(\theta)\big) \, d\theta \right], \tag{5.16}$$

where $q(\cdot)$ is uniquely specified by the requirements $q(t) = q$, and $\dot{q} = \alpha[z] =: v$. In other words, for $\theta \in [t, T]$,

$$q(\theta) = q + \int_t^\theta \alpha[z](\theta') \, d\theta'.$$

Note that we may write $\dot{q}(\theta) = \hat{H}_p\big(q(\theta), p(\theta); z(\theta)\big)$, where

$$p(\theta) = \hat{L}_v\big(q(\theta), \alpha[z](\theta); z(\theta)\big).$$

In terms of $p(\cdot)$, we have

$$L\big(q(\theta), \dot{q}(\theta); z(\theta)\big) = p(\theta) \cdot \dot{q}(\theta) - H\big(q(\theta), p(\theta); z(\theta)\big).$$

Note that when H is not convex in p, the relationship $v = H_p(q, p)$ is no longer invertible in p for a every q. However, if we specify z, then we can invert $p \mapsto \hat{H}_p(p, q; z)$. The role of the path $q(\cdot)$ is the same as the characteristic. The optimal path still solves the Hamiltonian ODE locally, but it is allowed to have corners. This is when we switch from one lael z to another.

Theorem 5.5 *The function u as in* (5.15) *is a viscosity solution of* (5.14).

The main ingredient for the proof of Theorem 5.5 is the following *dynamic programming optimality condition*:

Theorem 5.6 *For $s \in [t,T]$, we have*

$$\mathcal{V}_t^T(g)(q) = \sup_{\alpha\in\Delta(t,s)} \inf_{z\in Z(t,s)} \left[\mathcal{V}_s^T(g)(q(s)) - \int_t^s \hat{L}\big(q(\theta),\dot{q}(\theta);z(\theta)\big)\ d\theta\right]. \tag{5.17}$$

Proof Fix q. We write u and u' for the left and right hand sides of (5.6) respectively. We carry out the proof in two steps. First we pick $c < u'$. We wish to show that $c < u$. For this, first from $c < u'$, we know that there exists $\beta \in \Delta[t,s]$ such that for all $y \in Z(t,s)$, we have

$$c < \mathcal{V}_s^T(g)(q(s)) - \int_t^s \hat{L}\big(q(\theta),\dot{q}(\theta);z(\theta)\big)\ d\theta,$$

with $q(\theta) = q + \int_t^\theta \beta[y](\theta')\ d\theta'$. Now given $a = q(s)$, we can find $\gamma_a \in \Delta(s,T)$ such that for every $w \in Z(s,T)$, we have

$$c < g(q(T)) - \int_s^T \hat{L}\big(q(\theta),\dot{q}(\theta);w(\theta)\big)\ d\theta - \int_t^s \hat{L}\big(q(\theta),\dot{q}(\theta);y(\theta)\big)\ d\theta, \tag{5.18}$$

where

$$q(\theta) = q(s) + \int_s^\theta \gamma_{q(s)}[w](\theta')\ d\theta' = q + \int_t^s \beta[y](\theta')\ d\theta' + \int_s^\theta \gamma_{q(s)}[w](\theta')\ d\theta',$$

for $\theta \in [s,T]$. We now construct $\alpha \in \Delta(t,T)$ as follows: Given $z \in Z(t,T)$, we set

$$\alpha[z](\theta) = \begin{cases} \beta\big[z \restriction_{[t,s]}\big](\theta), & \theta \in [t,s] \\ \alpha_{q(s)}\big[z \restriction_{[s,T]}\big](\theta), & \theta \in (s,T], \end{cases}$$

where $q(s) = q + \int_t^s \beta\big[z \restriction_{[t,s]}\big](\theta)\ d\theta$. For this α, (5.18) means

$$c < g(q(T)) - \int_t^T \hat{L}\big(q(\theta),\dot{q}(\theta);z(\theta)\big)\ d\theta,$$

for every $z \in Z(t,T)$. This completes the proof of $u' \le u$.

We now turn to the proof of $u \le u'$. Pick $c < u$, and choose $\alpha \in \Delta(t,T)$ such that for every $z \in Z(t,T)$

$$\begin{aligned} c &< g(q(T)) - \int_t^T \hat{L}\big(q(\theta),\dot{q}(\theta);z(\theta)\big)\ d\theta \\ &= g(q(T)) - \int_s^T \hat{L}\big(q(\theta),\dot{q}(\theta);z(\theta)\big)\ d\theta - \int_t^s \hat{L}\big(q(\theta),\dot{q}(\theta);z(\theta)\big)\ d\theta. \end{aligned}$$

We the define $\beta \in \Delta(t,s)$ as follows: for every $y \in Z(s,t)$, we have $\beta[y] = \alpha[y']$, where $y' \in Z(t,T)$, is any extension of y. For this β, we wish to show that for every $y \in Z(t,s)$,

$$c < \mathcal{V}_s^T(g)(q(s)) - \int_t^s \hat{L}\big(q(\theta), \dot{q}(\theta); z(\theta)\big)\ d\theta,$$

where $q(\theta) = q + \int_t^\theta \beta[y](\theta')\ d\theta'$ for $\theta \in [t,s]$. Given $y \in Z(s,t)$, we need to come up with a family of strategies $\gamma_a \in \Delta(s,T)$ such that for every $w \in Z(s,T)$, we have

$$c < g(q(T)) - \int_s^T \hat{L}\big(q(\theta), \dot{q}(\theta); w(\theta)\big)\ d\theta - \int_t^s \hat{L}\big(q(\theta), \dot{q}(\theta); y(\theta)\big)\ d\theta,$$

This is achieved by setting

$$\gamma_{q(s)}[w] = \alpha[y \oplus w],$$

where

$$(y \oplus w)(\theta) = \begin{cases} y(\theta), & \theta \in [t,s], \\ w(\theta) & \theta \in [s,T]. \end{cases}$$

□

As our next step we show that we can always restrict α in (5.17) to those with bounded range:

Proposition 5.3 *If g is Lipschitz with Lipschitz constant r, then the supremum in* (5.17) *can be restricted to those α such that*

$$M(\alpha) := \sup_{z \in Z(t,T)} M(\alpha, z) := \sup_{z \in Z(t,T)} \left[\frac{1}{T-t}\int_t^T \big|\alpha[z](\theta)\big|^{\eta_0}\ d\theta\right]^{\frac{1}{\eta_0}} \leq C_0, \tag{5.19}$$

where

$$C_0 = C_0(r, \delta_0, \eta_0, a_0) = 2\alpha_0 + \left(\frac{r+1}{\delta_0}\right)^{\frac{1}{\eta_0 - 1}}.$$

Proof Assume that $g \in \Lambda_r$. Write

$$A\big(q; \alpha, z(\cdot)\big) := g(q(T)) - \int_t^T \hat{L}\big(q(\theta), \dot{q}(\theta); z(\theta)\big)\ d\theta,$$

with $q(\cdot)$ as in (5.16). We certainly have

$$\begin{aligned} A\big(q; \alpha, z(\cdot)\big) &\leq g(q) + r\left|\int_t^T \alpha[z]\ d\theta\right| + a_0(T-t) - \delta_0(T-t)M(\alpha)^{\eta_0} \\ &\leq g(q) + r(T-t)M(\alpha) + a_0(T-t) - \delta_0(T-t)M(\alpha)^{\eta_0}. \end{aligned}$$

On the other hand,

$$A\big(q;0,z(\cdot)\big) = g(q) - \int_t^T \hat{L}\big(q,0;z(\theta)\big)\ d\theta \geq g(q) - a_0(T-t).$$

In (5.16), we may ignore those α such that

$$\inf_{z\in Z(t,T)} A\big(q;\alpha,z(\cdot)\big) < g(q) - a_0(T-t).$$

For this, it suffices that for some $z(\cdot) \in Z(t,T)$, we have

$$r(T-t)M(\alpha,z) + a_0(T-t) - \delta_0(T-t)M(\alpha,z)^{\eta_0} < -a_0(T-t)$$

Equivalently,

$$\delta_0 M(\alpha,z)^{\eta_0} - rM(\alpha,z) - 2a_0 > 0.$$

This inequality is valid if

$$M(\alpha,z) > C_0 := 2\alpha_0 + \left(\frac{r+1}{\delta_0}\right)^{\frac{1}{\eta_0-1}}.$$

In summary, we may ignore those α such that

$$\sup_{z\in Z(t,T)} M(\alpha,z) > C_0.$$

We are done. □

With the aid of (5.19), we can show the regularity of of $u = \mathcal{V}_t(g)$.

Theorem 5.7 *Assume that $g \in \Lambda_r$. Then the following statements are true:*

(i) *The value of $u(q,t) = \big(\mathcal{V}_t^T g\big)(q)$ depends only on the restriction of g to the set*

$$B_{C_0(T-t)}(q) := \big\{q' :\ |q'-q| \leq C_0(T-t)\big\}.$$

(ii) *The value of $u(q,t) = \big(\mathcal{V}_t^T g\big)(q)$ depends only on the restriction of $\hat{H}$ to the set*

$$B_{C_0(T-t)}(q) \times \mathbb{R}^d \times Z = \big\{(q',p,z) \in \mathbb{R}^{2d} \times Z :\ |q'-q| \leq C_0(T-t)\big\}.$$

(iii) *We have*

$$-a_0(T-t) \leq u(q,t) - g(q) \leq C_1(T-t), \tag{5.20}$$

where $C_1 = C_1(r) = a_0 + c_1 r^{\eta_1}$, for constants $\eta_1 = (\eta_0-1)/\eta_0$, and $c_1 = c_1(\delta_0,\eta_0)$.

(iv) *Assume that* $s \in [t, T]$. *Then*

$$-a_0(s-t) \le u(q,t) - u(q,s) \le C_1(s-t). \tag{5.21}$$

(v) *For every* $t < T$, *and* $q, q' \in \mathbb{R}^d$, *we have*

$$|u(q',t) - u(q,t)| \le (C_1 + a_0 + r)|q' - q|. \tag{5.22}$$

Proof(i) The dependence of u on the final data is of the form $g(q(T))$ with

$$|q(T) - q| = \left| \int_t^T \alpha[z] \; d\theta \right| \le C_0(T-t),$$

by Proposition 5.19.

(ii) The spatial dependence of $\hat{L}$ is $q(\theta)$ with $\theta \in [t, T]$. We are done because $|q(\theta) - q| \le C_0(T-t)$.

(iii) By choosing the strategy $\alpha = 0$ in the definition of u we get

$$u(q,t) \ge g(q) - a_0(T-t).$$

On the other hand, by the Lipschitzness of g and (5.15),

$$\begin{aligned}
u(q,t) &\le g(q) + \sup_{\alpha \in \Delta(t,T)} \inf_{z \in Z(t,T)} \left[r|q(T) - q| - \int_t^T L_0\big(\dot{q}(\theta)\big) \; d\theta \right] \\
&\le g(q) + \sup_{\alpha \in \Delta(t,T)} \inf_{z \in Z(t,T)} \left[r|q(T) - q| - (T-t) L_0 \left(\frac{q(T) - q(t)}{T-t} \right) \right] \\
&= g(q) + \sup_Q \left[r|Q - q| - (T-t) L_0 \left(\frac{Q - q}{T-t} \right) \right] \\
&= g(q) + (T-t) \sup_{a \ge 0} \big[ra - \delta_0 a^{\eta_0} + a_0 \big] \\
&= g(q) + (T-t) \big[a_0 + c_1 r^{\eta_1} \big],
\end{aligned}$$

as desired.

(iv) Set $\delta = s - t$. From (5.17) and since $\hat{L}$ does not depend on t,

$$u(q,t) = \big(\mathcal{V}_{s-\delta}^T g\big)(q) = \big(\mathcal{V}_{s-\delta}^{T-\delta}\big(\mathcal{V}_{T-\delta}^T g\big)\big)(q) = \big(\mathcal{V}_s^T\big(\mathcal{V}_{T-\delta}^T g\big)\big)(q).$$

From this, $u(q,t) = \mathcal{V}_s^T g(q)$, and the contraction of the operator $\mathcal{V}_s^T$,

$$\inf \big(\mathcal{V}_{T-\delta}^T g - g\big) \le u(q,t) - u(q,s) \le \sup \big(\mathcal{V}_{T-\delta}^T g - g\big).$$

This and (5.20) yield (5.21).

(v) Set $\rho = |q - q'|$. First we assume that $\rho \geq T - t$. We then use (5.20) to write

$$\begin{aligned} u(q', t) - u(q, t) &\leq (C_1 + a_0)(T - t) + g(q') - g(q) \\ &\leq (C_1 + a_0)(T - t) + r|q' - q| \\ &\leq (C_1 + a_0 + r)|q' - q|. \end{aligned}$$

Hence

$$(5.23) \qquad |q' - q| \geq T - t \quad \Longrightarrow \quad |u(q', t) - u(q, t)| \leq (C_1 + a_0 + r)|q' - q|.$$

On the other hand, when $\rho < T - t$, we use (5.12) and Proposition 5.3 to write

$$u(q, t) = \sup_{\alpha \in \Delta(t, t+\rho)} \inf_{z \in Z(t, t+\rho)} \left[u\big(q(t + \rho), t + \rho\big) - \int_t^{t+\rho} \hat{L}\big(q(\theta), \dot{q}(\theta); z(\theta)\big)\, d\theta \right].$$

From this and (5.21) we learn

$$u(q, t) \geq \sup_{\alpha \in \Delta(t, t+\rho)} \inf_{z \in Z(t, t+\rho)} \left[u\big(q(t + \rho), t\big) - \int_t^{t+\rho} \hat{L}\big(q(\theta), \dot{q}(\theta); z(\theta)\big)\, d\theta \right] - C_1 \rho.$$

Pick a vector e and choose the constant strategy $\alpha[z] = e$ to assert

$$\begin{aligned} u(q, t) &\geq \inf_{z \in Z(t, t+\rho)} \left[u(q + \rho e, t) - \int_t^{t+\rho} \hat{L}\big(q + \theta e, e; z(\theta)\big)\, d\theta \right] - a_0 \rho \\ &\geq u(q + \rho e, t) - (C_1 + a_0)\rho. \end{aligned}$$

We now choose $e = (q' - q)/|q' - q|$ to conclude

$$u(q, t) - u(q', t) \geq -2a_0 \rho,$$

which yields

$$|q' - q| \leq T - t \quad \Longrightarrow \quad |u(q', t) - u(q, t)| \leq (C_1 + a_0)|q' - q|.$$

This and (5.23) yield (5.22). $\square$

Proof of Theorem 5.5 Fix (q_0, t_0), and assume that $\phi \in C^1$ with

$$u(q_0, t_0) = \phi(q_0, t_0), \quad u \leq \phi, \quad p_0 = \phi_q(q_0, t_0), \quad r_0 = \phi_t(q_0, t_0).$$

Pick $\delta > 0$, and write $\Delta'(t_0, t_0 + \delta)$ for the set of $\alpha \in \Delta(t_0, t_0 + \delta)$ such that

$$M(\alpha) := \sup_{z \in Z(t_0, t_0+\delta)} \left[\delta^{-1} \int_{t_0}^{t_0+\delta} \big|\alpha[z](\theta)\big|^{\eta_0} \, d\theta \right]^{\frac{1}{\eta_0}} \leq C_0.$$

By Theorem 5.6,

$$u(q_0, t_0) = \sup_{\alpha \in \Delta'(t_0, t_0+\delta)} \inf_{z \in Z(t_0, t_0+\delta)} \left[u\big(q(t_0+\delta), t_0+\delta\big) - \int_{t_0}^{t_0+\delta} \hat{L}\big(q(\theta), \dot{q}(\theta); z(\theta)\big) \, d\theta \right],$$

where $q(\theta) = q_0 + \int_{t_0}^{\theta} \alpha[z](\theta) \, d\theta$. To ease the notation, we write Δ'_δ and Z_δ for $\Delta'(t_0, t_0 + \delta)$ and $Z(t_0, t_0 + \delta)$. This implies

$$\begin{aligned}
0 &\leq \sup_{\alpha \in \Delta'_\delta} \inf_{z \in Z_\delta} \left[\phi\big(q(t_0+\delta), t_0+\delta\big) - \phi(q_0, t_0) - \int_{t_0}^{t_0+\delta} \hat{L}\big(q(\theta), \dot{q}(\theta); z(\theta)\big) \, d\theta \right] \\
&= \sup_{\alpha \in \Delta'_\delta} \inf_{z \in Z_\delta} \left[\int_{t_0}^{t_0+\delta} \Big(\phi_t(q(\theta), \theta) + \dot{q}(\theta) \cdot \phi_q(q(\theta), \theta) - \hat{L}\big(q(\theta), \dot{q}(\theta); z(\theta)\big) \Big) \, d\theta \right] \\
&\leq \sup_{\alpha \in \Delta'_\delta} \inf_{z \in Z_\delta} \left[\int_{t_0}^{t_0+\delta} \Big(\phi_t(q(\theta), \theta) + \hat{H}\big(q(\theta), \phi_q(q(\theta), \theta); z(\theta)\big) \Big) \, d\theta \right] \\
&\leq \sup_{\alpha \in \Delta'_\delta} \inf_{z \in Z} \left[\int_{t_0}^{t_0+\delta} \Big(\phi_t(q(\theta), \theta) + \hat{H}\big(q(\theta), \phi_q(q(\theta), \theta); z\big) \Big) \, d\theta \right],
\end{aligned}$$

where for the last inequality, we take the infimum over constant paths in $Z(t_0, t_0 + \delta)$. On the other hand, since $M(\alpha) \leq C_0$, with C_0 independent of δ,

$$|q(\theta) - q_0| \leq \int_{t_0}^{\theta} \big|\alpha[z](\theta')\big| \, d\theta' \leq \delta M(\alpha) \leq C_0 \delta, \tag{5.24}$$

for $\theta \in [t_0, t_0 + \delta]$. Hence, using the continuity of H as in (5.15),

$$\phi_t(q(\theta), \theta) + \hat{H}\big(q(\theta), \phi_q(q(\theta), \theta); z\big) \leq \phi_t(q_0, t_0) + \hat{H}\big(q_0, \phi_q(q_0, t_0); z\big) + c_1(\delta),$$

for a constant $c_1(\delta)$ such that $c_1(\delta) \to 0$ as $\delta \to 0$. As a result

$$\begin{aligned}
0 &\leq \delta \sup_{\alpha \in \Delta'_\delta} \inf_{z \in Z} \left[\phi_t(q_0, t_0) + \hat{H}\big(q_0, \phi_q(q_0, t_0); z\big) + c_1(\delta) \right] \\
&= \delta \inf_{z \in Z} \left[r_0 + \hat{H}(q_0, p_0; z) + c_1(\delta) \right] = \delta \left[r_0 + H(q_0, p_0) + c_1(\delta) \right].
\end{aligned}$$

We divide both sides by δ and send $\delta \to 0$ to arrive at $0 \leq r_0 + H(q_0, p_0)$, as desired. (Note that since we are solving a backward HJE, this is the correct inequality.)

We next assume that $\phi \in C^1$ with

$$u(q_0,t_0) = \phi(q_0,t_0), \quad u \le \phi, \quad p_0 = \phi_q(q_0,t_0), \quad r_0 = \phi_t(q_0,t_0).$$

After a repetition of what we did above, we now have

$$0 \ge \sup_{\alpha \in \Delta'_\delta} \inf_{z \in Z_\delta} \left[\int_{t_0}^{t_0+\delta} \Big(\phi_t(q(\theta),\theta) + \dot q(\theta) \cdot \phi_q(q(\theta),\theta) - \hat L\big(q(\theta), \dot q(\theta); z(\theta)\big) \Big) \ d\theta \right].$$

Using (5.22),

$$\phi_t(q(\theta),\theta) + H\big(q(\theta), \phi_q(q(\theta),\theta)\big) \ge r_0 + H(q_0,p_0) - c_1\delta,$$

for some constant c_1. From this and (5.21) we deduce

$$0 \ge \delta \left[r_0 + H(q_0,p_0) + c_1\delta \right].$$

We divide both sides by δ and send $\delta \to 0$ to arrive at $0 \ge r_0 + H(q_0,p_0)$, as desired. □

Example 5.5(i) When Z is a singleton, H is convex in p, the the set $\Delta(t,T)$ is isomorphic to the set $V(t,T)$, and (5.16) simply reads as

$$\begin{aligned} \mathcal{V}_t(g)(q) &= \sup_{\dot q \in V(t,T)} \left\{ g(q(T)) - \int_t^T \hat L\big(q(\theta), \dot q(\theta); z(\theta)\big) \ d\theta : \ q(0) = q \right\} \\ &= \sup_{q \in C^1(t,T)} \left\{ g(q(T)) - \int_t^T \hat L\big(q(\theta), \dot q(\theta); z(\theta)\big) \ d\theta : \ q(0) = q \right\}. \end{aligned}$$

(ii) We now assume that $Z = \big\{ z_1, \dots, z_k \big\}$ is finite, and that $H^i(p) = \hat H(p; z_i)$ are independent of position for each $i = 1, \dots, k$. We also write $L^i(v) = L(v; z_i)$. We make some definitions:

- We write $\mathbf{I}$ for the set of finite sequences of the form $\mathbf{i} = (i_0, \dots, i_\ell)$ of indices in $I = \{1, \dots, k\}$ such that $i_j \ne i_{j+1}$ for $j = 0, 1, \dots, \ell - 1$. We write $|\mathbf{i}| = \ell$ for the size of the sequence $\mathbf{i}$.
- We write $\mathbf{\Theta}(t,T)$ for the set of finite sequences of the form $\boldsymbol{\theta} = \big(\theta_0, \theta_1, \dots, \theta_\ell\big)$, with

  $$\theta_0 = t < \theta_1 < \dots < \theta_\ell < \theta_{\ell+1} = T.$$

  We write $|\boldsymbol{\theta}| = \ell$ for the size of the sequence $\boldsymbol{\theta}$.
- We write $\hat Z(t,T)$ for the set of pairs $(\mathbf{i}, \boldsymbol{\theta}) \in \mathbf{I} \times \mathbf{\Theta}(t,T)$ with $|\mathbf{i}| = |\boldsymbol{\theta}|$. By a slight abuse of notation, we think of $(\mathbf{i}, \boldsymbol{\theta})(\cdot) \in Z(t,T)$, with $(\mathbf{i}, \boldsymbol{\theta})(s) = z_{i_j}$, for $s \in [\theta_j, \theta_{j+1}), j = 0, 1, \dots, \ell$.

- We write $\mathbf{V}$ for the set of finite sequences of the form $\mathbf{v} = (v_0, \dots, v_\ell)$ of vectors in $\mathbb{R}^d$. We write $|\mathbf{v}| = \ell$ for the size of the sequence $\mathbf{v}$.
- We write $\hat{V}(t,T)$ for the set of pairs $(\mathbf{v}, \boldsymbol{\theta}) \in \mathbf{V} \times \boldsymbol{\Theta}(t,T)$ with $|\mathbf{v}| = |\boldsymbol{\theta}|$. By a slight abuse of notation, we think of $(\mathbf{v}, \boldsymbol{\theta})(\cdot) \in V(t,T)$, with $(\mathbf{v}, \boldsymbol{\theta})(s) = v_j$, for $s \in [\theta_j, \theta_{j+1}), j = 0, 1, \dots, \ell$.

Pick $\alpha \in \Delta(t,T)$, $(\mathbf{i}, \boldsymbol{\theta}) \in \hat{Z}(t,T)$, and write $v^j : [\theta_j, \theta_{j+1})$ for the restriction of $\alpha\big[(\mathbf{i}, \boldsymbol{\theta})(\cdot)\big]$ to the interval $[\theta_j, \theta_{j+1})$. We then have

$$\int_t^T L(\dot{q}(\theta); z(\theta))\; d\theta = \sum_{j=0}^{\ell} \int_{\theta_j}^{\theta_{j+1}} L^{i_j}(v^j(\theta))\; d\theta.$$

Recall that $q(s) = \int_t^s \alpha[\mathbf{z}](\theta)\; d\theta$, with $q(t) = t$ and $\mathbf{z} = (\mathbf{i}, \boldsymbol{\theta})$. We may define a sequence $q_0 = q, q_1, \dots, q_{\ell+1}$ inductively by

$$q_{j+1} = q_j + \int_{\theta_j}^{\theta_{j+1}} v^j(\theta)\; d\theta.$$

Since

$$\int_{\theta_j}^{\theta_{j+1}} L^{i_j}(v^j(\theta))\; d\theta \geq \big(\theta_{j+1} - \theta_j\big) L^{i_j}\left(\frac{q_{j+1} - q_j}{\theta_{j+1} - \theta_j}\right),$$

we learn that the action cannot decrease if we switch from the path $(v^0, v^1, \dots, v^k)$ to a collection of appropriate constant paths on the same intervals. Motivated by this, we now define $\hat{\Delta}$ for maps $\boldsymbol{\alpha} : \mathbf{I} \to \mathbf{V}$ such that the following two conditions hold:

- $|\mathbf{i}| = |\mathbf{v}|$.
- If $\mathbf{i}, \mathbf{i}' \in \mathbf{I}$, and $\mathbf{v} = \boldsymbol{\alpha}[\mathbf{i}], \mathbf{v}' = \boldsymbol{\alpha}[\mathbf{i}']$, with $i_r = i'_r$ for $r = 0, 1, \dots, m$, then $v_r = v'_r$ for $r = 0, 1, \dots, m$.

We now have a simpler expression for the viscosity solution:

$$\mathcal{V}_t(g)(q) = \sup_{\boldsymbol{\alpha} \in \hat{\Delta}} \inf_{(\mathbf{i}, \boldsymbol{\theta}) \in \hat{Z}(t,T)} \left\{ g\big(Q(q; \mathbf{i}, \boldsymbol{\theta})\big) - \sum_{j=0}^{|\mathbf{i}|} \big(\theta_{j+1} - \theta_j\big) L^{i_j}(v_j) :\; q(0) = q \right\},$$

where

$$Q(q; \mathbf{i}, \boldsymbol{\theta}) = q + \sum_{j=0}^{|\mathbf{i}|} \big(\theta_{j+1} - \theta_j\big) v_j, \qquad \mathbf{v} = \boldsymbol{\alpha}(\mathbf{i}).$$

(iii) If in **(ii)** we assume that $k = 2$, and write $L^{\pm}$ for L^1 and L^2, then $\mathbf{i} \in \mathbf{I}$ is fully determined by its length and i_0. Hence $\hat{Z}(t,T) = Z^+(t,T) \cup Z^-(t,T)$, where $Z^{\pm}(t,T)$ is isomorphic to $\{\pm\} \times \mathbf{\Theta}(t,T)$. □

Exercise(i) Assume that $d = 1$ and u is a (continuous) viscosity solution of (1.10). Let U be an open set in $\mathbb{R} \times (0,\infty)$ and assume that u is C^1 in $U \setminus \Gamma$, where

$$\Gamma = \big\{(a(t),t)) : \ t \in (t_0,t_1)\big\} \subset U,$$

with $a : (t_0,t_1) \to \mathbb{R}$ a C^1 function. Assume that $u = u^+$ and u^-, on the right and left side of Γ in U and both $u^{\pm}$ solve (1.10) classically. Use Example 5.3 to show the following:

- $\dot{a}(t) = H\big[u_q^+(a(t),t), u_q^-(a(t),t)\big]$.
- The pair $\big(u_q^-(a(t),t), u_q^+(a(t),t)\big)$ satisfies the Oleinink Condition for every $t \in (t_1,t_2)$.

□

6 Second Variation

Let M be a closed manifold and set $X = \mathcal{T}^*M$. Consider a Hamiltonian function $H : X \to \mathbb{R}$ and write $\phi_t^H = \phi_t$ for its flow. Consider a Lagrangian bundle $(L_x : \ x \in X)$ that is invariant for the flow. That is

$$\big(d\phi_t\big)_x L_x = L_{\phi_t(x)}.$$

We assume that L is a graph: for some symmetric $S_x : \mathcal{T}_q M \to \mathcal{T}_q^* M$, we have

$$L_x = \big\{(\hat{q}, S_q \hat{q}) : \ \hat{q} \in \mathcal{T}_q M\big\}.$$

If $\hat{x}(t) = (\hat{q}(t), \hat{p}(t)) = (d\phi_t)_x \hat{x}(0)$, then we have

$$\frac{d\hat{q}}{dt}(t) = H_{qp}(x(t))\hat{q}(t) + H_{pp}(x(t))\hat{p}(t), \qquad \frac{d\hat{p}}{dt}(t) = -H_{qq}(x(t))\hat{q}(t) - H_{qp}(x(t))\hat{p}(t).$$

Assuming $\hat{p} = S_x \hat{q}$,

$$\frac{d\hat{q}}{dt}(t) = \big(H_{qp}(x(t)) + H_{pp}(x(t))S_{x(t)}\big)\hat{q}(t),$$
$$\frac{d\hat{p}}{dt}(t) = -\big(H_{qq}(x(t)) + H_{qp}(x(t))S_{x(t)}\big)\hat{q}(t).$$

On the other hand,

$$\frac{d\hat{p}}{dt}(t) = \frac{d}{dt}\big(S_{x(t)}\big)\hat{q}(t) + S_{x(t)}\frac{d\hat{q}}{dt}(t) = \frac{d}{dt}\big(S_{x(t)}\big)\hat{q}(t) + S_{x(t)}\big(H_{qp}(x(t)) + H_{pp}(x(t))S_{x(t)}\big)\hat{q}(t).$$

As a result,

$$\frac{d}{dt}\big(S_{x(t)}\big) + S_{x(t)}H_{pp}(x(t))S_{x(t)} + S_{x(t)}H_{qp}(x(t)) + H_{qp}(x(t))S_{x(t)} + H_{qq}(x(t)) = 0.$$

Writing $S(t) = S_{x(t)}$, we get a Ricatti type equation

$$\dot{S} + SH_{pp}S + SH_{qp} + H_{qp}S + H_{qq} = 0. \tag{6.1}$$

We next take two such Lagrangian bundles L and L' associated with S and S', and set

$$S(t) = S_{\phi_t(x)}, \qquad S'(t) = S'_{\phi_t(x)}, \qquad x(t) = \phi_t(x).$$

We have that if $\hat{x}(t) = (\hat{q}(t), S(t)\hat{q}(t)) \in L_{x(t)}$, and $\hat{x}'(t) = (\hat{q}(t), S'(t)\hat{q}(t)) \in L'_{x(t)}$, then

$$\omega(t) := \omega_{x(t)}\big(\hat{x}(t), \hat{x}'(t)\big) = (S(t) - S'(t))\hat{q}(t) \cdot \hat{q}(t).$$

As a result

$$\begin{aligned}
\dot{\omega}(t) &= 2(S(t) - S'(t))\dot{\hat{q}}(t) \cdot \hat{q}(t) + (\dot{S}(t) - \dot{S}'(t))\hat{q}(t) \cdot \hat{q}(t) \\
&= 2(S(t) - S'(t))\big(H_{qp} + H_{pp}S(t)\big)\hat{q}(t) \cdot \hat{q}(t) \\
&\quad - \big(S(t)H_{pp}S(t) - S'(t)H_{pp}S'(t)\big)\hat{q}(t) \cdot \hat{q}(t) \\
&\quad + \big((S(t) - S'(t))H_{qp} + H_{qp}(S(t) - S'(t)\big)\hat{q}(t) \cdot \hat{q}(t) \\
&= (S(t) - S'(t))H_{pp}(x(t))(S(t) - S'(t))\hat{q}(t) \cdot \hat{q}(t).
\end{aligned}$$