1 Chapter 1

(iii) We have

$$\Delta u = \Delta_r u + r^{-2} u_{\theta\theta}, \qquad \text{where } \Delta_r u = u_{rr} + r^{-1} u_r.$$

If $-\lambda$ is an eigenvalue corresponding to the eigenfunction $w(x) = R(r)\Theta(\theta)$, then

$$-\lambda r^2 = r^2 \frac{\Delta_r R}{R}(r) + \frac{\Theta''}{\Theta}(\theta).$$

This is possible only if Θ''/Θ is a constant. Since Θ is 2π-periodic, we have

$$\Theta(\theta) = a_n \cos(n\theta) + b_n \sin(n\theta), \qquad n \in \mathbb{N}$$

for some constants a_n and b_n. This leads to the equation

$$\Delta_r R(r) + \left(\lambda - \frac{n^2}{r^2}\right) R = 0, \qquad R(a) = 0.$$

Note that since λ is an eigenvalue of $-\Delta$, we must have $\lambda > 0$ (because $\langle \Delta u, u \rangle < 0$ unless $u = 0$). To replace λ with 1, we rescale $\rho = \sqrt{\lambda} r$, $R(r) = \hat{R}(\rho)$, so that

$$\begin{aligned} &R_r = \sqrt{\lambda}\hat{R}_\rho, \qquad R_{rr} = \lambda \hat{R}_{\rho\rho}, \\ &\hat{R}_{\rho\rho} + \rho^{-1}\hat{R}_\rho + \left(1 - \frac{n^2}{\rho^2}\right)\hat{R} = 0, \qquad \hat{R}(a\sqrt{\lambda}) = 0. \end{aligned}$$

This is the Bessel's ODE of order $n \in \mathbb{N}$. This has a solution that does not blow up at the origin and satisfies $R(0) = 0$. Such a solution looks like a constant multiple of ρ^n near $\rho = 0$. A suitable choice of constant, namely $2^{-n}/(n!)$ leads to the Bessel function of order n that is denoted by $J_n(\rho)$. It has zeros of the form $0 < \lambda_{n1} < \cdots < \lambda_{nk} < \ldots$. We need to choose λ so that $\sqrt{\lambda} a = \lambda_{nk}$ for some k. In summary we have eigenvalues of the form

$$a^{-1}\lambda_{nk}^2, \qquad n, k \in \mathbb{N}.$$

The corresponding eigenfunctions can be chosen to be

$$J_n(\sqrt{\lambda_{nk}} r)\cos(n\theta), \qquad J_n(\sqrt{\lambda_{nk}} r)\sin(n\theta).$$

(v) Let $\bar{\lambda}$ be the minimium value of the RHS, and $\bar{w} \in H^1(\Omega)$ a minimizer. Then for any $v \in H^1(\Omega)$ such that $v \perp w_1, \ldots, w_{n-1}$, we have

$$\langle \nabla \bar{w}, \nabla v \rangle - \bar{\lambda} \langle \bar{w}, v \rangle = 0. \tag{1.1}$$

On the other hand, for $j = 1, \dots, n-1$,

$$\langle \nabla \bar{w}, \nabla w_j \rangle - \bar{\lambda} \langle \bar{w}, w_j \rangle = \langle \nabla \bar{w}, \nabla w_j \rangle = -\langle \bar{w}, \Delta w_j \rangle + \int_{\partial\Omega} \bar{w} \frac{\partial w_j}{\partial n} \, dS = \lambda_j \langle \bar{w}, w_j \rangle = 0,$$

because $\bar{w} \perp w_1, \dots, w_{n-1}$, and $\partial w_j / \partial n = 0$ on $\partial\Omega$. From this, (1.1) and Theorem 1.1 we learn that (1.1) is true for every v satisfying $\partial v / \partial n = 0$. In particular, we can choose an arbitrary function v that is identically 0 near boundary. For such v we can integrate by parts to deduce

$$\langle \Delta \bar{w} + \bar{\lambda} \bar{w}, v \rangle = 0.$$

This means that $\Delta \bar{w} + \bar{\lambda} \bar{w} = 0$ strictly inside Ω. Since $\bar{w} \in H^1(\Omega)$, we deduce that $\bar{w}$ has 3 weak derivatives, and after a bootstrap we show that $\bar{w}$ is smooth. In fact, we may use $-\Delta \bar{w} = \bar{\lambda} \bar{w}$ to assert that $\bar{w} \in H^2(\Omega)$, and as a result, we can make sense of $\partial \bar{w} / \partial n \in L^2(\partial\Omega)$. We now go back to (1.1) and integrate by parts to assert that for any smooth v,

$$\langle \Delta \bar{w} + \bar{\lambda} \bar{w}, v \rangle = \int_{\partial\Omega} v \frac{\partial \bar{w}}{\partial n} \, dS.$$

Since the LHS is zero, and v is arbitrary on $\partial\Omega$, we are done. (To make sense of $\partial \bar{w} / \partial n$ on $\partial\Omega$, take a collection of nested $\Omega_n \subset \Omega$ such that $\cup_n \Omega_n = \Omega$ and $\partial\Omega_n$ strictly inside Ω. If we write g_n for $\partial \bar{w} / \partial n$ on $\partial\Omega_n$, then we have a compact sequence in L^2 because $\bar{w} \in H^2(\Omega)$. Any limit point of this sequence serves as a candidate of $\partial \bar{w} / \partial n$.)

(vi) Note that if

$$\hat{E}(\lambda) = \cup_{k \in E(\lambda)} \prod_j [k_j - 1, k_j],$$

then

$$x \in \hat{E}(\lambda) \quad \Leftrightarrow \quad \lceil x \rceil \in Q.$$

Hence $\mathcal{N}^D(\lambda) = Vol(\hat{E}(\lambda)) \le Vol(Q)$. For (2.24) it suffices to show that $Q^- \subset \hat{E}(\lambda)$. Indeed

$$x \in Q^- \quad \Longrightarrow \quad x + (1, \dots, 1) \in Q \quad \Longrightarrow \quad \lceil x \rceil = \lfloor x + (1, \dots, 1) \rfloor \in Q \Longrightarrow \quad x \in \hat{E}(\lambda).$$

(viii) Since $Vol(Q^-) \le \mathcal{N}(\lambda) \le Vol(Q)$, it suffices to bound $Vol(Q) - Vol(Q^-)$. Define

$$\hat{Q} = \left\{ x = (x_1, \dots, x_d) : \; x_1, \dots x_d \ge 1, \; \sum_{j=1}^d \left(\frac{x_j}{\ell_j} \right)^2 \le \frac{\lambda}{\pi^2} \right\},$$

If $\tau(x_1, \dots, x_d) = (x_1 + 1, \dots, x_d + 1)$, then $\tau Q^- = \hat{Q}$. Evidently $Vol(Q^-) = Vol(\hat{Q})$, and $\hat{Q} \subset Q$. Hence

$$Vol(Q) - Vol(Q^-) = Vol(Q) - Vol(\hat{Q}) = Vol\big(Q \setminus \hat{Q}\big) = \sum_{i=1}^d Vol(Q_i),$$

where

$$
\begin{aligned}
Q_i &= \left\{ x = (x_1, \dots, x_d) : \ x_1, \dots x_d \geq 0, \ x_i \leq 1, \ \sum_{j=1}^d \left(\frac{x_j}{\ell_j} \right)^2 \leq \frac{\lambda}{\pi^2} \right\} \\
&\subset \left\{ x = (x_1, \dots, x_d) : \ x_1, \dots x_d \geq 0, \ x_i \leq 1, \ \sum_{j \neq i} \left(\frac{x_j}{\ell_j} \right)^2 \leq \frac{\lambda}{\pi^2} \right\} =: Q_i'.
\end{aligned}
$$

Note that Q_i' is a product of and the interval $[0,1]$, and a set Q_i'' which is the analog of Q for the variables $(x_j : \ i \neq i)$ in dimension $d-1$.

$$
Vol(Q_i) \leq Vol(Q_i') = Vol(Q_i'') = 2^{-d+1} \omega_{d-1} \prod_{j \neq i} \frac{\ell_j \sqrt{\lambda}}{\pi}.
$$

Hence

$$
\sum_{i=1}^d Vol(Q_i) \leq 2^{-1} (2\pi)^{-d+1} \ \omega_{d-1} \ \lambda^{(d-1)/2} \ Vol(\partial \Omega).
$$

(x) Note that if for $s \leq t$,

$$
h(x,s,t) = \int_\Omega S(x,y,t-s) f(y,s) \ dy,
$$

then $h_t = \alpha \Delta h$, and

$$
\lim_{s \to t} h(x,s,t) = \lim_{\delta \to 0} \int_\Omega S(x,y,\delta) f(y, t-\delta) \ dy \lim_{\delta \to 0} \int_\Omega S(x,y,\delta) f(y,t) \ dy = f(x,t).
$$

Here for the second equality, we have used the continuity of f. From this we learn that if

$$
w(x,t) = \int_0^t \int_\Omega S(x,y,t-s) f(y,s) \ dy \ ds,
$$

then

$$
\begin{aligned}
w_t(x,t) &= \frac{\partial}{\partial t} \int_0^t h(x,s,t) \ ds = f(x,t) + \int_0^t h_t(x,s,t) \ ds \\
&= f(x,t) + \alpha \int_0^t \Delta h(x,s,t) \ ds = f(x,t) + \alpha \Delta w(x,t).
\end{aligned}
$$

(x) Given u, the function $w(x,t) = u(x)$ satisfies

$$
w_t = \frac{1}{2} \Delta w + V w,
$$

where $V(x) = -\Delta u / (2u)$. On the other hand, by Feynman-Kac Formula (and the uniqueness of solution)

$$
u(x) = w(x,t) = \mathbb{E} \ u(x + B(t)) e^{-\int_0^t \frac{\Delta u}{2u}(x+B(s)) \ ds}.
$$