Lectures on Dynamical Systems

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Introduction

The main goal of the theory of dynamical system is the study of the global orbit structure of maps and flows. In these notes, we review some fundamental concepts and results in the theory of dynamical systems with an emphasis on differentiable dynamics.

Several important notions in the theory of dynamical systems have their roots in the work of Maxwell, Boltzmann and Gibbs who tried to explain the macroscopic behavior of fluids and gases on the basis of the classical dynamics of many particle systems. The notion of ergodicity was introduced by Boltzmann as a property satisfied by a Hamiltonian flow on its constant energy surfaces. Boltzmann also initiated a mathematical expression for the entropy and the entropy production to derive Maxwell’s description for the equilibrium states. Gibbs introduced the notion of mixing systems to explain how reversible mechanical systems could approach equilibrium states. The ergodicity and mixing are only two possible properties in the hierarchy of stochastic behavior of a dynamical system. Hopf invented a versatile method for proving the ergodicity of geodesic flows. The key role in Hopf’s approach is played by the hyperbolicity. Lyapunov exponents and Kolmogorov–Sinai entropy are used to measure the hyperbolicity of a system.
1 Invariant Measures and Ergodic Theorem

By a discrete dynamical system we mean a pair \((X, T)\), where \(X = (X, d)\) is a complete separable metric space (in short Polish space) with metric \(d\), and \(T : X \to X\) is a continuous map. By an orbit of \((X, T)\) we mean sequences of the form \(O(x) = (x_n = T^n(x) : n \in \mathbb{N}^*)\), where \(\mathbb{N}^*\) denotes the set of nonnegative integers. Here are some examples of dynamical systems that should be kept in mind for understanding various notions that will be developed in this Chapter.

Example 1.1(i) \(X = \mathbb{T}^d\) is the \(d\)-dimensional torus. We may regard \(\mathbb{T}\) as the interval \([0, 1]\) with \(0 = 1\). Given a vector \(\alpha = (\alpha_1, \ldots, \alpha_d) \in [0, 1)^d\), we define \(T : \mathbb{T}^d \to \mathbb{T}^d\) by \(T(x) = x + \alpha \mod 1\). In other words, when \(x \in [0, 1)^d\), then \(x + \alpha\) is understood as the sum of points \(x\) and \(\alpha\) in \(\mathbb{R}^d\). Though if \(x + \alpha \not\in [0, 1)^d\), then by dropping its integer parts of its coordinates, we obtain a point in \([0, 1)^d\). Alternatively, if we regard the circle \(\mathbb{T}\) as the set of complex numbers \(z = e^{2\pi i \theta}\) such that \(|z| = 1\), and set \(\beta = (\beta_1, \ldots, \beta_d)\), with \(\beta_j = e^{2\pi i \alpha_j}\), then \(T(z_1, \ldots, z_d) = (\beta_1 z_1, \ldots, \beta_d z_d)\).

(ii) Given an integer \(m \geq 2\), we define \(T = T_m : \mathbb{T} \to \mathbb{T}\), by \(T(x) = mx \mod 1\). Alternatively, if we regard the circle \(\mathbb{T}\) as the set of complex numbers \(z = e^{2\pi i \theta}\) such that \(|z| = 1\), then \(T(x) = z^m\).

(iii) Given a Polish space \(E\), set \(X = E^{\mathbb{N}}\) (respectively \(E^{\mathbb{Z}}\)) for the space of sequences \(\omega = (\omega_n : n \in \mathbb{N})\) (respectively \(\omega = (\omega_n : n \in \mathbb{Z})\)) in \(E\). Consider the shift map \(\tau : X \to X\) that is defined by \((\tau \omega)_n = \omega_{n+1}\).

Given a dynamical system \((X, T)\), we may wonder how often a subset of \(X\) is visited by an orbit of \(T\). For example, in the dynamical systems described in Example 1.1, most orbits (for “most” \(\alpha\) in part (i)) are dense and every nonempty open set is visited infinitely often for any such orbit. To measure the asymptotic fraction of times a set is visited, we may look at the limit points of the sequence

\[
\frac{1}{n} \sum_{j=0}^{n-1} \mathbb{I}_A(T^j(x))
\]

as \(n \to \infty\). More generally, we may wonder whether or not the limit

\[
\lim_{n \to \infty} \Phi_n(f)(x) := \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j(x))
\]

exists for a function \(f : X \to \mathbb{R}\). Let us write \(C_b(X)\) for the space of bounded continuous functions \(f : X \to \mathbb{R}\). Given \(x \in X\), if the limit of (1.2) exists for every \(f \in C_b(X)\), then the limit \(\ell_x(f)\) enjoys some obvious properties:
(i) \( f \geq 0 \Rightarrow \ell_x(f) \geq 0, \ell_x(\mathbb{1}) = 1 \).

(ii) \( \ell_x(f) \) is linear in \( f \).

(iii) \( |\ell_x(f)| \leq \sup_{y \in X} |f(y)| \).

(iv) \( \ell_x(f \circ T) = \ell_x(f) \).

If \( X \) is also locally compact, then we can use Riesz Representation Theorem to assert that there exists a unique (Radon) probability measure \( \mu \) such that \( \ell_x(f) = \int f \, d\mu \). Evidently, such a measure \( \mu(A) \) measures how often a set \( A \) is visited by the orbit \( O(x) \). Motivated by (iv), we make the following definition:

**Definition 1.1(i)** Given a Polish space \( X \), with a metric \( d \), we write \( \mathcal{B}(X) \) for the Borel \( \sigma \)-algebra of \( (X,d) \), and \( \mathcal{M}(X) \) for the set of Borel Radon probability measures on \( X \).

(ii) We write \( \mathcal{I}_T \) for the set of Radon probability measures \( \mu \) such that

\[
(1.3) \quad \int f \circ T \, d\mu = \int f \, d\mu,
\]

for every \( f \in C_b(X) \). Any such measure \( \mu \) is called an invariant measure of \( T \). Equivalently, \( \mu \in \mathcal{I}_T \) iff \( \mu(A) = \mu(T^{-1}(A)) \) for every \( B \in \mathcal{B}(X) \). \( \square \)

It seems natural that for analyzing the limit points of (1.1), we should first try to understand the space \( \mathcal{I}_T \) of invariant measures. Note that in (1.2), what we have is \( \int f \, d\mu_n \) where \( \mu_n^x = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{T^j(x)} \). We also learned that if (1.2) exists for every \( f \), then \( \mu_n^x \) has a limit and its limit is an invariant measure. Of course there is a danger that the limit (1.2) does not exist in general. This is very plausible if the orbit is unbounded and some of the mass of the measure \( \mu_n^x \) is lost as \( n \to \infty \) because \( T^j(x) \) goes off to infinity. This would not happen if we assume \( X \) is compact. To this end, let us review the notion of weak convergence for measures.

**Definition 1.2** We say a sequence \( \{ \mu_n \}_{n \in \mathbb{N}} \) in \( \mathcal{M}(X) \) converges weakly to \( \mu \in \mathcal{M}(X) \) (in short \( \mu_n \Rightarrow \mu \)), if

\[
(1.4) \quad \int f \, d\mu_n \to \int f \, d\mu,
\]

for every \( f \in C_b(X) \). \( \square \)

It turns out that for the weak convergence, we only need to verify (1.4) for \( f \in U_b(X) \) where \( U_b(X) \) denotes the space of bounded uniformly continuous functions. Since \( U_b(X) \) is separable, we can metrize the space of probability measures \( \mathcal{M}(X) \). (See for example [P].)

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Theorem 1.1 Suppose $X$ is a compact metric space.

(i) (Krylov–Bogobulov) Let $\{x_n\}$ be a sequence in $X$. Then any limit point of the sequence $\{\mu x_n\}$ is in $\mathcal{I}_T$. In particular, $\mathcal{I}_T \neq \emptyset$.

(ii) If $\mathcal{I}_T = \{\bar{\mu}\}$ is singleton, then

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j(x)) = \int f \, d\bar{\mu}
$$

uniformly for every $f \in C(X)$. In fact $D(\mu x_n, \bar{\mu}) \to 0$ uniformly in $x$.

(iii) If $\{\Phi_n(f)\}$ converges pointwise to a constant for functions $f$ in a dense subset of $C(X)$, then $\mathcal{I}_T$ is a singleton.

Proof

(i) This is an immediate consequence of Exercise 1.1(ii) and what we had in the beginning of this section.

(ii) Let $\{x_n\}$ be any sequence in $X$ and put $\nu_n = \mu x_n$. Since any limit point of $\{\nu_n\}$ is in $\mathcal{I}_T = \{\mu\}$, we deduce that $\nu_n \Rightarrow \bar{\mu}$. From this we can readily deduce that in fact $\mu x_n \Rightarrow \mu$ uniformly.

(iii) We are assuming that $\Phi_n(f)$ converges to a constant $\hat{f}$ for $f$ in a dense set $\mathcal{A} \subseteq C(X)$. Since the sequence $\{\Phi_n(f)\}$ is uniformly bounded for $f \in C(X)$, we deduce

$$
\int \Phi_n(f) \, d\mu = \int f \, d\mu,
$$

for every $\mu \in \mathcal{I}_T$. Hence the constant $\hat{f}$ can only be $\int f \, d\mu$. As a consequence, if $\mu, \nu \in \mathcal{I}_T$, then $\int f \, d\mu = \int f \, d\nu$ for a dense set of functions $f$. This implies that $\mu = \nu$ and we conclude that $\mathcal{I}_T$ is a singleton.

From Theorem 1.1 we learn that when $\mathcal{I}_T$ is a singleton, the statistics of the orbits are very simple. However, this is a rather rare situation and when it happens, we say that the transformation $T$ is uniquely ergodic.

Example 1.2(i) Consider the dynamical system of Example 1.1(i), when $d = 1$, and $\alpha = p/q$ a rational number with $p$ and $q$ coprime. Note that every orbit is periodic of period $q$. Moreover, for every $x \in \mathbb{T}$, the measure

$$
\mu^x = \frac{1}{q} \sum_{j=0}^{q-1} \delta_{T^j(x)},
$$
is invariant for $T$. One can show that any $\mu \in \mathcal{I}_T$ can be expressed as

$$\mu = \int_T \mu^x \tau(dx),$$

where $\tau$ is a probability measure on $\mathbb{T}$. To avoid repetition, we only need to take a probability measure that is concentrated on the interval $[0, q^{-1})$, or the interval $[0, q^{-1}]$, with $0 = q^{-1}$.

(iii) Consider the dynamical system of Example 1.1(i), when $d = 2$, and $\alpha = (\alpha_1, 0)$ with $\alpha_1 \notin \mathbb{Q}$. Let $\mu^{x_2}$ denotes the one-dimensional Lebesgue measure that is concentrated on the circle

$$\mathbb{T}_{x_2} = \{(x_1, x_2) \in \mathbb{T}^2 : x_1 \in \mathbb{T}\}.$$ 

Clearly this measure is invariant. In fact all invariant measures can be expressed as

$$\mu = \int_T \mu^{x_2} \tau(dx_2),$$
where \( \tau \) is a probability measure on \( T \). We also note
\[
\lim_{n \to \infty} \Phi_n(f) = \int_T f(x_1, x_2) \, dx_1.
\]

\[\Box\]

**Remark 1.1(i)** When \( d = 1 \), the condition (1.5) is equivalent to \( \alpha \notin \mathbb{Q} \). The fact that Lebesgue measure is the only invariant measure when \( \alpha \) is irrational is equivalent to that fact that the sequence \( \{n\alpha\} \) is dense on the circle \( T \). To see this, observe that if \( \mu \in \mathcal{I}_T \), then
\[
\int f(x + n\alpha) \mu(dx) = \int f(x) \mu(dx)
\]
for every continuous \( f \) and any \( n \in \mathbb{N} \). Since \( \{n\alpha\} \) is dense, we deduce that \( \mu \) is translation invariant. As is well known, the only translation invariant finite measure on \( T \) is the Lebesgue measure.

**(ii)** According to a classical result of Poincaré, if an orientation preserving homeomorphism \( T : T \to T \) has a dense orbit, then it is isomorphic to a rotation (i.e. there exists a change of coordinates \( h : T \to T \) such that \( h^{-1} \circ T \circ h \) is a rotation).

\[\Box\]

As we mentioned earlier, in most cases \( \mathcal{I}_T \) is not a singleton. There are some obvious properties of the set \( \mathcal{I}_T \) which we now state. Note that \( \mathcal{I}_T \) is always a convex and closed subset of \( \mathcal{M}(X) \). Also, \( \mathcal{I}_T \) is compact when \( X \) is compact because \( \mathcal{M}(X) \) is compact. Let us recall a theorem of Choquet that can be used to get a picture of the set \( \mathcal{I}_T \). Recall that if \( C \) is a compact convex set then a point \( a \in C \) is **extreme** if \( a = \theta b + (1 - \theta) c \) for some \( \theta \in [0, 1] \) and \( b, c \in C \) implies that either \( a = b \) or \( a = c \). According to Choquet’s theorem, if \( C \) is convex and compact, then any \( \mu \in C \) can be expressed as an average of the extreme points. More precisely, we can find a probability measure \( \theta \) on the set of extreme points of \( C \) such that
\[
\mu = \int_{C^{\text{ex}}} \alpha \theta(d\alpha).
\]

Motivated by (1.5) and Example 1.2, we formulate two natural concepts:

**Definition 1.3(i)** We write \( \mathcal{I}_T^{\text{ex}} \) for the set of extreme points of \( \mathcal{I}_T \).

**(ii)** Given \( \mu \in \mathcal{I}_T^{\text{ex}} \), we set
\[
X_\mu = \{x : \mu_n^x \Rightarrow \mu \text{ as } n \to \infty\}.
\]

\[\Box\]
Example 1.3 In Example 1.1(i), we have $\mathcal{I}^{xt}_T = \{\mu^x : x \in [0,q^{-1}]\}$. Example 1.2(iii), we have $\mathcal{I}^{xt}_T = \{\mu^{x_2} : x_2 \in T\}$.

Given $\mu \in \mathcal{I}^{xt}_T$, clearly the set $X_\mu$ is invariant under $T$. That is, if $x \in X_\mu$, then $T(x) \in X_\mu$. Also, if $\mu_1 \neq \mu_2 \in \mathcal{I}^{xt}_T$, then $X_{\mu_1} \cap X_{\mu_2} = \emptyset$. Our second Ergodic Theorem below implies that $\mu(X_\mu) = 1$. This confirms the importance of extreme measures among the invariant measures. Later we find more a practical criterion for the extremity in terms of the invariant sets and functions.

Theorem 1.2 (von Neumann) Let $T : X \to X$ be a Borel measurable transformation and let $\mu \in \mathcal{I}_T$. If $f \in L^2(\mu)$, then $\Phi_n(f) = \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^j$ converges in $L^2$-sense to $Pf$, where $Pf$ is the projection of $f$ onto the space of invariant functions $g$ satisfying $g \circ T = g$.

Proof. We note that $\Phi_n : L^2(\mu) \to L^2(\mu)$ is a bounded linear operator with

$$\|\Phi_n(f)\| \leq \|f\|,$$

because $\|f_k \circ T^j\|_{L^2} = \|f_k\|_{L^2}$ by invariance. Also observe that if $f = g \circ T - g$ for some $g \in L^2$, then $\Phi_n(f) \to 0$ as $n \to \infty$. Let $\mathcal{H}$ denote the linear space of gradient type functions $g \circ T - g$. If $f \in \mathcal{H}$, then we still have $\Phi_n(f) \to 0$ as $n \to \infty$. Indeed if $f_k \in \mathcal{H}$ converges to $f$ in $L^2$, then

$$\|\Phi_n(f)\|_{L^2} \leq \|\Phi_n(f_k)\|_{L^2} + \|f - f_k\|_{L^2},$$

Since $\|\Phi_n(f_k)\|_{L^2} \to 0$ as $n \to \infty$ and $\|f - f_k\|_{L^2} \to 0$ as $k \to \infty$, we deduce that $\Phi_n(f) \to 0$ as $n \to \infty$.

Given any $f \in L^2(\mu)$, write $f = g + h$ with $g \in \mathcal{H}$ and $h \perp \mathcal{H}$. If $h \perp \mathcal{H}$, then $\int h \varphi \circ T \, d\mu = \int h \varphi \, d\mu$, for every $\varphi \in L^2(\mu)$. Hence $\int (h \circ T - h)^2 \, d\mu = 0$. This means that $h \circ T = h$. As a result, $h$ is invariant and $\Phi_n(f) = \Phi_n(g) + \Phi_n(h) = \Phi_n(g) + h$. Since $\Phi_n(g) \to 0$, we deduce that $\Phi_n(f) \to h$ with $h = Pf$. □

Theorem 1.2 is also true in $L^1(\mu)$ setting. To explain this, let us first make sense of $Pf$ for $f \in L^1(\mu)$. For example, we may approximate any $f \in L^1(\mu)$ by the sequence $f_k = f \mathbb{1}_{(|f| \leq k)}$, and define

$$Pf = \lim_{k \to \infty} Pf_k.$$ 

The limit exists because the sequence $\{Pf_k\}$ is Cauchy in $L^1(\mu)$ (this is an immediate consequence of Exercise 1.2(ii) below). We are now ready to state and prove the Ergodic Theorem for $L^1$-functions.

Corollary 1.1 Suppose $\mu \in \mathcal{I}_T$ and $f \in L^1(\mu)$. Let $Pf$ be as above. Then $\Phi_n(f)$ converges to $Pf$ in $L^1$ sense.
Proof Clearly,
\[
\| \Phi_n(f) \|_{L^1(\mu)} \leq \| f \|_{L^1(\mu)},
\]
by the invariance of \( \mu \). From this and \( \int |P g| \, d\mu \leq \int |g| \, d\mu \) (see Exercise 1.2(ii) below) we learn
\[
\| \Phi_n(f) - Pf \|_{L^1(\mu)} \leq \| \Phi_n(f_k) - Pf_k \|_{L^1(\mu)} + \| \Phi_n(f - f_k) \|_{L^1(\mu)} + \| P(f - f_k) \|_{L^1(\mu)}
\]
where \( k \in \mathbb{N}, f_k = f 1(|f| \leq k) \). The proof follows because by Theorem 1.2 the first term goes to 0 as \( n \to \infty \), and by approximation, the second term goes to 0, as \( k \to \infty \).

Remark 1.2 We note that if \( \mu \in \mathcal{I}_T \), then the operator \( U f = f \circ T \) is an isometry of \( L^2(\mu) \) and the space of invariant functions \( \{ \varphi : \varphi \circ T = \varphi \} \) is the eigenspace associated with the eigenvalue one. Hence von Neumann’s theorem simply says that \( \frac{1}{n}(I + U + \cdots + U^{n-1}) \to P \). Note that if \( \lambda = e^{i\theta} \) is an eigenvalue of \( U \) and if \( \lambda \neq 1 \), then \( \frac{1}{n}(1 + \lambda + \cdots + \lambda^{n-1}) = \frac{\lambda^n - 1}{n(\lambda - 1)} \to 0 \) as \( n \to \infty \). This suggests that Theorem 1.2 may be proved using the Spectral Theorem for unitary operators. The above theorem also suggests studying the spectrum of the operator \( U \) for a given \( T \). Later we will encounter the notion of mixing dynamical systems. It turns out that the mixing condition implies that discrete spectrum of the operator \( U \) consists of the point 1 only.

From Theorem 1.2 we learn the relevance of the invariant (conserved) functions for a dynamical system. One possibility is that the only invariant function in the support of \( \mu \) is the constant function. In fact if there are non constant functions in \( \mu \), we may use them to decompose \( \mu \) into invariant measures with smaller support. The lack of nontrivial conserved functions is an indication of the irreducibility of our invariant measure. We may check such irreducibility by evaluating \( \mu \) at \( T \)-invariant subsets of \( X \). More precisely, we have the following definition.

Definition 1.4(i) We write \( \mathcal{F}_T \) for the set of Borel sets \( A \) that are invariant for \( T \); \( T^{-1}A = A \). Given \( \mu \in \mathcal{M}(X) \), we set
\[
\mathcal{F}_T^\mu = \{ A \in \mathcal{B}(X) : \mu (T^{-1}(A) \Delta A) = 0 \}.
\]
Equivalently, \( A \in \mathcal{F}_T \) (respectively \( \mathcal{F}_T^\mu \)), iff \( 1_A \circ T(x) = 1_A(x) \) (respectively \( 1_A \circ T = 1_A \), \( \mu \)-a.e.).

(ii) An invariant measure \( \mu \) is called ergodic if \( \mu(A) \in \{ 0, 1 \} \) for every \( A \in \mathcal{F}_T \). The set of ergodic invariant measures is denoted by \( \mathcal{I}_T^\mu \).

We will see in Exercise (vi) below that the sets in \( \mathcal{F}_T^\mu \) differ from sets in \( \mathcal{F}_T \) only in a set of measure 0. Also, we will see later that \( \mathcal{I}_T^e = \mathcal{I}_T^{ex} \). In view of (1.6), any \( \mu \in \mathcal{I}_T \) can be expressed as an average of ergodic ones.
Remark 1.3 By Theorem 1.2, we know that if $f \in L^2(\mu)$, then $Pf$ is the projection of $f$ onto the space of invariant functions. For $f \in L^1(\mu)$, we may define $Pf$ as the unique $\mathcal{F}_T^\mu$-measurable function such that

(1.9) \[ \int_A Pf \, d\mu = \int_A f \, d\mu \]

for every $A \in \mathcal{F}_T$. Note that since $Pf$ is $\mathcal{F}_T^\mu$-measurable, we have $Pf \circ T = Pf$, $\mu$-almost everywhere. Alternatively, $Pf$ is uniquely defined as the Radon–Nikodym derivative $f \mu$ with respect to $\mu$, if we restrict it to $\mathcal{F}_T - \sigma$-algebra. More precisely

$$ Pf = \frac{d(f \mu|_{\mathcal{F}_T})}{d\mu|_{\mathcal{F}_T}}. $$

□

As our next goal, we consider an almost everywhere mode of convergence.

**Theorem 1.3 (Birkhoff Ergodic Theorem)** Suppose $\mu \in \mathcal{I}_T$ and $f \in L^1(\mu)$. Then

$$ \mu \left\{ x : \lim_{n \to \infty} \Phi_n(f)(x) = Pf(x) \right\} = 1. $$

**Proof** Let $g = f - Pf - \epsilon$ for a fixed $\epsilon > 0$. Evidently $Pg \equiv -\epsilon < 0$ and $\Phi_n(f - Pf - \epsilon) = \Phi_n(f) - Pf - \epsilon$. Hence, it suffices to show

$$ \limsup_{n \to \infty} \Phi_n(g) \leq 0 \quad \mu \text{-a.e.} $$

We expect to have

$$ g + g \circ T + \cdots + g \circ T^{n-1} = -\epsilon n + o(n). $$

From this, it is reasonable to expect that the expression $g + \cdots + g \circ T^{n-1}$ to be bounded above $\mu$-a.e. Because of this, let us define $G_n = \max_{j \leq n} \sum_{i=0}^{j-1} g \circ T^i$. Set $A = \{ x : \lim_{n \to \infty} G_n(x) = +\infty \}$. Without loss of generality, we may assume that $g$ is finite everywhere. Clearly $A \in \mathcal{F}_T$ because $G_{n+1} = g + \max(0, G_n \circ T)$. Note also that if $x \notin A$, then $\limsup_{n \to \infty} \Phi_n(g) \leq 0$. To complete the proof, it remains to show that $\mu(A) = 0$. To see this, observe

$$ 0 \leq \int_A (G_{n+1} - G_n) \, d\mu = \int_A (G_{n+1} - G_n \circ T) \, d\mu $$

$$ = \int_A [g + \max(0, G_n \circ T) - G_n \circ T] \, d\mu = \int_A (g - \min(0, G_n \circ T)) \, d\mu. $$

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On the set \( A, -\min(0, G_n \circ T) \downarrow 0 \). On the other hand, if
\[
h_n = g - \min(0, G_n \circ T),
\]
then \( g \leq h_n \leq h_1 = g + (g \circ T)^- \). Hence by the Dominated Convergence Theorem, \( 0 \leq \int_A g d\mu = \int_A P g d\mu \leq -\epsilon \mu(A) \). Thus we must have \( \mu(A) = 0 \).

**Remark 1.4(i)** If \( \mu \) is ergodic, then the \( \sigma \)-algebra \( \mathcal{F}_T \) is trivial and any \( \mathcal{F}_T \) measurable function is constant. Hence \( Pf \) is constant and this constant can only be \( \int f d\mu \).

(ii) Since \( \mu_n^x \Rightarrow \mu \) iff \( \int f d\mu_n^x \to \int f d\mu \), for \( f \) in a countable dense set of continuous functions, we learn from Part (i) that \( \mu(X_\mu) = 1 \), where \( X_\mu \) was defined by (1.5). However, if \( \mu \) is not ergodic, then \( Pf \) is not constant in general and if \( \mu_T(x, dy) \) denotes the conditional distribution of \( \mu \) given \( \mathcal{F}_T \), then
\[
P f(x) = \int f(y)\mu_T(x, dy).
\]
From this we deduce that in this case,
\[
\mu \left\{ x : \lim_{n \to \infty} \mu_n^x = \mu_T(x, \cdot) \right\} = 1,
\]
Moreover, \( \mu_T(x, \cdot) \in \mathcal{I}_{\mu}^x \), for \( \mu \)-almost all \( x \).

(iii) If \( T \) is invertible, then we can have an ergodic theorem for \( T^{-1} \) as well. Since \( \mathcal{F}_T = \mathcal{F}_{T^{-1}} \), it is clear that \( P_T f = P_{T^{-1}} f \). As a consequence we have
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} f \circ T^j = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} f \circ T^{-j} = Pf.
\]
\( \mu \)-a.e.

We continue with more consequences of Theorem 1.6.

**Theorem 1.4** We have \( \mathcal{I}_{\mu}^x = \mathcal{I}_{\mu}^\mu \). Moreover, if \( \mu_1 \) and \( \mu_2 \) are two distinct ergodic measures, then \( \mu_1 \) and \( \mu_2 \) are mutually singular.

**Proof** Suppose that \( \mu \in \mathcal{I}_T \) is not ergodic and choose \( A \in \mathcal{F}_T \) such that \( \mu(A) \in (0, 1) \). If we define
\[
\mu_1(B) = \frac{\mu(A \cap B)}{\mu(A)}, \quad \mu_2(B) = \frac{\mu(A^c \cap B)}{\mu(A^c)},
\]
then \( \mu_1, \mu_2 \in \mathcal{I}_T \) and \( \mu = \alpha \mu_1 + (1 - \alpha)\mu_2 \) for \( \alpha = \mu(A) \). Hence \( \mathcal{I}_{\mu} \subseteq \mathcal{I}_{\mu_1} \) and \( \mathcal{I}_{\mu} \subseteq \mathcal{I}_{\mu_2} \). Therefore, \( \mathcal{I}_{\mu} \subseteq \mathcal{I}_{\mu_1} \cup \mathcal{I}_{\mu_2} \). Since \( \mu_1 \) and \( \mu_2 \) are ergodic, \( \mathcal{I}_{\mu_1} \) and \( \mathcal{I}_{\mu_2} \) are disjoint, we have \( \mathcal{I}_{\mu} = \mathcal{I}_{\mu_1} \cup \mathcal{I}_{\mu_2} \). Hence \( \mathcal{I}_{\mu} \) is not ergodic, contradiction. Therefore, \( \mu \) is ergodic.
Conversely, let $\mu$ be ergodic and $\mu = \alpha \mu_1 + (1 - \alpha) \mu_2$ for some $\mu_1, \mu_2 \in \mathcal{I}_T$ and $\alpha \in (0, 1)$. Note that we also have that $\mu_i(A) = 0$ or 1 if $A \in \mathcal{F}_T$ and $i = 1$ or 2. As a result, $\mu_1, \mu_2 \in \mathcal{I}_T^e$ and $\mu(X_\mu) = \mu_1(X_{\mu_1}) = \mu_2(X_{\mu_2}) = 1$. But $\mu = \alpha \mu_1 + (1 - \alpha) \mu_2$, and $\mu(X_\mu) = 1$, imply that $\mu_1(X_{\mu_1}) = \mu_2(X_{\mu_2}) = 1$. This is compatible with $\mu_1(X_{\mu_1}) = \mu_2(X_{\mu_2}) = 1$, only if $\mu = \mu_1 = \mu_2$.

Finally, if if $\mu_1$ and $\mu_2$ are two distinct ergodic measures, then $X_{\mu_1} \cap X_{\mu_2} = \emptyset$. This implies that $\mu_1 \perp \mu_2$, because $\mu_1(X_{\mu_1}) = \mu_2(X_{\mu_2}) = 1$. □

As we mentioned in the introduction, many important ergodic measures enjoy a stronger property known as mixing.

**Definition 1.5** A measure $\mu \in \mathcal{I}_T$ is called mixing if for any two measurable sets $A$ and $B$,

\begin{equation}
\lim_{n \to \infty} \mu(T^{-n}(A) \cap B) = \mu(A)\mu(B).
\end{equation}

□

**Remark 1.4** Mixing implies the ergodicity because if $A \in \mathcal{F}_T$, then $T^{-n}(A) = A$ and $T^{-n}(A) \cap A^c = \emptyset$. As a result,

$$\mu(A) = \lim_{n} \mu(T^{-n}(A) \cap A) = \mu(A)\mu(A),$$

which implies that either $\mu(A) = 0$ or $\mu(A) = 1$. Also note that if $\mu$ is ergodic, then

$$\mu \left\{ x : \frac{1}{n} \sum_{0}^{n-1} \mathbb{1}_A \circ T^j \to \mu(A) \right\} = 1,$$

which in turn implies

$$\lim_{n \to \infty} \int \left( \frac{1}{n} \sum_{0}^{n-1} \mathbb{1}_A \circ T^j \right) \mathbb{1}_B \ d\mu = \mu(A)\mu(B).$$

Hence the ergodicity means

\begin{equation}
\lim_{n \to \infty} \frac{1}{n} \sum_{0}^{n-1} \mu(T^{-j}(A) \cap B) = \mu(A)\mu(B).
\end{equation}

So, the ergodicity is some type of a weak mixing. □

**Example 1.4** Let $T : \mathbb{T}^d \to \mathbb{T}^d$ be a translation $T(x) = x + \alpha \ (\mod 1)$ with $\alpha$ satisfying (1.6). We now argue that $T$ is not mixing. To see this, take a set $B$ with $\tilde{\mu}(B) > 0$ and
assume that $B$ is not dense. Pick $x_0 \notin B$ and let $\delta = \text{dist}(x_0, B) > 0$. Take any set $A$ open with $\mu(A) > 0$ and $\text{diam}(A) < \delta/2$. By topological transitivity, $x_0 \in T^{-n}(A)$ for infinitely many $n \in \mathbb{N}$. Since $\text{diam}(T^{-n}(A)) = \text{diam}(A)$, we deduce that $T^{-n}(A) \cap B = \emptyset$ for such $n$'s. Clearly $\bar{\mu}(T^{-n}(A) \cap B) = 0$ does not converge to $\bar{\mu}(A)\bar{\mu}(B) \neq 0$ as $n \to \infty$. □

Before discussing examples of mixing systems, let us give an equivalent criterion for mixing.

**Proposition 1.1** A measure $\mu$ is mixing iff

$$
\lim_{n \to \infty} \int f \circ T^n g \, d\mu = \int f \, d\mu \int g \, d\mu
$$

for $f$ and $g$ in a dense subset of $L^2(\mu)$.

**Proof** If $\mu$ is mixing, then (1.10) is true for $f = 1_A$, $g = 1_B$. Hence (1.12) is true if both $f$ and $g$ are simple, i.e., $f = \sum_{j=1}^m c_j 1_{A_j}$, $g = \sum_{j=1}^m c_j' 1_{B_j}$. We now use the fact that the space of simple functions is dense in $L^2(\mu)$ to deduce that (1.12) is true for a dense set of functions.

For the converse, it suffices to show that if (1.12) is true for a dense set of functions, then it is true for every $f \in L^2(\mu)$. Observe that if $\|f - \hat{f}\|_{L^2}$ and $\|g - \hat{g}\|_{L^2}$ are small, then

$$
\left| \int f \circ T^n g \, d\mu - \int \hat{f} \circ T^n \hat{g} \, d\mu \right|
$$

is small. Indeed,

$$
\left| \int f \circ T^n g \, d\mu - \int \hat{f} \circ T^n \hat{g} \, d\mu \right| \leq \left| \int (f \circ T^n - \hat{f} \circ T^n) g \, d\mu \right| + \left| \int \hat{f} \circ T^n (g - \hat{g}) \, d\mu \right|
$$

by invariance and Schwartz Inequality. □

**Remark 1.5** We learn from Proposition 1.1 that $\mu$ is mixing iff for every $f \in L^2(\mu)$, we have $f \circ T^n \to \hat{f}$, as $n \to \infty$, where the constant $\hat{f}$ is $\int f \, d\mu$. □

**Example 1.5(i)** Let $(X, \tau)$ be as in Example 1.1(iii) and assume that the corresponding is a compact metric space. Given $\beta \in \mathcal{M}(E)$, let us write $\mu^\beta \in \mathcal{M}(X)$ for the product measure with marginals $\beta$. Clearly $\mu^\beta \in \mathcal{I}_\tau$. We now argue that indeed $\mu^\beta$ is mixing. To see this, write $\mathcal{A}_{loc}$ for the space of $L^2(\mu^\beta)$ functions that depend on finitely many coordinates. We claim that $\mathcal{A}_{loc}$ is dense in $L^2(\mu^\beta)$. This can be shown in two ways:
(1) Use Stone-Weirstrass Theorem to show that the space of $\mathcal{A}_{\text{loc}} \cap C(X)$ is dense in $C(X)$, and then apply Lusin’s Theorem to deduce that $\mathcal{A}_{\text{loc}}$ is dense in $L^2(\mu^\beta)$.

(2) Write $\mathcal{F}_k$ for the $\sigma$-algebra of sets that depend on the first $k$ coordinates. Now, given $f \in L^2(\mu^\beta)$, write $f_k = \mu^\beta(f|\mathcal{F}_k)$ for the conditional expectation of $f$, given $\mathcal{F}_k$. Note that $f_k \in \mathcal{A}_{\text{loc}}$. On the other hand $f_k \to f$ in $L^2(\mu^\beta)$ as $k \to \infty$ by the celebrated Martingale Convergence Theorem.

In view of Proposition 1.1, $\mu^\beta$ is mixing if we can show
\[
\lim_{n \to \infty} \int f \circ \tau^n \ g \ d\mu^\beta = \int f \ d\mu^\beta \int g \ d\mu^\beta,
\]
for every $f, g \in \mathcal{A}_{\text{loc}}$. Indeed if $f$ and $g$ depend on the first $k$ variables, then we simply have
\[
\int f \circ \tau^n \ g \ d\mu^\beta = \int f \ d\mu^\beta \int g \ d\mu^\beta,
\]
whenever $n > k$.

Also note that if $\bar{\omega} \in X$ is a periodic sequence of period exactly $k$, then
\[
\mu^{\bar{\omega}} = \frac{1}{k} \sum_{i=0}^{k-1} \delta_{\tau^i(\bar{\omega})} \in \mathcal{T}_r^{cr}.
\]

Though $\mu^{\bar{\omega}}$ is not mixing unless $k = 1$.

(ii) We next consider $T = T_m : \mathbb{T} \to \mathbb{T}$ of Example 1.1(ii). Observe that if $(X, \tau)$ is as in Part (i), for $E = \{0, \ldots, m - 1\}$, and $F : X \to [0, 1]$ is defined by
\[
F(\omega_1, \omega_2, \ldots, \omega_k, \ldots) = \sum_{i=1}^{\infty} \omega_i m^{-i},
\]
then $F$ is continuous. In fact, if we equip $X$ with the metric
\[
d(\omega, \omega') = \sum_{i=1}^{i} m^{-i} |\omega_i - \omega_i'|,
\]
then
\[
|F(\omega) - F(\omega')| \leq d(\omega, \omega').
\]

Given any $p = (p_0, \ldots, p_{m-1})$ with $p_j \geq 0$ and $p_0 + \cdots + p_{m-1} = 1$, we can construct a unique probability measure $\mu_p$ such that
\[
\mu_p(\omega_1 \ldots \omega_k, \omega_1 \ldots \omega_k + m^{-k}) = p_{\omega_1} p_{\omega_2} \cdots p_{\omega_k},
\]
where
\[ \omega_1 \omega_2 \ldots \omega_k = \omega_1 m^{-1} + \omega_2 m^{-2} + \ldots + \omega_k m^{-k}, \]

is a base \( m \) expansion with \( \omega_1, \ldots, \omega_k \in \{0, 1, \ldots, m-1\} \). It is not hard to show that \( \mu_p \) is an invariant measure for \( T_m \). In fact, if
\[ A = \{ x : x = \cdot \omega_1 \omega_2 \ldots \omega_k * * \ldots \}, \]

then
\[ T^{-1}(A) = \{ x : x = \cdot * \omega_1 \omega_2 \ldots \omega_k * * \ldots \} \]

and
\[ \mu_p(T^{-1}(A)) = \sum_{b=0}^{m-1} p_b p_{\omega_1} \ldots p_{\omega_k} = p_{\omega_1} \ldots p_{\omega_k} = \mu_p(A). \]

To show that each \( \mu_p \) is mixing observe that if
\[
\begin{align*}
A &= \{ x : x = \cdot \omega_1 \omega_2 \ldots \omega_k * * \ldots \}, \\
B &= \{ x : x = \cdot \omega_1 \omega_2 \ldots \omega_k * * \ldots \}
\end{align*}
\]

then
\[ T^{-n}(A) \cap B = \{ x : x = \cdot \underbrace{b_1 b_2 \ldots b_k}_{n} * * \ldots \omega_1 \ldots \omega_k * * \ldots \} \]

whenever \( n \geq k \), and
\[ \mu_p(T^{-n}(A) \cap B) = \mu_p(A) \mu_p(B). \]

This implies the mixing because the set of simple functions \( f = \sum_{j=1}^{\ell} c_j \mathbb{1}_{A_j} \) with \( A_j \) as in (1.11) is dense in \( L^2(\mu_p) \) and we can apply Proposition 1.1.

Here are some examples of \( \mu_p \):

(1) If \( p_j = 1 \) for some \( j \in \{0, 1, \ldots, m-1\} \), then \( \mu_p = \delta_{y_j} \) with
\[ y_j = j/(m-1) = jm/(m-1) \text{ (mod 1)}. \]

Note that \( y_0 = y_{m-1} \); otherwise, \( y_0, \ldots, y_{m-2} \) correspond to the (distinct) fixed points.

(2) If \( p_0 = \cdots = p_{m-1} = \frac{1}{m} \), then \( \mu_p \) is the Lebesgue measure.

(3) If for example \( m = 3 \) and \( p_0 = p_2 = 1/2 \), then the support of the measure \( \mu_p \) is the classical Cantor set of Hausdorff dimension \( \alpha = \log_2 3 \) (solving \( 3^{-\alpha} = 2^{-1} \)).
Note also that if $x$ is a periodic point of period $k$, then $\mu = \frac{1}{k} \sum_{j=0}^{k-1} \delta_{T^j(x)}$ is an ergodic measure. Such $\mu$ is never mixing unless $k = 1$. □

**Example 1.6** Consider a linear transformation on $\mathbb{R}^2$ associated with a $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $a, b, c, d \in \mathbb{Z}$, then $T(x) = Ax \mod 1$ defines a transformation on the 2-dimensional torus $\mathbb{T}^2$. Here we are using the fact that if $x = y \mod 1$, then $Ax = Ay \mod 1$. If we assume $\det A = 1$, then the Lebesgue measure $\lambda$ on $\mathbb{T}^2$ is invariant for the transformation $T$. To have $\lambda$ mixing, we need to assume that the eigenvalues of $T$ are real and different from 1 and $-1$. Let us assume that $A$ has eigenvalues $\alpha$ and $\alpha^{-1}$ with $\alpha \in \mathbb{R}$ and $|\alpha| < 1$. By Proposition 1.1, $\lambda$ is mixing if we can show that for every $n, m \in \mathbb{Z}^2$,

$$\lim_{n \to \infty} \int (\varphi_k \circ T^n) \varphi_m \, d\lambda = \int \varphi_k \, d\lambda \int \varphi_m \, d\lambda$$

where $\varphi_k(x) = \exp(2\pi i k \cdot x)$. If $m = 0$ or $k = 0$, then (1.12) is obvious. If $k, m \neq 0$, then the right-hand side of (1.12) is zero. We now establish (1.12) for $m, k \neq 0$ by showing that the left-hand side is zero for sufficiently large $n$. Clearly

$$\int \varphi_k \circ T^n \varphi_m \, d\lambda = \int \varphi((AT)^n k + m) \, d\lambda,$$

where $A^T$ denotes the transpose of $A$. To show that (1.13) is zero for large $n$, it suffices to show that $(AT)^n k + m \neq 0$ for large $n$. For this, it suffices to show that $\lim_{n \to \infty} (AT)^n k$ is either $\infty$ or 0. Write $v_1$ and $v_2$ for eigenvectors associated with eigenvalues $\alpha$ and $\alpha^{-1}$. We have $\lim_{n \to \infty} (AT)^n k = \infty$ if $k$ is not parallel to $v_1$ and $\lim_{n \to \infty} (AT)^n k = 0$ if $k$ is parallel to $v_1$. □

We end this section with some comments on the ergodicity of continuous dynamical system.

Given a flow $\{\phi_t : t \in \mathbb{R}\}$, let us define

$$\mathcal{I}_\phi = \left\{ \mu : \int f \circ \phi_t \, d\mu = \int f \, d\mu \quad \forall (f, t) \in C_b(X) \times \mathbb{R} \right\}.$$ 

Given $\mu \in \mathcal{I}_\phi$ and $f \in L^1(\mu)$, we would like to show

$$\mu \left\{ x : \lim_{t \to \infty} \frac{1}{t} \int_0^t f \circ \phi_\theta(x) \, d\theta \text{ exists} \right\} = 1.$$

To reduce this to the discrete case, let us define $\Omega = \prod_{j \in \mathbb{Z}} \mathbb{R}$ and $\Gamma : X \to \Omega$ by

$$\Gamma(x) = (\omega_j(x) : j \in \mathbb{Z}) = \left( \int_j^{j+1} f \circ \phi_\theta(x) \, d\theta : j \in \mathbb{Z} \right).$$

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We then define $T(\omega_j : j \in \mathbb{Z}) = (\omega_{j+1} : j \in \mathbb{Z})$. Clearly $\Gamma \circ \phi_1 = T \circ \Gamma$. Also, if $\mu \in \mathcal{I}_\phi$, then $\tilde{\mu}$ defined by $\tilde{\mu}(A) = \mu(\Gamma^{-1}(A))$ belongs to $\mathcal{I}_T$. Indeed,

$$\int g \circ T \, d\tilde{\mu} = \int g \circ T \circ \Gamma \, d\mu = \int g \circ \Gamma \circ \phi_1 \, d\mu = \int g \circ \Gamma \, d\mu = \int g \, d\tilde{\mu}.$$

We now apply Theorem 1.6 to assert

$$\tilde{\mu} \left\{ \omega : \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \omega_j \text{ exists} \right\} = 1.$$

Hence

$$\mu \left\{ x : \lim_{n \to \infty} \frac{1}{n} \int_0^n f \circ \phi_\theta(x) \, d\theta \text{ exists} \right\} = 1.$$

From this, it is straightforward to deduce

$$\mu \left\{ x : \lim_{t \to \infty} \frac{1}{t} \int_0^t f \circ \phi_\theta \, d\theta \text{ exists} \right\} = 1.$$

To see this, observe

$$\frac{1}{t} \int_0^t f \circ \phi_\theta \, d\theta = \left[ \frac{t}{t} \right] \frac{1}{t} \int_0^{\left[ \frac{t}{t} \right]} f \circ \phi_\theta \, d\theta + \frac{1}{t} \int_{\left[ \frac{t}{t} \right]}^t f \circ \phi_\theta \, d\theta.$$

Hence it suffices to show

\begin{equation}
(1.16) \quad \lim_{n \to \infty} \frac{1}{n} \int_n^{n+1} |f \circ \phi_\theta| \, d\theta = 0 \quad \mu - \text{a.e.}
\end{equation}

To prove this, observe

$$\lim_{n \to \infty} \frac{1}{n} \int_0^n |f \circ \phi_\theta| \, d\theta \text{ exists } \mu - \text{a.e.}$$

and this implies

$$\frac{1}{n} \int_n^{n+1} |f \circ \phi_\theta| \, d\theta = \frac{1}{n} \int_0^{n+1} |f \circ \phi_\theta| \, d\theta - \frac{1}{n} \int_0^n |f \circ \phi_\theta| \, d\theta = \frac{n+1}{n} \int_0^{n+1} |f \circ \phi_\theta| \, d\theta - \frac{1}{n} \int_0^n |f \circ \phi_\theta| \, d\theta$$

converges to $0 \mu - \text{a.e.}$, proving (1.16).
As before we can readily show that if \( \frac{1}{t} \int_0^t f \circ \phi_\theta \, d\theta \to Pf \), then \( Pf \circ \phi_t = Pf \mu \), a.e. for every \( t \), and that \( Pf \) is the projection of \( f \) onto the invariant sets. In particular, if \( \mu \) is ergodic, then \( Pf \equiv \int f \, d\mu \).

**Example 1.7** A prime example for a continuous dynamical system is a flow associated with an ODE \( \dot{x} = b(x) \) on a Riemannian manifold where \( b \) is a Lipschitz vector field. More precisely we write \( \phi_t(a) \) for a solution with initial condition \( x(0) = a \). Here are some examples for the invariant measures:

(i) If \( b(a) = 0 \), then \( \mu = \delta_a \) is an invariant measure. If \( \phi_T(a) = a \) for some \( T > 0 \), then

\[
\mu = T^{-1} \delta_{\phi_t(a)} \, dt,
\]

with support \( \gamma = \{ \phi_t(a) : t \in [0, T) \} \). In fact since \( b(\phi_t(a)) \, dt = d\ell \), is the length element, alternatively we may represents \( d\mu \) as the measure \( |b|^{-1} \, d\ell \) on \( \gamma \).

(ii) When \( \text{div } b = 0 \), the normalized volume measure is invariant. \( \square \)

**Exercises**

(i) Show that the topology associated with (1.4) is metrizable with the metric given by

\[
D(\mu, \nu) = \sum_{n=1}^{\infty} 2^{-n} \frac{\left| \int f_n \, d\mu - \int f_n \, d\nu \right|}{1 + \left| \int f_n \, d\mu - \int f_n \, d\nu \right|},
\]

where \( \{f_n : n \in N\} \) is a countable dense subset of \( U_b(X) \).

(ii) Let \( \mu_n \Rightarrow \mu \) and \( \mu(\partial A) = 0 \). Deduce that \( \mu_n(A) \to \mu(A) \). (Hint: For such \( A \), we can approximate the indicator function of \( A \) with continuous functions.)

(iii) Show that if \( X \) is a compact metric space, then \( M(X) \) is compact.

(iv) Let \( X \) be a Polish space. Suppose that \( \{\mu_N\} \) is a sequence of probability measures on \( X \). Assume that for every \( \delta > 0 \) there exists a compact set \( K_\delta \) such that \( \sup_N \mu_N(K_\delta^c) \leq \delta \). Show that \( \{\mu_N\} \) has a convergent subsequence.

(v) Show that if \( X \) is compact and \( f_n : X \to \mathbb{R} \) is a sequence of continuous functions, then \( f_n \to f \) uniformly iff

\[
x_n \to x \quad \Rightarrow \quad \lim_{n \to \infty} f_n(x_n) = f(x).
\]

(vi) Let \( A \) be a measurable set with \( \mu(A \Delta T^{-1}(A)) = 0 \). Show that there exists a set \( B \in F_T \) such that \( \mu(A \Delta B) = 0 \).

(vii) Show that \( \int |Pf| \, d\mu \leq \int |f| \, d\mu \).

(viii) Show that the decimal expansion of \( 2^n \) may start with any finite combination of digits. (Hint: use \( T : \mathbb{T} \to \mathbb{T} \) defined by \( T(x) = x + \alpha \, (\text{mod } 1) \) with \( \alpha = \log_{10} 2 \).)
In the case of an irrational rotation \( T : \mathbb{T} \to \mathbb{T}, T(x) = x + \alpha \, (\text{mod} \, 1) \), show that the operators \( \Phi_n = n^{-1}(I + U + \cdots + U^{n-1}) \) do not converge to the projection operator \( P \) strongly. More precisely,

\[
\liminf_{n \to \infty} \sup_{\|f\|_{L^2}=1} \|\Phi_n(f) - Pf\|_{L^2} > 0.
\]

Let \( a \) be a periodic point for \( T \) of period \( \ell \). Show that \( \mu = \frac{1}{\ell} \sum_{j=0}^{\ell-1} \delta_{T^j(x)} \) is not mixing if \( \ell > 1 \).

Show that if \( \mu \) is mixing and \( f \circ T = \lambda f \), then either \( \lambda = 1 \) or \( f = 0 \).

Show that the Lebesgue measure \( \lambda \) is ergodic for \( T(x,y) = (x + \alpha, x + y) \) (mod 1) iff \( \alpha \) is irrational. Show that \( \lambda \) is never mixing.

Let \( m > 1 \) be a prime number and assume that \( \mu \in \mathcal{I}_m \setminus \mathcal{I}_{m}^{*} \). Show that there exists a measurable set \( A \) such that \( \mu(A) = m^{-1} \), the collection \( \xi = \{ A, T^{-1}(A), \ldots, T^{-m+1}(A) \} \) is a partition of \( X \), and the set of \( T^m \) invariant sets \( \mathcal{F}_m \) is the \( \sigma \)-algebra generated by \( \xi \). □
2 Transfer Operator, Liouville Equation

In the previous section we encountered several examples of dynamical systems for which it was rather easy to find “nice” ergodic invariant measures. We also observed in the case of expanding map that the space of invariant measures is rather complex. One may say that the Lebesgue measure is the “nicest” invariant measure for an expanding map. Later in Section 3, we show how the Lebesgue measure stands out as the unique invariant measure of maximum entropy.

In general, it is not easy to find some natural invariant measure for our dynamical system. For example, if we have a system on a manifold with a Riemannian structure with a volume form, we may wonder whether or not such a system has an invariant measure that is absolutely continuous with respect to the volume form. To address and study this sort of questions in a systematic fashion, let us introduce an operator on measures that would give the evolutions of measures with respect to our dynamical system. This operator is simply the dual of the operator

\[ U_f = f \circ T. \]

More precisely, define \( A : \mathcal{M}(X) \to \mathcal{M}(X) \) by

\[ \int U_f d\mu = \int f \circ T d\mu = \int f \, dA\mu \]

for every \( f \in C_b(X) \). We certainly have

\[ (A\mu)(A) = U^* \mu(A) = \mu(T^{-1}(A)) \]

for every measurable \( A \). Even though we have some general results regarding the spectrum of \( U \), the corresponding questions for the operator \( A \) are far more complex. We can now cast the existence of an invariant measure with some properties as the existence of a fixed point of \( A \) with those properties. The operator \( A \) is called Perron–Frobenious, Perron–Frobenious–Ruelle or Transfer Operator, once an expression for it is derived when \( \mu \) is absolutely continuous with respect to the volume form. To get a feel for the operator \( A \), let us state a Proposition and examine some examples.

**Proposition 2.1** Set \( A_n = \Phi_n^* = n^{-1}(I + A + \cdots + A^{n-1}) \).

(i) \( A_n\delta_x = \mu_x^n \). Moreover any limit point of \( A_n\nu \) is an invariant measure.

(ii) A measure \( \mu \in \mathcal{I}_T \) iff \( A_n\nu \) converges to \( \mu \) in high \( n \) limit, for every \( \nu << \mu \).

(iii) A measure \( \mu \) is a mixing invariant measure iff \( A^n\nu \) converges to \( \mu \) in high \( n \) limit, for every \( \nu << \mu \).

The elementary proof of Proposition 2.1 is left as an exercise.

**Example 2.1(i)** \( T : \mathbb{T}^d \to \mathbb{T}^d, T(x) = x + \alpha \mod 1 \). The operator \( A \) simply translates a measure for the amount \( \alpha \). We assume that the numbers \( \alpha_1 \ldots \alpha_d \), and 1 are rationally
independent. We can study the asymptotic behavior of $A^n\mu$ for a given $\mu$. The sequence $\{A^n\mu\}$ does not converge to any limit as $n \to \infty$. In fact the set of limit points of the sequence $\{A^n\mu\}$ consists of all translates of $\mu$. However

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} A^j \mu = \lambda$$

where $\lambda$ denotes the Lebesgue measure. The proof of (2.1) follows from the unique ergodicity of $T$ that implies

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} A^j \mu = \lambda$$

uniformly for every continuous $f$. This implies

$$\lim_{n \to \infty} \int \Phi_n(f) d\mu = \lim_{n \to \infty} \int f \left( \frac{1}{n} \sum_{j=0}^{n-1} A^j \mu \right) = \int f d\lambda,$$

proving (2.2).

(ii) Let $(X, d)$ be a complete metric space and suppose $T : X \to X$ is a contraction. In other words, there exists a constant $\alpha \in (0, 1)$ such that $d(T(x), T(y)) \leq \alpha d(x, y)$. In this case $T$ has a unique fix point $\bar{x}$ and $\lim_{n \to \infty} T^n(x) = \bar{x}$ for every $x$ (the convergence is locally uniform). As a consequence we learn that $\lim_{n \to \infty} A^n\mu = \delta_{\bar{x}}$ for every measure $\mu \in M(X)$. For example, if $X = \mathbb{R}$ and $T(x) = \alpha x$ with $\alpha \in (0, 1)$, then $d\mu = \rho \, dx$ results in a sequence $A^n\mu = \rho_n \, dx$ with

$$\rho_n(x) = \alpha^{-n} \rho \left( \frac{x}{\alpha^n} \right).$$

In other words, the measure $\mu$ under $A$ becomes more concentrated about the origin.

(iii) Let $T : \mathbb{T} \to \mathbb{T}$ be the expansion $T(x) = 2x \pmod{1}$. If $d\mu = \rho \, dx$ and $A^n\mu = \rho_n \, dx$, then $\rho_1(x) = \frac{1}{2} \left( \rho \left( \frac{x}{2} \right) + \rho \left( \frac{x+1}{2} \right) \right)$ and

$$\rho_n(x) = \frac{1}{2^n} \sum_{j=0}^{2^n-1} \rho \left( \frac{x}{2^n} + j \frac{1}{2^n} \right).$$

From this, it is clear that if $\rho$ is continuous, then $\lim_{n \to \infty} \rho_n(x) \equiv 1$. Indeed

$$\lim_{n \to \infty} \left| \frac{1}{2^n} \sum_{j=0}^{2^n-1} \rho \left( \frac{j}{2^n} \right) \right| = \lim_{n \to \infty} \left| \frac{1}{2^n} \sum_{j=0}^{2^n-1} \left( \rho \left( \frac{x}{2^n} + j \frac{1}{2^n} \right) - \rho \left( \frac{j}{2^n} \right) \right) \right| = 0,$$

$$\lim_{n \to \infty} \frac{1}{2^n} \sum_{j=0}^{2^n-1} \rho \left( \frac{j}{2^n} \right) = \int \rho \, dx = 1.$$
This can also be seen by looking at the Fourier expansion of $\rho$. For the following argument, we only need to assume that $\rho \in L^2[0,1]$. If

$$\rho(x) = \sum_n a_n e^{2\pi i nx},$$

then $a_0 = 1$ and

$$\rho_1(x) = \sum_k a_{2k} e^{2\pi i kx},$$

and by induction,

$$\rho_n(x) = \sum_k a_{2^n k} e^{2\pi i kx}.$$

As a result,

$$\int_0^1 |\rho_n(x) - 1|^2 \, dx = \sum_{k \neq 0} a_{2^n k}^2 \to 0.$$

□

There is a couple of things to learn from Example 2.1. First, when there is a contraction, the operator $A$ makes measures more concentrated in small regions. Second, if there is an expansion then $A$ has some smoothing effect. In hyperbolic systems we have both expansion and contraction. In some sense, if we have more contraction than the expansion, then it is plausible that there is a fractal set that attracts the orbits as $n \to \infty$. If this happens, then there exists no invariant measure that is absolutely continuous with respect to the volume measure. Later in this section, we will see an example of such phenomenon. As a result, to have an absolutely continuous invariant measure, we need to make sure that, in some sense, the expansion rates and the contraction rates are balanced out. Let us first derive a formula for $A\mu$ when $\mu$ is absolutely continuous with respect to a volume form. As a warm up, first consider a transformation $T : \mathbb{R}^d \to \mathbb{R}^d$ that is smooth. We also assume that $T$ is invertible with a smooth inverse, i.e., $T$ is a diffeomorphism. We then consider $d\mu = \rho dx$. We have

$$\int_{\mathbb{R}^d} f \circ T \rho \, dx = \int_{\mathbb{R}^d} f \circ T^{-1} |JT^{-1}| \, dy$$

where $JT^{-1} = \det DT^{-1}$. As a result, if $A\mu = \hat{\rho} dx$, then $\hat{\rho} = |JT^{-1}| \rho \circ T^{-1} = \frac{\rho \circ T^{-1}}{|JT \circ T^{-1}|}$. We abuse the notation to write

$$A\rho(x) = \frac{\rho \circ T^{-1}}{|JT \circ T^{-1}|},$$

regarding $A$ as an operator acting on probability density functions. More generally, assume that $X$ is a smooth manifold and $T$ is $C^\infty$. Let $\omega$ be a volume form (non-degenerate $d$-form where $d$ is the dimension of $X$). Then $T^*\omega$, the pull-back of $\omega$ under $T$, is also a $k$-form and
we define $JT(x)$ to be the unique number such that $T^*\omega_x = JT(x)\omega_T$. More precisely, $T^*\omega_x(v_1 \ldots v_k) = \omega_{T(x)}(DT(x)v_1, \ldots, DT(x)v_k) = JT(x)\omega_{T(x)}(v_1 \ldots v_k)$. We then have

$$\int_X (f \circ T) \rho \, \omega = \int_X f(\rho \circ T^{-1})|JT^{-1}| \, \omega.$$ 

Hence (2.3) holds in general.

If $T$ is not invertible, one can show

$$\mathcal{A}_n \rho(x) = \sum_{y \in T^{-1}(\{x\})} \frac{\rho(y)}{|JT(y)|}.$$ 

The next proposition demonstrates how the existence of an absolutely continuous invariant measure forces a bound on the Jacobians.

**Proposition 2.2** Let $X$ be a smooth manifold with a volume form $\omega$. Let $T : X \to X$ be a diffeomorphism with $JT > 0$. The following statements are equivalent:

(i) There exists $\mu = \rho\omega \in \mathcal{I}_T$ for a bounded uniformly positive $\rho$.

(ii) The set $\{JT^n(x) : x \in X, n \in \mathbb{Z}\}$ is uniformly bounded.

**Proof** (i) $\Rightarrow$ (ii) Observe

$$\mathcal{A}_n \rho = \frac{\rho \circ T^{-n}}{JT^n \circ T^{-n}}, \quad n \in \mathbb{N}.$$ 

Also, $\mathcal{A}^{-1} \rho = (\rho \circ T)JT$, and by induction

$$\mathcal{A}^{-n} \rho = (\rho \circ T^n)JT^n \quad \Rightarrow \quad (\rho \circ T^n)JT^{-n} \circ T^n; \quad n \in \mathbb{N}.$$ 

Hence

$$\mathcal{A}_n \rho = \frac{\rho \circ T^{-n}}{JT^n \circ T^{-n}}; \quad n \in \mathbb{Z}.$$ 

If $\rho\omega$ is invariant, then $\mathcal{A}^n \rho = \rho$ for all $n \in \mathbb{Z}$. As a result, $(JT^n \circ T^{-n})\rho = \rho \circ T^{-n}$, or

$$JT^n = \frac{\rho}{\rho \circ T^n}; \quad n \in \mathbb{Z}.$$ 

Now it is clear that if $\rho$ is bounded and uniformly positive, then $\{JT^n(x) : n \in \mathbb{Z}, x \in X\}$ is uniformly bounded.
(ii) $\Rightarrow$ (i) Suppose $\{JT^n(x) : n \in \mathbb{Z} \text{ and } x \in X\}$ is bounded and define

$$
\rho(x) = \sup_{n \in \mathbb{Z}} JT^n(x).
$$

We then have

$$
JT(x)(\rho \circ T)(x) = \sup_{n \in \mathbb{Z}} (JT^n) \circ T(x)JT(x) = \sup_{n \in \mathbb{Z}} J(T^n \circ T)(x) = \rho(x).
$$

Hence $A\rho = \rho$. Evidently $\rho$ is bounded. Moreover

$$
1/\rho = \inf_n [1/\rho] = \inf_n JT^{-n} \circ T^n = \inf_n JT^n \circ T^{-n}
$$

is uniformly bounded by assumption. □

Recall that expansions are harmless and have smoothing effect on $A\rho$. As a test case, let us consider an expansion of $[0,1]$ given by

$$
T(x) = \begin{cases} 
T_1(x) & x \in [0,\theta_0) = I_1 \\
T_2(x) & x \in [\theta_0, 1] = I_2
\end{cases}
$$

with $T_1, T_2$ smooth functions satisfying $|T'_i(x)| \geq \lambda$ for $x \in I_i$. We assume $\lambda > 1$ and that $T_i(I_i) = [0,1]$. In this case

$$
(2.7) \quad A\rho(x) = \frac{\rho \circ T^{-1}_1(x)}{T'_1 \circ T^{-1}_1(x)} + \frac{\rho \circ T^{-1}_2(x)}{T'_2 \circ T^{-1}_2(x)}.
$$

**Theorem 2.1** If $T_1, T_2 \in C^2$, then there exists $\mu \in \mathcal{L}_T$ of the form $d\mu = \rho dx$ with $\rho$ of finite variation.

**Proof** Write $S_i = T^{-1}_i$ so that

$$
A\rho = (\rho \circ S_1)S'_1 + (\rho \circ S_2)S'_2.
$$

Using $S'_i \leq \frac{1}{\lambda}$, we learn

$$
\int_0^1 |(A\rho)'| dx \leq \lambda^{-1} \int_0^1 |A|\rho' | dx + \beta_0 \int_0^1 A\rho dx,
$$

where

$$
\beta_0 = \max_x \max_{i \in \{1,2\}} \frac{|S''_i(x)|}{S'_i(x)}.
$$
Hence
\[ \int_0^1 |(A\rho)'| dx \leq \lambda^{-1} \int_0^1 |\rho'| dx + \beta_0. \]
By induction,
\[ \int_0^1 |(A^n \rho)'| dx \leq \lambda^{-n} \int_0^1 |\rho'| dx + \beta_0 \frac{1 - \lambda^{-n}}{1 - \lambda^{-1}}. \]
From this we learn that
\[ \sup_n \|A^n \rho\|_{BV} < \infty. \]
Hence \( A^n \rho \) has convergent subsequences in \( L^1[0,1] \). But a limit point may not be an invariant density. To avoid this, let us observe that we also have
\[ \sup_n \left\| \frac{1}{n} \sum_{j=0}^{n-1} A^j \rho \right\|_{BV} < \infty. \]
Hence the sequence \( \{\rho_n = n^{-1} \sum_{j=0}^{n-1} A^j \rho\}_n \) has convergent subsequences by Helley Selection Theorem. If \( \bar{\rho} \) is a limit point, then \( A\bar{\rho} = \bar{\rho} \) by Proposition 2.1. Also, for every periodic \( J \in C^1 \),
\[ \left| \int_0^1 J' \bar{\rho} \, dx \right| = \lim_{n \to \infty} \left| \int_0^1 J' \rho_n \, dx \right| \leq \|J\|_{L^\infty} \limsup_{n \to \infty} \|\rho_n\|_{BV} \leq \frac{\beta_0}{1 - \lambda^{-1}} \|J\|_{L^\infty}. \]
Hence \( \bar{\rho} \in BV. \)

A review of the proof of Theorem 2.1 reveals that in fact
\[ AB_a \subseteq B_{a\lambda^{-1} + \beta_0}, \]
where \( B_a \) denotes the space of probability densities \( \rho \) such that \( \int_0^1 |\rho'| \, dx \leq a \). In particular, if \( a \) is sufficiently large, then \( a\lambda^{-1} + \beta_0 < a \) and \( A \) maps the set \( B_a \) to a strictly smaller subset of \( B_a \). From this, we may wonder whether \( A \) is a contraction with respect to a suitable metric on \( B_a \). Such a contraction for sure guarantees the existence of a fixed point and the convergence of \( A^n \rho \), and this is exactly what we are looking for. Instead of working on the space \( B_a \), we would rather work on smaller space which yields the convergence of \( A^n \rho \) even with respect to the uniform topology. Let us consider the following function space:
\[ (2.8) \quad \mathcal{C}_a = \{ e^\theta : |g(x) - g(y)| \leq a|x - y| \text{ for } x, y \in [0,1] \}. \]
We note that \( \rho \in \mathcal{C}_a \cup \{0\} \), iff \( \rho \geq 0 \) and for all \( x, y \in [0,1] \),
\[ \rho(x) \leq \rho(y) e^{a|x - y|}. \]
Recall that \( S_i = T_i^{-1} \) and \( \beta_0 = \max_{i \in \{1,2\}} \max_x \frac{|S_i''(x)|}{S_i'(x)}. \)
Lemma 2.1 We have that $A\mathcal{C}_a \subseteq \mathcal{C}_{a\sigma}$, whenever $a \geq \frac{\beta_0}{\sigma - \lambda - 1}$ and $\sigma > \lambda^{-1}$.

Proof Let $\rho = e^g \in \mathcal{C}_a$. Then

$$A\rho(x) = \sum_{i=1}^{2} \rho \circ S_i(x)S'_i(x)$$

$$\leq \sum_{i=1}^{2} \rho \circ S_i(y)e^{\alpha|S_i(x) - S_i(y)|}S'_i(y)$$

$$= \sum_{i=1}^{2} \rho \circ S_i(y)e^{\alpha|S_i(x) - S_i(y)|}e^{\log(S'_i(x)) - \log(S'_i(y))}$$

$$\leq \sum_{i=1}^{2} \rho \circ S_i(y)|S'_i(y)|e^{\alpha\lambda^{-1}|x-y|}e^{\beta_0|x-y|}$$

$$= A\rho(y)e^{(\alpha\lambda^{-1} + \beta_0)|x-y|}.$$}

As a result, $A\mathcal{C}_a \subseteq \mathcal{C}_{a\lambda^{-1}+\beta_0} \subseteq \mathcal{C}_{a\sigma}$.

What we learn from Lemma 2.1 is that if $\sigma \in (\lambda^{-1}, 1]$, then we can find a function space $\mathcal{C}_a$ that is mapped into itself by $A$. Note that indeed $\mathcal{C}_a$ is a cone in the sense that

$$\begin{cases} 
\text{if } \rho \in \mathcal{C}_a, & \text{then } \lambda \rho \in \mathcal{C}_a \text{ for } \lambda > 0, \\
\text{if } \rho_1, \rho_2 \in \mathcal{C}_a, & \text{then } \rho_1 + \rho_2 \in \mathcal{C}_a.
\end{cases}$$

Define a partial order

(2.9) $\rho_1 \preceq \rho_2$ iff $\rho_2 - \rho_1 \in \mathcal{C}_a \cup \{0\}.$

In other words, $\rho_1 \preceq \rho_2$ iff $\rho_1 \leq \rho_2$ and

(2.10) $\rho_2(x) - \rho_1(x) \leq (\rho_2(y) - \rho_1(y))e^{\alpha|x-y|}, \ x, y \in [0, 1].$

Hilbert metric associated with our cone $\mathcal{C}_a$ is defined as

(2.11) $d_a(\rho_1, \rho_2) = \log(\beta_a(\rho_1, \rho_2)\beta_a(\rho_2, \rho_1)),$

where $\beta_a(\rho_1, \rho_2) = \inf\{\lambda \geq 0 : \rho_2 \preceq \lambda \rho_1\}$. By convention, $\beta_a(\rho_1, \rho_2) = \infty$ if there exists no such $\lambda$. We certainly have

(2.12) $d_a(\rho_1, \rho_2) = \inf_{\alpha} \inf_{\beta} \left\{ \frac{\log \beta}{\alpha} : \alpha \rho_1 \preceq \rho_2 \preceq \beta \rho_1 \right\}$

$$= \inf_{\gamma} \{ \gamma : \alpha \rho_1 \preceq \rho_2 \preceq e^{\gamma} \alpha \rho_1 \text{ for some } \alpha > 0 \} \geq 0.$$
Lemma 2.2 (i) \( \beta_a(\rho_1, \rho_2) = \sup_{x \neq y} e^{a|x-y|} \rho_2(y) - \rho_2(x) \geq \sup_{x} \rho_2(x) \).

(ii) \( d_a \) is a quasi-metric with \( d_a(\rho_1, \rho_2) = 0 \) iff \( \rho_1 = \lambda \rho_2 \) for some \( \lambda > 0 \).

(iii) If \( a_1 \leq a_2 \) then \( d_{a_1}(\rho_1, \rho_2) \geq d_{a_2}(\rho_1, \rho_2) \) for \( \rho_1, \rho_2 \in C_{a_1} \).

Proof (i) If \( \rho_2 \preceq \lambda \rho_1 \), then \( \rho_2 \leq \lambda \rho_1 \) and

\[-\rho_2(x) + \lambda \rho_1(x) \leq e^{a|x-y|}(-\rho_2(y) + \lambda \rho_1(y)),
-\rho_2(x) + e^{a|x-y|} \rho_2(y) \leq \lambda(-\rho_1(x) + e^{a|x-y|} \rho_1(y)).\]

From this we deduce

\[\beta_a(\rho_1, \rho_2) = \max\left\{ \sup_{x} \rho_2(x), \sup_{x \neq y} e^{a|x-y|} \rho_2(y) - \rho_2(x) \right\} \cdot \]

Note that if \( \sup_x \rho_2(x) = \rho_2(\bar{x}) \), then

\[\frac{e^{a|x-\bar{x}|} \rho_2(\bar{x}) - \rho_2(x)}{e^{a|x-\bar{x}|} \rho_1(\bar{x}) - \rho_1(x)} = \frac{e^{a|x-\bar{x}|} \rho_1(\bar{x}) \rho_2(\bar{x}) - \rho_1(x) \rho_2(x)}{e^{a|x-\bar{x}|} \rho_1(\bar{x}) - \rho_1(x)} \geq \frac{\rho_2(x)}{\rho_1(x)} \cdot \]

This completes the proof of (i)

(ii) The triangle inequality is a consequence of the fact that if \( \rho_2 \preceq \lambda_1 \rho_1 \) and \( \rho_3 \preceq \lambda_2 \rho_2 \), then \( \rho_3 \preceq \lambda_1 \lambda_2 \rho_1 \).

(iii) First observe \( C_{a_1} \subseteq C_{a_2} \). Hence \( \rho_2 \preceq \lambda \rho_1 \) in \( C_{a_1} \) implies the same inequality in \( C_{a_2} \).

Recall that we are searching for a fixed point for the operator \( \mathcal{A} \). By Lemma 2.1, if \( \sigma \in (\lambda^{-1}, 1) \) and \( a > \frac{2b}{\sigma-\lambda^{-1}} \), then \( \mathcal{A}(C_{a}) \subseteq C_{a} \subseteq C_{a} \). As our next step., we show that \( \mathcal{A} \) is a contraction on \( C_{a} \). But first let us demonstrate that in fact that the set \( C_{a} \) is a bounded subset of \( C_{a} \).

Lemma 2.3 \( \text{diam } C_{a} = \sup_{\rho_1, \rho_2 \in C_{a}} d_a(\rho_1, \rho_2) \leq \log \frac{1+\sigma}{1-\sigma} + 2a\sigma . \)

Proof From \( \rho_2(x) \leq \rho_2(y)e^{-a|x-y|} \) and \( \rho_1(x) \leq \rho_1(y)e^{a|x-y|} \) we deduce

\[\beta_a(\rho_1, \rho_2) \leq \sup_{x,y} \frac{e^{a|x-y|} - e^{-a|x-y|} \rho_2(y)}{e^{a|x-y|} - e^{a|x-y|} \rho_1(y)} \cdot \]

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To calculate this, set $z = a|x - y|$. Then $z \geq 0$ and
\[
\lim_{z \to 0} \frac{e^z - e^{-\sigma z}}{e^z - e^{\sigma z}} = \frac{1 + \sigma}{1 - \sigma}.
\]
On the other hand,
\[
\frac{e^z - e^{-\sigma z}}{e^z - e^{\sigma z}} \leq \frac{1 + \sigma}{1 - \sigma}
\]
which is the consequence of the convexity of the exponential function;
\[
e^{\sigma z} \leq \frac{2\sigma}{1 + \sigma} e^z + \frac{1 - \sigma}{1 + \sigma} e^{-\sigma z}.
\]
As a result,
\[
\beta_a(\rho_1, \rho_2) \leq \frac{1 + \sigma}{1 - \sigma} \sup_y \rho_2(y) \leq \frac{1 + \sigma}{1 - \sigma} \frac{\rho_2(y_0) e^{\alpha \sigma/2}}{\rho_1(y_0) e^{\alpha \sigma/2}} = \frac{\rho_2(y_0) e^{\alpha \sigma} 1 + \sigma}{\rho_1(y_0) 1 - \sigma}
\]
for $y_0 = \frac{1}{2}$. Hence
\[
\beta_a(\rho_1, \rho_2) \beta_a(\rho_2, \rho_1) \leq \left(\frac{1 + \sigma}{1 - \sigma}\right)^2 e^{2\alpha \sigma},
\]
completing the proof of lemma. \hfill \Box

We are now ready to show that $\mathcal{A}$ is a contraction.

**Lemma 2.4** For every $\rho_1, \rho_2 \in \mathcal{C}_a$,
\[
d_a(\mathcal{A} \rho_1, \mathcal{A} \rho_2) \leq \tanh \left(\frac{b}{4}\right) d_a(\rho_1, \rho_2).
\]

**Proof** By Lemma 2.3, $\text{diam}(\mathcal{A} \mathcal{C}_a) \leq b$. As a consequence if $\beta \rho_1 \succ \rho_2 \succ \alpha \rho_1$, then
\[
d_a(\mathcal{A}(\rho_2 - \alpha \rho_1), \mathcal{A}(\beta \rho_1 - \rho_2)) \leq b
\]
for every $\rho_1, \rho_2 \in \mathcal{C}_a$ and $\alpha, \beta \geq 0$. This means that we can find two constants $\lambda_1, \lambda_2 \geq 0$ such that $\log \frac{\lambda_2}{\lambda_1} \leq b$ and
\[
\frac{\beta + \alpha \lambda_2}{1 + \lambda_2} \mathcal{A} \rho_1 \preceq \mathcal{A} \rho_2 \preceq \frac{\beta + \alpha \lambda_1}{1 + \lambda_1} \mathcal{A} \rho_1.
\]
As a result,
\[
d_a(\mathcal{A} \rho_1, \mathcal{A} \rho_2) \leq \log \frac{\beta + \alpha \lambda_1}{1 + \lambda_1} \frac{1 + \lambda_2}{\beta + \alpha \lambda_2} = \log \frac{\beta + \lambda_1}{\alpha + \lambda_2} + \log \frac{1 + \lambda_2}{1 + \lambda_1}.
\]
Minimizing over $\beta/\alpha$ yields
\[
d_a(A\rho_1, A\rho_2) \leq \log \frac{e^{d_a(\rho_1, \rho_2)} + \lambda_1}{e^{d_a(\rho_1, \rho_2)} + \lambda_2} - \log \frac{1 + \lambda_1}{1 + \lambda_2} = \int_0^{d_a(\rho_1, \rho_2)} \frac{e^\theta (\lambda_2 - \lambda_1)}{(e^\theta + \lambda_1)(e^\theta + \lambda_2)} d\theta \leq d_a(\rho_1, \rho_2) \frac{\sqrt{\lambda_2} - \sqrt{\lambda_1}}{\sqrt{\lambda_2} + \sqrt{\lambda_1}}.
\]
because $\max_{x \geq 1} \frac{x(\lambda_2 - \lambda_1)}{(x + \lambda_1)(x + \lambda_2)} = \frac{\sqrt{\lambda_2} - \sqrt{\lambda_1}}{\sqrt{\lambda_2} + \sqrt{\lambda_1}}$. Finally from $\log \frac{\lambda_1}{\lambda_1} \leq b$ we obtain
\[
d_a(A\rho_1, A\rho_2) \leq d_a(\rho_1, \rho_2) \frac{e^{\frac{1}{2}b} - 1}{e^{\frac{1}{2}b} + 1} = d_a(\rho_1, \rho_2) \tanh \left( \frac{b}{4} \right).
\]

This evidently gives us a contraction on $C_\alpha$ for any $a \geq \frac{\beta_0}{\sigma - \lambda T}$ provided that $\sigma \in (\lambda^{-1}, 1)$, because $\tanh \left( \frac{b}{4} \right) < 1$ always. We may minimize the rate of contraction $\tanh \left( \frac{b}{4} \right)$ by first choosing the best $a$, namely $a = \frac{\beta_0}{\sigma - \lambda T}$, and then minimizing $b$ in $\sigma$ as $\sigma$ varies in $(\lambda^{-1}, 1)$. Our goal is to show that $\lim_{n \to \infty} A^n \rho$ converges to a unique invariant density $\rho$. For this, let us establish an inequality connecting $d_a(\rho_1, \rho_2)$ to $\|\rho_1 - \rho_2\|_{L^1}$.

**Lemma 2.5** For every $\rho_1, \rho_2 \in C_\alpha$, with $\int_0^1 \rho_1 \, dx = \int_0^1 \rho_2 \, dx = 1$, we have
\[
\int_0^1 |\rho_1 - \rho_2| \, dx \leq (e^{d_a(\rho_1, \rho_2)} - 1), \quad |\rho_1 - \rho_2| \leq (e^{d_a(\rho_1, \rho_2)} - 1) \rho_1.
\]

**Proof** Let us write $d_a(\rho_1, \rho_2) = \log \frac{\beta}{\alpha}$ with $\alpha \rho_1 \leq \rho_2 \leq \beta \rho_1$. This in particular implies that $\alpha \rho_1 \leq \rho_2 \leq \beta \rho_1$. Integrating this over $[0, 1]$ yields $\alpha \leq 1 \leq \beta$, which in turn implies that $\alpha \rho_1 \leq \rho_1 \leq \beta \rho_1$. As a result,
\[
(\alpha - \beta) \rho_1 \leq \rho_2 - \rho_1 \leq (\beta - \alpha) \rho_1.
\]
Thus
\[
|\rho_1 - \rho_2| \leq (\beta - \alpha) \rho_1 \leq (\beta/\alpha - 1) \rho_1 \leq (e^{d_a(\rho_1, \rho_2)} - 1) \rho_1,
\]
\[
\int_0^1 |\rho_2 - \rho_1| \, dx \leq |\beta - \alpha| = \frac{\beta - \alpha}{\alpha} = 1 - e^{d_a(\rho_1, \rho_2)} - 1.
\]

We are now ready to state and prove the main result of this section.
Theorem 2.2 Let $a = \frac{b \sigma}{\sigma - \lambda - 1}$ and $\sigma \in (\lambda^{-1}, 1)$. Then for every $\rho \in C_a$ with $\int_0^1 \rho \, dx = 1$, $\lim_{n \to \infty} A^n \rho = \bar{\rho}$ exists uniformly and $\bar{\rho} \, dx \in \mathcal{I}_T$ with $\bar{\rho} \in C_{a\sigma}$. Moreover, there exists a constant $\bar{c}_1$ such that

$$
\left| \int_0^1 f \circ T^n \, g \, dx - \int_0^1 f \, g \, dx \int_0^1 f \, \bar{\rho} \, dx \right| \leq \bar{c}_1 \lambda^n \| f \|_{L^1} (\|g\|_{L^1} + \|g^\prime\|_{L^\infty})
$$

where $\hat{\lambda} = \tanh \left( \frac{b}{2} \right)$, $b = 2 \log \frac{\exp \left( \frac{\hat{\lambda}}{2} \right)}{\hat{\lambda} - \hat{\lambda}^\prime}$, $f \in L^1$, and $g$ is Lipschitz.

An immediate consequence of Theorem 2.2 is the mixing property of $\bar{\rho}$ because we may choose $g = h \bar{\rho} / \int h \bar{\rho}$ to deduce

$$
\lim_{n \to \infty} \int_0^1 f \circ T^n \, h \bar{\rho} \, dx = \int_0^1 f \, \bar{\rho} \, dx \int_0^1 h \bar{\rho} \, dx.
$$

**Proof of Theorem 2.2** We first show that if $\rho \in C_a$, then $A^n \rho$ converges to a function $\bar{\rho} \in C_a$ in $L^1$-sense. Indeed

$$
\|A^{n+m} \rho - A^n \rho\|_{L^1} \leq \exp(d_a(A^{n+m} \rho, A^n \rho)) - 1 \leq \exp(\lambda^{n-1} d_a(A^{n+1} \rho, A \rho) + 1) \leq e^{\lambda^{n-1} b} - 1 \leq \lambda^{n-1} b e^{\lambda^{n-1} b} \leq c_0 \lambda^{n-1}
$$

for a constant $c_0$ that depends on $b$ only. This implies that $A^n \rho$ is Cauchy in $L^1$. Let $\bar{\rho} = \lim \rho_n$ where $\rho_n = A^n \rho$. Since $\rho_n(x) \leq \rho_n(y)e^{\alpha |x-y|}$ and $\rho_{n_k} \to \bar{\rho}$ a.e. for a subsequence, we deduce that $\bar{\rho}(x) \leq \bar{\rho}(y)e^{\alpha |x-y|}$ for a.e. $x$ and $y \in [0,1]$. By modifying $\bar{\rho}$ on a set of zero Lebesgue measure if necessary, we deduce that $\bar{\rho} \in C_{a\sigma}$. Note that $\bar{\rho}$ is never zero, because if $\bar{\rho}(x_0) = 0$ for some $x_0$, then $\bar{\rho}(x) \leq \bar{\rho}(x_0)e^{\alpha |x-x_0|}$ implies that $\bar{\rho}(x) = 0$ for every $x$. But $\int_0^1 \rho \, dx = 1$ implies that $\int_0^1 \bar{\rho} \, dx = 1$. So $\bar{\rho} > 0$, completing the proof of $\bar{\rho} \in C_a$.

We now show that $A^n \rho \to \bar{\rho}$ uniformly. Indeed from $A^n \rho \to \bar{\rho}$ in $L^1$ we deduce that $\int f \circ T^n \rho \, dx \to \int f \, \bar{\rho} \, dx$ for every bounded $f$, which implies that $A \bar{\rho} = \bar{\rho}$. Moreover

$$
|A^n \rho - \bar{\rho}| = |A^n \rho - A^n \bar{\rho}| \leq (e^{d_a(A^n \rho, A^n \bar{\rho})} - 1) A^n \bar{\rho} \leq (e^{\lambda^{n-1} d_a(A \rho, \bar{\rho})} - 1) \bar{\rho} \leq (e^{\lambda^{n-1} b} - 1) \bar{\rho} \leq \lambda^{n-1} b e^{\lambda^{n-1} b} \bar{\rho} \leq c_0 \lambda^n \bar{\rho}
$$

with $c_0$ depending on $b$ only. From this we learn that

$$
\|A^n \rho - \bar{\rho}\|_{L^\infty} \leq c_0 \lambda^n \|\bar{\rho}\|_{L^\infty},
$$

for every $\rho \in C_a$ with $\int_0^1 \rho \, dx = 1$. 

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We now turn to the proof of (2.13). Without loss of generality, we may assume that
\( g \geq 0 \). Given such a function \( g \), find \( l > 0 \) large enough so that
\( \rho = g + l \bar{\rho} \in C_a \). Indeed, for \( y > x \), we have that
\( \rho(y) \leq g(y) + l \bar{\rho}(x) \exp(a \sigma(y-x)) =: \exp(h(y)) \). On the other hand
\[
h'(y) = \frac{g'(y) + la \sigma \bar{\rho}(x)e^{a \sigma(y-x)}}{g(y) + l \bar{\rho}(x)e^{a \sigma(y-x)}} \leq \frac{\|g'\|_{L^\infty}}{l \inf \bar{\rho}} + a \sigma \bar{\rho} \rho(x)e^{a \sigma(y-x)} \leq \frac{\|g'\|_{L^\infty}}{l \inf \bar{\rho}} + a \sigma.
\]
This is at most \( a \) if we choose \( l = \frac{\|g'\|_{L^\infty}}{a(1 - \sigma) \inf \bar{\rho}} \).

Hence
\[
\left\| A^n g + l \bar{\rho} \right\|_{L^\infty} \leq c_0 \hat{\lambda}^n \| \bar{\rho} \|_{L^\infty}
\]
where \( Z = \int_0^1 (g + l \bar{\rho}) dx \). Since \( A \bar{\rho} = \bar{\rho} \), we deduce
\[
\left\| \frac{A^n g}{Z} + \frac{l \bar{\rho} - \bar{\rho}}{Z} \right\|_{L^\infty} \leq c_0 \hat{\lambda}^n \| \bar{\rho} \|_{L^\infty},
\]
\[
\left\| A^n g - (Z - l) \bar{\rho} \right\|_{L^\infty} \leq c_0 \hat{\lambda}^n \| \bar{\rho} \|_{L^\infty} Z.
\]

Hence
\[
\left\| A^n g - \bar{\rho} \int_0^1 g dx \right\|_{L^\infty} \leq c_1 \hat{\lambda}^n \left[ \int g dx + l \right] \leq c_2 \hat{\lambda}^n \left[ \int g dx + \| g' \|_{L^\infty} \right].
\]
From this, we can readily deduce (2.13). \( \square \)

**Example 2.2** Let
\[
T(x) = \begin{cases} 
\frac{x}{1-x} & \text{for } x \in \left[0, \frac{1}{2}\right), \\
2x - 1 & \text{for } x \in \left[\frac{1}{2}, 1\right]. 
\end{cases}
\]
Note that for this example, the condition \( |T'(x)| > 1 \) is violated at a single point \( x = 0 \). It turns out \( T \) has no invariant measure which is absolutely continuous with respect to Lebesgue measure. We omit the proof and refer the reader to [LaYo]. \( \square \)

As our next scenario, let us study an example of a 2-dimensional system that has expanding and contracting direction but there is no absolutely continuous invariant measure. As a toy model for such a phenomenon, we consider a (generalized) baker’s transformation:
\[
T : \mathbb{T}^2 \to \mathbb{T}^2, \quad T(x_1, x_2) = \begin{cases} 
\left( \frac{x_1}{\alpha}, \beta x_2 \right) & \text{if } 0 \leq x_1 \leq \alpha, \\
\left( \frac{x_1}{\beta}, \beta + \alpha x_2 \right) & \text{if } \alpha < x_1 \leq 1.
\end{cases}
\]
with $\alpha, \beta > 0$ and $\alpha + \beta = 1$. Note

$$|JT(x_1, x_2)| = \begin{cases} \frac{\beta}{\alpha} & \text{if } 0 \leq x \leq \alpha, \\ \frac{\alpha}{\beta} & \text{if } \alpha < x \leq 1. \end{cases}$$

As we will see later, the transformation $T$ does not have an absolutely continuous invariant measure unless $\alpha = \beta = \frac{1}{2}$. To analyze Perron–Frobenious operator, let us define

$$F(x_1, x_2) = F_\mu(x_1, x_2) = \mu([0, x_1] \times [0, x_2]).$$

If $\hat{F} = F_{\mathcal{A}_\mu}$, then

$$T^{-1}(x_1, x_2) = \begin{cases} (\alpha x_1, \frac{x_2}{\beta}) & \text{if } 0 \leq x_2 \leq \beta, \\ (\alpha + \beta x_1, \frac{x_2 - \beta}{\alpha}) & \text{if } \beta < x_2 \leq 1. \end{cases}$$

To see this, recall that $\hat{F}(x_1, x_2) = \mu(T^{-1}([0, x_1] \times [0, x_2]))$. Also

$$\mu\left([0, \alpha x_1] \times \left(0, \frac{x_2 - \beta}{\alpha}\right]\right) = F\left(\alpha + \beta x_1, \frac{x_2 - \beta}{\alpha}\right) - F\left(\alpha, \frac{x_2 - \beta}{\alpha}\right),$$

completing the proof of (2.14). Since the expanding and contracting directions are the $x, y$-axis, we may separate variable to solve the equation $\hat{\mathcal{A}}F := \hat{F} = F$. In other words, we
search for a function \( F(x_1, x_2) = F_1(x_1)F_2(x_2) \) such that \( \hat{A}F = F \). Since the Lebesgue measure is invariant for the map \( S : \mathbb{T} \to \mathbb{T} \), that is given by

\[
S(x_1) = \begin{cases} 
\frac{x_1}{\alpha} & \text{if } 0 \leq x_1 \leq \alpha, \\
\frac{x_1 - \alpha}{\beta} & \text{if } \alpha < x_1 \leq 1,
\end{cases}
\]

we may try \( F_1(x_1) = x_1 \). Substituting this in \( \hat{A}F \) yields \( \hat{A}(x_1, x_2) = x_1 \hat{F}_2(x_2) \) where

\[
\hat{F}_2(x) = \begin{cases} 
\alpha F_2 \left( \frac{x}{\beta} \right) & 0 \leq x_2 \leq \beta, \\
\alpha + \beta F_2 \left( \frac{x_2 - \beta}{\alpha} \right) & \beta < x_2 \leq 1.
\end{cases}
\]

Here we are using \( F_2(1) = 1 \). We are now searching for \( F_2 \) such that \( \hat{A}F_2 = F_2 \). It turns out that this equation has a unique solution \( F_2 \) that has zero derivative almost everywhere. Hence our invariant measure \( \mu = \lambda_1 \times \lambda_2 \) with \( \lambda_1 \) the Lebesgue measure and \( \lambda_2 \) a singular measure. One can show that the support of the measure \( \lambda_2 \) is of fractal dimension \( \alpha \log \alpha + \beta \log \beta =: \Delta \).

To explain this heuristically, let us first propose a definition for the fractal dimension of a measure. To motivate our definition, observe that if a measure \( \mu \) is absolutely continuous with respect to the Lebesgue measure \( \lambda = \lambda_1 \), with \( d\mu = f d\lambda_1 \), then by a classical theorem of Lebesgue,

\[
\lim_{\delta \to 0} (2\delta)^{-1} \mu(I_n(x)) = f(x),
\]

for almost all \( x \). This means that if \( I_n(x) = (x - n^{-1}, x + n^{-1}) \), then

\[
\lim_{n \to \infty} \frac{\mu(I_n(x))}{\lambda_1(I_n(x))} = f(x),
\]

for almost all \( x \). In the support of \( \mu \), we can assert

\[
0 < \lim_{n \to \infty} \frac{\mu(I_n(x))}{\lambda_1(I_n(x))} < \infty,
\]

because \( f(x) > 0 \), \( \mu \)-almost everywhere. We may say that a measure \( \mu \) is of fractal dimension \( D \) if

\[
0 < \limsup_{\delta \to 0} \frac{\mu(I_n(x))}{\lambda_1(I_n(x))}^D \leq \liminf_{\delta \to 0} \frac{\mu(I_n(x))}{\lambda_1(I_n(x))}^D < \infty,
\]

\( \mu \)-almost everywhere. This requires that the fractal measure \( \mu \) to be rather regular and if true, (2.16) is often hard to establish. To have a less ambitious definition, we may require

\[
\lim_{n \to \infty} \frac{\log \mu(I_n(x))}{\log \lambda_1(I_n(x))} = D, \quad \mu - a.e.
\]
In fact, we may try to establish (2.17) for a sequence of intervals \( I_n(x) \) with \( x \in I_n(x) \), \( \cap_n I_n(x) = \{x\} \). With this definition in mind, let us write \( A \) for the set of points \( x \) such that

\[
\lim_{n \to \infty} \frac{\log \lambda_2(I_n(x))}{\log \lambda_1(I_n(x))} = \Delta.
\]

We wish to show that \( \lambda_2(A) = 1 \). To construct \( I_n \), let us first define a family of intervals \( I_{a_1,\ldots,a_n} \), with \( a_1,\ldots,a_n \in \{0,1\} \), so that \( I_0 = [0,\beta), I_1 = [\beta,1) \), and if \( I_{a_1,\ldots,a_n} = [p,q) \), then \( I_{a_{1,\ldots,a_n}0} = [p,p + \beta(q-p)), I_{a_{1,\ldots,a_n}1} = [p + \beta(q-p),q) \). By induction on \( n \), it is not hard to show

\[
\lambda_2(I_{a_1,\ldots,a_n}) = \alpha^{L_n} \beta^{R_n}, \quad \lambda_1(I_{a_1,\ldots,a_n}) = \beta^{L_n} \alpha^{R_n},
\]

where \( L_n \) and \( R_n \) denote the number of 0 and 1 in the sequence \( a_1,\ldots,a_n \), respectively. Given \( x \), we can find a sequence \( \omega(x) = (a_1,\ldots,a_n,\ldots) \in \Omega = \{0,1\}^\mathbb{N} \), such that \( x \in I_{a_1,\ldots,a_n} \) for every \( n \). The transformation \( x \mapsto \omega(x) \) pushes forward the measure \( \lambda_2 \) to the product measure \( \lambda_2' \) such that each \( a_n \) is 0 with probability \( \alpha \). If \( L_n(x) \) and \( R_n(x) \) denote the number of 0 and 1 in \( a_1,\ldots,a_n \) with \( \omega(x) = (a_1,\ldots,a_n,\ldots) \), then by Birkhoff Ergodic Theorem

\[
\lambda_2\left(\left\{x : \lim_{n \to \infty} \frac{L_n(x)}{n} = \alpha, \lim_{n \to \infty} \frac{R_n(x)}{n} = \beta\right\}\right) = 1.
\]

From this and (2.16) we can readily deduce that \( \lambda_2(A) = 1 \). Note that the support of \( \tilde{\mu} \) is of dimension \( 1 + \Delta \). Evidently \( \Delta < 1 \) unless \( \alpha = \beta = \frac{1}{2} \).

What we have constructed is the Sinai–Ruelle–Bowen (SRB) measure \( \tilde{\mu} \) of our baker’s transformation \( T \). Note that this measure is absolutely continuous with respect to the expanding direction \( x_1 \)-axis. A remarkable result of Sinai–Ruelle–Bowen asserts

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j(x)) = \int f d\tilde{\mu}
\]

for almost all \( x \) with respect to the Lebesgue measure. This is different from Birkhoff’s ergodic theorem that only gives the convergence for \( \tilde{\mu}\text{-a.e.} \) and \( \tilde{\mu} \) is singular with respect to the Lebesgue measure. We may define the SRB measure as the invariant measure of the maximum metric entropy. The metric entropy will be discussed thoroughly in Chapter 4.

We end this section with a discussion regarding the flow-analog of Perron–Frobenious equation. Given a flow \( \phi_t \) associated with the ODE \( \frac{dx}{dt} = f(x) \), let us define

\[
T_t g = g \circ \phi_t.
\]
This defines a group of transformations on the space of real-valued functions \( g \). The dual of \( T_t \) acts on measures. More precisely, \( T_t^* \mu \) is defined by

\[
\int T_t f \, d\mu = \int f T_t^* \mu,
\]
or equivalently \( T_t^* \mu(A) = \mu(\phi_t^{-1} A) = \mu(\phi_{-t}(A)) \). The following theorem of Liouville gives an infinitesimal description of \( T_t^* \mu \) when \( \mu \) is absolutely continuous with respect to Lebesgue measure.

**Theorem 2.3** Suppose that there exists a differentiable function \( \rho(x,t) \) such that \( d(T_t^* \mu) = \rho(x,t) dx \). Then \( \rho \) satisfies the Liouville’s equation

\[
\rho_t + \text{div}(f \rho) = 0.
\]

**Proof** Let \( g \) be a differentiable function of compact support. We have

\[
\int g(y) \rho(y, t + h) dy = \int g(\phi_{t+h}(x)) \rho(x,0) dx \\
= \int g(\phi_h(\phi_t(x))) \rho(x,0) dx \\
= \int g(\phi_h(y)) \rho(y,t) dy \\
= \int g(y) \rho(y,t) dy + h \int \nabla g(y) \cdot f(y) \rho(y,t) dy + o(h).
\]

This implies that \( \frac{d}{dt} \int g(y) \rho(y, t) dy = \int f(y) \cdot \nabla g(y) \rho(y, t) dy \). After an integration by parts,

\[
\frac{d}{dt} \int g(y) \rho(y, t) dy = \int g(\rho_t + \text{div}(f \rho)) dy.
\]

Since \( g \) is arbitrary, we are done. \( \square \)

In particular a measure \( \rho dx \) is invariant if

\[
\text{div}(f \rho) = 0,
\]
or equivalently \( \rho \nabla f + \rho \text{ div } f = 0 \). The generalization of this to manifolds is straightforward. If \( \mathcal{L}_f \) denotes the Lie derivative and \( f \) is the velocity of the flow, then \( \rho \omega \) is invariant if and only if

\[
\mathcal{L}_f \rho + \rho \text{ div } f = 0.
\]
Exercises

(i) Prove Proposition 2.1.

(ii) Show that the baker’s transformation is reversible in the following sense: If \( \Phi(x, y) = (1 - y, 1 - x) \) then \( \Phi^2 = \text{identity} \) and \( T^{-1} = \Phi T \Phi \).

(iii) Show that if \( \mu \in \mathcal{I}_T \), then \( \mu \Phi \in \mathcal{I}_{T^{-1}} \) where \( \mu \Phi \) is defined by \( \int f d(\mu \Phi) = \int f \circ \Phi d\mu \).

(iv) Let \( T : (0, 1] \to (0, 1] \) by \( T(x) = \{ \frac{1}{2} x \} \) where \( \{ \cdot \} \) means the fractional part. Derive the corresponding Perron–Frobenious equation. Show that \( \rho(x) = \frac{1}{\log 2} \frac{1}{1 + x} \) is a fixed point for the corresponding Perron–Frobenious operator.

(v) Let \( T : [0, 1] \to [0, 1] \) by \( T(x) = 4x(1 - x) \). Derive the corresponding Perron–Frobenious equation and show that \( \rho(x) = \pi^{-1}(x(1 - x))^{-1/2} \) is a fixed point.

(vi) Let \( u(x, t) = T_t g(x) = g(\phi_t(x)) \). Show that \( u \) satisfies \( u_t = \mathcal{L} u \) where \( \mathcal{L} u = f(x) \cdot \frac{\partial u}{\partial x} \).

(vii) Show that \( \mu \in \mathcal{I}_\phi \) iff \( \int \mathcal{L} g \, d\mu = 0 \) for every \( g \in C^1 \) of compact support. □

Notes The proof of Theorem 2.2 was taken from [Li]. The example of the generalized baker’s transformation was taken from [D].
3 Entropy

Roughly speaking, the entropy measures the exponential rate of increase in dynamical complexity as a system evolves in time. We will discuss two notions of entropy in this section, the topological entropy and (Kolmogorov–Sinai) metric entropy. We start with the definition of the topological entropy.

**Definition 3.1** Let \((X,d)\) be a compact metric space and \(T : X \to X\) be a continuous transformation. Define

\[
d_n(x,y) = d_n^T(x,y) = \max\{d(x,y), d(T(x), T(y)), \ldots, d(T^{n-1}(x), T^{n-1}(y))\},
\]

\[
B^n(x,r) = B^n_{T,d}(x,r) = \{y : d_n(x,y) < r\}.
\]

We then define two numbers. First \(S^n_{T,d}(r)\) is defined as the smallest number \(k\) for which we can find a set \(A\) of cardinality \(k\) such that \(X = \bigcup_{x \in A} B^n_{T,d}(x,r)\). We also define \(N^n_{T,d}(r)\) to be the maximal number of points in \(X\) with pairwise \(d_n\)-distances at least \(r\). Set

\[
h_{\text{top}}(T; d) = h_{\text{top}}(T) = \lim_{r \to 0} \limsup_{n \to \infty} \frac{1}{n} \log S^n_{T,d}(r),
\]

\[
\bar{h}_{\text{top}}(T; d) = \bar{h}_{\text{top}}(T) = \lim_{r \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N^n_{T,d}(r).
\]

We will see below that \(\bar{h}_{\text{top}} = h_{\text{top}}\) and we call \(h_{\text{top}}(T)\), the topological entropy of \(T\). We will see that \(h_{\text{top}}(T; d)\) is independent of the choice of the metric and depends on the topology of the underlying space. In some sense, “higher entropy” means “more orbits”. But the number of orbits is usually uncountably infinite. Hence we fix a “resolution” \(r\), so that we do not distinguish points that are of distance less than \(r\). Hence \(N^n(r)\) represents the number of distinguishable orbits of length \(n\), and this number grows like \(e^{nh_{\text{top}}(T)}\). Here are some properties of the topological entropy.

**Proposition 3.1** (i) If the metrics \(d\) and \(d'\) induce the same topology, then \(h_{\text{top}}(T; d) = h_{\text{top}}(T; d')\).

(ii) If \(F : X \to Y\) is a homeomorphism, \(T : X \to X\), \(S : Y \to Y\), and \(S \circ F = F \circ T\), then \(h_{\text{top}}(T) = h_{\text{top}}(S)\).

(iii) \(h_{\text{top}}(T^n) = nh_{\text{top}}(T)\). Moreover, if \(T\) is a homeomorphism, then \(h_{\text{top}}(T) = h_{\text{top}}(T^{-1})\).

(iv) \(h_{\text{top}}(T) = \bar{h}_{\text{top}}(T)\). Moreover \(\limsup\) in the definition of \(h_{\text{top}}\) and \(\bar{h}_{\text{top}}\) can be replaced with \(\liminf\).
Proof (i) Set $\eta(\epsilon) = \min\{d'(x, y) : d(x, y) \geq \epsilon\}$. Then
\[d'(x, y) < \eta(\epsilon) \Rightarrow d(x, y) < \epsilon.\]
As a result, $\lim_{\epsilon \to 0} \eta(\epsilon) = 0$ and $B_{T, d'}^n(x, \eta(\epsilon)) \subseteq B_{T, d}^n(x, \epsilon)$. Hence $S_{T, d'}^n(\eta(\epsilon)) \geq S_{T, d}^n(\epsilon)$. Thus
\[h_{\text{top}}(T, d) \leq h_{\text{top}}(T, d').\]

(ii) Given a metric $d$ on $X$, define a metric $d'$ on $Y$ by $d'(x, y) = d(F^{-1}(x), F^{-1}(y))$. Evidently $h_{\text{top}}(T; d) = h_{\text{top}}(S; d')$.

(iii) Evidently $B_{T, d}^n(x, r) \subseteq B_{T, d'}^n(x, r)$. Hence
\[S_{T, d}^n(r) \geq S_{T, d'}^n(r), \quad h_{\text{top}}(T^n) \leq n h_{\text{top}}(T).\]
For the converse, use the continuity of $T$ to find a function $\eta : (0, \infty) \to (0, \infty)$ such that $\eta(r) \leq r$ and $B_d(x, \eta(r)) \subseteq B_{T, d}^n(x, r)$. Then $B_{T, d'}^n(x, \eta(r)) \subseteq B_{T, d}^n(x, r)$. This implies that
\[S_{T, d}^n(\eta(r)) \geq S_{T, d'}^n(r),\]
which in turn implies
\[\frac{1}{k} \log S_{T, d}^n(\eta(r)) \geq \frac{n}{k} \max_{(k-1)n \leq \ell \leq kn} \frac{1}{\ell} \log S_{T, d}^\ell(r).\]
From this, it is not hard to deduce that $h_{\text{top}}(T^n) \geq n h_{\text{top}}(T)$.

For $h_{\text{top}}(T^{-1}) = h_{\text{top}}(T)$, observe that $d^n_T(x, y) = d^{T^{-1}}(T^n, T^{-1})$. This means that $T^{-1}(B_{T, d}^n(x, r)) = B_{T, d}^n(T^{-1}(x), r)$. Hence $X = \bigcup_{j=1}^k B_{T, d}^n(x_j, r)$ is equivalent to $X = \bigcup_{j=1}^k B_{T, d}^{n-1}(x_j, r)$. From this we deduce $S_{T, d}^n(r) = S_{T, d}^n(r)$. This implies that $h_{\text{top}}(T^{-1}) = h_{\text{top}}(T)$.

(iii) The equality $h_{\text{top}}(T) = \tilde{h}_{\text{top}}(T)$, is an immediate consequence of the following straightforward inequalities:
\[N_{T, d}^n(2r) \leq S_{T, d}^n(r) \leq N_{T, d}^n(r)\]
The second inequality follows from the fact that if $N^n(r) = L$ and $\{x_1, \ldots, x_L\}$ is a maximal set, then $X = \bigcup_{j=1}^L B_{d}^n(x_j, r)$. The first inequality follows from the fact that no $d_{n}$-ball of radius $r$ can contain two points that are $2r$-apart.

To replace $\limsup$ with $\liminf$ in the definition of $h_{\text{top}}$ and $\tilde{h}_{\text{top}}$, we first give an alternative way of calculating the entropy by allowing arbitrary open sets for our covering. More precisely, we let $\hat{S}_{T, d}^n(r)$ to denote the smallest number $k$ for which we can find an open cover $O$ of $X$ cardinality $k$ such that for every $A \in O$, the diameter of the set $A$ with respect to the metric $d_n$ is at most $2r$. Evidently
\[S_{T, d}^n(2r) \leq \hat{S}_{T, d}^n(r) \leq S_{T, d}^n(r),\]
because if $O$ is an open cover of $X$ such that for every $A \in O$, the diameter of the set $A$ with respect to the metric $d_n$ is at most $2r$, then we can pick $a_A \in A$ and form a cover of the form
\[X = \bigcup_{A \in O} B_{T, d}^n(a_A, 2r).\]
We are done if we can show that in fact
\[
\lim_{n \to \infty} \frac{1}{n} \log \hat{S}_m(r),
\]
exists for every \( r > 0 \). This is achieved by showing that the sequence \( a_n = \log \hat{S}_m(r) \) is subadditive (see Lemma 3.2 below). More precisely, we have that for each \( n \) and \( m \in \mathbb{N} \),
\[
\hat{S}_{m+n}(r) \leq \hat{S}_m(r) \hat{S}_n(r).
\]
To see this, take two collections of sets \( \mathcal{O} \) and \( \mathcal{O}' \) with \( \mathcal{O} \) (respectively \( \mathcal{O}' \)) an open cover of \( X \) with respect to the metric \( d_m \) (respectively \( d_n \)). Then \( \mathcal{O}'' \) consisting of the sets of the form \( A \cap T^{-m}(B) \) with \( A \in \mathcal{O} \) and \( B \in \mathcal{O}' \) is an open cover of \( X \) with respect to the metric \( d_{m+n} \). \( \Box \)

**Lemma 3.1** Let \( a_n \) be a sequence of numbers such that \( a_{n+m} \leq a_n + a_m \). Then \( \lim_{n \to \infty} \frac{1}{n} a_n = \inf_n \frac{a_n}{n} \).

**Proof** Evidently \( \lim \inf_{n \to \infty} \frac{a_n}{n} \geq \inf_n \frac{a_n}{n} \). On the other hand, if \( n = \ell m + r \) with \( m, \ell \in \mathbb{N} \), \( r \in [0, m) \), then
\[
a_n = a_{\ell m + r} \leq a_{\ell m} + a_r \leq \frac{\ell m a_m}{n} + \frac{a_r}{n}.
\]
After sending \( n \to \infty \), we obtain,
\[
\limsup_{n \to \infty} \frac{a_n}{n} \leq \frac{a_m}{m}
\]
for every \( m \in \mathbb{Z}^+ \). This completes the proof. \( \Box \)

**Example 3.1(i)** Let \( T : \mathbb{T}^d \to \mathbb{T}^d \) be a translation. Since \( T \) is an isometry, \( d_n(x, y) = d(x, y) \) for \( d(x, y) = |x - y| \). Thus \( S_n(r) \) is independent of \( n \) and \( h_{\text{top}}(T) = 0 \).

(ii) Let \( X = \{0, 1, \ldots, m-1\}^\mathbb{Z} \). Given \( \omega = (\omega_j : j \in \mathbb{Z}) \in X \), define \( (\tau \omega)_j = \omega_{j+1} \). Consider the metric
\[
d(\omega, \omega') = \sum_{j \in \mathbb{Z}} \eta^{-|j|} |\omega_j - \omega'_j|,
\]
with \( \eta > 1 \). Fix \( \alpha \in X \) and take any \( \omega \in X \). Evidently
\[
\sum_{|j| > \ell} \eta^{-|j|} |\alpha_j - \omega_j| \leq 2(m - 1) \sum_{\ell+1} \eta^{-\ell} = \frac{2(m - 1)}{\eta^{\ell-1}}.
\]
Also, if \( \omega_j \neq \alpha_j \) for some \( j \in \{-\ell, \ldots, \ell\} \), then
\[
\sum_{|j| \leq \ell} \eta^{-|j|} |\alpha_j - \omega_j| \geq \eta^{-\ell}.
\]

Evidently \( d \) induces the product topology on \( X \) no matter what \( \eta \in (1, \infty) \) we pick. Choose \( \eta \) large enough so that \( \frac{2(m-1)}{\eta-1} < 1 \). For such a choice of \( \eta \),
\[
B_d \left( \alpha, \eta^{-\ell} \right) = \{ \omega : \omega_j = \alpha_j \text{ for } j \in \{-\ell, \ldots, \ell\} \}.
\]

Since
\[
\{ \omega : d(\tau^i(\omega), \tau^i(\alpha)) < \eta^{-\ell} \} = \{ \omega : \omega_{j+i} = \alpha_{j+i} \text{ for } j \in \{-\ell, \ldots, \ell\} \},
\]
we deduce
\[
B_{d_n} \left( \alpha, \eta^{-\ell} \right) = \{ \omega : \omega_j = \alpha_j \text{ for } j \in \{-\ell, \ldots, \ell+n-1\} \}.
\]
Evidently every two \( d_n \)-balls of radius \( \eta^{-\ell} \) are either identical or disjoint. As a result,
\[
S^n_{\tau,d} \left( \eta^{-\ell} \right) = m^{2\ell+n}.
\]

Thus
\[
h_{\text{top}}(T) = \lim_{\ell \to \infty} \limsup_{n \to \infty} \frac{1}{n} \log m^{2\ell+n} = \log m.
\]

(iii) Let \( T_m : \mathbb{T} \to \mathbb{T} \) be the expansion map as in Example 1.5(ii). From Part (ii) and Exercise 3.1(i), we deduce that \( h_{\text{top}}(T_m) \leq \log m \). We will see later in Example 3.4 below that in fact \( h_{\text{top}}(T_m) = \log m \).

(iv) Let \((X, \tau)\) be as in the Part (ii) and let \( A = [a_{ij}] \) be an \( m \times m \) matrix with \( a_{ij} \in \{0,1\} \) for all \( i, j \in \{0,1,\ldots,m-1\} \). Set
\[
X_A = \{ \omega \in X : a_{\omega_i, \omega_{i+1}} = 1 \text{ for all } i \in \mathbb{Z} \}.
\]

Evidently \( X_A \) is an invariant set and the restriction of \( \tau \) to \( X_A \) gives a dynamical system. Write \( \tau_A \) for this restriction. To have an irreducible situation, we assume that each row and column of \( A \) contains at least one 1 (if for example \( a_{0j} = 0 \) for all \( j \), we may replace \( X \) with \( \{1,2,\ldots,m-1\}^\mathbb{Z} \)). For such \( A \),
\[
S^n_{\tau_A,d} \left( \eta^{-\ell} \right) = \# \text{ of balls of radius } \eta^{-\ell} \text{ with nonempty intersection with } X_A
\]
\[
= \# \text{ of } (\alpha_{-\ell}, \ldots, \alpha_{\ell+n-1}) \text{ with } a_{\alpha_{i}, \alpha_{i+1}} = 1 \text{ for } -\ell \leq i < \ell + n - 1
\]
\[
= \sum_{r,s=0}^{m-1} \# \{ (\alpha_{-\ell}, \ldots, \alpha_{\ell+n-1}) : a_{\alpha_{i}, \alpha_{i+1}} = 1 \text{ for } -\ell \leq i < \ell + n - 1 \}
\]
\[
\text{and } \alpha_{-\ell} = r, \alpha_{\ell+n-1} = s
\]
\[
= \sum_{r,s=0}^{m-1} a_{2n+\ell-1} = \| A^{2n+\ell-1} \|.
\]

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where \(a_{r,s}^k\) is the \((r,s)\) entry of the matrix \(A^k\), and \(\|A\|\) denotes the norm of \(A\), i.e., \(\|A\| = \sum_{r,s} |a_{r,s}|\). We now claim

\[
h_{\text{top}}(\tau_A) = \lim_{\ell \to \infty} \limsup_{n \to \infty} \frac{1}{n} \log \|A^{2\ell+n-1}\| = \log r(A),
\]

where \(r(A) = \max\{|\lambda| : \lambda \text{ an eigenvalue of } A\}\). To see this, first observe that if \(Av = \lambda v\), then \(A^kv = \lambda^k v\). Hence

\[
|\lambda|^k \max_j |v_j| \leq |\lambda|^k \sum_j |v_j| \leq \sum_{i,j} |a_{i,j}^k||v_i| \leq \|A^k\| \max_j |v_j|.
\]

As a result, \(\|A^k\| \geq |\lambda|^k\). This shows that \(h_{\text{top}}(\tau_A) \geq \log r(A)\). For the converse, we choose a basis so that the off-diagonal entries in Jordan normal form of \(A\) become small. Using this we can show that \(|Av| \leq (r(A) + \delta)|v|\) which in turn implies that \(|A^kv| \leq (r(A) + \delta)^k|v|\). From this we deduce that \(h_{\text{top}}(\tau_A) \leq \log (r(A) + \delta)\). Finally send \(\delta \to 0\) to deduce that \(h_{\text{top}}(\tau_A) \leq \log r(A)\). This completes the proof of \(h_{\text{top}}(\tau_A) = \log r(A)\).

(b) Let \(T : \mathbb{T}^2 \to \mathbb{T}^2\) is given by \(T(x) = Bx \pmod{1}\), where \(B\) is an integer-valued matrix with eigenvalues \(\lambda_1, \lambda_2\) satisfying \(|\lambda_2| < 1 < |\lambda_1| = |\lambda_2|^{-1}\). For the sake of definiteness, let us take \(B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\) with eigenvalues \(\lambda_1 = \frac{3+\sqrt{5}}{2}, \lambda_2 = \frac{3-\sqrt{5}}{2}\) and eigenvectors \(v_1 = \begin{bmatrix} 1 \\ \sqrt{5}-1 \end{bmatrix}\), \(v_2 = \begin{bmatrix} 1 \\ -\sqrt{5}-1 \end{bmatrix}\). \(T\) is a contraction along \(v_2\) and an expansion along \(v_1\). We now draw the eigen lines from the origin and let them intersect several times to separate torus into disjoint rectangles. Let us write \(R_1\) and \(R_2\) for these rectangles and study \(T(R_1)\) and \(T(R_2)\). We set

\[
T(R_1) \cap R_1 = Z_0 \cup Z_1, \quad T(R_1) \cap R_2 = Z_3, \quad R_1 = Z_0 \cup Z_1 \cup Z_2.
\]

We then define \(Z_4\) so that \(R_2 = Z_3 \cup Z_4\). One can then show that \(T(R_2) = Z_2 \cup Z_4\). We now define \(X = \{0, 1, 2, 3, 4\}\) and \(F : X_A \to \mathbb{T}^2\) with

\[
A = [a_{ij}] = \begin{bmatrix}
1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1
\end{bmatrix},
\]

where \(F(\omega) = x\) for \(\{x\} = \bigcap_{n \in \mathbb{Z}} T^{-n}(Z_{\omega_n})\). In other words, \(F(\omega) = x\) iff \(T^n(x) \in Z_{\omega_n}\) for all \(n \in \mathbb{Z}\). If \(\tau_A\) denotes the shift on \(X_A\), then we have \(T \circ F = F \circ \tau_A\). Here we are using the fact that if \(x \in Z_i\) and \(T(x) \in Z_j\), then \(a_{ij} = 1\). This also guarantees that \(\bigcap_{n \in \mathbb{Z}} T^{-n}(Z_{\omega_n}) \neq \emptyset\). Also, since \(T\) is contracting in \(v_2\)-direction and \(T^{-1}\) is contracting in
$v_1$-direction, then $\bigcap_{n \in \mathbb{Z}} T^{-n}(Z_{\omega_n})$ has at most one point. Clearly the transformation $F$ is onto. However, $h$ is not one-to-one. For example if $\bar{\alpha}$ denotes $\bar{\alpha} = (\omega_n : n \in \mathbb{Z})$ with $\omega_n = \alpha$ for all $n$, then $\bar{0}, \bar{1}, \bar{4} \in X_A$ (but not $\bar{2}$ and $\bar{3}$). Moreover $\tau_A(\bar{0}) = \bar{0}$, $\tau_A(\bar{1}) = \bar{1}$, $\tau_A(\bar{4}) = \bar{4}$. On the other hand the only $x$ with $T(x) = x$ is $x = 0$. In fact $F(\bar{0}) = F(\bar{1}) = F(\bar{4})$ is equal to the origin. From $T \circ F = F \circ \hat{T}$, Exercise 3.1(i) and Example 3.1(iv) we conclude that $h_{\text{top}}(T) \leq h_{\text{top}}(\tau_A) = \log r(A)$. A straightforward calculation yields $r(A) = \lambda_1 = \frac{3 + \sqrt{5}}{2}$. Later we discuss the metric entropy, and using the metric entropy of $T$ we will show in Example 3.4 below that indeed $h_{\text{top}}(T) = \log \frac{3 + \sqrt{5}}{2}$.

The metric entropy is the measure-theoretic version of the topological entropy. As a preparation, we make a definition.

**Definition 3.2** Let $T : X \to X$ be a measurable transformation and take $\mu \in \mathcal{I}_T$.

(i) A countable collection $\xi$ of measurable subsets of $X$ is called a $\mu$-partition if $\mu(A \cap B) = 0$ for every two distinct $A, B \in \xi$, and $\mu \left( X \setminus \bigcup_{A \in \xi} A \right) = 0$. We also write $C_\xi(x)$ for the unique $A \in \xi$ such that $x \in A$. Note that $C_\xi(x)$ is well-defined for $\mu$-almost all $x$.

(ii) If $\xi$ and $\eta$ are two $\mu$-partition, then their common refinement $\xi \vee \eta$ is the partition $\xi \vee \eta = \{ A \cap B : A \in \xi, B \in \eta, \mu(A \cap B) > 0 \}$.

Also, if $\xi$ is a $\mu$-partition, then we set

$$T^{-1}\xi = \{ T^{-1}(A) : A \in \xi \},$$

which is also a $\mu$-partition because $\mu \in \mathcal{I}_T$.

(iii) For $m < n$, we define

$$\xi^T(m, n) = \xi(m, n) = T^{-m}\xi \vee T^{-m-1}\xi \vee \cdots \vee T^{-n}\xi.$$

As we discussed in the introduction, the metric entropy measures the exponential gain in the information. Imagine that we can distinguish two points $x$ and $y$ only if $x$ and $y$ belong to different elements of the partition $\xi$. Now if the orbits up to time $n - 1$ are known, we can use them to distinguish more points. The partition $\xi^T(0, n - 1)$ represents the accumulated information gained up to time $n - 1$. Except for a set of zero $\mu$-measure, each $x$ belongs to a unique element

$$C_n(x) = C_\xi(x) \cap T^{-1}(C_\xi(T(x))) \cap \cdots \cap T^{1-n}(C_\xi(T^{n-1}(x))) \in \xi(0, n - 1).$$

Let’s have an example.
Example 3.2(i) Let \((X, \tau)\) be as in Example 1.1(iii), with \(E = \{0, 1, \ldots, m - 1\}\). Choose 
\[\xi = \{A_0, \ldots, A_{m-1}\},\] 
with \(A_i = \{\omega: \omega_1 = i\}\). We have 
\[C_n(\alpha) = \{\omega: \omega_i = \alpha_i \text{ for } i = 1, \ldots, n\}.\]

Given \(p = (p_0, \ldots, p_{m-1})\), with \(p_j \geq 0\), \(\sum_j p_j = 1\), recall \(\mu_p \in \mathcal{I}\) is the product measure with \(\mu_p(A_i) = p_i\). We certainly have 
\[\mu_p(C_n(\alpha)) = p_{\alpha_1} \cdots p_{\alpha_n}\]
and 
\[\frac{1}{n} \log \mu_p(C_n(\alpha)) = \frac{1}{n} \sum_j \log p_{\alpha_j} = \frac{1}{n} \sum_{j=0}^{n-1} \log f(T^j(\alpha))\]
where \(f(\alpha) = p_{\alpha_1}\). By the Ergodic Theorem,
\[(3.1) \quad \lim_{n \to \infty} \frac{1}{n} \log \mu_p(C_n(\alpha)) = \sum_{j=0}^{m-1} p_j \log p_j.\]

(ii) Let \((T, T_m)\) be the expansion map as in Example 1.5(ii). Let 
\[\xi = \left\{ \left[ \frac{j}{m}, \frac{j+1}{m} \right): j = 0, \ldots, m - 1 \right\}.\]

Then 
\[\eta_n = \xi^T(0, n - 1) = \left\{ [a_1 \ldots a_n, a_1 \ldots a_n + m^{-n}): a_1 \ldots a_n \in \{0, 1, \ldots, m - 1\} \right\}.\]

Given \(x\), let \(\cdot a_1 a_2 \ldots a_n \ast \ldots \) denote its base \(m\) expansion. Note that for points on the boundary of the intervals in \(\eta_n\), we may have two distinct expansions. Since we have chosen closed-open intervals in \(\xi\), we dismiss expansions which end with infinitely many \(m\). In other words, between \(.a_1 \ldots a_K(m - 1)(m - 1)\ldots\), with \(a_k < m - 1\) and \(.a_1 \ldots a'_K 00\ldots\) for \(a'_k = a_k + 1\), we choose the latter. we have 
\[C_{\eta_n}(x) = [a_1 \ldots a_n, a_1 \ldots a_n + m^{-n}).\]

For \(\mu_p\) as in Example 1.5(iii), we have 
\[\mu_p(C_{\eta_n}(x)) = p_{a_1} \cdots p_{a_n}\]
and 
\[\frac{1}{n} \log \mu_p(C_{\eta_n}(x)) = \frac{1}{n} \sum_j \log p_{a_j} = \frac{1}{n} \sum_{j=0}^{n-1} \log f(T^j(x))\]
where \(f(a_1 a_2 \ldots) = p_{a_1}\). By the Ergodic Theorem,
\[(3.2) \quad \lim_{n \to \infty} \frac{1}{n} \log \mu_p(C_{\eta_n}(x)) = \sum_{j=0}^{m-1} p_j \log p_j.\]
In general, since we are interested in the amount of information the partition \( \eta_n = \xi(0, n - 1) \) carries out, perhaps we should look at \( \mu(C_n(x)) \) where \( C_n(x) = C_{\eta_n}(x) \). This is typically exponentially small in \( n \). Motivated by Example 3.2, we make the following definition.

**Definition 3.3(i)** Let \( \mu \in \mathcal{M} \) and \( \xi \) be a \( \mu \)-partition. The *entropy* of \( \xi \) with respect to \( \mu \) is defined by

\[
H_\mu(\xi) = \int I_\xi(x) \mu(dx) = - \sum_{C \in \xi} \mu(C) \log \mu(C),
\]

where \( I_\xi(x) = -\log \mu(C_\xi(x)) \).

**(ii)** Given two \( \mu \)-partitions \( \eta \) and \( \xi \), the *conditional entropy* of \( \xi \), given \( \eta \) is defined by

\[
H_\mu(\xi \mid \eta) = \int I_{\xi \mid \eta}(x) d\mu = - \sum_{A \in \xi, B \in \eta} \mu(A \cap B) \log \frac{\mu(A \cap B)}{\mu(B)},
\]

where

\[
I_{\xi \mid \eta}(x) = -\log \mu(C_\xi(x) \mid C_\eta(x)) = -\log \frac{\mu(C_\xi(x) \cap C_\eta(x))}{\mu(C_\eta(x))}.
\]

**(iii)** Given a dynamical system \((X, T)\), an invariant measure \( \mu \in \mathcal{I}_T \), and a \( \mu \)-partition \( \xi \), we define

\[
(3.3) \quad h_\mu(T, \xi) = \lim_{n \to \infty} \frac{1}{n} H_\mu(\xi(0, n - 1)),
\]

The existence of the limit follows from the subadditivity of \( a_n = H_\mu(\xi(0, n - 1)) \), which is an immediate consequence of Proposition 3.2(ii) below.

**(iv)** The entropy of \( T \) with respect to \( \mu \in \mathcal{I}_T \) is defined by

\[
h_\mu(T) = \sup_{\xi} h(T, \xi),
\]

where the supremum is over all finite \( \mu \)-partitions.

**(v)** We write \( \alpha \leq \beta \) when \( \beta \) is a refinement of \( \alpha \). This means that for every \( B \in \beta \), there exists a set \( A \in \alpha \) such that \( \mu(B \setminus A) = 0 \).

**Proposition 3.2** Let \( \xi, \eta, \) and \( \gamma \) be three \( \mu \)-partitions.

**(i)** We have

\[
I_{\xi \wedge \eta} = I_\eta + I_{\xi \mid \eta}, \quad H_\mu(\xi \wedge \eta) = H_\mu(\eta) + H_\mu(\xi \mid \eta).
\]
More generally,

\[ I_{(\xi \vee \eta)\gamma} = I_{\eta\gamma}(x) + I_{\xi(\eta \vee \gamma)}, \quad H_\mu(\xi \vee \eta \mid \gamma) = H_\mu(\eta \mid \gamma) + H_\mu(\xi \mid \eta \vee \gamma) \]

(ii) \( H_\mu(\xi \vee \eta) \leq H_\mu(\xi) + H_\mu(\eta) \).

(iii) We have \( H_\mu(T^{-1}\xi) = H_\mu(\xi) \). More generally, \( H_\mu(T^{-1}\xi \mid T^{-1}\eta) = H_\mu(\xi \mid \eta) \).

(iv) If \( \eta \leq \gamma \), then \( H_\mu(\eta) \leq H_\mu(\gamma), \quad H_\mu(\xi \mid \eta) \geq H_\mu(\xi \mid \gamma) \).

**Proof**

(i) By definition,

\[
I_{(\xi \vee \eta)\gamma}(x) = -\log \left( \frac{\mu(C_{\xi \vee \eta}(x) \cap C_\gamma(x))}{\mu(C_\gamma(x))} \right) = -\log \left( \frac{\mu(C_\xi(x) \cap C_\eta(x) \cap C_\gamma(x))}{\mu(C_\gamma(x))} \right) = I_{\eta\gamma}(x) + I_{\xi(\eta \vee \gamma)}(x).
\]

(ii) On account of (i), it suffices to show that \( H_\mu(\xi \mid \eta) \leq H_\mu(\xi) \). Set \( \phi(x) = x \log x \) and use the convexity of \( \phi \) to assert

\[
\phi(\mu(A)) = \phi \left( \sum_{B \in \eta} \mu(B) \frac{\mu(A \cap B)}{\mu(B)} \right) \leq \sum_{B \in \eta} \mu(B) \phi \left( \frac{\mu(A \cap B)}{\mu(B)} \right) = \sum_{B \in \eta} \mu(A \cap B) \log \frac{\mu(A \cap B)}{\mu(B)}.
\]

(iii) This is obvious because \( \mu(T^{-1}(A)) = \mu(A) \) for every \( A \in \xi \).

(iv) We only prove the second inequality as the proof of the first inequality is similar. By definition,

\[
H_\mu(\xi \mid \alpha) = -\sum_{A \in \alpha, C \in \xi} \mu(A \cap C) \log \frac{\mu(A \cap C)}{\mu(A)} = -\sum_{A \in \alpha, C \in \xi} \mu(A) \phi \left( \frac{\mu(A \cap C)}{\mu(A)} \right).
\]

Fix \( A \) and choose a family \( J \subseteq \beta \), so that

\[
\mu(A \Delta \cup \{ B : B \in J \}) = 0.
\]
Hence
\[ \varphi \left( \frac{\mu(A \cap C)}{\mu(A)} \right) = \varphi \left( \sum_{B \in J} \frac{\mu(B)}{\mu(A)} \frac{\mu(C \cap B)}{\mu(B)} \right) \leq \sum_{B \in J} \frac{\mu(B)}{\mu(A)} \varphi \left( \frac{\mu(C \cap B)}{\mu(B)} \right). \]

From this we deduce \( H_\mu(\xi \mid \alpha) \geq H_\mu(\xi \mid \beta). \)

We now show that the limits (3.1) and (3.2) in Example 3.2 are always true if \( \mu \) is ergodic.

**Theorem 3.1 (Shannon–McMillan–Breiman)** If \( \mu \in \mathcal{I}_T^\circ \), then
\[ \lim_{n \to \infty} \int \left| \frac{1}{n} \log \mu(C_n(x)) + h_\mu(T, \xi) \right| d\mu = 0. \]

**Proof** Recall \( \xi(n, m) = T^{-n}\xi \lor T^{-n-1}\xi \lor \cdots \lor T^{-m}\xi \) whenever \( n < m \). We have
\[ I_\xi(0, n-1) = I_\xi \lor I_\xi(1, n-1) = I_\xi(1, n-2) \lor T + I_\xi(1, n-1), \]
because \( C_{T^{-1}\eta}(x) = C_\eta(T(x)) \). Applying this repeatedly, we obtain
\[ \frac{1}{n} I_{\xi(0, n-1)} = \frac{1}{n} \left[ I_{\xi(1, n-1)} + I_{\xi(1, n-2)} \lor T + \cdots + I_{\xi(1, 2)} \lor T^{n-3} + I_{\xi(1, 1)} \lor T^{n-2} + I_{\xi} \lor T^{n-1} \right] \]
If it were not for the dependence of \( I_{\xi(1, n-j)} \mid n-j \), we could have used the Ergodic Theorem to finish the proof. However, if we can show that \( \lim_{n \to \infty} I_{\xi(1, n)} = \hat{I} \) exists, say in \( L^1(\mu) \)-sense, then we are almost done because we can replace \( I_{\xi(1, n-j)} \) with \( \hat{I} \) with an error that is small in \( L^1(\mu) \)-sense. We then apply the ergodic theorem to assert
\[ \lim_{n \to \infty} \frac{1}{n} I_{\xi(0, n-1)} = \int \hat{I} \, d\mu. \]

Note that if we write \( \mathcal{F}_\eta \) for the \( \sigma \)-algebra generated by \( \eta \), then \( \mu(C_\xi(x) \mid C_\eta(x)) \) is nothing other than
\[ \mu(C_\xi \mid \mathcal{F}_\eta)(x) = \sum_{A \in \xi} \mu(A \mid \mathcal{F}_\eta)(x) \mathbb{1}_A(x), \]
i.e. the conditional expectation of the indicator function of the set \( C_\xi \), given the \( \sigma \)-field \( \mathcal{F}_\eta \). Hence, we simply have
\[ \hat{I}(x) = -\log \left\{ \lim_{n \to \infty} \sum_{A \in \xi} \mu(A \mid \xi(1, n))(x) \mathbb{1}_A(x) \right\} = -\sum_{A \in \xi} \log \left\{ \lim_{n \to \infty} \mu(A \mid \xi(1, n))(x) \right\} \mathbb{1}_A(x). \]
This suggests studying \( \lim_{n \to \infty} \mu(A \mid \xi(1, n)) \). The existence and interpretation of the limit involve some probabilistic ideas. We may define \( \mathcal{F}_{1,n} \) to be the \( \sigma \)-algebra generated by the
partition $\xi(1, n)$. We then have $F_{1,2} \subseteq F_{1,3} \subseteq \ldots$ and if $F_{1,\infty}$ is the $\sigma$-algebra generated by all $\xi(1, n)$'s, then

$$\lim_{n \to \infty} \mu(A \mid \xi(1, n)) = \mu(A \mid F_{1,\infty}),$$

$\mu$-almost surely and in $L^1(\mu)$-sense. The right-hand side is the conditional measure of $A$ given the $\sigma$-algebra $F_{1,\infty}$. The proof of convergence follows the celebrated martingale convergence theorem. We refer the reader to any textbook on martingales for the almost sure convergence.

For our purposes, we need something stronger, namely

$$\lim_{n \to \infty} \log \mu(A \mid F_{1,n}) = \log \mu(A \mid F_{1,\infty}),$$

in $L^1(\mu)$. This would follow from (3.5) provided that we can show

$$\int_A \left( \sup_n (-\log \mu(A \mid F_{1,n})) \right) d\mu \leq -\mu(A) \log \mu(A) + \mu(A).$$

Indeed if we pick $\ell > 0$ and define

$$A_n = \{ x : \mu(A \mid F_{1,n})(x) < e^{-\ell}, \quad \mu(A \mid F_{1,k})(x) \geq e^{-\ell} \quad \text{for} \quad k = 1, 2, \ldots, n-1 \},$$

then $A_n \in F_{1,n}$ and we can write

$$\mu \left\{ x \in A : \sup_n (-\log \mu(A \mid F_{1,n})(x)) > \ell \right\} = \mu(A \cap \bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A \cap A_n)$$

$$= \sum_{n=1}^{\infty} \int_{A_n} \mu(A \mid F_{1,n}) d\mu \leq \sum_{n=1}^{\infty} \int_{A_n} e^{-\ell} d\mu = e^{-\ell} \sum_{n=1}^{\infty} \mu(A_n) \leq e^{-\ell}.$$

From this we deduce

$$\int_A \left( \sup_n (-\log \mu(A \mid F_{1,n})) \right) d\mu = \int_0^\infty \mu \left\{ x \in A : \sup_n (-\log \mu(A \mid F_{1,n})(x)) > \ell \right\} d\ell$$

$$\leq \int_0^\infty \min \{ \mu(A), e^{-\ell} \} d\ell = -\mu(A) \log \mu(A) + \mu(A).$$

This completes the proof of (3.7).

The proof of Theorem 3.1 suggests an alternative formula for the entropy. In some sense $h_\mu(T, \xi)$ is the entropy of the “present” $\xi$ relative to its “past” $\xi(1, \infty)$. To make this rigorous, first observe that by Proposition 3.2(i),

$$H_\mu(\xi(0, n-1)) = \sum_{j=1}^{n-1} H_\mu(\xi \mid \xi(1, j))$$

where $H_\mu(\xi \mid \xi(1, 1))$ means $H_\mu(\xi)$. In fact we have
Proposition 3.3 \( h_\mu(T, \xi) = \inf_n H_\mu(\xi | \xi(1,n)) \) and the sequence \( H_\mu(\xi | \xi(1,n)) \) is nondecreasing.

**Proof** The monotonicity of the sequence \( a_n = H_\mu(\xi | \xi(1,n)) \) follows from Proposition 3.2(iii). We then use (3.8) to assert

\[
\lim_{n \to \infty} \frac{1}{n} H_\mu(\xi(0,n-1)) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n-1} H_\mu(\xi | \xi(1,j)) = \lim_{n \to \infty} H_\mu(\xi(1,n)) = \inf_n H_\mu(\xi | \xi(1,n)).
\]

\( \square \)

We continue with some basic properties of the entropy.

Proposition 3.4

(i) \( h_\mu(T^k) = kh_\mu(T) \) and if \( T \) is invertible, then \( h_\mu(T) = h_\mu(T^{-1}) \).

(ii) If \( \theta \in [0,1] \), \( \mu \perp \nu \) and \( \mu, \nu \in \mathcal{I}_T \), then \( h_{\theta \mu + (1-\theta)\nu}(T) = \theta h_\mu(T) + (1-\theta) h_\nu(T) \).

**Proof** (i) We have

\[
\frac{k}{n^k} H_\mu \left( \bigvee_{r=0}^{n-1} T^{-r} \xi \right) = \frac{1}{n} H_\mu \left( \bigvee_{j=0}^{n-1} (T^k)^{-j}(\xi \vee T^{-1} \xi \vee \cdots \vee T^{-k+1} \xi) \right).
\]

Hence \( kh_\mu(T, \xi) = h_\mu(T^k, \eta) \) where \( \eta = \xi \vee T^{-1} \xi \vee \cdots \vee T^{-k+1} \xi \). Since \( \eta \geq \xi \), we deduce that \( kh_\mu(T) = h_\mu(T^k) \).

The claim \( h_\mu(T^{-1}) = h_\mu(T) \) follows from the invariance of \( \mu \) and the fact

\[
\xi(0,n-1) = \xi \vee \cdots \vee T^{-n+1} \xi = T^{-n+1}(\xi \vee \cdots \vee T^{n-1} \xi).
\]

(ii) Let \( A \) be such that \( \mu(A) = 1 \), \( \nu(A) = 0 \). Set \( B = \bigcup_{n=0}^\infty \bigcap_{n \geq m} T^{-n}(A) \). We can readily show that \( T^{-1} B = B \) and that \( \mu(B) = 1 \), \( \nu(B) = 0 \). Set \( \beta = \{B, X \setminus B\} \) and given a partition \( \xi \), define \( \hat{\xi} = \xi \vee \beta \). If \( \gamma = \theta \mu + (1-\theta) \nu \), then

\[
H_\gamma(\hat{\xi}_n) = \theta H_\mu(\xi_n) + (1-\theta) H_\nu(\xi_n) - \theta \log \theta - (1-\theta) \log(1-\theta),
\]

where \( \xi_n = \xi(0,n-1) \) and \( \hat{\xi}_n = \hat{\xi}(0,n-1) \). To see this, observe that if \( C \in \hat{\xi}_n \) and \( \phi(z) = z \log z \), then

\[
\phi(\gamma(C)) = \begin{cases} 
\theta \mu(C) \log(\theta \mu(C)) & \text{if } C \subseteq B, \\
(1-\theta) \nu(C) \log((1-\theta) \nu(C)) & \text{if } C \subseteq X \setminus B.
\end{cases}
\]
This clearly implies (3.9). Hence,

\[ h_{\gamma}(T, \hat{\xi}) = \theta h_\mu(T, \xi) + (1 - \theta) h_\nu(T, \xi). \]

From this we deduce

\[ h_{\gamma}(T) \leq \theta h_\mu(T) + (1 - \theta) h_\nu(T). \]

This and Exercise (viii) complete the proof. \(\square\)

In practice, we would like to know whether \(h_\mu(T) = h_\mu(T, \xi)\) for a partition \(\xi\). In the next theorem, we provide a sufficient condition for this.

**Theorem 3.2** Let \(\xi\) be a finite \(\mu\)-partition and assume that the \(\sigma\)-algebra consisting of \(T^{-n}(C), n \in \mathbb{N}, C \in \xi\), equals to the Borel \(\sigma\)-algebra. Then \(h_\mu(T) = h_\mu(T, \xi)\).

As a preparation we prove an inequality.

**Lemma 3.2** For every pair of finite partitions \(\eta\) and \(\xi\) we have

\[ h_\mu(T, \xi) \leq h_\mu(T, \eta) + H_\mu(\xi | \eta). \]

**Proof** Recall \(\xi(m, n) = T^{-m}\xi \vee \cdots \vee T^{-n}\xi\). We certainly have

\[ H_\mu(\xi(0, n - 1)) \leq H_\mu(\eta(0, n - 1)) + H_\mu(\xi(0, n - 1) | \eta(0, n - 1)). \]

We are done if we can show that \(H_\mu(\xi(0, n - 1) | \eta(0, n - 1)) \leq nH_\mu(\xi | \eta)\). Indeed using Proposition 3.2(i), we can assert

\[
H_\mu(\xi(0, n - 1) | \eta(0, n - 1)) \leq H_\mu(\xi | \eta(0, n - 1)) + H_\mu(\xi(1, n - 1) | \eta(0, n - 1) \vee \xi) \\
\leq H_\mu(\xi | \eta) + H_\mu(\xi(1, n - 1) | \eta(1, n - 1)) \\
\leq H_\mu(\xi | \eta) + H_\mu(T^{-1}\xi(0, n - 2) | T^{-1}\eta(0, n - 2)) \\
= H_\mu(\xi | \eta) + H_\mu(\xi(0, n - 2) | \eta(0, n - 2)) \\
\cdots \\
\leq nH_\mu(\xi | \eta). \\
\]

\(\square\)

**Proof of Theorem 3.2** For a given partition \(\eta\), we apply Lemma 3.1 to assert

(3.10) \[ h_\mu(T, \eta) \leq h_\mu(T, \xi \vee \cdots \vee T^{-n+1}\xi) + H_\mu(\eta | \xi \vee \cdots \vee T^{-n+1}\xi). \]
From the definition, it is not hard to see that indeed $h_\mu(T, \xi \vee \cdots \vee T^{-n+1} \xi) = h_\mu(T, \xi)$. From this and (3.9), it suffices to show that for every partition $\eta$,

$$
\lim_{n \to \infty} H_\mu(\eta \mid \xi \vee \cdots \vee T^{-n+1} \xi) = 0.
$$

(3.11)

To believe this, observe that if $\eta \leq \alpha$, then $H_\mu(\eta \mid \alpha) = 0$ because

$$
I_{\eta \mid \alpha}(x) = -\log \frac{\mu(C_\eta(x) \cap C_\alpha(x))}{\mu(C_\alpha(x))} = 0.
$$

Now if the $\sigma$-algebra generated by all $\xi_n = \xi \vee \cdots \vee T^{-n+1} \xi$, $n \in \mathbb{N}^*$ is the full $\sigma$-algebra, then $\eta \leq \xi_n$ at least asymptotically. We may prove this by the Martingale Convergence Theorem. In fact if $F_n$ is the $\sigma$-algebra generated by $\xi_n$, then

$$
\mu(C_\eta(x) \mid C_{\xi_n}(x)) = \sum_{A \in \eta} \mathbb{I}_A(x) \mu(A \mid F_n)(x) \to \sum_{A \in \eta} \mathbb{I}_A(x) \mu(A \mid F_\infty)(x) = \sum_{A \in \eta} \mathbb{I}_A(x) \mathbb{I}_A(x) = 1.
$$

This and (3.6) imply that $H_\mu(\eta \mid \xi_n) = -\int \log \mu(C_\eta(x) \mid C_{\xi_n}(x)) \mu(dx) \to 0$, which is simply (3.11).

\[\square\]

**Example 3.3(i)** Consider the dynamical system of Example 3.2. Let $\xi$ be as in Example 3.2. The condition of Theorem 3.2 is satisfied for such $\xi$ and we deduce

$$
h_{\mu_p}(T) = -\sum_{0}^{m-1} p_j \log p_j.
$$

(ii) Consider a translation $T(x) = x+\alpha \pmod{1}$ in dimension 1. If $\alpha \in \mathbb{Q}$, then $T^m$ = identity for some $m \in \mathbb{N}$. This implies that $h_\mu(T) = \frac{1}{m} h_\mu(T^m) = 0$ where $\mu$ is the Lebesgue measure. If $\alpha$ is irrational, then set $\xi = \{[0, 1/2), [1/2, 1]\}$. By the denseness of $\{T^{-n}(a) : n \in \mathbb{N}\}$ for $a = 0$ and $1/2$, we deduce that $\xi$ satisfies the condition of Theorem 3.2. As a result, $h_\mu(T) = h_\mu(T, \xi)$. On the other hand $\xi_n := \xi \vee \cdots \vee T^{-n+1} \xi$ consists of $2n$ elements. To see this, observe that if we already know that $\xi_n$ has $2n$ elements, then as we go to $\xi_{n+1}$, we produce two more elements because $T^n(0)$ and $T^n(0)$ bisect exactly two intervals in $\xi_n$. From this and Exercise (vi), $H_\mu(\xi(0, n-1)) \leq \log(2n)$. As a result, $h_\mu(T, \xi) = 0$, which in turn implies that $h_\mu(T) = 0$.

\[\square\]

In fact we can show that the entropy of a translation is zero using the fact that the topological entropy of a translation zero. More generally we always have the following fundamental formula.

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**Theorem 3.3** For any compact metric space $X$ and continuous transformation $T$, we have

$$h_{\text{top}}(T) = \sup_{\mu \in \mathcal{I}_T} h_\mu(T) = \sup_{\mu \in \mathcal{I}_T^e} h_\mu(T).$$

Note that by Proposition 3.4(ii), (3.12) is equivalent to

$$h_{\text{top}}(T) = \sup_{\mu \in \mathcal{I}_T^e} h_\mu(T).$$

To prepare for the proof of (3.13), let us make some definitions.

**Definition 3.4** Given $r, \delta > 0$, we define $S^n_{T,d}(r, \delta)$ to be the smallest $k$ such that there exists a set $E$ with $\#E = k$ and $\mu\left(\bigcup_{x \in E} B^n_{T,d}(x, r)\right) > 1 - \delta$. We then define

$$\hat{h}_\mu(T) = \lim_{\delta \to 0} \lim_{r \to 0} \limsup_{n \to \infty} \frac{1}{n} \log S^n_{T,d}(r, \delta).$$

Evidently

$$\hat{h}_\mu(T) \leq h_{\text{top}}(T).$$

Moreover,

**Theorem 3.4** (Katok) For every ergodic $\mu \in \mathcal{I}_T$, we have $h_\mu(T) \leq \hat{h}_\mu(T)$.

**Proof** Let $\xi = \{C_1, \ldots, C_\ell\}$ be a $\mu$-partition. Choose compact sets $K_1, \ldots, K_\ell$ with $K_j \subseteq C_j$ such that $\mu(C_j \setminus K_j) \leq \epsilon$ for $j = 1, \ldots, \ell$. Let $K_0 = X \setminus (K_1 \cup \cdots \cup K_\ell)$ and put $\eta = \{K_0, K_1, \ldots, K_\ell\}$. Evidently $\eta$ is a partition and

$$H_\mu(\xi \mid \eta) = -\sum_{i,j} \mu(C_i \cap K_j) \log \frac{\mu(C_i \cap K_j)}{\mu(K_j)}$$

$$= -\sum_i \mu(C_i \cap K_0) \log \frac{\mu(C_i \cap K_0)}{\mu(K_0)}$$

$$= -\mu(K_0) \sum_i \frac{\mu(C_i \cap K_0)}{\mu(K_0)} \log \frac{\mu(C_i \cap K_0)}{\mu(K_0)}$$

$$\leq \mu(K_0) \log \ell \leq \epsilon \ell \log \ell$$

by Exercise (vi). From this and Lemma 3.2 we deduce,

$$h_\mu(T, \xi) \leq h_\mu(T, \eta) + \epsilon \ell \log \ell.$$

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Set $\eta_n = \eta(0, n - 1)$. Recall that by Theorem 3.1,
\[ \lim_{n \to \infty} \frac{1}{n} \log \mu(C_n(x)) = -h_\mu(T, \eta) \]
in $L^1$-sense, when $C_n(x) = C_{\eta_n}(x)$. Pick $\varepsilon' > 0$ and choose a subsequence \{\(n_j : j \in \mathbb{N}\)\} such that
\[ X_N = \left\{ x \in X : \frac{1}{n_j} \log \mu(C_{n_j}(x)) \leq -h_\mu(T, \eta) + \varepsilon' \text{ for } n_j > N \right\}, \]
then $\mu(X_N) \to 1$ as $N \to \infty$. Pick $\delta > 0$, and find $N$ such that $\mu(X_N) > 1 - \delta$. Let
\[ r = \frac{1}{2} \min \{ \text{dist}(K_i, K_j) : i \neq j, i, j \in \{1, \ldots, \ell\} \}. \]

Clearly a ball $B_d(x, r)$ intersects at most two elements of $\eta$, one $K_j$ with $j \in \{1, \ldots, n\}$ and perhaps $K_0$. We now argue that $B_{d_n}(x, r)$ intersects at most $2^n$ elements of $\eta_n$. To see this, observe
\[ B_{d_n}(x, r) = B_d(x, r) \cap T^{-1}(B_d(T(x), r)) \cap \cdots \cap T^{-n+1}(B_d(T^{n-1}(x), r)). \]
Also, if $A \in \eta_n$, then $A = A_0 \cap T^{-1}(A_1) \cap \cdots \cap T^{-n+1}(A_{n-1})$ with $A_j \in \eta$. Now if $B_{d_n}(x, r) \cap A \neq \emptyset$, then $T^{-j}(B_d(T^j(x), r)) \cap T^{-j}(A_j) \neq \emptyset$ for $j = 0, \ldots, n-1$. Hence $B_d(T^j(x), r) \cap A_j \neq \emptyset$ for $j = 0, \ldots, n-1$. As a result, there are at most $2^n$-many choices for $A$. Now assume that $\mu(\bigcup_{x \in E} B_{d_n}(x, r)) > 1 - \delta$. We would like to bound $\#E$ from below. First observe
\[ 1 - 2\delta \leq \mu \left( \bigcup_{x \in E} B_{d_n}(x, r) \cap X_N \right) \leq \sum_{x \in E} \mu(B_{d_n}(x, r) \cap X_N), \]
\[ = \sum_{x \in E} \sum_{A \in \eta_n} \mu(B_{d_n}(x, r) \cap X_N \cap A). \]

But if $B_{d_n}(x, r) \cap X_N \cap A \neq \emptyset$ for $n = n_j > N$, then
\[ \mu(B_{d_n}(x, r) \cap X_N \cap A) \leq \mu(A) \leq e^{-n(h_\mu(T, \eta) - \varepsilon')} \]
As a result,
\[ 1 - 2\delta \leq 2^n e^{-n(h_\mu(T, \eta) - \varepsilon')} (\#E). \]
Hence
\[ h_\mu(T, \eta) \leq \limsup_{n_j \to \infty} \frac{1}{n_j} \log S_{T,d}^{n_j}(r, \delta) + \varepsilon' + \log 2. \]
From this we deduce that $h_\mu(T, \eta) \leq h_\mu(T) + \varepsilon' + \log 2$. From this and (3.13) we learn that
\[ h_\mu(T, \xi) \leq h_\mu(T) + \varepsilon \log \ell + \varepsilon' + \log 2. \] By sending $\varepsilon, \varepsilon' \to 0$ and taking supremum over $\xi$ we deduce
\[ (3.16) \quad h_\mu(T) \leq h_\mu(T) + \log 2. \]
We wish to modify (3.17) by getting rid of \( \log 2 \). To achieve this, we would like to replace \( T \) with \( T_m \) in the equation (3.17). A repetition of the proof of Proposition 3.1(iii) yields
\[
\frac{1}{m} \hat{h}_\mu(T^m) = \hat{h}_\mu(T).
\]
which is desirable because of the factor \( m^{-1} \) in front of \( \log 2 \); this factor goes to 0 in large \( m \) limit. However, it is possible that \( \mu \not\in \mathcal{I}_{T,m} \). If this is the case, then we may apply Exercise (viii) of Chapter 1, to assert that if \( m \) a prime number, then all \( T^m \) invariant sets come from a finite partition \( \zeta \) with exactly \( m \) elements. This suggests replacing the partition \( \eta \) with \( \hat{\eta} = \eta \wedge \zeta \) so that we still have
\[
\lim_{n \to \infty} n^{-1} \log \mu(C_n(x)) = -h_\hat{\eta}(\mu),
\]
where \( C_n(x) = C_{\hat{\eta}_n}(x) \), and \( \hat{\eta}_n = \eta \wedge T^{-m} \hat{\eta} \wedge \cdots \wedge T^{-m(1-n)} \hat{\eta} \). Here we are using the fact that the ergodic theorem is applicable because the limit is constant on members of the partition \( \zeta \). Repeating the above proof for \( T^m \), we can only assert that \( B_d(x,r) \) can intersect at most \( 2m \) elements of \( \hat{\eta} \), and that \( B_{d_n}(x,r) \) can intersect at most \( (2m)^n \) elements of \( \hat{\eta}_n \). This leads to the bound
\[
h_\mu(T^m, \xi) \leq \hat{h}_\mu(T^m) + \varepsilon \ell \log \ell + \varepsilon' + \log(2m),
\]
which in turn yields
\[
h_\mu(T) \leq \hat{h}_\mu(T) + m^{-1} \log(2m),
\]
for every prime number \( m \). We arrive at \( h_\mu(T) \leq \hat{h}_\mu(T) \), by sending \( m \) to infinity. \( \square \)

**Remark 3.1** Theorem 3.4 provides us with a rather local recipe for calculating the entropy. It turns out that there is another local recipe for calculating the entropy that is related to \( \hat{h}_\mu(T) \). A theorem of Brin and Katok\[BK\] asserts that if \( \mu \in \mathcal{I}_T \) is ergodic, then
\[
\frac{1}{n} \log \mu(B_{d_n}(x,r)) \approx h_\mu(T).
\]
for \( \mu \)-almost all \( x \).

We are now ready to establish Theorem 3.3.

**Proof of Theorem 3.3** On account of Theorem 3.4, it remains to show
\[
(3.17) \quad h_{\text{top}}(T) \leq \sup_{\mu \in \mathcal{I}_T} h_\mu(T).
\]
First observe that it suffices to show that for every $r > 0$, there exists a partition $\xi$ and an invariant measure $\bar{\mu}$ such that

$$\limsup_{n \to \infty} \frac{1}{n} \log N^n(r) \leq h_{\bar{\mu}}(T, \xi),$$

(3.18)

$N^n(r) = N^n_{T,d}(r)$. For each $n$, pick a set $E_n$ such that $\#E_n = N^n_{T,d}(r) = N^n(r)$. In other words, $E_n$ is a maximal set satisfying $d_{n}(x, y) \geq r$ for $x, y \in E_n$ with $x \neq y$. Define

$$\mu_n = \frac{1}{N^n(r)} \sum_{x \in E_n} \delta_x.$$

First observe that $H_{\mu_n}(\eta) = \log N^n(r)$ provided that $\text{diam}_n(C) < r$ for every $C \in \eta$, where $\text{diam}_n(A)$ denotes the diameter of $A$ with respect to $d_n$. This is because $\mu_n(A) = 0$ or $N^n(r)^{-1}$ for every $A \in \xi_n$ simply because each such $A$ contains at most one element of $E_n$. As a result

$$H_{\mu_n}(\xi_n) = N^n(r) \log N^n(r) N^n(r)^{-1} = \log N^n(r).$$

Using this, we would like to estimate from below $H_{\bar{\mu}}(\xi_m)$ for a suitable invariant measure $\bar{\mu}$ and a partition $\xi$. (Recall that we write $\xi_m$ for $\xi \vee \cdots \vee T^{1-m}\xi$.) We are tempted to try any partition $\xi$ such that $\text{diam}(C) < r$ for every $C \in \xi$ and select a limit point of the sequence $\{\mu_n\}$ for $\bar{\mu}$. This would not work because such a limit point may not be an invariant measure. Moreover, the function $H_{\mu}(\eta)$ is not a continuous function of $\mu$ with respect to the weak topology. To treat the former issue, we define

$$\hat{\mu}_n = \frac{1}{n} \sum_{j=0}^{n-1} A^j \mu_n,$$

where $A\nu(A) = \nu(T^{-1}(A))$, as in Chapter 2. Equivalently

$$\int f \, d\hat{\mu}_n = \frac{1}{n} \sum_{j=0}^{n-1} \int f(T^j(x)) \mu_n(dx) = \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{N^n(r)} \sum_{x \in E_n} f(T^j(x)).$$

Let us choose an increasing subsequence $n_i \to \infty$ such that

$$\limsup_{n \to \infty} \frac{1}{n} \log N^n(r) = \lim_{i \to \infty} \frac{1}{n_i} \log N^{n_i}(r), \quad \text{and} \quad \lim_{i \to \infty} \hat{\mu}_{n_i} =: \bar{\mu} \quad \text{exists.}$$

It is not hard to show that $\bar{\mu} \in \mathcal{I}_T$ because

$$A\bar{\mu} - \bar{\mu} = \lim_{n \to \infty} (A^n \mu_n - \mu_n)/n = 0.$$
Pick a partition $\xi$ such that $\text{diam}(C) < r$ for every $C \in \xi$. We wish to use

$$H_{\mu_n}(\xi_n) = \log N^n(r),$$

to deduce

$$\lim_{i \to \infty} \frac{1}{n_i} \log N^{n_i}(r) \leq h(\mu, T, \xi).$$

To achieve this, pick $k$ and $m$ such that $0 \leq k < m < n = n_i$ and set $a(k) = \left\lfloor \frac{n-k}{m} \right\rfloor$ so that we can write

$$\{0, 1, \ldots, n-1\} = \{k + tm + i : 0 \leq t < a(k), \ 0 \leq i < m\} \cup R$$

with $R = \{0, 1, \ldots, k - 1\} \cup \{k + ma(k), k + ma(k) + 1, \ldots, n-1\} =: R_1 \cup R_2$. Clearly $\#R_1 \leq m, \#R_2 \leq m$. We then write

$$\xi_n = \bigvee_{t=0}^{a(k) - 1} T^{-(tm+k)}(\xi) \vee \cdot \cdot \cdot \vee T^{-m+1}(\xi) \vee \bigvee_{i \in R} T^{-i}(\xi).$$

Using Proposition 3.2(ii) and Exercise (iv) below we learn,

$$\log N^n(r) = H_{\mu_n}(\xi_n) \leq \sum_{t=0}^{a(k) - 1} H_{\mu_n}(T^{-(tm+k)}\xi_m) + \sum_{i \in R} H_{\mu_n}(T^{-i}\xi)$$

$$= \sum_{t=0}^{a(k) - 1} H_{A^{tm+k}\mu_n}(\xi_m) + \sum_{i \in R} H_{\mu_n}(T^{-i}\xi)$$

$$\leq \sum_{t=0}^{a(k) - 1} H_{A^{tm+k}\mu_n}(\xi_m) + 2m \log(\#\xi).$$

This is true for every $k$. Hence

$$m \log N^n(r) \leq \sum_{k=0}^{m-1} \sum_{t=0}^{a(k) - 1} H_{A^{tm+k}\mu_n}(\xi_m) + 2m^2 \log(\#\xi)$$

$$\leq \sum_{j=0}^{n-1} H_{A^{j}\mu_n}(\xi_m) + 2m^2 \log(\#\xi)$$

$$\leq n H_{\tilde{\mu}_n}(\xi_m) + 2m^2 \log(\#\xi),$$

where for the last inequality we used Exercise (viii) below. As a result,

$$\frac{1}{n} \log N^n(r) \leq \frac{1}{m} H_{\tilde{\mu}_n}(\xi_m) + \frac{2m}{n} \log(\#\xi).$$

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We now send \( n = n_i \) to infinity to deduce
\[
\limsup_{n \to \infty} \frac{1}{n} \log N^n(r) = \lim_{j \to \infty} \frac{1}{n_j} \log N^{n_j}(r) \leq \frac{1}{m} H_\mu(\xi_m),
\]
provided that we have
\[
(3.21) \quad \lim_{i \to \infty} H_{\hat{\mu}_m}(\xi_m) = H_\mu(\xi_m),
\]
for every \( m \). We now send \( m \) to infinity to deduce
\[
\limsup_{n \to \infty} \frac{1}{n} \log N^n(r) \leq h_\mu(T, \xi) \leq \sup_{\mu \in \mathcal{I}_T} h_\mu(T),
\]
as desired.

It remains to verify (3.21). For this we need to be more selective about the partition \( \xi \). We first would like to find a partition \( \xi = \{C_1 \ldots C_\ell\} \) such that \( \text{diam}(C_j) \leq \delta := r/2 \) for \( j = 1, \ldots, \ell \), and \( \mu(\partial C_j) = 0 \) where \( \partial C_j \) denotes the boundary of \( C_j \). The construction of such a partition \( \xi \) is straightforward. First, if \( B_d(x,a) \) is a ball of radius \( a \), then we consider
\[
\bigcup \{ \partial B_d(x,a') : a - \epsilon \leq a' \leq a \},
\]
to observe that there exists \( a' \in (a - \epsilon, a) \) such that \( \mu(\partial B_d(x,a')) = 0 \). From this, we learn that we can cover \( X \) by finitely many balls \( B_j, \ j = 1, \ldots, \ell \) of radius at most \( \delta/2 \) such that \( \mu(\partial B_j) = 0 \) for \( j = 1, \ldots, \ell \). We finally define \( \xi = \{C_1 \ldots C_\ell\} \) by \( C_1 = \bar{B}_1, \ C_2 = \bar{B}_2 \setminus \bar{B}_1, \ldots, C_\ell = \bar{B}_\ell \setminus \bigcup_{j=1}^{\ell-1} \bar{B}_j \). Since \( \partial C_j \subseteq \bigcup_{k=1}^\ell \partial B_k \), we are done. We now argue that the partition \( \xi_n = \xi \vee \ldots \vee T^{-n+1} \xi \) enjoys the same property; \( \mu(\partial C) = 0 \) if \( C \in \xi_n \). This is because \( \partial C \subseteq \bigcup_{A \in \xi} \bigcup_{k=0}^{n-1} T^{-j}(\partial A) \) and by invariance, \( \mu(T^{-j}(\partial A)) = \mu(\partial A) = 0 \). For such a partition we have (3.21) because by Exercise (ii) in Chapter 1, \( \mu_n(A) \to \hat{\mu}(A) \) for every \( A \in \xi_m \). \( \square \)

**Example 3.4** Consider \( T : \mathbb{T}^2 \to \mathbb{T}^2, \ T x = Ax \ (\mod 1) \) with \( A \) an integer matrix with \( \det A = 1 \). We assume that \( A \) is symmetric and its eigenvalues \( \lambda_1, \lambda_2 = \lambda_1^{-1} \) satisfy \( |\lambda_1| > 1 > |\lambda_2| \). We claim that if \( \mu \) is the Lebesgue measure, then \( h_\mu(T) \geq \log |\lambda_1| \). In case of \( T = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \), we can use our result \( h_{\text{top}}(T) \leq \log |\lambda_1| \) from Example 3.6 to conclude that in fact \( h_{\mu}(T) = h_{\text{top}}(T) = \log |\lambda_1| \).

For \( h_\mu(T) \geq \log |\lambda_1| \), observe that by the invariance of \( \mu \) with respect to \( T \), \( H_\mu(T^{-n} \xi \vee \ldots \vee T^n \xi) = H_\mu(\xi \vee \ldots \vee T^{-2n} \xi) \). Hence it suffices to study \( \lim_{n \to \infty} \frac{1}{2^n} H_\mu(T^{-n} \xi \vee \ldots \vee T^n \xi) \). For estimating this, we show that the area of each \( C \in \eta_n = T^{-n} \xi \vee \ldots \vee T^n \xi \) is exponentially small. This is achieved by showing that \( \text{diam}(C) = O(|\lambda_1|^{-n}) \). For example, let us choose \( \xi = \{Z_0, \ldots, Z_4\} \) where \( Z_i \)'s are the rectangles of Example 3.6. It is not hard to see that if
the side lengths of $Z_i$'s are all bounded by a constant $c$, then $\eta_n$ consists of rectangles with side lengths bounded by $c\lambda_1^{-n}$. Hence $\mu(A) \leq c^2 \lambda_1^{2n}$ for every $A \in \eta_n$.

This evidently implies that $\frac{1}{2^n} H_\mu(\eta_n) \geq \log |\lambda_1| + o(1)$, and as $n \to \infty$ we deduce that $h_\mu(T) \geq \log |\lambda_1|$.

Motivated by the thermodynamics formalism in statistical mechanics, we may formulated a variant of the variational problem (3.12) for which the maximizing measure if exists is a variant of the SRB measure; it may be regarded as the analog of Gibbs Measures with respect to the SRB measure. For this we need a variant of Definition 3.1:

**Definition 3.4** Let $(X, d)$ be a compact metric space and $T : X \to X$ and $f : X \to \mathbb{R}$ be two continuous functions. Define

$$S_{T,d}^n(r; f) = \min \left\{ \sum_{x \in A} e^{n\Phi_n(f)} : X = \bigcup_{x \in A} B_{T,d}^n(x, r) \right\},$$

$$N_{T,d}^n(r; f) = \max \left\{ \sum_{x \in A} e^{n\Phi_n(f)} : a, b \in A, a \neq b \implies d_n(a, b) > r \right\},$$

$$h_{\text{top}}(T; f) = \lim_{r \to 0} \limsup_{n \to \infty} \frac{1}{n} \log S_{T,d}^n(r; f) = \sup_{r > 0} \limsup_{n \to \infty} \frac{1}{n} \log S_{T,d}^n(r; f),$$

$$\bar{h}_{\text{top}}(T; f) = \lim_{r \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N_{T,d}^n(r; f) = \sup_{r > 0} \limsup_{n \to \infty} \frac{1}{n} \log N_{T,d}^n(r; f).$$

**Theorem 3.5** For every $X, T$ and $f$ as in Definition 3.4, we have

$$h_{\text{top}}(T; f) = \sup_{\mu \in \mathcal{I}_T} \left( \int f \, d\mu + h_\mu(T) \right).$$

**Remark 3.2** It turns out that for a nice hyperbolic system a lot more can be said. For example, if $X$ is a manifold with a volume measure $m$, then there exists a unique $\bar{\mu} = \mu_{\text{SRB}} \in \mathcal{I}_T$ such that $h_{\text{top}}(T) = h_{\bar{\mu}}(T)$, and if $I(\mu) = h_{\text{top}}(T) - h_\mu(T) = h_{\bar{\mu}}(T) - h_\mu(T)$, then, we roughly have

$$m \{ x : \mu_n^x \text{ is near } \mu \} \approx e^{-nI(\mu)}.$$

This is known in probability theory as a large deviation principle. Recall that the entropy $h_\mu(T)$ is affine in $\mu$. Hence $I$ is affine, and its convex conjugate, the pressure,

$$\Psi(f) = \sup_{\mu \in \mathcal{I}_T} \left( \int f \, d\mu - I(\mu) \right),$$

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satisfies

\[ (3.23) \quad \Psi(f) = \lim_{n \to \infty} \frac{1}{n} \log \int \exp \left( \sum_{j=0}^{n-1} f(T^j(x)) \right) m(dx). \]

Indeed (3.23) was established by Donsker and Varadhan for the dynamical system of Example 3.1(ii) (see for example [R]). Recall that in this example, \( X = E^Z \), for \( E = \{0,1,\ldots,m-1\} \), and \( \tau \) is the shift operator. As we saw before, if we choose \( \xi = \{A_0,\ldots,A_{m-1}\} \), with \( A_i = \{\omega : \omega_0 = i\} \), then \( h_\mu(\tau) = h_\mu(\tau,\xi) \). In this case the invariant measure of maximum entropy is the produc measure \( \bar{\mu} \) with \( \bar{\mu}(A_i) = m^{-1} \), and \( h_{\text{top}}(\tau) = \log m \).

Note that \( \xi(0, n-1) \) consists of sets of the form \( A(a_0,\ldots,a_{n-1}) \), with

\[ A(a_0,\ldots,a_{n-1}) = \{\omega : \omega_0 = a_0, \omega_1 = a_1,\ldots,\omega_{n-1} = a_{n-1}\}. \]

Since \( \bar{\mu}(A) = m^{-n} \), for every \( A \in \xi_n := \xi(0, n-1) \), we deduce

\[ I(\mu) = \log m - h_\mu(\tau,\xi) = \lim_{n \to \infty} \frac{1}{n} \sum_{A \in \xi_n} \log \left( \frac{\mu(A)}{\mu(A)} \right) \mu(A). \]

Indeed if we write \( \mathcal{F}_n \) for the \( \sigma \)-algebra generated by \( \xi_n \), and write \( h_n \) for the Radon-Nikodym derivative \( \frac{d\mu}{d\bar{\mu}} \) for the restriction of \( \bar{\mu} \) and \( \mu \) to \( \mathcal{F}_n \), then

\[ I(\mu) = \lim_{n \to \infty} \frac{1}{n} H_{\mathcal{F}_n}(\mu|\bar{\mu}). \]

when

\[ H_{\mathcal{F}_n}(\mu|\bar{\mu}) = \int h_n \log h_n \, d\bar{\mu}, \]

represents the relative entropy of \( \mu \) with respect to \( \bar{\mu} \) in \( \mathcal{F}_n \). \( \square \)

\section*{Exercises}

(i) Let \( F : X \to Y \) be a continuous function with \( F(X) = Y \). Let \( T : X \to X \), \( T' : Y \to Y \) be continuous and \( F \circ T = T' \circ F \). show that \( h_{\text{top}}(T') \leq h_{\text{top}}(T) \).

(ii) Let \( (X_1,d_1), (X_2,d_2) \) be two compact metric spaces and let \( T_i : X_i \to X_i, \, i = 1,2 \) be two continuous functions. show that \( h_{\text{top}}(T_1 \times T_2) = h_{\text{top}}(T_1) + h_{\text{top}}(T_2) \).

\textit{Hint:} For \( T = T_1 \times T_2 \) and a suitable choice of a metric \( d \) for \( X_1 \times X_2 \), show that

\[ S^n_{T,d}(r) \leq S^n_{T_1,d_1}(r) S^n_{T_2,d_2}(r), \quad N^n_{T,d}(r) \geq N^n_{T_1,d_1}(r_1) N^n_{T_2,d_2}(r_2). \]

(iii) Let \( A \) be as in Example 3.1(v). show that \( r(A) = \frac{3 + \sqrt{5}}{2} \).
(iv) Show that if \( \tau_m \) denotes the shift map of Example (iv) on \( m \) many symbols, then \( \tau_m^k \) may be regarded as a shift map on a set of \( m^k \) many symbols. Use this to show that if \( h(m) = h_{\text{top}}(\tau_m) \), then \( h(m^k) = kh(m) \).

(v) According to the Perron-Frobenius Theorem, for any matrix \( A \) with non-negative entries we can find a nonnegative eigenvalue with a corresponding nonnegative eigenvector. Use this theorem to show that the matrix \( A \) in Example 3.1(iv) has a real eigenvalue \( \lambda \geq 1 \).

(vi) If \( \xi \) has \( m \) elements, then \( 0 \leq H_\mu(\xi) \leq \log m \).

(vii) If \( \alpha \leq \beta \), then \( H_\mu(\alpha) \leq H_\mu(\beta) \) and \( h_\mu(T, \alpha) \leq h_\mu(T, \beta) \).

(viii) If \( \mu_1, \mu_2 \in \mathcal{I}_T \) and \( \theta \in [0, 1] \), then
\[
H_{\theta \mu_1 + (1-\theta) \mu_2}(\xi) \geq \theta H_{\mu_1}(\xi) + (1-\theta) H_{\mu_2}(\xi),
\]
\[
h_{\theta \mu_1 + (1-\theta) \mu_2}(T, \xi) \geq \theta h_{\mu_1}(T, \xi) + (1-\theta) h_{\mu_2}(T, \xi),
\]
\[
h_{\theta \mu_1 + (1-\theta) \mu_2}(T) \geq \theta h_{\mu_1}(T) + (1-\theta) h_{\mu_2}(T).
\]

(ix) (Rokhlin Metric) Define \( d(\eta, \xi) = H_\mu(\eta | \xi) + H_\mu(\xi | \eta) \). Show that \( d \) is a metric on the space of \( \mu \)-partitions.

(x) We say that the matrix \( A \) in Example 3.1(iv) is irreducible and aperiodic or primitive if \( A^{n_0} \) has positive entries for some \( n_0 \in \mathbb{N} \). In the case of a primitive \( A \), Perron-Frobenius Theorem asserts that the largest eigenvalue \( \lambda > 1 \) of \( A \) is of multiplicity 1 and the corresponding right and left eigenvector \( u^r \) and \( u^l \) can be chosen to have positive components. We may assume that \( u^r \cdot u^l = 1 \). Define measure \( \mu \) on \( X_A \) with the following recipe:
\[
\mu \left( X_A \cap \{ \omega : \omega_1 = \alpha_1, \ldots, \omega_k = \alpha_k \} \right) = \pi(\alpha_1) \prod_{i=1}^{k-1} p(\alpha_i, \alpha_{i+1}),
\]
where
\[
\pi(i) = u_i^r u_i^l, \quad p(i, j) = \frac{a_{ij} u_j^r}{\lambda u_i^r}.
\]
Show that the measure \( \mu \) is well-defined and is invariant for \( \tau \). Show
\[
h_\mu(T) = - \sum_{i,j=0}^{m-1} \pi(i)p(i, j) \log p(i, j) = \log \lambda.
\]

(xi) Work out the measure \( \mu \) of part (x) in the case of Example 3.1(v). \( \square \)
4 Lyapunov Exponents

For the expanding map of Examples 3.1(iii) and 3.3(i), and Arnold’s map of Example 3.4 we saw that the entropy was indeed the logarithm of the expansion rate. In this chapter, we learn how to define the exponential rates of expansions and contractions of a transformation with respect to its invariant measures. These rates are called the Lyapunov Exponents, and can be used to bound the entropy. In fact for the so-called hyperbolic dynamical system, the entropy can be expressed as the sum of positive Lyapunov exponents by Pesin’s formula. In general an inequality of Ruelle implies that the entropy is bounded above by the sum of positive Lyapunov exponents.

Consider a transformation $T : X \to X$ where $X$ is a compact $C^1$ manifold and $T$ is a $C^1$ transformation. We also assume that $M$ is a Riemannian manifold. This means that for each $x$ there exists an inner product $\langle \cdot, \cdot \rangle_x$ and (an associated norm $\|x\|$) that varies continuously with $x$. To study the rate of expansion and contraction of $T$, we may study $$(dT^n)_x : T_x X \to T_{T^n(x)} X.$$ We certainly have

$$\tag{4.1} (dT^n)_x = (dT)_{T^{n-1}(x)} \circ \cdots \circ (dT)_{T(x)} \circ (dT)_x.$$ 

If we write $A(x) = (dT) : T_x X \to T_{T(x)} X$, then (4.1) can be written as

$$\tag{4.2} A_n(x) := (dT^n)_x = A(T^{n-1}(x)) \circ \cdots \circ A(T(x)) \circ A(x).$$

Here we are interested in the long time behavior of the dynamical system associated with $dT : TX \to TX$ that is defined by $dT(x, v) = (T(x), (dT)_x(v)) = (T(x), A(x)v)$.

The formula (4.1) suggests an exponential growth rate for $(dT^n)_x$. Let us examine some examples first.

**Example 4.1(i)** Consider the dynamical system $T : T^d \to T^d$ that is given by $T(x) = Ax \pmod 1$, where $A$ is a $d \times d$ matrix. Identifying $T T^d$ with $T^d \times \mathbb{R}^d$, we learn that $A(x) = A$ is constant and $A_n(x) = A^n$. We may use a Jordan normal form to express $A$ as a diagonal block matrix. More precisely, we can express

$$\mathbb{R}^d = G_1 \oplus \cdots \oplus G_q,$$

where each $G_j$ corresponds to an eigenvalue $\lambda_j$ of $A$ and $r_j = \dim G_j$ represents the multiplicity of $\lambda_j$. If $\lambda_j$ is complex, we use the same $G_j$ for both $\lambda_j$ and its complex conjugate, and $r_j$ is twice the multiplicity. For real $\lambda_j$, the space $G_j$ is the generalized eigenspace associated with $\lambda_j$:

$$G_j = \{ v \in \mathbb{R}^d : (A - \lambda_j)^r v = 0 \text{ for some } r \in \mathbb{N} \}.$$ 

In the case of a complex pair of eigenvalues $\alpha_j \pm i\beta_j$, the space $G_j$ is spanned by real and imaginary parts of the generalized eigenvectors. In the case of real $\lambda_j$, the restriction of the
map $x \mapsto Ax$ to $G_j$ has a diagonal block matrix representation, with each block of the form

$$A' = \begin{bmatrix}
\lambda_j & 0 & \ldots & 0 & 0 \\
1 & \lambda_j & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & \lambda_j
\end{bmatrix}$$

In the case the complex eigenvalue $\alpha_j \pm i\beta_j$, in $A'$ we replace $\lambda_j$ by the $2 \times 2$ matrix

$$R_j = \begin{bmatrix}
\alpha_j & -\beta_j \\
\beta_j & \alpha_j
\end{bmatrix}$$

and the 1’s below the diagonal are replaced by the $2 \times 2$ identity matrix. For our purposes, we would like to replace the off-diagonal entries with some small number $\delta$. (When $A'$ is $\ell \times \ell$, we make the change of coordinates $(x_1, \ldots, x_\ell) \mapsto (\delta^{-1} x_1, \ldots, \delta^{-\ell} x_\ell)$.) From this, it is not hard to show

$$\left( |\lambda_j| - \delta \right)^n \leq |A^n v| \leq \left( |\lambda_j| + \delta \right)^n.$$  

By sending $\delta$ to 0 we learn that for $v \in G_j \{0\}$,

$$\lim_{n \to \infty} \frac{1}{n} \log |A^n v| = \log |\lambda_j|. \quad (4.4)$$

(ii) Consider a dynamical system $(X, T)$, with $X$ a smooth Riemmanian manifold of dimension $d$. Let us take a fixed point $a$ of $T$ so that $A = A(a) = (dT)_a$ and $A_n(a) = A^n$ map $T_a X$ to itself. We may identify $T_a X$ with $\mathbb{R}^d$ and represent $A$ as a $d \times d$ matrix. Using the decomposition of Part (i), and (4.4), we deduce that for $v \in G_j \{0\}$,

$$\lim_{n \to \infty} \frac{1}{n} \log |A^n(a) v| = \log |\lambda_j|.$$  

(iii) Let us now assume that $T : X \to X$ is a diffeomorphism, and the orbit associated with $a$ is periodic of period $N$. This means that $T^N(a) = a$ and $(dT^N)_a$ maps $T_a X$ to itself. If we write $n = mN + r$ with $m \in \mathbb{N}$ and $r \in \{0, \ldots, N - 1\}$, Then we have $A_n(a) = C \circ B^m$, where

$$B = (dT^N)_a = A(T^{N-1}(a)) \circ \cdots \circ A(a),$$

$$C = \begin{cases} A(T^{r-1}(a)) \circ \cdots \circ A(a) & \text{for } r > 0, \\
I & \text{for } r = 0.\end{cases}$$

From this, we learn

$$c_0^{-1} |B^m v| \leq |A_n(a) v| \leq c_0 |B^m v|.$$
for $c_0 = \left( \max(1, \| A \|) \right)^{N-1}$. We use the generalized eigenspaces of $B$ to decompose $T_aX \equiv \mathbb{R}^d$ as in (4.4). We now have that for $v \in G_j \setminus \{0\}$,

$$
\lim_{n \to \infty} \frac{1}{n} \log |A^n(a)v| = \frac{1}{N} \log |\lambda_j|,
$$

where $\lambda_j$ is an eigenvalue of $B$. □

As a preparation for the definition of Lyapunov exponents, let us observe that if we set $S_n(x) = \log \| A_n(x) \|$, then $S_0 = 0$ and

$$(4.5) \quad S_{n+m}(x) \leq S_n(x) + S_m(T^n(x)),$$

by (4.2). The following theorem guarantees the existence of the largest Lyapunov exponent.

**Theorem 4.1** Let $T$ be a diffeomorphism and assume that $\mu \in \mathcal{I}_T^x$. Then there exists $\lambda \in \mathbb{R}$ such that

$$
\lim_{n \to \infty} \frac{1}{n} \log \| (dT^n)_x \| = \lambda,
$$

for $\mu$-almost all $x$.

This theorem is an immediate consequence of the Kingman’s subadditive ergodic theorem:

**Theorem 4.2** Let $\mu \in \mathcal{I}_T^x$ and suppose that $\{S_n(\cdot) : n = 0, 1, \ldots \}$ is a sequence of $L^1(\mu)$ functions satisfying (4.5). Then

$$
\lim_{n \to \infty} \frac{1}{n} S_n(x) = \lambda := \inf_m \left\{ \frac{1}{m} \int S_m \, d\mu \right\} \in [-\infty, +\infty),
$$

for $\mu$-almost all $x$.

**Proof of Theorem 4.1** On the account of Theorem 4.2, we only need to show $\lambda \neq -\infty$. Clearly,

$$
id = (dT^{-n})_{T^n(x)} \circ (dT^n)_x : 1 \leq \left\| (dT^{-n})_{T^n(x)} \right\| \left\| (dT^n)_x \right\|.
$$

On the other hand, if we write

$$
\alpha := \sup_x \left\| (dT^{-1})_x \right\|,
$$

then

$$
\left\| (dT^{-n})_{T^n(x)} \right\| = \left\| (dT^{-1})_x \circ \ldots \circ (dT^{-1})_{T^{n-1}(x)} \circ (dT^{-1})_{T^n(x)} \right\| \leq \alpha^n.
$$

Hence $\| (dT^n)_x \| \geq \alpha^{-n}$ which implies that $\lambda \geq -\log \alpha$. □
Proof of Theorem 4.2  Fix \( m > 0 \). Any \( n > m \) can be written as \( n = qm + r \) for some \( q \in \mathbb{N}^* \) and \( r \in \{0,1,\ldots,m-1\} \). As a result, if \( k \in \{0,\ldots,m-1\} \), then \( n = k + q'm + r' \) with \( q' = q'(k) = \begin{cases} q & \text{if } r \geq k \\ q-1 & \text{if } r < k \end{cases} \) \( r' = r'(k) = \begin{cases} r - k & \text{if } r \geq k \\ r - k + m & \text{if } r < k \end{cases} \) By subadditivity,

\[
S_n(x) \leq S_k(x) + S_{q'm}(T^k(x)) + S_{r'}(T^{k+q'm}(x)) \\
\leq S_k(x) + \sum_{j=0}^{q'-1} S_{m}(T^{k+jm}(x)) + S_{r'}(T^{k+q'm}(x)).
\]

We now sum over \( k \) to obtain

\[
S_n(x) \leq \frac{1}{m} \sum_{0}^{m-1} S_k(x) + \frac{1}{m} \sum_{0}^{n-1} S_{m}(T^i(x)) + \frac{1}{m} \sum_{0}^{m-1} S_{r'(k)}(T^{k+jm}(x)),
\]

where \( S_0 = 0 \). Hence

\[
\frac{1}{n} S_n(x) \leq \frac{1}{n} \sum_{0}^{n-1} \frac{1}{m} S_{m}(T^i(x)) + R_{n,m}(x),
\]

where \( \|R_{n,m}\|_{L^1} \leq \text{constant} \times \frac{n}{m} \), because \( \int |S_l(T^r)|d\mu = \int |S_l|d\mu \). By the Ergodic Theorem,

\[
\limsup_{n \to \infty} \frac{1}{n} S_n(x) \leq \int \frac{S_{m}}{m}d\mu.
\]

Since \( m \) is arbitrary,

\[
(4.6) \quad \limsup_{n \to \infty} \frac{1}{n} S_n(x) \leq \lambda,
\]

almost everywhere and in \( L^1 \)-sense.

For the converse, we only need to consider the case \( \lambda > -\infty \). Let us take a function \( \varphi : \mathbb{R}^n \to \mathbb{R} \) that is nondecreasing in each of its arguments. We certainly have

\[
\int \varphi(S_1,\ldots,S_n)d\mu = \int \varphi(S_1 \circ T^k,\ldots,S_n \circ T^k)d\mu \\
\geq \int \varphi(S_{k+1} - S_k,S_{k+2} - S_k,\ldots,S_{k+n} - S_k)d\mu
\]

for every \( k \). Hence

\[
(4.7) \quad \int \varphi(S_1,\ldots,S_n)d\mu \geq \frac{1}{N} \sum_{k=0}^{N-1} \int \varphi(S_{k+1} - S_k,\ldots,S_{k+n} - S_k)d\mu \\
= \int \varphi(S_{k+1} - S_k,\ldots,S_{k+n} - S_k)d\mu \nu_N(dk)
\]

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where $\nu_N = \frac{1}{N} \sum_{k=0}^{N-1} \delta_k$. We think of $k$ as a random number that is chosen uniformly from 0 to $N - 1$. To this end let us define $\Omega = \mathbb{R}^{\mathbb{Z}^+} = \{w : \mathbb{Z}^+ \to \mathbb{R}\}$ and $T : M \times N \to \Omega$ such that $T(x, k) = w$ with $w(j) = S_{k+j}(x) - S_{k+j-1}(x)$. We then define a measure $\mu_N$ on $\Omega$ by $\mu_N(A) = (\mu \times \nu_N)(T^{-1}(A))$. Using this we can rewrite (4.7) as

\[ \int \varphi(S_1, \ldots, S_n) d\mu \geq \int \varphi(w(1), w(1) + w(2), \ldots, w(1) + \cdots + w(n)) \mu_N(dw). \]

We want to pass to the limit $N \to \infty$. Note that $\Omega$ is not a compact space. To show that $\mu_N$ has a convergent subsequence, observe

\[ \int w(j)^+ \mu_N(dw) = \int (S_{k+j}(x) - S_{k+j-1}(x))^+ \mu(dx) \nu_N(dx) \]
\[ = \frac{1}{N} \sum_{j=0}^{N-1} \int (S_{k+j}(x) - S_{k+j-1}(x))^+ \mu(dx) \]
\[ \leq \frac{1}{N} \sum_{j=0}^{N-1} \int (S_1(T^{k+j-1}(x)))^+ \mu(dx) = \int S_1^+ d\mu, \]

\[ \int w(j) \mu_N(dw) = \frac{1}{N} \sum_{j=0}^{N-1} \int (S_{k+j}(x) - S_{k+j-1}(x)) \mu(dx) \]
\[ = \frac{1}{N} \int (S_{j+N-1} - S_{j-1}) d\mu \geq \frac{j + N - 1}{N} - \frac{1}{N} \int S_{j-1} d\mu \]
\[ > -\infty, \]

uniformly in $N$. As a result $\int w(j)^- \mu_N(dw)$ is uniformly bounded. Hence

\[ \sup_N \int |w_j| d\mu_N = \beta_j < \infty \]

for every $j$. We now define

\[ K_\delta = \left\{ w : |w_j| \leq \frac{2^{j+1} \beta_j}{\delta} \right\}. \]

The set $K_\delta$ is compact and

\[ \mu_N(K_\delta) \leq \frac{1}{2} \sum_j 2^{-j} \beta_j^{-1} \delta \beta_j = \delta. \]

From this and Exercise 1.1.(iv) we deduce that $\mu_N$ has a convergent subsequence. Let $\bar{\mu}$ be a limit point and set $\bar{S}_j = w(1) + \cdots + w(j)$. By (4.8),

\[ \int \varphi(S_1, \ldots, S_n) d\mu \geq \int \varphi(\bar{S}_1, \ldots, \bar{S}_n) d\bar{\mu}. \]
for every continuous monotonically decreasing \( \varphi \). We now define \( \tau : \Omega \rightarrow \Omega \) by \( \tau(w)(j) = w(j + 1) \). It is not hard to see \( \bar{\mu} \in \mathcal{I} \). By Ergodic Theorem, \( \frac{1}{n} \sum_{k=0}^{n} \tau^{k} \rightarrow \mathbb{E} \) for almost all \( w \). Moreover, \( \int Zd\bar{\mu} = \int w(1)d\bar{\mu}(dw) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N} \tau^{k}(S_{N} - S_{0})d\mu = \lambda \). We use (4.9) to assert that for every bounded continuous increasing \( \psi \),

\[
\int \psi \left( \min_{k \leq n \leq l} \frac{S_{n}}{n} \right) d\mu \geq \int \psi \left( \min_{k \leq n \leq l} \frac{S_{n}}{n} \right) d\bar{\mu}.
\]

We now apply the bounded convergence theorem to deduce

\[
\int \psi(S) d\mu \geq \int \psi(Z) d\bar{\mu}
\]

where \( S = \liminf_{n \rightarrow \infty} \frac{S_{n}}{n} \). Choose \( \psi(z) = \psi^{r,l}(z) = (zv(-l)) \wedge r \), \( \psi_{l}(z) = zv(-l) \). We then have

\[
\int \psi_{l}(S) d\mu \geq \int \psi^{r,l}(S) d\mu \geq \int \psi^{r,l}(Z) d\bar{\mu}.
\]

After sending \( r \rightarrow \infty \), we deduce

\[
(4.10) \quad \int \psi_{l}(S) d\mu \geq \int Zd\bar{\mu} = \lambda, \text{ or } \int (\psi_{l}(S) - \lambda) d\mu \geq 0.
\]

Recall \( \underline{S} \leq \limsup_{n \rightarrow \infty} \frac{S_{n}}{n} \leq \lambda \). But (4.10) means

\[
\int_{\underline{S} \geq -l} (\underline{S} - \lambda) d\mu + (-l - \lambda) \mu \{ \underline{S} \leq -l \} \geq 0.
\]

Since \( \lambda > -\infty \), we can choose \( l \) large enough to have \( -l - \lambda < 0 \). For such \( l \), \( \underline{S} - \lambda = 0 \) on the set \( \{ \underline{S} \geq -l \} \). By sending \( l \rightarrow +\infty \) we deduce \( \underline{S} = \lambda \) almost everywhere, and this completes the proof. \( \Box \)

We now state the Oseledets Theorem that guarantees the existence of a collection of Lyapunov exponents.

**Theorem 4.3** Let \( T : X \rightarrow X \) be a \( C^{1} \)-diffeomorphism with \( \dim X = d \) and let \( \mu \in \mathcal{I}_{T}^{v} \). Let \( A \) be a measurable function such that \( A(x) : T_{x}X \rightarrow T_{T(x)}X \) is linear for each \( x \) and \( \log^{+} \| A(x) \| \in L^{1}(\mu) \). Define \( A_{n}(x) = A(T^{n-1}(x)) \circ \cdots \circ A(T(x)) \circ A(x) \). Then there exists a set \( X' \subseteq X \) with \( \mu(X') = 1 \), numbers \( l_{1} < l_{2} < \cdots < l_{k} \) and \( n_{1}, \ldots, n_{k} \in \mathbb{N} \) with \( n_{1} + \cdots + n_{k} = d \), and a linear decomposition \( T_{x}X = E_{x}^{1} \oplus \cdots \oplus E_{x}^{k} \) with \( x \mapsto (E_{x}^{1}, \ldots, E_{x}^{k}) \) measurable such that

\[
\lim_{n \rightarrow \infty} \frac{1}{n} \log \| A_{n}(x)v \| = l_{j}
\]

for \( x \in X' \) and \( v \in F_{x}^{j} \setminus F_{x}^{j-1} \), where \( F_{x}^{j} : = E_{x}^{1} \oplus \cdots \oplus E_{x}^{j} \).
Example 4.2(i) Let $T : T^m \to T^m$ be a translation. Then $dT^n = id$ and the only Lyapunov exponent is zero.

(ii) Let $T : T^m \to T^m$ be given by $T(x) = Ax \pmod{1}$ with $A$ a matrix of integer entries. Let $\lambda_1, \ldots, \lambda_q$ denote the eigenvalues of $A$. Let $l_1 < l_2 < \cdots < l_k$ be numbers with \{log $|\lambda_1|$, $\ldots$, log $|\lambda_q|$\}. We also write $n_j$ for the sum of the multiplicities of eigenvalues $\lambda_i$ with log $|\lambda_i| = l_j$. The space spanned by the corresponding generalized eigenvectors is denoted by $E_j$. We certainly have that if $v \in E_j$ then $\lim_{n \to \infty} \frac{1}{n} \log |A^n v| = l_j$.

(iii) If $a \in X$ is a periodic point of period $N$, then $\mu = N^{-1} \sum_{j=0}^{N-1} \delta_{T^j(a)}$ is an ergodic invariant measure. In this case the Oseledets Theorem follows from our discussion in Example 4.1(iii). Indeed if $\lambda_1, \ldots, \lambda_q$ denote the eigenvalues of $B = (dT^N)_a$, then $\ell_1 < \cdots < \ell_k$ are chosen so that \{log $|\lambda_1|$, $\ldots$, log $|\lambda_q|$\} and $E^u_a = \oplus_i \{V_i : N^{-1} \log |\lambda_i| = \ell_j \}$ where $G_i = \{v \in T_a M; (A(a) - \lambda_i)^r v = 0 \mbox{ for some } r \}$ is the generalized eigenspace associated with $\lambda_i$.

(iv) When $d = 1$, Theorem 4.3 (or 4.1) is an immediate consequence of the Ergodic Theorem and the only Lyapunov exponent is $l_1 = \int \log |A(x)| \mu(dx)$.

Remark 4.1(i) The identity $A_n(T(x)) A(x) v = A_{n+1}(x) v$ implies $$A(x) F^j_x \subseteq F^j_T(x),$$ for $j = 1, \ldots, k$. By invertibility, we can also show that $$A(x) F^j_x \supseteq F^j_T(x).$$

(ii) By Ergodic Theorem, $$\frac{1}{n} \log |\det A_n(x)| = \frac{1}{n} \sum_{j=0}^{n-1} \log |\det A(T^j(x))| \to \int \log |\det(dT)_x| \ d\mu,$$ As we will see later, $$\int \log |\det(dT)_x| d\mu = \sum_{j=1}^k n_j l_j.$$

(iii) It turns out that the most challenging part of Theorem 4.3 is the existence of the limit. Indeed if we define $$l(x, v) = \limsup_{n \to \infty} \frac{1}{n} \log |A_n(x)v|,$$ then we can show that as in Theorem 4.3 there exists a splitting $T_x X = E^1_x \oplus \cdots \oplus E^k_x$ with $l(x, v) = l_j$ for $v \in F^j_x(x)$.
Proof of Theorem 4.3 for \(d = 2\) We only prove Theorem 4.3 when \(A = dT\). The proof of general case is similar. By Theorem 4.1, there exist numbers \(l_1\) and \(l_2\) such that if
\[
X_0 = \left\{ x : \lim_{n \to \infty} \frac{1}{n} \log \| (dT^n)_x \| = l_2, \quad \lim_{n \to \infty} \frac{1}{n} \log \| (dT^{-n})_x \| = -l_1 \right\}
\]
then \(\mu(X_0) = 1\). Evidently \(|A_n v|^2 = \langle A_n^* A_n v, v \rangle = |B_n v|^2\) where \(B_n = (A_n^* A_n)^{1/2}\). Clearly \(A_n^* A_n \geq 0\) and \(B_n\) is well-defined. Since \(B_n \geq 0\), we can find numbers \(\mu_2^n(x) \geq \mu_1^n(x) \geq 0\) and vectors \(a^n_1(x), a^n_2(x)\) such that \(|a^n_1| = |a^n_2| = 1, \langle a^n_1, a^n_2 \rangle_x = 0\) and \(B_n a^n_j = \mu_j^n a^n_j\) for \(j = 1, 2\).

Note that since \(\|A_n(x)\| = \|B_n(x)\|\),
\[
(4.11) \quad l_2 = \lim_{n \to \infty} \frac{1}{n} \log \mu_2^n.
\]
To obtain a similar formula for \(l_1\), first observe that
\[
(dT^n)_{T^{-n}(x)} \cdots (dT^{-n})_x = id \Rightarrow A_{-n}(x) := (dT^{-n})_x = (A_n(T^{-n}(x)))^{-1}.
\]
If we set \(S_{-n}(x) = \log \|A_{-n}(x)\|\) and \(R_n(x) = \log \|A_n(x)^{-1}\|\) then both \(\{S_{-n}(x) : n \in \mathbb{N}\}\) and \(\{R_n(x) : n \in \mathbb{N}\}\) are subadditive;
\[
S_{-n-m} \leq S_{-n} \circ T^{-m} + S_{-m}, \quad R_{n+m} \leq R_n \circ T^m + R_m.
\]
Clearly,
\[
-l_1 = \lim_{n \to \infty} \frac{1}{n} S_{-n} = \inf_{n} \frac{1}{n} \int S_{-n} d\mu, \quad -l_2 = \lim_{n \to \infty} \frac{1}{n} R_n = \inf_{n} \frac{1}{n} \int R_n d\mu.
\]
Since \(S_{-n} = R_n \circ T^{-n}\), we have \(\int R_n d\mu = \int S_{-n} d\mu\). This in turn implies that \(\hat{l} = -l_1\). As a result,
\[
-l_1 = \lim_{n \to \infty} \frac{1}{n} \log \|A_{-n}^{-1}\| = \lim_{n \to \infty} \frac{1}{n} \log \|A_n^{* -1}\|.
\]
(Recall that \(\|A\| = \|A^*\|\).) We then have
\[
(4.12) \quad -l_1 = \lim_{n \to \infty} \frac{1}{n} \log \|(A_n^* A_n)^{-1/2}\| = \lim_{n \to \infty} \frac{1}{n} \log \|B_n^{-1}\|
\]
\[
= - \lim_{n \to \infty} \frac{1}{n} \log (\mu_1^n \land \mu_2^n) = - \lim_{n \to \infty} \frac{1}{n} \log \mu_1^n.
\]
Naturally we expect \(E_x^2\) to be the limit of the lines \(\{ta^n_2 : t \in \mathbb{R}\}\) as \(n \to \infty\). For this though we need to assume that \(l_1 < l_2\). To see this, let us first estimate \(|a_{n+1}^2(x) - a_2^n(x)|\). We may assume that \(\langle a_{n+1}^2, a_2^n \rangle_x \geq 0\) for all \(n\). Indeed if this is true for all \(n < m\) but \(\langle a_{m+1}^2, a_2^m \rangle_x < 0\), replace \(a_{m+1}^2\) with \(-a_{m+1}^2\).
We certainly have
\[ |a_2^{n+1} - a_2^n|^2 = 2 - 2\langle a_2^{n+1}, a_2^n \rangle, \quad 1 = |a_2^{n+1}|^2 = \langle a_2^{n+1}, a_2^{n+1} \rangle + \langle a_2^{n+1}, a_2^n \rangle. \]
From this and the elementary inequality \( 1 - z^2 \leq \sqrt{1 - z^2} \), we learn
\[ |a_2^{n+1} - a_2^n|^2 = 2 - 2(1 - \langle a_2^{n+1}, a_1^{n+1} \rangle)^{1/2} \leq 2\langle a_2^{n+1}, a_1^n \rangle^2 = 2\langle a_2^{n+1}, a_1^n \rangle^2 \]
\[ \leq 2\mu_n^{n+1-2}|B_{n+1}a_n^n|^2 = 2\mu_n^{n+1-2}|A_{n+1}a_n^n|^2 \]
\[ = 2\mu_n^{n+1-2}|A(T_n(x))A_n(x)a_1^n(x)|^2 \leq 2\mu_n^{n+1-2}c_0|A_n(x)a_1^n(x)|^2 \]
\[ = 2\mu_n^{n+1-2}c_0|B_n|a_1^n|^2 = 2c_0(\mu_n^{n+1} - \mu_n^n)^2 \]
for \( c_0 = \max_x |A(x)| \). From this, (4.11) and (4.12) we deduce
\[ \limsup_{n \to \infty} \frac{1}{n} \log |a_2^{n+1} - a_2^n| \leq -(l_2 - l_1). \]
Let us now assume that \( l_2 - l_1 > \delta > 0 \). We then have that for constants \( c_1, c_2 \),
\[ |a_2^{n+1} - a_2^n| \leq c_1 e^{-\delta n}, \quad |a_2^n - a_2^n + a_2^n - a_2^n| \leq c_2 e^{-\delta n} \]
for all positive \( n \) and \( r \). As a result, \( \lim_{n \to \infty} a_2^n = b_2 \) exists for \( x \in X \) and
\[ |a_2^n - b_2| \leq c_2 e^{-\delta n} \]
for all \( n \). This being true for all \( \delta \in (0, l_2 - l_1) \), means
\[ \limsup_{n \to \infty} \frac{1}{n} \log |a_2^n - b_2| \leq -(l_2 - l_1). \]
We now define \( E_2^T = \{ t b_2(x) : t \in \mathbb{R} \} \). To show that \( \lim_{n \to \infty} \frac{1}{n} \log |A_n(x)b_2(x)| = l_2 \), observe
\[ |A_n b_2| \leq |A_n a_2^n| + |A_n(a_2^n - b_2)| \leq |B_n a_2^n| + \| A_n \| |a_2^n - b_2| = \mu_2^n + \| A_n \| |a_2^n - b_2|, \]
\[ |A_n b_2| \geq |A_n a_2^n| - |A_n(a_2^n - b_2)| \geq |B_n a_2^n| - \| A_n \| |a_2^n - b_2| = \mu_2^n - \| A_n \| |a_2^n - b_2| \]
From this and (4.11)-(4.13) we deduce
\[ \limsup_{n \to \infty} \frac{1}{n} \log |A_n b_2| \leq \max \left( \limsup_{n \to \infty} \frac{1}{n} \log \mu_2^n, \limsup_{n \to \infty} \frac{1}{n} \log (\| A_n \| |a_2^n - b_2|) \right) \]
\[ \leq \max(l_2, l_1) = l_2, \]
\[ l_2 = \lim_{n \to \infty} \frac{1}{n} \log \mu_2^n \leq \max \left( \liminf_{n \to \infty} \frac{1}{n} \log |A_n b_2|, \limsup_{n \to \infty} \frac{1}{n} \log (\| A_n \| |a_2^n - b_2|) \right) \]
\[ \leq \max \left( \liminf_{n \to \infty} \frac{1}{n} \log |A_n b_2|, l_1 \right). \]
From this we can readily deduce
\[
\lim_{n \to \infty} \frac{1}{n} \log |A_n(x)b_2| = l_2,
\]
for \(x \in X\).

To find \(E^x_1\), replace \(T\) with \(T^{-1}\) in the above argument. This completes the proof when \(l_1 \neq l_2\).

It remains to treat the case \(l_1 = l_2\). We certainly have
\[
|A_n v|^2 = |B_n v|^2 = \langle v, a_1^n \rangle^2 (\mu_1^n)^2 + \langle v, a_2^n \rangle^2 (\mu_2^n)^2.
\]
Hence
\[
\mu_1^n |v| \leq |A_n v| \leq \mu_2^n |v|.
\]
We are done because \(\lim \frac{1}{n} \log \mu_2^n = \lim \frac{1}{n} \log \mu_1^n = l_1 = l_2\). \(\square\)

From (4.11) and (4.12) we learned that when \(d = 2\), we may use the eigenvalues of the matrix
\[
A_n(x) = (A_n(x)^* \circ A_n(x))^{\frac{1}{2}} = B_n(x)\frac{1}{2} : T_x X \to T_x X,
\]
to find the Lyapunov exponents. In fact the same is true in any dimension. To explain this, observe that
\[
\lim_{n \to \infty} \frac{1}{n} \log \det B_n(x) = \lim_{n \to \infty} \frac{1}{n} \log |\det A_n(x)|,
\]
should yield a way of getting the sum of all Lyapunov exponents. In dimension 2, and when there are two Lyapunov exponents, we can use the determinant to get our hand on \(l_1\), because by Theorem 4.1 we already have a candidate for \(l_2\). To generalize this idea to higher dimension, we use the notion of the exterior power of a linear transformation and a vector space.

**Definition 4.1(i)** Given a vector space \(V\), its \(r\)-fold exterior power \(\wedge^r V\) is a vector space consisting of
\[
\wedge^r V = \{v_1 \wedge \cdots \wedge v_r : v_1, \ldots, v_r \in V\}.
\]
By convention, \(v_1 \wedge \cdots \wedge v_r = 0\) if \(v_1, \ldots, v_r\) are not linearly independent. The wedge product is characterized by two properties: it is multilinear and alternative. By the former we mean that for all scalars \(c\) and \(c'\),
\[
(cv_1 + c'v'_1) \wedge v_2 \wedge \cdots \wedge v_r = c(v_1 \wedge v_2 \wedge \cdots \wedge v_r) + c'(v'_1 \wedge v_2 \wedge \cdots \wedge v_r).
\]
By the latter we mean that interchanging two vectors in \(a = v_1 \wedge \cdots \wedge v_r\) changes the sign of \(a\). If \(\{e_1, \ldots, e_d\}\) is a basis for \(V\), then
\[
\{e_{i_1,i_2, \ldots, i_r} := e_{i_1} \wedge \cdots \wedge e_{i_r} : i_r < \cdots < i_r\},
\]
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is a basis for $\wedge^r V$. In particular $\dim \wedge^r V = \binom{d}{r}$.

(ii) If $\langle \cdot, \cdot \rangle$ is an inner product on the vector space $V$, then we equip $\wedge^r V$ with the inner product

$$\langle v_1 \wedge \cdots \wedge v_r, v'_1 \wedge \cdots \wedge v'_r \rangle = \det \left[ \langle v_i, v'_j \rangle \right]_{i,j=1}^r.$$ 

The quantity

$$\|v_1 \wedge \cdots \wedge v_r\|^2 = \langle v_1 \wedge \cdots \wedge v_r, v_1 \wedge \cdots \wedge v_r \rangle = \det \left[ \langle v_i, v_j \rangle \right]_{i,j=1}^r,$$

represents the $r$-dimensional volume of the parallelepiped generated by vectors $v_1, \ldots, v_r$.

(iii) Let $V$ and $V'$ be two vector spaces and assume that $A : V \to V'$ is a linear transformation. We define

$$\wedge^r A : \wedge^r V \to \wedge^r V',$$

by

$$\wedge^r A(v_1 \wedge \cdots \wedge v_r) = (Av_1) \wedge \cdots \wedge (Av_r).$$

(vi) (Grassmanian of a vector space) Given a vector space $V$ of dimension $d$, we write $Gr(V, r)$ for the set of $r$-dimensional linear subspaces of $V$. If $V$ is equipped with an inner product and the corresponding norm is denoted by $| \cdot |$, then we may define a metric $d_{Gr}$ on $Gr(V, r)$ as follows: Given $W, W' \in Gr(V, r)$, write

$$\hat{W} = \{ w \in W : |w| = 1 \}, \quad \hat{W}' = \{ w' \in W' : |w'| = 1 \},$$

and set

$$d_{Gr}(W, W') = \max \left( \max_{x \in \hat{W}} \min_{y \in \hat{W}'} |x - y|, \max_{y \in \hat{W}'} \min_{x \in \hat{W}} |x - y| \right).$$

(v) Given a smooth $d$-dimensional Riemannian manifold $X$, the manifold $\wedge^r X$ is a vector bundle that assign to each point $x \in X$, the vector space $\wedge^r_x X = \wedge^r T_x X$. The metric on $X$ induces a metric on $\wedge^r X$ by using $\langle \cdot, \cdot \rangle_x$ to produce an the inner on product $\wedge^r_x X$ as Part (ii).

What we have in mind is that the $r$-vector $v_1 \wedge \cdots \wedge v_r$ represents the $r$-dimensional linear subspace that is spanned by vectors $v_1, \ldots, v_r$. We list a number of straightforward properties of the $r$-vectors in Proposition 4.1 below. The elementary proof of this proposition is omitted.

**Proposition 4.1** (i) Two sets of linearly independent vectors $\{v_1, \ldots, v_r\}$ and $\{v'_1, \ldots, v'_r\}$ span the same vector space iff $v_1 \wedge \cdots \wedge v_r = \lambda v'_1 \wedge \cdots \wedge v'_r$ for some nonzero scalar $\lambda$. 

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(ii) If $\langle \cdot, \cdot \rangle$ is an inner product on the vector space $V$, and $\{a_1, \ldots, a_d\}$ is an orthonormal basis for $V$, then the set $\{a_{i_1} \wedge \cdots \wedge a_{i_r} : 1 \leq i_1 < \cdots < i_r\}$ is an orthonormal basis for $\wedge^r V$.

(iii) If $V, V', V''$ are three vector spaces and $A : V \to V'$, $B : V' \to V''$ are linear, then $\wedge^r (B \circ A) = (\wedge^r B \circ \wedge^r A)$. If $A$ is invertible, then $\wedge^r A^{-1} = (\wedge^r A)^{-1}$. If $V$ and $V'$ are inner product spaces and $A^* : V' \to V$ is the transpose of $A$, then $\wedge^r A^* = (\wedge^r A)^t$.

(iv) If $A : \mathbb{R}^d \to \mathbb{R}^d$ is represented by a $d \times d$ matrix, then the transformation $\wedge^r A$ is represented by a $\binom{d}{r} \times \binom{d}{r}$ matrix we obtain by taking the determinants of all $r \times r$ submatrices of $A$.

(v) Suppose that $V$ is an inner product space of dimension $d$, and $\{a_1, \ldots, a_d\}$ is an orthonormal basis consisting of eigenvectors, associated with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_d$, then the set $\{a_{i_1} \wedge \cdots \wedge a_{i_r} : 1 \leq i_1 < \cdots < i_r\}$ is an orthonormal basis consisting of eigenvectors of $\wedge^r A$ associated with eigenvalues $\{\lambda_{i_1} \cdots \lambda_{i_r} : 1 \leq i_1 < \cdots < i_r\}$.

(v) For an inner product space $V$, the space $(\text{Gr}(V,r), d_{Gr})$ is a compact metric space.

**Proof of Theorem 4.3 for general $d$**

**Step 1.** Recall that if $B_n = (A_n^* A_n)^{1/2}$, then $|A_n v| = |B_n v|$ with $B_n \geq 0$. Let us write $\mu_1^n \leq \cdots \leq \mu_d^n$ for the eigenvalues of $B_n$ and $a_1^n, \ldots, a_d^n$ for the corresponding eigenvectors with $|a_i^n| = 1$, $<a_i^n, a_j^n> = 0$ for $i \neq j$. We now claim that

(4.15) $\lim_{n \to \infty} n^{-1} \log \mu_i^n(x) =: m_i,$

exists $\mu$-almost surely. Indeed since

$A_{m+n}(x) = A_n(T^m(x)) \circ A_m(x),$

we may apply Proposition 4.1(ii) to assert

$\wedge^r A_{m+n}(x) = (\wedge^r A_n)(T^m(x)) \circ \wedge^r A_m(x).$

This in turn implies

$\|\wedge^r A_{m+n}(x)\| \leq \|\wedge^r A_n(T^m(x))\| + \|\wedge^r A_m(x)\|.$

This allows us to apply Subadditive Ergodic Theorem (Theorem 4.2) to deduce

(4.16) $\lim_{n \to \infty} \frac{1}{n} \log \|\wedge^r A_n(x)\| := \lambda^r.$
exists for $\mu$-almost all $x$ and every $r \in \{1, \ldots, d\}$. On the other hand, Proposition 4.1 allows us to write
\[ \|\land^r A_n(x)\| = \left\| (\land^r A_n(x)^* \circ A_n(x))^{1/2} \right\| = \left\| \left( \land^r (A_n(x)^* \circ A_n(x)) \right)^{1/2} \right\| = \|\land^r B_n(x)\| = \mu_d \ldots \mu_{d-r+1}. \]

From this and (4.15) we deduce,
\[ \lim_{n \to \infty} \left[ \sum_{i=d-r+1}^{d} n^{-1} \log \mu_i^n(x) \right], \]
exists for $\mu$-almost all $x$ and every $r \in \{1, \ldots, d\}$. This certainly implies (4.15)

**Step 2.** Choose $l_1 < \cdots < l_k$ so that $\{m_1, \ldots, m_d\} = \{l_1, \ldots, l_k\}$. We also set
\[ L_j = \{ i : m_i \leq l_j \}. \]

We define
\[ F_{x,n}^j = \left\{ \sum_{i \in L_j} c_i a_i^n(x) : c_i \in \mathbb{R} \text{ for } i \in L_j \right\} \subseteq T_x X. \]

We wish to show that the sequence $\{F_{x,n}^j\}$ is convergent with respect to $d_{Gr}$, so that we can define
\[ (4.17) \quad F_x^j = \lim_{n \to \infty} F_{x,n}^j. \]

We only prove this for $j = k - 1$; the proof for other $j$ can be carried out in an analogous way.

To establish (4.17), we show first that if $b_n \in F_{x,n}^{k-1}$, with $|b_n| = 1$, then there exists $u_{n+1} \in F_{x,n}^{k-1,n+1}$ such that
\[ (4.18) \quad \limsup_{n \to \infty} \frac{1}{n} \log \left| b_n - \frac{u_{n+1}}{|u_{n+1}|} \right| \leq -(l_k - l_{k-1}). \]

For this, it suffice to show
\[ (4.19) \quad \limsup_{n \to \infty} \frac{1}{n} \log |b_n - u_{n+1}| \leq -(l_k - l_{k-1}), \]

because
\[ \left| b_n - \frac{u_{n+1}}{|u_{n+1}|} \right| \leq |b_n - u_{n+1}| + \left| u_{n+1} - \frac{u_{n+1}}{|u_{n+1}|} \right| \leq |b_n - u_{n+1}| + |1 - |u_{n+1}|| \leq 2 |b_n - u_{n+1}|. \]
In fact \( u_{n+1} \) is simply the projection of \( b_n \) onto \( F^{k-1,n+1}_x \). More precisely, we write \( b_n = u_{n+1} + v_{n+1} \) with \( u_{n+1} \in F^{k-1,n+1}_x, v_{n+1} \perp F^{k-1,n+1}_x \), and show

\[
(4.20) \quad \limsup_{n \to \infty} \frac{1}{n} \log |v_{n+1}| \leq -(l_k - l_{k-1}),
\]

which implies (4.19). For (4.20), observe that we can find scalars \( c_j^n \) and \( c_j^{n+1} \) such that

\[
b_n = \sum_{i \in L_{k-1}} c_i^n a_i^n, \quad v_{n+1} = \sum_{i \notin L_{k-1}} c_i^{n+1} a_i^{n+1}.
\]

As a result,

\[
|A_{n+1}v_{n+1}|^2 = |B_{n+1}v_{n+1}|^2 = \left| \sum_{i \notin L_{k-1}} c_i^{n+1} \mu_i^{n+1} a_i^{n+1} \right|^2 = \sum_{i \notin L_{k-1}} (c_i^{n+1})^2 (\mu_i^{n+1})^2 \geq \left( \min_{i \notin L_{k-1}} (\mu_i^{n+1})^2 \right) \sum_{i \notin L_{k-1}} (c_i^{n+1})^2 = \left( \min_{i \notin L_{k-1}} (\mu_i^{n+1})^2 \right) |v_{n+1}|^2.
\]

In summary,

\[
(4.21) \quad \left( \min_{i \notin L_{k-1}} (\mu_i^{n+1})^2 \right) |v_{n+1}| \leq |A_{n+1}v_{n+1}|.
\]

On the other hand, observe

\[
\langle A_{n+1}u_n, A_{n+1}v_n \rangle = \langle A_{n+1}^* \circ A_{n+1}u_n, v_n \rangle = \langle B_{n+1}^2 u_n, v_n \rangle = 0,
\]

because the space \( F^{k-1,n+1}_x \) is invariant under the action of \( B_{n+1} \). This and \( A_{n+1}b_{n+1} = A_{n+1}u_{n+1} + A_{n+1}v_{n+1} \) imply

\[
(4.22) \quad |A_{n+1}b_n|^2 = |A_{n+1}u_n|^2 + |A_{n+1}v_n|^2 \geq |A_{n+1}v_n|^2.
\]

Furthermore,

\[
|A_{n+1}b_n|^2 = |A(T^n(x)) \circ A_n(x)b_n|^2 \leq |A(T^n(x))|^2 |A_n(x)b_n|^2 \leq C_0^2 |A_n(x)b_n|^2 = C_0^2 |B_n(x)b_n|^2 = C_0^2 \sum_{i \in L_{k-1}} (c_i^n)^2 (\mu_i^n)^2 \leq C_0^2 \left( \max_{i \in L_{k-1}} (\mu_i^n)^2 \right) \sum_{i \in L_{k-1}} (c_i^n)^2 = C_0^2 \left( \max_{i \in L_{k-1}} (\mu_i^n)^2 \right).
\]
where $C_0$ is the maximum of the function $\|A\|$ over $X$. From this, (4.21) and (4.22) we deduce

$$|v_{n+1}| \leq C_0^2 \left( \max_{i \in L_{k-1}} \mu_i^0 \right) \left( \min_{i \in L_{k-1}} \mu_i^{n+1} \right)^{-1}. $$

This implies (4.18).

**Step 3.** Repeating the above argument and using the boundedness of $\|A^{-1}\|$ on $X$, we can show that if $b_{n+1} \in F^{k-1,n+1}$, and $u'_n$ is the projection of $b_{n+1}$ onto $F^{k-1,n}$, then

$$\limsup_{n \to \infty} \frac{1}{n} \log \left| b_{n+1} - \frac{u'_n}{|u'_n|} \right| \leq -(l_k - l_{k-1}).$$

From this and (4.17) we deduce

$$\limsup_{n \to \infty} \frac{1}{n} \log d_{Gr}(F^{k-1,n}, F^{k-1,n+1}) \leq -(l_k - l_{k-1}).$$

This in turn implies that the limit

$$F^{k-1} := \lim_{n \to \infty} F^{k-1,n},$$

exists.

**Step 4.** We wish to show that if $v \in F^k \setminus F^{k-1}$, $|v| = 1$, then

$$\lim_{n \to \infty} \frac{1}{n} \log |A_n v| = l_k,$$

$\mu$-a.e. We already know,

$$\limsup_{n \to \infty} \frac{1}{n} \log |A_n v| = \limsup_{n \to \infty} \frac{1}{n} \log |B_n v| \leq \limsup_{n \to \infty} \frac{1}{n} \log (||B_n|| |v|)$$

$$= \limsup_{n \to \infty} \frac{1}{n} \log ||B_n|| = \limsup_{n \to \infty} \frac{1}{n} \log \mu_d^n = l_k. $$

Hence, for (4.25), we only need to show

$$\liminf_{n \to \infty} \frac{1}{n} \log |A_n v| \geq l_k,$$

For this we decompose $v$ as

$$v = u_n + v_n, \quad u_n \in F_x^{k-1,n}, \quad v_n = \sum_{i \in L_{k-1}} c_i^n a_i^n \downarrow F_x^{k-1,n}. $$
Since \( v \notin F^{k-1} \), we must have

\[
(4.27) \quad \limsup_{n \to \infty} \frac{1}{n} \log |v_n| \geq 0.
\]

Simply because if (\ref{eq:4.27}) were not true, then

\[
\lim_{n \to \infty} \left| v - \frac{u_n}{|u_n|} \right| = 0,
\]

which contradicts \( v \notin F^{k-1} \). On the other hand, since \( A_n u_n \perp A_n v_n \) as in (4.19), we can assert

\[
|A_n v|^2 \geq |A_n v_n|^2 = \left| B_n v_n \right|^2 = \left| \sum_{i \notin L_{k-1}} c_i^n \mu_i^n u_i^n \right|^2 = \sum_{i \notin L_{k-1}} \left( c_i^n \right)^2 \left( \mu_i^n \right)^2 \geq \left( \min_{i \notin L_{k-1}} \mu_i^n \right)^2 \sum_{i \notin L_{k-1}} \left( c_i^n \right)^2 = \left( \max_{i \in L_{k-1}} \mu_i^n \right)^2 |v_n|^2.
\]

This and (\ref{eq:4.27}) imply (\ref{eq:4.27}), completing the proof of (\ref{eq:4.27}).

**Final Step.** Inductively, we can construct other \( F^j \). For example, for \( j = k - 2 \), the analog of (4.25) is the statement that if \( v \in F^{k-1} \), then

\[
(4.28) \quad \limsup_{n \to \infty} \frac{1}{n} \log |A_n(x) v| \leq l_{k-1},
\]

\( \mu \)-a.e. To prove (4.28), we use the definition of \( F^{k-1} \) and (4.23), to find a sequence \( \{u_n\} \) such that \( u_n \in F^{k-1,n} \) and

\[
\limsup_{n \to \infty} \frac{1}{n} \log |v - u_n| \leq -(l_k - l_{k-1}).
\]

As a result,

\[
\limsup_{n \to \infty} \frac{1}{n} \log |A_n v| \leq \max \left\{ \limsup_{n \to \infty} \frac{1}{n} \log |A_n u_n|, \limsup_{n \to \infty} \frac{1}{n} \log (\|A_n\| |v - u_n|) \right\} \leq \max \{l_{k-1}, \lambda_k + l_{k-1} - l_k\} = l_{k-1},
\]

proving (4.28).

We now state and prove an inequality of Ruelle.
Theorem 4.4 Let $T : X \to X$ be $C^1$ and $\mu \in I^p_\mu$. Then
\[
h_\mu(T) \leq \sum_{j=1}^{k} n_{j} l_{j}^*.
\]

Proof We only present the proof when $\dim X = d = 2$. First we would like to divide $X$ into “small squares”. For this we take a triangulation of $X$: $X = \bigcup_{i} \Delta_i$ where each $\Delta_i$ is a diffeomorphic copy of a triangle in $\mathbb{R}^2$ and $\Delta_i \cap \Delta_j$ is either empty, or a common vertex, or a common side. We then divide each triangle into squares of side length $\varepsilon$ and possibly triangles of side length at most $\varepsilon$ (we need these triangles near the boundary of $\Delta_i$’s). The result is a covering of $X$ that is denoted by $\xi^\varepsilon$. Note that we may choose members of $\xi^\varepsilon$ such that $\mu(\partial A) = 0$ for $A \in \xi^\varepsilon$. (If this is not the case, move each element of $\xi^\varepsilon$ by small amount and use the fact that for some translation of boundary side we get zero measure.) As a result, $\xi^\varepsilon$ is a $\mu$-partition. It is not hard to show
\[
(4.29) \quad h_\mu(T) = \lim_{\varepsilon \to 0} h_\mu(T, \xi^\varepsilon).
\]

Recall that $h_\mu(T, \xi^\varepsilon) = \lim_{k \to \infty} \int I_{\xi^\varepsilon} d\mu$ where $\xi^\varepsilon k = T^{-1}(\xi^\varepsilon) \cup T^{-2}(\xi^\varepsilon) \cup \ldots \cup T^{-k}(\xi^\varepsilon)$ and
\[
I_{\xi^\varepsilon} = - \sum_{A \in \xi^\varepsilon} \sum_{B \in \xi^\varepsilon k} \frac{\mu(A \cap B)}{\mu(B)} \log \frac{\mu(A \cap B)}{\mu(B)} \mathbb{1}_B.
\]

Given $x$, let $B = B_{\varepsilon, k}(x)$ be the unique element of $\xi^\varepsilon k$ such that $x \in B$. Such $B$ is of the form $T^{-1}(C_1) \cap \cdots \cap T^{-k}(C_k)$ with $C_1 \cdots C_k \in \xi^\varepsilon$, where $C_j = C_{\xi^\varepsilon}(T^j(x))$. Let us write simply write $C_1(x)$ for $C_{\xi^\varepsilon}(T(x))$. We have
\[
(4.30) \quad I_{\xi^\varepsilon k}(x) \leq \log \# \{ A \in \xi^\varepsilon : A \cap B_{\varepsilon, k}(x) \neq \emptyset \}
\]
\[
\leq \log \# \{ A \in \xi^\varepsilon : A \cap T^{-1}(C_1(x)) \neq \emptyset \}.
\]

Each $C_1(x)$ is a regular set; either a diffeomorphic image of a small square or a triangle. Since the volume of $C$ is of order $O(\varepsilon^2)$, we have
\[
\text{vol}(T^{-1}(C)) \leq c_1 \varepsilon^2 \max_{z \in C} | \det (dT)_z^{-1} |,
\]
for a constant $c_1$. If $A \cap T^{-1}(C) \neq \emptyset$, then for a constant $\alpha_0$,
\[
A \subseteq \{ y : | y - x_0 | \leq \alpha \varepsilon \text{ for some } x_0 \in T^{-1}(C) \} := D.
\]

We now want to bound $\text{vol}(D)$. The boundary of $T^{-1}(C)$ is a regular curve. Hence its length is comparable to the diameter of $T^{-1}(C)$, and this is bounded above by a multiple of the norm of $dT^{-1}$. Using this we obtain
\[
(4.31) \quad \text{vol}(D) \leq c_2 \max_{z \in C} \left( 1 + | (dT)_z^{-1} | + | \det (dT)_z^{-1} | \right) \varepsilon^2.
\]
for a constant $c_2$. (We could have bounded $\text{vol}(A)$ by $(||(dT)^{-1}||\varepsilon)^2$ but (4.31) is a better bound.)

We now use (4.31) to obtain an upper bound for the right-hand side (4.30). Indeed

$$(4.32) \# \{ A \in \xi^\varepsilon : A \cap T^{-1}(C) \neq \emptyset \} \leq c_3 \max_{z \in C} \left( 1 + \left\| (dT)^{-1}_z \right\| + \left| \det(dT)^{-1}_z \right| \right)$$

for a constant $c_3$. This is because the union of such $A$’s is a subset of $D$, for two distinct $A, B$, we have $\mu(A \cap B) = 0$, and for each $A \in \xi^\varepsilon$ we have that $\text{vol}(A) \geq c_4 \varepsilon^2$ for some positive constant $c_4$. From (4.31) and (4.32) we learn

$$I_{\xi|\xi^\varepsilon,k}(x) \leq c_5 + \log \max_{z \in C} \left( 1 + \left\| (dT)^{-1}_z \right\| + \left| \det(dT)^{-1}_z \right| \right)$$

for $C = C_1(x)$. By sending $k \to \infty$ we deduce

$$(4.33) \ h_\mu(T, \xi^\varepsilon) \leq c_5 + \int \log \max_{z \in C \setminus \{T(x)\}} \left( 1 + \left\| (dT)^{-1}_z \right\| + \left| \det(dT)^{-1}_z \right| \right) \ d\mu.$$

By the invariance of $\mu$,

$$h_\mu(T, \xi^\varepsilon) \leq c_5 + \int \log \max_{z \in C \setminus \{T(x)\}} \left( 1 + \left\| (dT)^{-1}_z \right\| + \left| \det(dT)^{-1}_z \right| \right) \ \mu(dx).$$

Send $\varepsilon \to 0$ to yield

$$h_\mu(T) \leq c_5 + \int \left( 1 + \left\| (dT)^{-1}_z \right\| + \left| \det(dT)^{-1}_z \right| \right) \ d\mu(dx).$$

The constant $c_5$ is independent of $T$. This allows us to replace $T$ with $T^{-n}$ to have

$$nh_\mu(T) \leq c_5 + \int \log \left( 1 + \left\| (dT^n) \right\| + \left| \det(dT^n) \right| \right) \ d\mu.$$  

First assume that there are two Lyapunov exponents. Since

$$\frac{1}{n} \log \left\| d(T^n) \right\| \to l_2, \quad \frac{1}{n} \log \left| \det d(T^n) \right| \to l_1 + l_2,$$

$\mu$-a.e., we deduce

$$(4.34) \ h_\mu(T) \leq \max(0, l_2, l_1 + l_2) \leq l_1^+ + l_2^+.$$  

In the same way we treat the case of one Lyapunov exponent. $\square$

The bound (4.34) may appear surprising because $h_\mu(T) > 0$ would rule out the case $l_1, l_2 < 0$. In fact we cannot have $l_1, l_2 < 0$ because we are assuming $T$ is invertible. An invertible transformation cannot be a pure contraction. Moreover if $h_\mu(T) > 0$ we must have a hyperbolic transformation in the following sense:
Corollary 4.1 If \( \dim X \geq 2 \) and \( h_\mu(T) > 0 \), then there exists a pair of Lyapunov exponents \( \alpha, \beta \) such that \( \alpha > 0 \), \( \beta < 0 \). In particular, if \( \dim X = 2 \) and \( h_\mu(T) > 0 \), then \( l_1 < 0 < l_2 \).

Proof Observe that if \( l_1 < \cdots < l_k \) are Lyapunov exponents of \( T \), then \( -l_k < \cdots < -l_1 \) are the Lyapunov exponents of \( T^{-1} \). Simply because if \( A_n(x) = D_x T^n \), then \( A_{-n} \circ T^n = A_n^{-1} \). Now by Theorem 4.7,

\[
h_\mu(T) = h_\mu(T^{-1}) \leq \sum_i n_i (-l_i)^+ = \sum_i n_i l_i^-,
\]

\[
h_\mu(T) \leq \sum_i n_i l_i^+.
\]

From these we deduce that \( -\sum_i l_i^- < 0 < \sum_i l_i^+ \) whenever \( h_\mu(T) > 0 \). □

Pesin’s theorem below gives a sufficient condition for having equality in Theorem 4.7. We omit the proof of Pesin’s formula.

Theorem 4.5 Let \( X \) be a \( C^1 \)-manifold and assume \( T : X \to X \) is a \( C^1 \) diffeomorphism. Assume \( DT \) is Hölder continuous. Let \( \mu \in \mathcal{I}_T \) be an ergodic measure that is absolutely continuous with respect to the volume measure of \( X \). Then

\[
h_\mu(T) = \sum_i n_i l_i^+.
\]

In the context of Theorem 4.3, the Lyapunov exponents of \( T^{-1} \) are \( -l_k < \cdots < -l_1 \). Let us write

\[
\mathcal{T}_x X = \bigoplus_i \hat{E}_x^i,
\]

for the splitting associated with \( T^{-1} \). It is natural to define

\[
E_x^s = \bigoplus_{l_i < 0} E_x^i, \quad E_x^u = \bigoplus_{l_i > 0} \hat{E}_x^i,
\]

If there is no zero Lyapunov exponent, we have \( \mathcal{T}_x X = E_x^s \oplus E_x^u \), \( \mu \)-almost everywhere. If we write \( l^\pm = \min_i l_i^\pm \), then we have

\[
\lim_{n \to \infty} \frac{1}{n} \log \left| (dT^{-n})_x v \right| \leq -l^+
\]

for \( v \in E_x^u \setminus \{0\} \), and

\[
\lim_{n \to \infty} \frac{1}{n} \log \left| (dT^n)_x v \right| \leq -l^-
\]

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for \( v \in E^s_x - \{0\} \), \( \mu \)-almost everywhere. If this happens in a uniform fashion, then we say that \( \mu \) is an \textit{Anosov measure}. More precisely, we say that the measure \( \mu \in \mathcal{I}_x^T \) is anosov if there exists a decomposition \( \mathcal{T}_x X = E^u_x \oplus E^s_x \) and constants \( K > 0 \) and \( \alpha \in (0, 1) \) such that

\[
(dT)_x E^u_x = E^u_{T(x)}, \quad (dT)_x E^s_x = E^s_{T(x)},
\]

\[
\| (dT)_x^n v \| \leq K \alpha^n \| v \| \quad \text{for } v \in E^s_x,
\]

\[
\| (dT^{-n})_x v \| \leq K \alpha^n \| v \| \quad \text{for } v \in E^u_x.
\]

If we define

\[
W^s(x) = \left\{ y : \lim_{n \to \infty} d(T^n(x), T^n(y)) = 0 \right\},
\]

\[
W^u(x) = \left\{ y : \lim_{n \to \infty} d(T^{-n}(x), T^{-n}(y)) = 0 \right\},
\]

with \( d \) a metric on \( X \), then we have a nice foliation of \( X \). In fact

\[
W^s(x) \cap W^s(y) \neq \emptyset \quad \Rightarrow \quad W^s(x) = W^s(y),
\]

\[
W^u(x) \cap W^u(y) \neq \emptyset \quad \Rightarrow \quad W^u(x) = W^u(y),
\]

\[
E^u_x = \mathcal{T}_x W^u(x), \quad E^s_x = \mathcal{T}_x W^s(x).
\]

We also have a simple formula for the topological entropy:

\[
h_{\text{top}}(T) = \int \log \| (dT)_x \| \, \mu(dx) = \sum \lambda_i \log \lambda_i,
\]

where \( (dT)_x \) denotes the restriction of \( (dT)_x \) to \( E^u_x \). An obvious example of an Anosov transformation is the Arnold cat transformation.

In the next section we study the Lyapunov exponents for Hamiltonian systems. As a prelude, we show that the Lyapunov exponents for a Hamiltonian flow come in a pair of numbers of opposite signs.

In the case of a Hamiltonian system, we have a \textit{symplectic transformation} \( T : X \to X \). This means that \( X \) is equipped with a \textit{symplectic form} \( \omega \) and if \( A(x) = (dT)_x \), then

\[
\omega_x(a, b) = \omega_{T(x)}(A(x)a, A(x)b).
\]

By a symplectic form we mean a closed non-degenerate 2-form. As is well-known, \( \dim X = 2d \) is always even, and the volume form associated with \( \omega \) (namely the \( d \)-wedge product of \( \omega \)) is preserved under \( T \). An example of a symplectic manifold is \( X = \mathbb{R}^{2d} \) that is equipped with the standard form \( \tilde{\omega} \): \( \omega_x(a, b) = \tilde{\omega}(a, b) \) with \( \tilde{\omega}(a, b) =Ja \cdot b \), and

\[
J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \]

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where $I$ is the $d \times d$ identity matrix. In this case we may represent $A$ as a matrix and the symplectic property means

$$A(x)^t J A(x) = J.$$  

As is well-known, this in particular implies that $\det A(x) = 1$. Of course we already know this for Hamiltonian systems by Liouville’s theorem, namely the volume is invariant under a Hamiltonian flow.

**Theorem 4.6** Let $(X, \omega)$ be a closed symplectic manifold of dimension $2d$. Then the Lyapunov exponents $l_1 < l_2 < \cdots < l_k$ satisfy $l_j + l_{k-j+1} = 0$ and $n_j = n_{k-j+1}$ for $j = 1, 2, \ldots, \lfloor k/2 \rfloor$. Moreover the space $F_x^j := \bigoplus_{i=1}^j E_x^i$ is $\omega$-orthogonal complement of $E_x^{k-j+1}$.

**Proof.** Write $l(x,v) = \lim_{n \to \infty} \frac{1}{n} \log |A_n(x)v|$ where $A_n(x) = (dT^n)_x$ and $v \in T_x X$. Note that since $X$ is compact, we can find a constant $c_0$ such that

$$|\omega_x(a, b)| \leq c_0 |a||b|$$

for all $a, b \in T_x M$ and all $x \in M$. As a result,

$$|\omega_x(a, b)| = |\omega_{T^n(x)}(A_n(x)a, A_n(x)b)| \leq c_0 |A_n(x)a||A_n(x)b|,$$

and if $\omega_x(a, b) \neq 0$, then

$$l(x,a) + l(x,b) \geq 0.$$  

(4.36)

By Theorem 4.4, we can find numbers $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_{2d}$ and spaces

$$\{0\} = V_0 \subseteq V_1(x) \subseteq \cdots \subseteq V_{2d-1}(x) \subseteq V_{2d}(x) = T_x X$$

such that $\dim V_j(x) = j$ and if $v \in V_{j+1}(x) \setminus V_j(x)$, then $l(x,v) = \beta_j$. Of course $l_1 < \cdots < l_k$ are related to $\beta_1 \leq \cdots \leq \beta_{2d}$ by $\{l_1, \ldots, l_k\} = \{\beta_1, \ldots, \beta_{2d}\}$ and $n_j = \#\{s: \beta_s = l_j\}$. Note that if $W$ is a linear subspace of $T_x M$ and

$$W^\perp = \{ b \in T_x X : \omega(a, b) = 0 \text{ for all } a \in W \},$$

then one can readily show that $\dim W + \dim W^\perp = 2d$. As a result, we can use $\dim V_j + \dim V_{2d-j+1} = 2d + 1$ to deduce that there exist $a \in V_j$ and $b \in V_{2d-j+1}$ such that $\omega(a, b) \neq 0$. Indeed the set

$$\Lambda = \{ (a,b) \in (T_x X)^2 : a \in V_j, \ b \in V_{2d-j+1}, \ \omega_x(a, b) \neq 0 \}$$

is a nonempty open subset of $V_j \times V_{2d-j+1}$. Hence

$$\tilde{\Lambda} = \{ (a,b) \in (T_x X)^2 : a \in V_j, \ b \in V_{2d-j+1}, \ \omega_x(a, b) \neq 0 \}.$$

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is also nonempty. As a result, we can use (4.36) to assert
\[ \beta_j + \beta_{2d-j+1} \geq 0, \]
for \( j \in \{1, 2, \ldots, d\} \). On the other hand
\[ \sum_{j=1}^{d} (\beta_j + \beta_{2d-j+1}) = \sum_{i} n_i l_i = 0, \]
by (4.14) because the volume is preserved. From this and (4.37) we deduce that
\[ \beta_j + \beta_{2d-j+1} = 0. \]

From this we can readily deduce that
\[ l_j + l_{k-j+1} = 0 \] and \( n_j = n_{k-j+1} \).
For the last claim, observe that since \( l_j + l_{k-j+1} = 0 \), we have \( l_j + l_i < 0 \) whenever \( i+j \leq k \). From this and (4.31) we learn that if \( i+j \leq k \) and \((a, b) \in E_x^i \times E_x^j\), then \( \omega_x(a, b) = 0 \).

We end this section by describing an approach that would allow us to approximate \( E_u \) and \( E_s \). Recall that \( (dT_n) X E_x \subset E_x T_n(x) \). For simplicity, let us assume that \( d = 2 \), so that if we have two Lyapunov exponents \( l_1 < 0 < l_2 \), then both \( E_x^u \) and \( E_x^s \) are straight lines. Now imagine that we can find a sector \( C_x \) such that
\[ E_x^u \subset C_x, \quad E_x^s \cap C_x = \{0\}, \quad (dT)_x C_x \subset C_T(x). \]
(Recall that the \( E_x^a \) component of each vector \( v \in C_x \) shrinks under the transformation \( A(x) = (dT)_x \).) In other words, \( A(x) C_x \) is a slimmer sector than \( C_T(x) \). As we repeat this, we get a very slim sector \((dT^n)_x C_x\) at the point \( T^n(x) \) that is approximating \( E_x T^n(x) \). To approximate \( E_x^u \), we may try
\[ C_x^m := (dT^n)_{T^{-n}(x)} C_{T^{-n}(x)}. \]

Observe
\[ C_x^{n+1} = (dT^{n+1})_{T^{-n-1}(x)} C_{T^{-n-1}(x)} = (dT^n)_{T^{-n}(x)} (dT)_{T^{-n-1}(x)} C_{T^{-n-1}(x)} \]
\[ \subset (dT^n)_{T^{-n}(x)} C_{T^{-n}(x)} = C_x^{(n)}. \]
From this we expect 

\[ E^u_x = \cap_{n=0}^{\infty} C^u_x. \]

Similarly, if we can find a sector \( C'_x \) such that 

\[ E^s_x \subset C'_x, \quad E^u_x \cap C'_x = \{0\}, \quad (dT^{-1})_x C^s_x \subset C_{T^{-1}(x)}, \]

then we expect to have 

\[ E^s_x = \cap_{n=0}^{\infty} C^{-n}_x, \]

where 

\[ C^{-n}_x := (dT^{-n})_{T^n(x)} C_{T^n(x)}. \]

The existence of such sectors also guarantee that we have nonzero Lyapunov exponents. To see how this works in principle, let us examine an example.

**Example 4.3** Consider a matrix-valued function \( A(x), x \in \mathbb{T}^2 \) such that for almost all \( x, A \) has positive entries and \( \det A(x) = 1 \). Let \( T : \mathbb{T}^2 \to \mathbb{T}^2 \) be invariant with respect to the Lebesgue measure \( \mu \) and define \( l(x,v) = \lim_{n \to \infty} \frac{1}{n} \log |A_n(x)v| \), where 

\[ A_n(x) = A(T^{n-1}(x))A(T^{n-2}(x)) \cdots A(T(x))A(x). \]

Define the sector \( C(x) \equiv C = \left\{ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} : v_1 v_2 > 0 \right\} \). Note that if \( \begin{bmatrix} v'_1 \\ v'_2 \end{bmatrix} = A(x) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \), and 

\[ A(x) = \begin{bmatrix} a(x) & b(x) \\ c(x) & d(x) \end{bmatrix}, \]

then 

\[ Q(v'_1, v'_2) = v'_1 v'_2 = (av_1 + bv_2)(cv_1 + dv_2) = (1 + 2bc)v_1 v_2 + acv_2^2 + bdv_2^2 \geq (1 + 2bc)v_1 v_2 + 2\sqrt{acbd} \ v_1 v_2 =: \lambda Q(v_1, v_2). \]

We can also show that \( A \) maps \( C \) onto a sector which lies strictly inside \( C \). If 

\[ A_n(x) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1^{(n)} \\ v_2^{(n)} \end{bmatrix}, \]

then 

\[ |A_n(x)v|^2 \geq 2v_1^{(n)} v_2^{(n)} \geq 2v_1 v_2 \prod_{i=0}^{n-1} \left[ \lambda(T^i(x)) \right], \]

\[ \lim \inf_{n \to \infty} \frac{1}{n} \log |A_n(x)v| \geq \frac{1}{2} \int \log \lambda \ d\mu =: \bar{l} > 0, \]

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whenever \( v_1 v_2 > 0 \). This implies that \( l_2 > 0 \). Since \( \det A_n \equiv 1 \), we know that \( l_1 + l_2 = 0 \). So \( l_1 < 0 < l_2 \). As for \( E_x^u \), let us write \( \theta_n(x) \) for the slope of the lower boundary of \( C_x^u \). Since

\[
C_x^{n+1} = A(T^{-1}(x))C_x^{n(T-1(x))},
\]

we learn

\[
\theta_{n+1}(x) = F(T^{-1}(x), \theta_n(T^{-1}(x))),
\]

where

\[
F(x, \theta) = \frac{c(x) + d(x)\theta}{a(x) + b(x)\theta} = \frac{c(x)}{a(x)} + \frac{1}{a(x) b(x) + a(x)\theta - 1}.
\]

This follows from the fact that if \( \theta = v_2/v_1 \) and \( \theta' = v'_2/v'_1 \), then \( \theta' = F(x, \theta) \). The slope of \( E_x^u \) is given by

\[
\theta^* := \lim_{n \to \infty} \theta_n(x),
\]

with \( \theta_0(x) = 0 < \theta_1(x) = c(x)/a(x) \). The last expression of (4.39) may be used to give an expression for \( \theta^* \) as a continued fraction. Observe that the sequence \( \theta_n \) is strictly increasing with \( \theta_n < (d/b) \circ T^{-1} \) for all \( n \). The latter follows from the fact that \( F \) is increasing in \( \theta \): If we already know that \( \theta_{n-1} < \theta_n \), then

\[
\theta_{n+1}(x) = F(T^{-1}(x), \theta_n(T^{-1}(x))) < F(T^{-1}(x), \theta_{n-1}(T^{-1}(x))) = \theta_n(x).
\]

The bound \( \theta_n < (d/b) \circ T^{-1} \) is an immediate consequence of the monotonicity \( F(x, \theta) < F(x, \infty) = d/b \). We may use the whole past history \( \{T^{-n}(x) : n \in \mathbb{N}\} \) to express \( \theta^* \) as a continued fraction

\[
\theta^* = A_1 + \frac{1}{B_1 + \frac{C_1}{A_2 + \frac{1}{B_2 + \frac{C_2}{A_3 + \frac{1}{\ddots}}}}},
\]

where

\[
A_i = \frac{c}{a} \circ T^{-i}, \quad B_i = (ab) \circ T^{-i}, \quad C_i = a^2 \circ T^{-i}.
\]

In the continuous case the Lyapunov exponents are defined likewise. Consider a group of \( C^1 \)-transformations \( \{\phi_t : t \in \mathbb{R}\} \). Here each \( \phi_t \) is from an \( m \)-dimensional manifold \( M \) onto
itself. We then pick an ergodic measure \( \mu \in \mathcal{I} \) and find a splitting \( T_x M = E^1_x \oplus \cdots \oplus E^k_x \) such that for \( v \in E^1_x \oplus \cdots \oplus E^j_x \setminus E^1_x \oplus \cdots \oplus E^{j-1}_x \),

\[
\lim_{t \to \infty} \frac{1}{t} \log |(d\phi_t)_x v| = l_j.
\]

It turns out that we always have a zero Lyapunov exponent associated with the flow direction. More precisely, if \( \frac{d}{dt} \phi_t(x)|_{t=0} = \xi(x) \), then

\[
\lim_{t \to \infty} \frac{1}{t} \log |(d\phi_t)_x \xi| = 0.
\]

Intuitively this is obvious because two phase points that lie close to each other on the same trajectory do not separate exponentially.

**Exercise (i)** Verify the following properties of \( l(x,v) \) of Remark 4.1(iii) directly (without using Theorem 4.3):

(i) \( l(x, \alpha v_1) = l(x, v_1) \), \( l(x, v_1 + v_2) \leq \max(l(x, v_1), l(x, v_2)) \) for every \( x \), \( v_1 \), and \( v_2 \) and scalar \( \alpha \neq 0 \).

(ii) \( l(T(x), A(x)v) = l(x, v) \)

(iii) We have \( l(x, v) \in [-\infty, +\infty) \).

(iv) The space \( \{v : l(x, v) \leq t\} = V_x(t) \) is linear and that \( V_x(s) \subseteq V_x(t) \) for \( s \leq t \), \( A(x)V_x(t) \subseteq V_{T(x)}(t) \).

(v) There exists \( k(x) \in \mathbb{N} \), numbers \( l_1(x) < l_2(x) < \cdots < l_{k(x)}(x) \) and splitting \( T_x M = E^1_x \oplus \cdots \oplus E^{k(x)}_x \) such that if \( v \in E^1_x \oplus \cdots \oplus E^j_x \setminus E^1_x \oplus \cdots \oplus E^{j-1}_x \) then \( l(x, v) = l_j \). Indeed \( E^1_x \oplus \cdots \oplus E^j_x = V_x(l_j) \).

(ii) Prove Parts (i) and (v) of Proposition 4.1. \( \square \)
5 Lorentz Gases and Billiards

So far we have discussed various statistical notions such as ergodicity, entropy and Lyapunov exponents for dynamical systems. We have examined these notions for a rather limited number of examples, namely toral automorphisms, translations (or free motions) and one-dimensional expansions. In this section we study examples coming from classical mechanics. A Lorentz gas is an example of a gas in which heavy molecules are assumed to be immobile and light particles are moving under the influence of forces coming from heavy particles. The dynamics of a light particle with position $q(t)$ is governed by the Newton's law

$$\frac{d^2q}{dt^2} = -\nabla V(q),$$

where $V(q) = \sum_j W(|q - q_j|)$ with $q_j$ denoting the center of immobile particles and $W(|z|)$ represents a central potential function. For simplicity we set the mass of the light particle to be 1. We may rewrite (5.1) as

$$\begin{align*}
\frac{dq}{dt} &= p, \\
\frac{dp}{dt} &= -\nabla V(q).
\end{align*}$$

Recall that the total energy $H(q,p) = \frac{1}{2}|p|^2 + V(q)$ is conserved. Because of this, we may wish to study the ergodicity of our system restricted to an energy shell

$$\{(q,p) : H(q,p) = E\}.$$  

When $W$ is of compact support, we may simplify the model by taking

$$W(|z|) = \begin{cases} 0 & \text{if } |z| > \varepsilon, \\ \infty & \text{if } |z| \leq \varepsilon. \end{cases}$$

To interpret (5.1) for $W$ given by (5.2), let us first assume that the support of $W(|q - q_i|), i \in \mathbb{Z}$ are non overlapping. Assume a particle is about to enter the support of $W(||q - q_i||)$. For such a scenario, we may forget about other heavy particles and assume that the potential energy is simply given by $W(|q - q_i|)$. For such a potential we have two conservation laws:

conservation of energy: $\frac{d}{dt} \left( \frac{1}{2}|p|^2 + V(|q - q_i|) \right) = 0$

conservation of angular momentum: $\frac{d}{dt} p \times (q - q_i) = 0$.

Let us assume that a particle enters the support at a position $q$ with velocity $p$ and exits the support at a position $q'$ with velocity $p'$. For a support choose a ball of center $q_i$ and diameter $\varepsilon$. If $n = \frac{q - q_i}{|q - q_i|}$ and $n' = \frac{q' - q_i}{|q' - q_i|}$, then we can use the above conservation laws to
conclude that $|p'| = |p|$ and the angle between $(p, n)$ is the negation of the angle between $(p', n')$.

The same conservation laws hold for the case (5.2). We are now ready for interpretation of dynamics when $W$ is given by (5.2). Draw a ball of diameter $\varepsilon$ and center $q_i$ for each $i$. Then the phase space is

$$X = \left(\mathbb{R}^d \setminus \bigcup_i B_{\varepsilon/2}(q_i)\right) \times \mathbb{R}^d = \{(q, p) : |q - q_i| \geq \varepsilon \text{ for all } i, \text{ and } p \in \mathbb{R}^d\}.$$  

For $q \notin \partial X$ we simply have $\frac{dq}{dt} = p$. When $|q - q_i| = \varepsilon$ then the dynamics experiences a jump discontinuity in $p$-component. More precisely

$$|q(t) - q_i| = \varepsilon \quad \text{implies} \quad p(t+) = p(t-) - 2p(t-) \cdot n_i(t)n_i(t),$$

where $n_i(t) = \frac{q(t) - q_i}{|q(t) - q_i|}$. As our state, we may consider

$$X = \{q : |q - q_i| \geq \varepsilon \text{ for all } i\} \times \{p : |p| = 1\} =: Y_{\varepsilon} \times S^{d-1}.$$  

Classically two possibilities for the configurations of $q_i$’s are considered. As the first possibility, imagine that the $q_i$’s are distributed periodically with period 1. Two cases may occur: Either $\varepsilon < 1$ which corresponds to an infinite horizon because a light particle can go off to infinity; or $\varepsilon \geq 1$ which corresponds to a finite horizon.

As our second possibility we distribute $q_i$’s randomly according to a Poissonian probability distribution.

In this section we will study Lorentz gases on tori. In the periodic case of an infinite horizon, we simply have a dynamical system with phase space

$$M = \left(\mathbb{T}^d \setminus B_{\varepsilon}\right) \times S^{d-1} =: Y_{\varepsilon} \times S^{d-1},$$

where $\mathbb{T}^d \setminus B_{\varepsilon}$ represents a torus from which a ball of radius $\varepsilon/2$ is removed. In the case of finite horizon our $M = Y_{\varepsilon} \times S^{d-1}$ but now $Y_{\varepsilon}$ is a region confined by 4 concave arcs. In the random case we may still restrict the dynamics to a torus. For example, we select $N$ points $q_1, \ldots, q_j$ randomly and uniformly from the set

$$X_{\varepsilon} = \{(q_1, \ldots, q_N) : |q_i - q_j| > \varepsilon \text{ for } i \neq j\},$$

and then we set

$$Y_{\varepsilon} = \{q : |q - q_i| \geq \varepsilon \text{ for } i = 1, \ldots, N\}.$$  

A Lorentz gas can be regarded as an example of a billiard. For the sake of definiteness let us focus on the case of finite horizon that can be recast as a billiard in a bounded domain
with piecewise smooth boundary. More generally, let us take a bounded region $Y$ in $\mathbb{R}^d$ with piecewise smooth boundary and set $X = \bar{Y} \times \mathbb{S}^{d-1}$. We assume that $Y$ is closed and set

$$\partial^\pm X = \{(q,p) : q \in \partial Y, \pm(p \cdot \nu(q)) \geq 0\},$$

where $\nu(q)$ represents the unit outward normal to $\partial Y$ at $q$. Points in $\partial^- X$ and $\partial^+ X$ represent the pre and post collisional states respectively in our billiard. We now define two closely related dynamical systems.

(i) A continuous dynamical system $\phi_t(q,p) = (q(t),p(t))$, that is defined in the following way: so long as $q(t) \in Y^o = Y \setminus \partial Y$, we have

$$\frac{dq}{dt}(t) = p(t), \quad \frac{dp}{dt}(t) = 0.$$

However, when $q(t)$ reaches a boundary point, $p(t)$ experiences a jump discontinuity. More precisely, whenever $q(t) \in \partial Y$ with $p(t) \cdot \nu(q(t)) \leq 0$, then

$$p'(t) := p(t+) = R_{q(t)} p(t-) := p(t-) - 2(p(t-) \cdot \nu(q(t))) \nu(q(t)).$$

(ii) A discrete dynamical system on $\partial^+ X$ associated with a map $T : \partial^+ X \to \partial^+ X$. The map $T(q,p) = (Q,P)$ is defined by the following recipe:

$$Q = q + \tau(q,p)p, \quad P = p - 2(p \cdot \nu(Q)) \nu(Q),$$

where $\tau(q,p)$ is the smallest $\tau > 0$ such that $\phi_{\tau}(q,p)$ reaches the boundary.

Next we find an invariant measure for the dynamical system $(q(t),p(t))$ and the map $T$. Regarding the flow $\phi_t$ as a Hamiltonian flow, we expect that the normalized Lebesgue measure $m(dx) = Z^{-1} dq \, dp$ where $Z$ is a normalizing constant, $dq$ is the Lebesgue measure on $Y$, and $dp$ is the standard volume measure on $\mathbb{S}^{d-1}$ (compare with Example 1.7(ii)). To prove the invariance of $m$, let us take a smooth test function $J : X \to \mathbb{R}$ such that $J(q, R_q p) = J(q,p)$ whenever $(q,p) \in \partial^- X$. Such a test function produces

$$(T_t J)(x) = u(x,t) = J(\phi_t(x)),$$

that is continuous in $(x,t)$. In fact $u$ satisfies a Liouville-type equation with boundary conditions:

$$\begin{aligned}
\begin{cases}
u = p \cdot u, & x \in X \setminus \partial X; \\
u(q, R_q p, t) = u(q,p,t), & t \geq 0, \ (q,p) \in \partial^- X.
\end{cases}
\end{aligned}$$

We expect (5.4) to be true weakly; if $K$ is a smooth function, then the expression

$$-\frac{d}{dt} \int_X u(x,t)K(x) \, dqdp,$$
equals

(5.5) \[ \int_X u(x,t)p \cdot K_q(x) \ dq \ dp + \int_{\partial X} u(x,t)K(x)(p \cdot \nu(q)) \ \sigma(dq)dp, \]

where \( \sigma(dq) \) represents the surface integration on \( \partial Y \). To verify this, let us write

\( \tau_0(x) = 0 < \tau_1(x) < \tau_2(x) < \ldots \)

for a sequence of functions, such that \( \phi_t(x) \in X \setminus \partial X \) for \( t \in (\tau_j(x), \tau_{j+1}(x)) \), \( \phi_{\tau_j(x)}(x) \in \partial X \) if \( j > 0 \), and each finite interval \([0,T]\) can have only finitely many \( \tau_i \)'s. Let us explain this further.

Note that \( u(x,t) = J(\phi_t(x)) \) is as smooth as \( J \) in \((x,t)\) provided that \( \phi_t(x) \notin \partial X \). This means that \( u \) is as smooth as \( J \) with \( u_t = p \cdot u_q \), provided \((x,t) \in X \times (0,\infty) \setminus \bigcup_j S_j \), where

\[ S_j = \{(x,t) : \tau_j(x) = t\}. \]

Note that when \( t \) is restricted to a finite interval \([0,T]\), then finitely many \( S_j \)'s are relevant, each \( S_j \) is of codimension 1 in \( X \times (0,T) \), and different \( S_j \)'s are well-separated. It is a general fact that if \( u \) is continuous and \( u_t = p \cdot u_q \) off \( \bigcup_j S_j \), then \( u_t = p \cdot u_q \) weakly in \( X \). To see this, take a test function \( R(x,t) \) with support in an open set \( U \) such that exactly one of the \( S_j \)'s bisect \( U \) into \( U^+ \) and \( U^- \). We then have \( \int u(R_t - p \cdot R_q)dx \ dt = \int_{U^+} + \int_{U^-} \) and that if we integrate by parts on each \( U^\pm \) we get two contributions. One contribution comes from carrying out the differentiation on \( u \), i.e., \( \int_{U^\pm} (-u_t + p \cdot u_q)R \ dx \ dt \), which is 0 because \( u_t = p \cdot u_q \) in \( U^\pm \). The other contribution comes from the boundary of \( U^\pm \), and they cancel each other out by the continuity of \( u \).

As a consequence of (5.4) we have that the Lebesgue measure the measure \( m \) is invariant. In fact if initially \( x \) is selected according to a probability measure \( d\mu = f^0(x)dx \), then at later times \( x(t) \) is distributed according to \( d\mu_t = f(x,t)dx \) where \( f(x,t) = f^0(\phi_{-t}(x)) \). To see this observe that if we choose \( K \equiv 1 \) in (5.5), we have

(5.6) \[ \frac{d}{dt} \int J(\phi_t(x))dx = - \int_{\partial X} u(q,p)(p \cdot \nu(q)) \ \sigma(dq)dp. \]

If we integrate over \( p \) first and make a change of variable \( p \mapsto p' = p - 2p \cdot n \ n \), for \( n = \nu(q) \), then \( u \) does not change and \( p \cdot n \) becomes \( p' \cdot n = -p \cdot n \). Also the Jacobian of such a transformation is 1. As a result, the right-hand side of (5.7) is equal to its negation. This implies

(5.7) \[ \int J(\phi_t(x))dx = \int J(x)dx, \]

for every \( t \) and every \( J \) continuous with \( J(q,p') = J(q,p) \) on \( \partial X \). If \( K \) and \( f^0 \) have the same property and we choose

\[ J(x) = f^0(\phi_{-t}(x))K(x), \]

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then we deduce
\[
\int K(x)f^0(\phi_t(x))dx = \int K(\phi_t(x))f^0(x)dx.
\]
From this we conclude
\[
(5.8) \quad f(x,t) = f^0(\phi_t(x)),
\]
as was claimed before.

**Remark 5.1** If \(\phi_t\) is the flow of the ODE (5.1), and \(f(x,t) = f^0(\phi_t(x))\), then the function \(f\) satisfies the Liouville’s equation
\[
f_t + p \cdot f_q - \nabla V(q) \cdot f_p = 0.
\]
The partial derivatives \(\alpha = f_q\) and \(\beta = f_p\) satisfy
\[
\begin{align*}
\alpha_t + \alpha_q p - \alpha_p \nabla V(q) &= D^2 V(q) \beta, \\
\beta_t + \beta_q p - \beta_p \nabla V(q) &= -\alpha.
\end{align*}
\]
Note that if \(Q(x,t) = f_q(x,t) \cdot f_p(x,t) = \alpha(x,t) \cdot \beta(x,t)\), then
\[
Q_t + p \cdot Q_q - \nabla V(q) \cdot Q_p = D^2 V(q) \beta \cdot \beta - |\alpha|^2
\]
or equivalently
\[
\bar{Q}_t = D^2 V(q(t)) \bar{\beta} \cdot \bar{\beta} - |\bar{\alpha}|^2
\]
for \(\bar{Q}(x,t) = Q(\phi_t(x),t)\). In the case of a billiard, the function \(f\) satisfies
\[
\begin{align*}
\begin{cases}
f_t + p \cdot f_q = 0 & \text{inside } Y \times \mathbb{R}^d, \\
f(q,p,t-) = f(q,p',t+) & \text{on } \partial Y \times \mathbb{R}^d,
\end{cases}
\end{align*}
\]
where \(q \in \partial Y\) and \(t\) is a collision time. Setting \(a(q,p,t) = f_q(q,p,t), b(q,p,t) = f_p(q,p,t)\), we then have
\[
\begin{align*}
a_t + pD_q a &= 0, \\
b_t + pD_p a &= -a.
\end{align*}
\]
Later in this chapter we will learn how to relate \(\alpha(q,p,t-)\) to \(\beta(q,p',t+)\) on the boundary \(\partial X\) as we study the evolution of \(d\phi_t\). □

Identity (5.7) suggests that the invariant measure \(m\) of \(\phi_t\) induces an invariant measure
\[
d\mu = (p \cdot \nu(q)) \sigma(dq)dp,
\]
on $\partial^+ X$. To explain this, let us is let us define

$$\hat{X} = \{(x, t) = (q, p, t) : (q, p) \in \partial^+ X, \ 0 \leq t < \tau(q, p)\},$$

and $F : \hat{X} \to X$ by $F(x, t) = \phi_t(x)$. It is not hard to see that $F$ is invertible. In fact $F$ is an automorphism between the measure spaces $(X, dm)$ and $(\hat{X}, \mu(dx)dt)$. This simply follows from the fact that the Jacobian of the transformation $(q, t) \mapsto q + pt$, equals $p \cdot \nu(q)$. The transformation $F$ provides us with a useful representation of points in $X$. Using this representation we can also represent our dynamical system in a special form that is known as special flow representation. Let us study $F^{-1} \circ \hat{\phi}_\theta \circ F$:

$$\hat{\phi}_\theta := F^{-1} \circ \hat{\phi}_\theta \circ F(x, t) = \begin{cases} (x, \theta + t) & \theta + t < \tau(x) \\ (T(x), \theta + t - \tau_1(y)) & \theta + t - \tau(x) < \tau(T(x)) \end{cases}$$

The measure $\mu(dx)dt$ is an invariant measure for the flow $\hat{\phi}_\theta$. We choose $A$ sufficiently small in diameter so that we can find $\theta_1, \theta_2$ and $\theta_3$ with the following property:

$$t \in [\theta_1, \theta_2] \Rightarrow \tau(x) < \theta_3 + t < \tau(T(x))$$

for every $x \in A$. This means

$$\hat{\phi}_\theta(A \times [\theta_1, \theta_2]) = \{(T(x), \theta_3 + t - \tau_1(y)) : y \in A, \ t \in [\theta_1, \theta_2]\}.$$ 

Since $\hat{\phi}_\theta$ has $d\mu dt$ for an invariant measure,

$$(\theta_2 - \theta_1)\mu(A) = (\theta_2 - \theta_1)\mu(T(A)).$$

Since $T$ is invariant, we deduce that $\mu$ is invariant.

We say a billiard table $Y$ is dispersive if there exists $\delta > 0$ such that

$$(d\nu)_q \geq \delta I,$$

for every $q \in \partial Y$ at which $\nu$ is differentiable. As we will see below all dispersive billiards have some positive Lyapunov exponents. This was shown by Sinai when $d = 2$, and by Chernov and Sinai when $d \geq 3$.

**Theorem 5.1** Let $Y$ be a dispersive billiard table satisfying (5.10).

(i) There exists a $T$-invariant function $g \geq 0$ with $\int g \ d\mu > 0$, such that for $\mu$-almost every $(q, p) \in \partial^+ X$ and any $(\hat{q}, \hat{p}) \in T_{(q,p)} \partial^+ X$, with $\hat{q} \cdot \hat{p} > 0$, we have

$$\liminf_{n \to \infty} n^{-1} \log |(dT^n)_{(q,p)}(\hat{q}, \hat{p})| \geq g(q, p).$$
(ii) There exists a $\phi$-invariant function $h \geq 0$ with $\int h \, dm > 0$, such that for $m$-almost every $(q, p) \in X$ and any $(\hat{\xi}, \hat{\pi}) \in \mathbb{R}^{2d}$, with $p \cdot \hat{\pi} = 0$ and $\hat{\xi} \cdot \hat{\pi} > 0$, we have

$$\liminf_{t \to \infty} t^{-1} \log \left| (d\phi_t)(q, p)(\hat{\xi}, \hat{\pi}) \right| \geq h(q, p).$$

Note that since $\phi_t$ is discontinuous, it is not clear that $d\phi_t$ and $dT$ are well-defined. As we will see in Proposition 5.1 both $d\phi_t$ and $dT$ can be defined almost everywhere with respect to the invariant measure $m$ and $\mu$ respectively. In fact we have a very precise meaning for the flow $d\phi_t$ that will be described shortly.

Recall that if $\phi_t$ is the flow associated with an ODE of the form $\frac{dx}{dt} = b(x)$, then the matrix-valued function $(d\phi_t)_x$ solves

$$\frac{dA}{dt} = [(d\phi_t)_x b] A.$$

Hence, $\dot{x}(t) = A(x, t) \dot{x}$ solves the equation

$$\frac{d\dot{x}}{dt}(t) = B(x, t) \dot{x}(t),$$

where $B(x, t) = (d\phi_t)_x b(x)$. In the case of a Hamiltonian flow of the form (5.2), we have $b(q, p) = (p, -\nabla V(q))$ and we simply that $\dot{x}(t) = (\dot{q}(t), \dot{\pi}(t))$ solves

$$\frac{d\dot{q}}{dt}(t) = \dot{\pi}(t), \quad \frac{d\dot{\pi}}{dt}(t) = -D^2 V(q(t)) \dot{q}(t).$$

For our billiard model, some care is needed because $\phi_t(x)$ is not even continuous. We wish to derive an evolution equation for

$$\dot{x}(t) = (d\phi_t)_x (\dot{x}(0)).$$

We think of $(x, \dot{x}) \in TX$ as the initial data for the path

$$(\phi_t(x), (d\phi_t)_x(\dot{x})).$$

For our purposes, we take a path $(x^*(\theta) : \theta \in (-\delta_0, \delta_0))$ with $x^*(0) = x$, $\dot{x}^*(0) = \dot{x}$, and keep track of

$$(5.13) \quad \dot{x}(t) = \dot{\phi}_t(\dot{x}) := x^*_\theta(t, 0).$$

where $x^*(t, \theta) = \phi_t(x^*(\theta))$. We use (5.13) to define $\dot{x}(t)$. In the same fashion, we define $dT$:

Take a path $x^*(\theta) = (q^*(\theta), p^*(t\theta))$, that lies on $\partial^+ X$, such that $q^*(0) = q, p^*(0) = p$, and
\[ \dot{q}(0) = \dot{q} \in \partial Y, \dot{p}(0) = \dot{p} \in p^\perp, \] and if we write \( T(q, p) = (Q, P) \), then \( Q = q + \tau(q, p)p, \) and \( P = R_Q p. \) We then set \((Q(\theta), P(\theta)) = T(x^*(\theta))\), and define

\[ (dT)_{(q, p)}(\dot{q}, \dot{p}) := (\dot{Q}(0), \dot{P}(0)). \]

In analogy with the Riemannian geometry, we may regard \( \dot{x}(t) \) as the Jacobi field associated with \( x(t) \), and we wish to derive the corresponding Jacobi’s equation. This will be achieved in the following Proposition.

**Proposition 5.1 (i)** Let \( \dot{x}(t) = (\dot{q}(t), \dot{p}(t)) \) be as in (5.13). Then in between collisions, we simply have

\[ \frac{d\dot{q}}{dt} = \dot{p}, \quad \frac{d\dot{p}}{dt} = 0. \]

Moreover, at a collision, the precollisional coordinates \((q, p, \dot{q}, \dot{p})\), with \((q, p) \in \partial^- X\), become \((q, p', \dot{q}', \dot{p}')\) right after collision, with \( \dot{q}' = R_q \dot{q}, \) and

\[ \dot{p}' = R_q \dot{p} + 2(p \cdot \nu(q)) - (R_q V^t_{q,p}(d\nu)_q V_{q,p}) \dot{q}, \]

where

\[ R_q = I - 2\nu(q) \otimes \nu(q), \quad V_{q,p} = I - \frac{p \otimes \nu(q)}{p \cdot n}. \]

**Proof (i)** If we take a path \((x^*(\theta) : \theta \in (-\delta_0, \delta_0))\) with \( x^*(0) = x, \dot{x}^*(0) = \dot{x} \), and write \((q^*(t, \theta), p^*(t, \theta))\), for \( \dot{\phi}_\theta(x^*(\theta)) \), then in between collisions, \( p^*(t, \theta) = p^*(\theta) \) does not change, and \( q^*(t, \theta) = q^*(\theta) + tp^*(\theta) \). This implies (5.15).

For the dynamics at a collision, let us write \( \tau(\theta) \), for the first time \( q^*(t, \theta) \) reaches the boundary of \( Y \). Without loss of generality, we may assume that \( \tau(0) = 0 \) and that \( \tau(\theta) > 0 \) for \( \theta > 0 \). We also write

\[ Q(\theta) = q^*(\tau(\theta), \theta) = q^*(\theta) + \tau(\theta)p^*(\theta), \]

for the hitting location at time \( \tau \), and \( n^*(\theta) = \nu(Q(\theta)) \), for the unit normal at the collision. Differentiating these equations and evaluating the derivatives at \( \theta = 0 \) yield

\[ \dot{Q} = \dot{q} + \dot{\tau} \dot{p}, \quad \dot{n} = (d\nu)_q \dot{Q}, \]

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where $\hat{\tau}$ and $\hat{Q}$ denote $\dot{\tau}(0)$ and $\dot{Q}(0)$ respectively. Since the path $(Q(\theta) : \theta \in (-\delta_0, \delta_0))$ lies on $\partial Y$, we have $n \cdot \hat{Q} = 0$, where $n = n(0)$. From this and (5.21) we learn

\begin{equation}
(5.19) \quad \hat{\tau} = -\frac{\hat{q} \cdot n}{p \cdot n}, \quad \hat{Q} = V_{q,p} \hat{q} = \left( I - \frac{p \otimes n}{p \cdot n} \right) \hat{q}, \quad \hat{n} = (d\nu)_q V \hat{Q}.
\end{equation}

We are now ready to determine $\hat{p}'$ and $\hat{q}'$.

Observe that for $\theta > 0$ and $t > \tau(\theta)$,

$q^*(t, \theta) = Q(\theta) + (t - \tau(\theta)) \hat{q} - \tau(\theta) p^*(\theta) + tp^*(\theta) := b(\theta) + tp^*(\theta).$

As we differentiate with respect to $\theta$, and set $\theta = 0$, we deduce,

$\hat{q}(t) = \dot{b}(0) + t \hat{p}'.

We learn from this $\hat{q}' = \dot{b}(0) = \hat{Q} - \hat{n} \cdot \hat{p}' = \hat{q} - \hat{\tau} \hat{p}' = \hat{q} - \hat{\tau}(p - 2(p \cdot n)n) = \hat{q} - 2(\hat{q} \cdot n)n.$

Hence $\hat{q}' = R_q \hat{q}$.

We now turn our attention to $\hat{p}'$. From differentiating

$p^*(\theta) = Rp^*(\theta) := (I - 2n(\theta) \otimes n(\theta)) p^*(\theta),

and evaluating the derivative at $\theta = 0$, we arrive at

$\hat{p}' = R \hat{p} - 2(\hat{n} \otimes n + n \otimes \hat{n}) p.$

On the other hand,

$(\hat{n} \otimes n + n \otimes \hat{n}) p = (n \cdot n) \hat{n} + (n \otimes p) \hat{n} = (p \cdot n) \left( I + \frac{n \otimes p}{p \cdot n} \right) \hat{n} = (p \cdot n) \hat{V} \hat{n}.$

As a result,

\begin{equation}
(5.20) \quad \hat{p}' = R \hat{p} - 2(p \cdot n) \hat{V} (d\nu)_q V \hat{q} =: R \hat{p} - 2A \hat{q}.
\end{equation}

Note that $|\nu| = 1$ implies that $(d\nu)_Q$ map $n^\perp$ onto $n^\perp$. Also the range of $V$ is $n^\perp$ and $V : p^\perp \to n^\perp$ is an isomorphism. Moreover, $\hat{V}$ restricted to $n^\perp$ equals $I - \frac{n \otimes p}{p \cdot n}$, and that $\hat{V} : n^\perp \to p'^\perp$ is an isomorphism, which simply $n$-projects onto $p'^\perp$. Indeed since $Rn = -n$ and $R = I$ on $n^\perp$,

$R \hat{V} = R + R \frac{n \otimes p}{n \cdot p} = R - \frac{n \otimes p}{n \cdot p}, \quad R \hat{V} |_{n^\perp} = I - \frac{n \otimes p}{n \cdot p}.$
and $R V = V^t$ is the transpose of $V$ because
\[ w \cdot (R V)w' = w \cdot \left( I - \frac{n \otimes p}{n \cdot p} \right) w' = w \cdot w' - \frac{(p \cdot w')(n \cdot w)}{n \cdot p} = (V w) \cdot w', \]
for every $w, w' \in n^\perp$. As a result,
\[ A = (p \cdot n) RV^t (d \nu)_Q V \dot{q}. \]
This and (5.20) imply (5.16).

(ii) If we take a path $x^*(\theta) = \left( q^*(\theta), p^*(t \theta) \right)$, that lies on $\partial^+ X$, such that $q^*(0) = q, p^*(0) = p,$ and $\dot{q}(0) = \dot{q} \in \partial Y, \dot{p}(0) = \dot{p} \in p^\perp$, and if we write $T(q, p) = (Q, P)$, then $Q = q + \tau p, P = R_Q p$, and
\[ (d T)_{(q, p)}(\dot{q}, \dot{p}) = \left( V(\dot{q} + \tau \dot{p}), R \dot{p} - 2 A(\dot{q} + \tau \dot{p}) \right), \]
where $\tau = \tau(q, p), R = R_Q, V = V_{Q, p}$, and
\[ A = A_Q, p = (p \cdot \nu(Q)) R_Q V_{Q, p}^t (d \nu)_Q V_{Q, p}. \]

We note that the operator $V$ is the $p$-projection onto $n^\perp$. That is $(I - V) \dot{q}$ is parallel to $p$ and $V \dot{q} \cdot n = 0$ always. The operator $d \nu$ is known as the shape operator or the Gauss map of $\partial Y$.

**Proof of Theorem 5.1(i)** Fix $x = (q, p) \in \partial^+ X$, set $(\dot{Q}, \dot{P}) = T(q, p)$, and $\tau = \tau(q, p)$. To explore the dispersive behavior of a dispersive billiard, we study the evolution of the quadratic form
\[ Q(\dot{q}, \dot{p}) = Q_x(\dot{q}, \dot{p}) = \dot{q} \cdot \dot{p}, \]
along a $T$ orbit. Here $Q_x$ is defined for $\dot{q} \in T_q Y$, and $\dot{p} \in p^\perp$. By (5.17),
\[ Q((d T)_{(q, p)}(\dot{q}, \dot{p})) = [V_{Q, p}(\dot{q} + \tau \dot{p})] \cdot \left[ R_Q \dot{p} + 2 (p \cdot \nu(Q)) \right] - (R_Q V_{Q, p}^t (d \nu)_Q V_{Q, p})(\dot{q} + \tau \dot{p}) \]
\[ = (\dot{q} + \tau \dot{p}) \cdot \dot{p} + 2 (p \cdot \nu(Q)) \cdot [V_{Q, p}(\dot{q} + \tau \dot{p})] \cdot [(d \nu)_Q V_{Q, p}(\dot{q} + \tau \dot{p})] \]
\[ = Q(\dot{q}, \dot{p}) + \tau |\dot{p}|^2 + 2(p \cdot \nu(Q)) \cdot [V_{Q, p}(\dot{q} + \tau \dot{p})] \]
\[ \geq Q(\dot{q}, \dot{p}) + \tau |\dot{p}|^2 + 2(\nu(Q)) \cdot [V_{Q, p}(\dot{q} + \tau \dot{p})] \]
Here for the second equality we used $R_Q V_{Q, p} = V_{Q, p}$ (which is true because the restriction of $R_Q$ to $T_q Y$ is identity), and that $V_{Q, p} z \cdot \dot{p} = z \cdot \dot{p}$ (which is true because $V_{Q, p} z - z$ is parallel to $p$, and $p \cdot \dot{p} = 0$), and for the third equality we used $V_{Q, p}^2 = V_{Q, p}$. Observe that if $\dot{q} = a + b$ with $a \in p^\perp$, and $b$ parallel to $p$, then $V_{Q, p}(\dot{q} + \tau \dot{p}) = V_{Q, p}(a + \tau \dot{p})$, and
\[ |V_{Q, p}(\dot{q} + \tau \dot{p})|^2 = |V_{Q, p}(a + \tau \dot{p})|^2 \geq |a + \tau \dot{p}|^2 \geq 4 \tau a \cdot \dot{p} = 4 \tau Q(\dot{q}, \dot{p}). \]
Hence,

\[(5.22)\quad Q \left( (dT)_{(q,p)}(\hat{q}, \hat{p}) \right) \geq f(q,p)Q(\hat{q}, \hat{p}),\]

where

\[
f(q,p) = 1 + 8\delta(p \cdot \nu(Q))^\tau(q,p),
\]

for \( Q = q + \tau(q,p)p \). Inductively \((5.22)\) yields

\[
Q \left( (dT^n)_{(q,p)}(\hat{q}, \hat{p}) \right) \geq \prod_{i=0}^{n-1} f(T^i(q,p))Q(\hat{q}, \hat{p}),
\]

provided that \( Q(\hat{q}, \hat{p}) > 0 \). From this and the Ergodic Theorem,

\[(5.23)\quad \lim \inf_{n \to \infty} n^{-1} \log Q \left( (dT^n)_{(q,p)}(\hat{q}, \hat{p}) \right) \geq \lim \inf_{n \to \infty} n^{-1} \sum_{i=0}^{n-1} \log f(T^i(q,p)) =: g_0(q,p),\]

where \( g_0 = P_{\mu}f \), with

\[
\int g_0 \, d\mu = \int f \, d\mu > 0.
\]

Finally, given any nonzero \((\hat{q}, \hat{p})\) with \( Q(\hat{q}, \hat{p}) > 0 \), use \((5.23)\) and

\[
\left| (dT^n)_{(q,p)}(\hat{q}, \hat{p}) \right| \geq \sqrt{2}Q \left( (dT^n)_{(q,p)}(\hat{q}, \hat{p}) \right)^{1/2},
\]

to deduce \((5.11)\) for \( g = 2^{-1}g_0 \).

\[(ii)\quad \text{By Proposition 5.1, the quantity } \lambda(t) := p(t) \cdot \dot{q}(t) \text{ is independent of time because in between collisions } \lambda(t) = p(t) \cdot \dot{p}(t) = 0, \text{ and at a collision time } t,
\]

\[
\lambda(t+) = R_{q(t)}p(t-) \cdot R_{q(t)}\dot{q}(t-) = p(t-) \cdot \dot{q}(t-) = \lambda(t-).
\]

As a result, if initially \( p \cdot \dot{q} = 0 \), then \( p(t) \cdot \dot{q}(t) = 0 \), for all \( t \geq 0 \). If we write \( Q(t) = Q(\dot{q}(t), \dot{p}(t)) \), then in between collisions, \( \frac{dQ}{dt} = |\dot{p}|^2 \) and at a collision,

\[
\begin{align*}
Q(t+) &= \dot{q}' \cdot \dot{p}' = R_{q}q' \cdot \left[ R_{q}p + 2(p \cdot \nu(q))^-(R_{q}V_{q,p}(dv\nu)_qV_{q,p})\dot{q} \right] \\
&= Q(t-) + 2(p \cdot \nu(q))^-(V_{q,p}\dot{q}) \cdot ((dv\nu)_qV_{q,p}) \\
&\geq Q(t-) + 2\delta(p \cdot \nu(q))^-(V_{q,p}\dot{q})^2,
\end{align*}
\]

where \((q,p) = (q(t), p(t))\). We note that since \( V_{q,p}\dot{q} - \dot{q} \) is parallel to \( p \), and \( p \perp \dot{q} \), we learn

\[
|V_{q,p}\dot{q}|^2 \geq |\dot{q}|^2.
\]
As a result,
\[
Q(t+) \geq Q(t-) + 2\delta (p(t-) \cdot \nu(q(t))) - |\dot{q}(t-)|^2
\]

Let us write \( \tau_i \), for the time at which \( q(t) \) has reached the boundary for the \( i \)-th times, and set \( \bar{\tau}_i = \tau_i - \tau_{i-1} \). By (5.24),
\[
Q(\tau_{i+1} +) \geq Q(\tau_i +) + \int_{\tau_i}^{\tau_{i+1}} |\hat{p}(t)|^2 dt + 2\delta (p_i \cdot n_{i+1}) - |\hat{q}(\tau_{i+1} -)|^2
\]
\[
\geq Q(\tau_i +) + 2\delta (p_i \cdot n_{i+1}) - |\hat{q}_i + \bar{\tau}_i \hat{p}_i|^2
\]
\[
\geq Q(\tau_i +) + 8\delta \bar{\tau}_i (p_i \cdot n_{i+1}) - (\hat{q}_i \cdot \hat{p}_i)
\]
\[
= \left( 1 + 8\delta \bar{\tau}_i (p_i \cdot n_{i+1}) \right) Q(\tau_i +),
\]

where \( q_{i+1} = q(\tau_{i+1}), p_i = p(\tau_i), \hat{q}_i = \hat{q}(\tau_i), \hat{p}_i = \hat{p}(\tau_i) \) and \( n_{i+1} = \nu(q_{i+1}) \). From this, we can readily deduce that for a function \( h_0 \) with \( \int h_0 \, dm > 0 \),
\[
\liminf_{n \to \infty} n^{-1} \log Q(\tau_n +) \geq g_0(\tau_1).
\]

It is not hard to show
\[
\lim_{n \to \infty} n^{-1} \tau_n = k,
\]
exists and is positive \( m \)-almost everywhere. This and (5.25) imply
\[
\liminf_{t \to \infty} t^{-1} \log Q(t) \geq h_1,
\]
for a nonnegative and nonzero function \( h_1 \). This in turn implies (5.12) as in Part (i). \( \square \)

**Remark 5.2(i)** Write \( V \) for \( V_{q,p} \) with \( q \in \partial Y \). The operator \( W = V^t V \) on \( p^\perp \) has a simple geometric interpretation. For \( \hat{q} \in p^\perp \),
\[
W \hat{q} = \left( I - \frac{n \otimes p}{p \cdot n} \right) \left( I - \frac{p \otimes n}{p \cdot n} \right) \hat{q} = \left( I - \frac{n \otimes p}{p \cdot n} \right) \left( \hat{q} - \frac{n \cdot \hat{q}}{n \cdot p} \frac{n \cdot p}{n \cdot n} \right)
\]
\[
= \hat{q} - \frac{n \cdot \hat{q}}{n \cdot p} \left( p - \frac{|p|^2}{n \cdot n} \right) = \hat{q} - \frac{n \cdot \hat{q}}{n \cdot p} V^t p
\]
\[
= \left( I - \frac{(V^t p) \otimes n}{p \cdot n} \right) \hat{q},
\]
where \( V^t p \) is the \( n \)-projection of \( p \) onto \( p^\perp \). Moreover,
\[
|V \hat{q}|^2 = W \hat{q} \cdot \hat{q} = |\hat{q}|^2 + \left( \frac{n \cdot \hat{q}}{n \cdot p} \right)^2 |p|^2.
\]
(ii) For the flow $\phi_t$, we have a zero Lyapunov exponent in the direction of the flow. To avoid this direction, we assume that $\dot{q} \cdot p = 0$ in Theorem 5.1. This suggests that we restrict $(\dot{q}, \dot{p})$ to $W(x) = \{(\dot{q}, \dot{p}) : \dot{q} \cdot p = \dot{p} \cdot p = 0\} = p^\perp$ for $x = (q, p)$. Note that if $(\dot{q}, \dot{p}) \in W(x)$ initially, then $(\dot{q}(t), \dot{p}(t)) \in W(\phi_t(x))$ at later times. Let us define a sector

$$C_x = \{(\dot{q}, \dot{p}) \in W(x) : \dot{q} \cdot \dot{p} > 0\}.$$ 

What we have learned so far is that

$$\tag{5.26} (d\phi_t)_x(C_x) \subsetneq C_{\phi_t(x)}.$$ 

Note that $\dot{q}$ is gaining in size in between collisions. However the gain in the size of $\dot{p}$ is occurring only at collisions.

(iii) As we mention in Remark 5.1, the function $f(q, p, t) = f^0(\phi_{-t}(q, p))$, satisfies the equation $f_t + p \cdot f_q = 0$ strictly inside $X$. To derive an equation for the evolution of $\alpha = f_q$ and $\beta = f_p$, observe that if $\Lambda_0(c) = \{x : f^0(x) = c\}$, then

$$\Lambda_t(c) = \{(q, p) : f(q, p, t) = c\} = \phi_t(\Lambda_0(c)).$$

This means that if $x \in \Lambda_t(c)$, then $x = \phi_t(y)$, for some $y \in \Lambda_0(c)$, and the normal $z(0) := \nabla f^0(y)$ is transported to $z(t) = (\alpha(x, t), \beta(x, t)) = \nabla f(x, t)$ after $t$ units of time. This suggests studying the evolution of a normal vector to a surface of codimension one that evolves with $\phi_t$. More generally, take a surface $\Lambda$ of codimension one in $X$. The manifold $\mathcal{T}\Lambda \subseteq \mathcal{T}X$ evolves to $\mathcal{T}\phi_t(\Lambda)$ and we study its evolution by keeping track of its corresponding unit normal. If $z(t) = (a(t), b(t)) \in \mathcal{T}X$ is normal to $\mathcal{T}(\phi_t(\Lambda))$ at all times, then we would like to derive an evolution equation for it. The vector $(a, b)$ is chosen so that for every $t$,

$$a(t) \cdot \dot{q}(t) + b(t) \cdot \dot{p}(t) = 0,$$

where $(\dot{q}(t), \dot{p}(t)) \in \mathcal{T}_{\phi_t(\Lambda)}$ with $\Lambda_t = \phi_t(\Lambda)$. In between collisions, $\dot{x}(t) = (\dot{q} + t\dot{p}, \dot{p})$ and $a(t) \cdot (\dot{q} + t\dot{p}) + b(t) \cdot \dot{p} = 0$, or $a(t) \cdot \dot{q} + (ta(t) + b(t)) \cdot \dot{p} = 0$. Hence if initially $(a(0), b(0)) = (a, b)$, then $a(t) = a$ and $b(t) = b - ta$. So in between collisions we simply have $\frac{da}{dt} = 0$, $\frac{db}{dt} = -a$.

At a collision $(a, b)$ experiences a jump discontinuity. If after a collision the normal vector is given by $(a', b')$, then

$$a' \cdot (R\dot{q}) + b' \cdot (R\dot{p} - 2A\dot{q}) = 0,$$

$$(Ra') \cdot \dot{q} + (Rb') \cdot \dot{p} - 2(A' b') \cdot \dot{q} = 0.$$ 

This suggests

$$\tag{5.27} b' = Rb, \quad a' = Ra + 2RA'Rb =: Ra + 2Bb.$$
Note that if \( Q(t) = a(t) \cdot b(t) \), then in between collisions,
\[
\frac{dQ}{dt} = -|a|^2,
\]
and at a collision
\[
Q(t+) = a' \cdot b' = (R a + 2 R A^t R b) \cdot R b
= Q(t-) + 2 A^t R b \cdot b
= Q(t-) + 2 b \cdot R A b
= Q(t-) + 2(p \cdot n)(d\nu)Q(V b) \cdot (V b),
\]
and in the case of a dispersive billiard,
\[
Q(t+) - Q(t-) \leq 2\delta(p \cdot n)|V b|^2 < 0.
\]
Hence \( Q(t) \) is decreasing. \( \square \)

As we mentioned in Chapter 4, we may use sectors to find the stable and unstable directions for a dynamical systems. We wish to use Proposition 5.1 to determine how certain sectors get slimmer along the flow. As a first step, let us observe that since the flow \( \phi_t \) or the map are symplectic, then both the stable and unstable subspaces are Lagrangian. Recall that if \( \hat{x} = (\hat{q}, \hat{p}) \) and \( \hat{x}' = (\hat{q}', \hat{p}') \), then
\[
\omega(\hat{x}, \hat{x}') = \hat{p} \cdot \hat{q}' - \hat{q} \cdot \hat{p}'.
\]
Also recall for \( x = (q, p) \),
\[
W_x = \{ (\hat{q}, \hat{p}) : \hat{q} \cdot p = \hat{p} \cdot p = 0 \} = p^\perp \times p^\perp.
\]
We think of \( W = (W_x : x \in X) \) as a vector bundle of dimension \( 2d - 2 \). A sub-bundle \( L = (L_x : x \in X) \) is called Lagrangian if \( \omega(a, b) = 0 \) for every \( a, b \in L_x \). Here we think of \( \omega \) as a symplectic form on \( W \).

**Proposition 5.2** (i) The billiard flow is symplectic.

(ii) Both stable and unstable bundles \( E^{s(u)} \) are Lagrangian.

(iii) If there exist symmetric linear maps \( S^s_x, S^u_x : p^\perp \to p^\perp \) such that
\[
E^{u(s)}_x = \{ (\hat{q}, \hat{p}) : \hat{p} = S^u_x(\hat{q}), \hat{q} \in p^\perp \},
\]
then in between collisions, , or equivalently
\[
\frac{d}{dt} S^{s(u)}_{\phi_t(x)} + (S^{s(u)}_{\phi_t(x)})^2 = 0,
\]
\[
S^{s(u)}_{\phi_t(x)} = S^s_x(I + t S^{s(u)}_x)^{-1} = (t I + (S^{s(u)}_x)^{-1})^{-1},
\]
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where the last equality holds whenever \( S_x^{s(u)} \) is invertible. At a collision, \( S_x^{s(u)} \) changes to \( S_x^{s(u)} \), where

\[
S_x^{s(u)} = R_q S_x^{s(u)} R_q + R_q V_{q,p}^t (dv)_q V_{q,p} R_q =: R_q S_x^{s(u)} R_q + \Gamma_{q,p}.
\]

(iv) Assume that there exist positive symmetric linear maps \( S_x^s, S_x^r \) : \( \mathcal{T}_q \partial^* Y \rightarrow p^\perp \) such that

\[
E_x^{s(u)} = \{ (\hat{q}, \hat{p}) : \hat{p} = S_x^s \hat{q}, \hat{q} \in \mathcal{T}_q \partial^* Y \}.
\]

Then

\[
S_{T(x)}^s = \Gamma_{Q,P} + R_Q \left( \tau(x) I + (S_x^{s(u)})^{-1} \right)^{-1} R_Q,
\]

where \( T(q, p) = (Q, P) \) and \( x = (q, p) \).

**Proof(i)** For symplectic property observe that if \( \hat{x} = (\hat{q}, \hat{p}), \hat{x}_s = (\hat{q}_s, \hat{p}_s) \in W_x \) change to \( \hat{x}' = (\hat{q}', \hat{p}'), \hat{x}'_s = (\hat{q}'_s, \hat{p}'_s) \in W_x \) at a collision, then

\[
\omega(\hat{x}', \hat{x}'_s) = \hat{p}' \cdot \hat{q}_s' - \hat{q}' \cdot \hat{p}_s'
\]

\[
= (R_q \hat{p} + 2(p \cdot n)^{-} R_q V_{q,p}^t (dv)_q V_{q,p} \hat{q}) \cdot (R_q \hat{q}_s)
\]

\[
- (R_q \hat{p}_s + 2(p \cdot n)^{-} R_q V_{q,p}^t (dv)_q V_{q,p} \hat{q}_s) \cdot (R_q \hat{q})
\]

\[
= (\hat{p} + 2(p \cdot n)^{-} V_{q,p}^t (dv)_q V_{q,p} \hat{q}) \cdot \hat{q}_s - (\hat{p}_s + 2(p \cdot n)^{-} V_{q,p}^t (dv)_q V_{q,p} \hat{q}_s) \cdot \hat{q}
\]

\[
= \omega(\hat{x}, \hat{x}_s),
\]

where for the last equality, we used the symmetry of \( V_{q,p}^t (dv)_q V_{q,p} \).

(ii) \( a, b \in E_x^s \), then by symplectic property

\[
\omega_x(a, b) = \lim_{t \rightarrow \infty} \omega_{\phi_t(x)}((d\phi_t)_x a, (d\phi_t)_x b) = 0,
\]

because both \( |(d\phi_t)_x a| \) and \( |(d\phi_t)_x b| \) decay exponentially fast.

(iii) Note that since \( \hat{p}(t) \) stays constant in between collisions, we

\[
0 = \frac{d}{dt} S(t) \hat{q}(t) = \hat{S}(t) \hat{q}(t) + S(t) \frac{d\hat{q}}{dt}(t) = \hat{S}(t) \hat{q}(t) + S(t) \hat{p}(t) = \hat{S}(t) \hat{q}(t) + S(t)^2 \hat{q}(t),
\]

where \( S(t) \) denotes \( S_x^{s(u)}(\hat{q}) \). This implies the first equation (5.29) because \( \hat{q}(t) \) can take any vector. The second equation of (5.29) also implies the first by differentiation.

Since \( (d\phi_t)_x E_x^{s(u)} = E_x^{s(u)}(\phi_t(x), (d\phi_t)_x (\hat{q}, \hat{p}) = (\hat{q} + t\hat{p}, \hat{p}) \), we must have

\[
S_x^{s(u)}(\hat{q} + t S_x^{s(u)} \hat{q}) = S_x^{s(u)} \hat{q}.
\]

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This is exactly the second equation in (5.29). As for (5.30), we use (5.16) to write
\[ S_x^{\text{s(u)}}(R_q \dot{q}) = R_q S_x^{\text{s(u)}} \dot{q} + R_q V_{\dot{q},q}(d\nu)_q V_{\dot{q},q} \dot{q}, \]
which yields (5.30).

(iv) The formula (5.32) is an immediate consequence of (5.30) and (5.29).

\[ \square \]

**Remark 5.3(i)** A particularly nice example of \( \Lambda \) as in Remark 5.2(ii) is a normal bundle of a \( q \)-surface. More precisely, suppose \( \Theta \) is a surface of codimension one in \( Y \) and set
\[ \Lambda = \{ (q,p) : q \in \Theta, \ p \text{ is the normal vector at } q \}. \]
Here we are assuming that \( \Lambda \) is orientable and a normal vector \( p \) at each \( q \in \Theta \) is specified.

In this case \((q,p,\hat{q},\hat{p}) \in T\Lambda \) means that \( \hat{q} \in T_q \Theta \) and that \( \hat{p} = C_q \hat{q} \) for a suitable matrix \( C_q(q) \) which is known as the curvature matrix. (If \( p = \nu(q) \) is the normal vector, then \( C_q = (d\nu)_q \).) The evolution of \( C \) along an orbit is governed by (5.29) and (5.30).

(ii) If the billiard map is hyperbolic, we would have \( \dim E^u = \dim E^s = d - 1 \). This is the highest dimension a Lagrangian bundle can have because \( \dim W = 2d - 2 \). In fact the assumptions (5.28) and (5.31) are based on the fact that if we assume that a Lagrangian subspace \( L \) is non-degenerate, then it must be a graph of a symmetric matrix. To guess what the stable fiber at a point \( x = (q,p) \) is, observe that if we replace \( x \) with \( T^{-1}(x) \) in (??), we obtain
\[ S^s_x = \Gamma_{q,p_1} + R_q \left( \tau_1(x) I + (S^s_{T^{-1}(x)})^{-1} \right)^{-1} \frac{I}{\tau_1 I + \frac{I}{S^s_{x_1}}} R_q, \]
where \( x_1 = T^{-1}(x) = T(q,-p) = (q_1,p_1) \) and \( \tau_1 = \tau(T^{-1}(x)) \). From this we guess
\[ S^s_x = \hat{\Gamma}_1 + \hat{R}_0 \left( \frac{I}{\tau_1 I + \frac{I}{S^s_{x_1}}} \right) \hat{R}_0. \]

Here we write \( x_i = (q_i,p_i) \) for \( T^{-i}(x) \), and \( \tau_i = \tau(x_i) \), \( \hat{R}_i = R_{q_i} \), and \( \hat{\Gamma}_i = \Gamma_{p_i,q_i} \). To verify (5.33), we need to make sure that the continued fraction is convergent. This is more or less equivalent to the convergence
\[ S^s_{x,n} = (dT^n)_{T^{-n}(x)} S^s_{T^{-n}(x)}. \]

(iii) It is instructive to compare the billiard flow with our Example 4.3. In Example 4.3, we had the sector \( C_x = \{ (\dot{q},\dot{p}) : \dot{q} \dot{p} \geq 0 \} \), that got slimmer under \( dT^n \). Its lower boundary
$L = \{(\hat{q},0) : \hat{q} \in \mathbb{R}\}$ yielded a sequence of lines $L^n_x = (dT^n)_{T^{-n}(x)}L$ with increasing slopes $\theta_n(x) \to \theta^s$. The limit $\theta^s$ gave the slope of the stable line. We have a similar scenario for our billiard: the sector

$$C_x = C_{q,p} = \{(\hat{q},\hat{p}) : \hat{q},\hat{p} \in \mathbb{R}\},$$

yields a family

$$C^t_x := (d\phi_{t})_{\phi^{-1}(x)}C_{\phi^{-1}(x)},$$

of nested sectors as $t \to \infty$. Its lower boundary

$$L^t_x := (d\phi_{t})_{\phi^{-1}(x)}L_x,$$

with $L_x = \{(\hat{q},0) : \hat{q} \in \mathbb{R}\}$,

is a $d$-dimensional Lagrangian subspace that is expected to converge to the stable fiber $E^s_x$. Similarly, the sector

$$C_x = C_{q,p} = \{(\hat{q},\hat{p}) : \hat{q} \in T_q\partial Y, \hat{p} \in \mathbb{R}\},$$

yields a family

$$C^n_x := (dT^n)_{T^{-n}(x)}C_{T^{-n}(x)},$$

of nested sectors as $n \to \infty$. Its boundaries are given by

$$L^n_{x}^{-} = (dT^n)_{T^{-n}(x)}L_{x}^{-}, \quad \text{with} \quad L_{x}^{-} = \{(\hat{q},0) : \hat{q} \in T_q\partial Y\},$$

$$L^n_{x}^{+} = (dT^n)_{T^{-n}(x)}L_{x}^{+}, \quad \text{with} \quad L_{x}^{+} = \{(0,\hat{p}) : \hat{p} \in \mathbb{R}\}.$$  

These are $d$-dimensional Lagrangian subspaces that are expected to converge to the stable fiber $E^s_x$ for the billiard map $T$. \hfill \Box

To establish the convergence of $C^n_x$ as $n \to \infty$, we may use the strict nested property of these sequences. For this, we need to define a nice metric for the set of Lagrangian subspaces so that the strict nested property guarantees the convergence. Indeed if we write

$$(dT^n)_{T^{-n}(x)} = \begin{bmatrix} A_n(x) & B_n(x) \\ C_n(x) & D_n(x) \end{bmatrix},$$

then the monotonicity

$$(dT)_{x}C_{x} \subsetneq C_{T(x)}, \quad C^{n+1}_{x} \subsetneq C^{n}_{x},$$

means $C^{n}_{x}B_{n} > 0$, and if $\lambda^{n}_{1}$ denotes its smallest eigenvalue, then $\lambda^{n}_{1} \to \infty$ as $n \to \infty$. We then define a metric $\mathcal{D}$ on the set of Lagrangian subspaces such that

$$\mathcal{D}(L_{x}^{-n},L_{x}^{+n}) \leq c_0(\lambda^{n}_{1})^{-1},$$

for a constant $c_1$

**Theorem 5.2** For $\mu$-almost all $x$, we have that $\cap_n C^n_x =: E^s_x$ is a Lagrangian subspace.

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Proof Since 

\[ C^{n+1}_x = (dT)T^{-1}(x)C^n_{T^{-1}(x)}, \]

we learn 

\[ L^{\pm,n+1}_x = (dT)T^{-1}(x)L^{\pm,n}_{T^{-1}(x)}. \]

If 

\[ L^{\pm,n}_x = \{ (\hat{q}, S^{\pm,n}_x \hat{q}) : \hat{q} \in T_q \partial Y \}, \]

then from (5.31) we learn 

\begin{equation}
(5.35) \quad S^{\pm,n+1}_{Q,P} = \Gamma_{Q,p} + R_Q \left( \tau(q,p)I + \left( S^{\pm,n}_{(q,p)} \right)^{-1} \right) R_Q =: G \left( (q,p), S^{\pm,n}_{(q,p)} \right),
\end{equation}

where \( T^{-1}(Q, P) = (q, p) \) so that \( Q = q + \tau(q,p) \), and \( P = R_Q p \). (Compare this to (4.38).)

Note that the function \( G(x, A) \) is monotonically increasing in \( A \). Since \( S^{-0} = 0 \), and \( S^{-1} > 0 \), we deduce that the sequence \( \{ S^{-n} \}_n \) is increasing.

We now define a metric on the set of Lagrangian subspaces with respect to the sequence \( \{ S^{\pm,n} \}_n \) converges. Since we are dealing with Lagrangian subspaces associated with positive definite matrices, we define a metric on the set of positive matrices. This metric yields a metric on the set of corresponding Lagrangian. We write 

\[ L(A) = \{ (a, Aa) : a \in \mathbb{R}^d \}, \]

for the Lagrangian space associated with \( A \). We define 

\[ D(\mathcal{L}(A), \mathcal{L}(B)) = D(A, B) = \sup_{a \neq 0} \left| \log \frac{Aa \cdot a}{Ba \cdot a} \right|. \]

We next study the effect of an invertible symplectic matrix on such Lagrangian subspaces. If 

\[ G = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \]

is symplectic, then the the Lagrangian subspace associated with \( S \) is mapped to a subspace associated with 

\[ S' = (C + DS)(A + BS)^{-1}. \]
Lemma 5.1 There exist invertible matrices $F_1$ and $F_2$ and a diagonal matrix $K$ such that

\[
\begin{bmatrix}
F_1^{-1} & I \\ I & F_2^* \\
\end{bmatrix}
\begin{bmatrix}
A & B \\ C & D \\
\end{bmatrix}
\begin{bmatrix}
F_1 & I \\ I & F_2 \\
\end{bmatrix} = \begin{bmatrix}
I & I \\ K & I + K \\
\end{bmatrix} =: GF_1 \Lambda(K)GF_2.
\]

Moreover the matrix $K$ has the same eigenvalues as $C^*B$.

Accepting this Lemma for now, and the elementary facts,

\[
G_F \mathcal{L}(A) = \mathcal{L}(F^*AF), \quad \mathcal{D}(F^*AF, F^*BF) = \mathcal{D}(A, B),
\]

we can assert

\[
\mathcal{D}(G \mathcal{L}(A), G \mathcal{L}(B)) = \mathcal{D}(G_F \Lambda(K)GF_2 \mathcal{L}(A), G_F \Lambda(K)GF_2 \mathcal{L}(B)) \\
= \mathcal{D}(\Lambda(K) \mathcal{L}(F_2^*AF_2), \Lambda(K) \mathcal{L}(F_2^*BF_2)) \\
=: \mathcal{D}(\Lambda(K) \mathcal{L}(A'), \Lambda(K) \mathcal{L}(B')) = \mathcal{D}(\Lambda(K)A', \Lambda(K)B')
\]

In particular

\[
\mathcal{D}(GL_x^-, GL_x^+) = \mathcal{D}(\Lambda(K)L_x^-, \Lambda(K)L_x^+) = \mathcal{D}(K, I + K) \\
= \mathcal{D}(I + K^{-1}, I) = \log (1 + \lambda_1(G)^{-1}),
\]

where $\lambda_1(G)$ is the smallest eigenvalue of $C^*B$. On the other hand, since $Q(GF\hat{x}) = Q(\hat{x})$,

\[
\inf_{\hat{x} \in C_x} \frac{Q(G\hat{x})}{Q(\hat{x})} = \inf_{\hat{x} \in C_x} \frac{Q(\Lambda(K)\hat{x})}{Q(\hat{x})} = \inf_{\hat{x} \in C_x} \frac{K(\hat{q} + \hat{p}) \cdot (\hat{q} + \hat{p}) + \hat{p} \cdot (\hat{q} + \hat{p})}{Q(\hat{x})} \\
\leq \inf_{a, b > 0} \frac{\lambda_1(a + b)(a + b) + b(a + b)}{ab} = \inf_{b > 0} \left[ \lambda_1 \left( b + b^{-1} \right)^2 + b \left( b + b^{-1} \right) \right] \\
= \inf_{b > 0} \left[ (1 + \lambda_1)b^2 + \lambda_1 b^{-2} + 2\lambda_1 + 1 \right] = 2\sqrt{\lambda_1(\lambda_1 + 1) + 2\lambda_1 + 1}
\]

Now if

\[
G = G_n = (dT^n)_{T^{-n}(x)},
\]

we are done if we can show

\[
\lim_{n \to \infty} \lambda_1(G_n) = \infty.
\]

For this it suffices to show

(5.36) \[
\lim_{n \to \infty} \sigma(G_n) := \lim_{n \to \infty} \inf_{\hat{x} \in C_x} \frac{Q(G_n\hat{x})}{Q(\hat{x})} = \infty.
\]

Since the set

\[
\{ \hat{x} \in C_x : Q(\hat{x}) = 1 \},
\]

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is compact, (5.36) follows if we can show that for each $\hat{x} \in C_x$,

$$\lim_{n \to \infty} \frac{Q(G_n \hat{x})}{Q(\hat{x})} = \infty.$$ 

This is a consequence of Theorem 5.1. □

Exercises

(i) Consider a billiard inside a planar disc $D$. Write down a explicit formula for the billiard map $T : \partial^+ X \to \partial^+ X$, where $X = D \times S^{d-1}$. Write $(q, \theta)$ for a point on $\partial^+ X$, with $q \in \partial D$, and $\theta \in (-\pi, \pi)$ for the angle $p$ makes with the inward normal. Show that the set of $\{(q, \theta) : q \in \partial D\}$ is invariant for $T$.

(ii) Consider a billiard inside a planar domain $D$, and assume that the set $D$ is strictly convex. Write $(q, \theta)$ for a point on $\partial^+ X$, with $q \in \partial D$, and $\theta \in (-\pi, \pi)$ for the angle $p$ makes with the tangent vector at $q$. Given $q, Q \in \partial D$, write $S(q, Q) = |q - Q|$. Show that if $T(q, \theta) = (Q, \Theta)$, then $S_q = -\cos \theta$ and $S_Q = \cos \Theta$. □
6 Ergodicity of Hyperbolic Systems

Lyapunov exponents can be used to measure the hyperbolicity of dynamical systems. Anosov measures (systems) are examples of uniformly or strongly hyperbolic systems which exhibit chaotic and stochastic behavior. In reality, dynamical systems are rarely strongly hyperbolic and those coming from Hamiltonian systems are only weakly (or even partially) hyperbolic.

An argument of Hopf shows that hyperbolicity implies ergodicity. We examine this argument for two models in this section: Examples 6.1 and 6.2. To explain Hopf’s argument, let us choose the simplest hyperbolic model with expansion and contraction, namely Arnold cat transformation, and use this argument to prove its ergodicity. In fact in Example 1.6 we showed the mixing of Arnold’s cat transformation which in particular implies the ergodicity. But our goal is presenting a second proof of ergodicity which is the key idea in proving ergodicity for examples coming from Hamiltonian systems.

Example 6.1 Let $A = \begin{bmatrix} 1 & \alpha \\ \alpha & 1 + \alpha^2 \end{bmatrix}$ with $\alpha \in \mathbb{Z}$. Let $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ be the projection $\pi(a) = a(\text{mod } 1)$ and define $T : \mathbb{T}^2 \to \mathbb{T}^2$ by $T \circ \pi = \pi \circ T^{\hat{a}}$ where $T^{\hat{a}}(a) = Aa$. Since $\alpha \in \mathbb{Z}$ and $\det A = 1$, we know that $T$ is continuous and that the normalized Lebesgue measure $\mu$ on $\mathbb{T}^2$ is invariant for $T$. The eigenvalues of $A$ are

$$\lambda_1 = \lambda(\alpha) = \frac{1}{2}[2 + \alpha^2 - \alpha\sqrt{4 + \alpha^2}] < 1 < \lambda_2 = (\lambda(\alpha))^{-1},$$

provided that $\alpha > 0$. The corresponding eigenvectors are denoted by $v_1$ and $v_2$. Define

$$W^s(a) = \{a + tv_1 : t \in \mathbb{R}\}, \quad W^u(a) = \{a + tv_2 : t \in \mathbb{R}\}.$$

We then have that $W^s(x)$ and $W^u(x)$ defined by

$$W^s(\pi(a)) = \pi(\hat{W}^s(a)), \quad W^u(\pi(a)) = \pi(\hat{W}^u(a))$$

are the stable and unstable manifolds. Take a continuous periodic $\hat{f} : \mathbb{R}^2 \to \mathbb{R}$. This induces a continuous $f : \mathbb{T}^2 \to \mathbb{R}$ such that $f \circ \pi = \hat{f}$. We have that $f \circ T^n \circ \pi = \hat{f} \circ T^n$. Define $\hat{X}^\pm$ to be the set of points $a$ such that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \hat{f}(\hat{T}^j(a)) =: \hat{f}^\pm(a)$$

exists. Then $\pi(\hat{X}^\pm) = X^\pm$ with $X^\pm$ consisting of points $x$ such that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j(x)) =: f^\pm(x)$$

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exists with $f^\pm = \hat{f}^\pm \circ \pi$. Evidently $f^\pm \circ T = f^\pm$ on $X^\pm$ and $\hat{f}^\pm \circ \hat{T} = \hat{f}^\pm$ on $\hat{X}^\pm$. From definition, we see that if $b \in \hat{W}^s(a)$ (resp. $b \in \hat{W}^u(a)$), then

$$|\hat{T}^n(b) - \hat{T}^n(a)| = \lambda^n|a - b|,$$

(resp. $|\hat{T}^{-n}(b) - \hat{T}^{-n}(a)| = \lambda^n|a - b|$).

for $n \in \mathbb{N}$. Hence $a \in \hat{X}^+$ (resp. $\hat{X}^-$) implies that $\hat{W}^s(a) \subseteq \hat{X}^+$ (resp. $\hat{W}^u(a) \subseteq \hat{X}^-$). Let $d(\cdot, \cdot)$ be the standard distance on the torus. More precisely,

$$d(x, y) = \min\{|a - b|: \pi(a) = x, \pi(b) = y\}.$$

Again if $y \in W^s(x)$ (resp. $y \in W^u(x)$), then

$$d(T^n(x), T^n(y)) = \lambda^n d(x, y),$$

(resp. $d(T^{-n}(x), T^{-n}(y)) = \lambda^n d(x, y)$).

for $n \in \mathbb{N}$. Similarly $x \in X^+$ (resp. $X^-$) implies that $W^s(x) \subseteq X^+$ (resp. $W^u(x) \subseteq X^-$). Let $Y$ (respectively $\hat{Y}$) denote the set of points $x \in X^- \cap X^+$ (respectively $x \in \hat{X}^- \cap \hat{X}^+$) such that $f^+(x) = f^-(x)$ (respectively $\hat{f}^+(x) = \hat{f}^-(x)$). By Remark 1.3(iii), $\mu(Y) = 1$ and the Lebesgue measure of the complement of $\hat{Y}$ is zero. Choose a point $x_0$ such that $\hat{W}^s(x_0) \setminus \hat{Y}$ is a set of 0 length. The function $\hat{f}^+$ is constant on $\hat{W}^s(x_0)$. The function $\hat{f}^-$ is constant on $W^u(y)$ for every $y \in W^s(x_0) \cap \hat{Y}$ and this constant coincides with the value $\hat{f}^+$ at $y$. Hence $\hat{f}^+ = \hat{f}^-$ is a constant on the set

$$\bigcup_{y \in W^s(x_0) \cap \hat{Y}} \hat{W}^u(y).$$

But this set is of full measure. So $\hat{f}^+ = \hat{f}^-$ is constant a.e. and this implies that $f^+ = f^-$ is constant a.e. \hfill \square

Let us call a discrete dynamical system hyperbolic if its Lyapunov exponents are nonzero. According to a result of Pesin, a hyperbolic diffeomorphism with a smooth invariant measure has at most countably many ergodic components. Pesin’s theory also proves the existence of stable and unstable manifolds for hyperbolic systems.

Sinai studied the issue of ergodicity and hyperbolicity for a system of colliding balls in the late 60’s. These systems can be regarded as hyperbolic systems with discontinuities. To get a feel for Sinai’s method, we follow a work of Liverani and Wojtkowski [LiW] by studying a toral transformation as in Example 6.1 but now we assume that $\alpha \notin \mathbb{Z}$ so that the induced transformation is no longer continuous. As we will see below, the discontinuity
of the transformation destroys the uniform hyperbolicity of Example 6.1 and, in some sense our system is only weakly hyperbolic.

**Example 6.2** As in Example 6.1, let us write \( \pi : \mathbb{R}^2 \to \mathbb{T}^2 \) for the (mod 1) projection onto the torus and set \( \hat{T}(a) = Aa \), for

\[
A = \begin{bmatrix}
1 & \alpha \\
\alpha & 1 + \alpha^2
\end{bmatrix}.
\]

This induces \( T : \mathbb{T}^2 \to \mathbb{T}^2 \), with \( T \circ \pi = \pi \circ \hat{T} \). If \( 0 < \alpha < 1 \), then \( T \) is discontinuous. However the Lebesgue measure \( \mu \) is still invariant for \( T \). To understand \( T \), let us express \( \hat{T} = \hat{T}_1 \circ \hat{T}_2 \), \( T = T_1 \circ T_2 \), \( \hat{T}_i(a) = A_i a \) for \( i = 1, 2 \), where

\[
A_1 = \begin{bmatrix}
1 & 0 \\
\alpha & 1
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
1 & \alpha \\
0 & 1
\end{bmatrix}.
\]

If we regard \( \mathbb{T} \) as \([0, 1]\) with \( 0 = 1 \), then \( T_i \) is discontinuous on the circle \( x_i \in \{0, 1\} \). As a result, \( T \) is discontinuous on the circle \( x_2 \in \{0, 1\} \) and on the curve \( x_1 + \alpha x_2 \in \mathbb{Z} \). One way to portray this is by introducing the sets

\[
\Gamma^+ = \{(x_1, x_2) : 0 \leq x_1 + \alpha x_2 \leq 1, \ 0 \leq x_2 \leq 1\}
\]

\[
\Gamma^- = \{(x_1, x_2) : 0 \leq x_1 \leq 1, \ 0 \leq -\alpha x_1 + x_2 \leq 1\}
\]

and observing that \( \hat{T} \) maps \( \Gamma^+ \) onto \( \Gamma^- \) but \( T \) is discontinuous along \( S^+ = \partial \Gamma^+ \). Moreover \( \hat{T}^{-1} = \hat{T}_2^{-1} \circ \hat{T}_1^{-1} \) with \( \hat{T}_i^{-1}(a) = A_i^{-1} a \) for \( i = 1, 2 \), where

\[
A_1^{-1} = \begin{bmatrix}
1 & 0 \\
-\alpha & 1
\end{bmatrix}, \quad A_2^{-1} = \begin{bmatrix}
1 & -\alpha \\
0 & 1
\end{bmatrix}.
\]

Since \( T_1^{-1} \) is discontinuous on the circle \( x_2 \in \{0, 1\} \) and \( T_2^{-1} \) is discontinuous on the circle \( x_1 \in \{0, 1\} \), we deduce that \( T^{-1} \) is discontinuous on \( S^- = \partial \Gamma^- \). Note that the line \( x_2 = 0 \) is mapped onto the line \( x_2 = ax_1 \) and the line \( x_2 = 1 \) is mapped onto the line \( x_2 = ax_1 + 1 \). Also note that distinct points on \( S^+ \) which correspond to a single point on \( \mathbb{T}^2 \) are mapped to distinct points on \( \mathbb{T}^2 \).

We now examine the stable and unstable manifolds. For the unstable manifold, we need to have that if \( y \in W^u(x) \), then \( d(T^{-n}(x), T^{-n}(y)) \to 0 \) as \( n \to +\infty \). We may try

\[
W^u_0(x) = \{ \pi(a + v_t) : t \in \mathbb{R} \}
\]
where $a$ is chosen so that $\pi(a) = x$ and $v_2$ is the expanding direction. This would not do
the job because of the discontinuity. Indeed the discontinuity set $S^-$ cut the set $W^u_0(x)$ into pieces. Let us write $W^u_t(x)$ for the connected component of $W^u_0(x)$ inside $\Gamma^-$. Since
crossing $W^-$ into pieces. Let us write

Motivated by this, let us pick $x$ such that $d(T^{-n}(x), T^{-n}(y)) \neq 0$ if $y \in W^u_n(x) \setminus W^u_n(x)$. However note that if $y \in W^u_n(x)$, then $d(T^{-1}(x), T^{-1}(y)) = \lambda d(x, y)$. As a result, $d(T^{-1}(x), T^{-1}(y))$ gets smaller than $d(x, y)$ by a factor of size $\lambda$. To have

d(T^{-n}(x), T^{-n}(y)) = \lambda^n d(x, y), we need to make sure that the segment joining $T^{-n}(x)$ to $T^{-n}(y)$ is not cut into pieces by $S^-$. That is, the segment $xy$ does not intersect $T^n(S^-)$. Motivated by this, let us pick $x \in T^2 \setminus \cup_{i=0}^{\infty} T^i(S^-)$ and define $W^u(x)$ to be the component of $W^u_0(x)$ which avoids $\cup_{i=0}^{\infty} T^i(S^-)$. We now claim that for $\mu$-almost all points, $W^u(x) = \cap_{j=0}^{\infty} W^u_j(x)$ is still a nontrivial segment. (This would be our unstable manifold.) More

precisely, we show that for $\mu$-almost all $x$, there exists a finite $N(x)$ such that

$$W^u(x) = \bigcap_{j=0}^{\infty} W^u_j(x) = \bigcap_{j=0}^{N(x)} W^u_j(x).$$

To see this, let us observe that for example

$$W^u_0(x) = T(T^{-1}W^u_1(x) \cap W^u_1(T^{-1}(x))).$$

In other words, we take $W^u_0(x)$ which is a line segment with endpoints in $S^-$. We apply $T^{-1}$
on it to get a line segment $T^{-1}W^u_1(x)$ with $T^{-1}(x)$ on it. This line segment is shorter than
$W^u_0(x)$; its length is $\lambda$ times the length of $W^u_0(x)$. If this line segment is not cut by $S^-$, we
set $W^u_2(x) = W^u_1(x)$; otherwise we take the connected component of $T^{-1}W^u_1(x)$ which lies
inside $S^-$ and has $T^{-1}(x)$ on it. This connected component lies on $W^u_0(T^{-1}(x))$. We then
map this back by $T$. Note that $W^u_2(x) \neq W^u_1(x)$ only if $d(T^{-1}(x), S^-) = \text{distance of } T^{-1}(x)$
from $S^-$ is less than

$$\text{length}(T^{-1}W^u_1(x)) = \lambda \text{length}(W^u_1(x)).$$

More generally,

$$W^u_{i+1}(x) = T(T^{-i}W^u_i(x) \cap W^u_i(T^{-i}(x))),$$

and $W^u_{i+1}(x) \neq W^u_i(x)$ only if

$$d(T^{-i}(x), S^-) < \lambda^i \text{length } (W^u_i(x)).$$

Since length $(W^u_i(x)) \leq \text{length } (W^u_i(x)) =: c_0$, we learn that if $W^u_i(x) = \{x\}$, then

$$d(T^{-i}(x), S^-) < c_0 \lambda^i.$$
for infinitely many $i$. Set $S^{-}_\delta = \{ x \in \Gamma^{-} : d(x, S^{-}) < \delta \}$. We can write

$$\{ x : W^u(x) = \{ x \} \} \subseteq \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} T^i(S^{-}_{c_0 \lambda_i}),$$

$$\mu \left( \{ x : W^u(x) = \{ x \} \} \right) \leq \lim_{n \to \infty} \sum_{i=n}^{\infty} \mu(T^i(S^{-}_{c_0 \lambda_i}))$$

$$= \lim_{n \to \infty} \sum_{i=n}^{\infty} \mu(S^{-}_{c_0 \lambda_i})$$

$$\leq \lim_{n \to \infty} \sum_{i=n}^{\infty} c_1 c_0 \lambda_i = 0$$

for some constant $c_1$. From this we deduce that for $\mu$-almost all points $x$, the set $W^u(x)$ is an interval of positive length with endpoints in $\bigcup_{i=0}^{\infty} T^i(S^{-})$. Moreover, if $y \in W^u(x)$, then

$$d(T^{-n}(y), T^{-n}(x)) = \lambda^n d(x, y) \to 0$$
as $n \to \infty$. In the same fashion, we construct $W^s(x)$.

We now apply the Hopf’s argument. To this end, let us take a dense subset $A$ of $C(T^2)$ and for $f \in C(T^2)$ define $f^\pm$ as in Example 6.1. Set $Y = \cap \{ Y_f : f \in A \}$, where

$$Y_f = \{ x \in T^2 : f^\pm(x), W^s(x), W^u(x) \text{ are well-defined and } f^+(x) = f^-(x) \}$$

So far we know that $\mu(Y) = 1$. Regarding $T^2$ as $[0, 1]^2$ with $0 = 1$ and slicing $T^2$ into line segments parallel to $v_i$ for $i = 0, 1$, we learn that each stable or unstable leaf intersects $Y$ on a set of full length, except for a family of leaves of total $\mu$-measure 0. Let us pick a leaf $W^s(x_0)$ which is not one of the exceptional leaf and define

$$Z(x_0) = \bigcup \{ W^u(y) : y \in W^s(x_0) \text{ and } y \in Y \}.$$

Since $W^u(y)$ is of positive length, for each $y \in W^s(x)$, we deduce that $\mu(Z(x_0)) > 0$. On the other hand $f^+$ is constant on $W^s(x_0)$ and $f^-$ is constant on each $W^u(y)$, $y \in W^s(x_0) \cap Y$. Since $f^+ = f^-$ on $W^s(x_0)$, we deduce that $f^+ = f^-$ is constant on $Z(x_0)$ for every $f \in A$.

With the aid of Hopf’s argument, we managed to show that $f^\pm$ is constant on a set of positive $\mu$-measure. But for ergodicity of $\mu$, we really need to show this on a set of $\mu$-full measure. This is where Hopf’s argument breaks down, however it does show that $\mu$ has at most countably many ergodic components. Indeed if we define

$$Z'(x_0) = \{ x : f^\pm(x) \text{ exist and } f^+(x) = f^-(x_0) \},$$
then $\mu(Z'(x_0)) > 0$ because $Z'(x_0) \supseteq Z(x_0)$. Since this is true for $\mu$-almost all $x_0$, we deduce that $\mu$ can only have countably many ergodic components.

We now explain how Sinai’s method can be used to prove the ergodicity of $\mu$. To this end, let us take a box $B$ with boundary lines parallel to $v_1$ and $v_2$ and define

$$W^u(B) = \{ y \in B \cap Y' : W^u(y) \cap Y \text{ is of full length and } W^u(y) \text{ reaches the boundary of } B \text{ on both ends} \}$$

where

$$Y' = Y'(f) = \{ y : f^+(y) \text{ and } f^-(y) \text{ are defined and } f^+(y) = f^-(y) \}.$$ 

In the same fashion we define $W^s(B)$. We now claim that $f^+$ is constant on $W^s(B)$, $f^-$ is constant on $W^u(B)$, and these constants coincide. To see this, we fix $W^u(y) \subseteq W^u(B)$ and take all $z \in W^u(y) \cap Y'$. We have that $f^-$ is constant on $W^u(y)$ and that $f^-(z) = f^+(z)$ for such $z \in W^u(y) \cap Y'$. Since $f^+$ is constant on each $W^s(z)$, we deduce that $f^+$ is constant on $\bigcup_{z \in W^u(y) \cap Y'} W^s(z) \cap Y')$ and this constant coincides with $f^-(y)$. By varying $y \in W^u(B)$, we obtain the desired result. (Here we are using the fact that if $W^u(y) \subseteq W^u(B)$ and $W^s(z) \subseteq W^s(B)$, then $W^u(y)$ and $W^s(z)$ intersect.)

Let us regard the vertical and horizontal axis as the stable and unstable directions. We now take two boxes which overlap. For example, imagine that $B_1 = I_1 \times J_1$, $B_2 = I_2 \times J_2$ in the $(v_2, v_1)$ coordinates, where either $J_1 = J_2$ and $I_1 \cap I_2 \neq \emptyset$, or $I_1 = I_2$ and $J_1 \cap J_2 \neq \emptyset$. We wish to show that the constant $f^\pm$ of $W^{u(s)}(B_1)$ equal the constant $f^\pm$ of $W^{u(s)}(B_2)$. We know that $f^+$ is constant on $W^s(B_1) \cap W^u(B_2)$ and that $f^-$ is constant on $W^u(B_1) \cap W^u(B_2)$. We also know that $f^+ = f^-$ in $Y'$. Clearly if $J_1 = J_2$, $I_1 \cap I_2 \neq \emptyset$ and $W^s(B_1) \cap W^u(B_2) \neq \emptyset$ (respect. $I_1 = J_2$, $J_1 \cap J_2 \neq \emptyset$ and $W^u(B_1) \cap W^u(B_2) \neq \emptyset$), then the constant $f^+$ (respect. $f^-$) for $W^s(B_1)$ (respect. $W^u(B_1)$) coincides with the constant $f^+$ (respect. $f^-$) for $W^s(B_2)$ (respect. $W^u(B_2)$). Let us identify a scenario for which $\mu(W^s(B_1) \cap W^u(B_2)) > 0$. Given $\beta \in (0, 1)$, let us call a box $B$ $\beta$-unconnected if the set

$$B^u = \{ x \in B : W^u(x) \text{ is defined and reaches the boundary of } B \text{ on both ends} \}$$

satisfies $\mu(B^u) > \beta \mu(B)$. The set $B^s$ is defined in a similar way and we say that $B$ is $\beta$-connected if $\mu(B^s) > \beta \mu(B)$. Note that if $\mu(B^u(s)) > \beta \mu(B)$, then $\mu(W^u(s)(B)) > \beta \mu(B)$ because $Y'$ is of full-measure. (Here we are using Fubini’s theorem to write the measures of $Y'$ as an integral of the lengths of $v_1$ or $v_2$ slices of $Y'$.) Now assume that $B_1$ and $B_2$ satisfy the following conditions:

- $B_1 = I_1 \times J_1$, $B_2 = I_2 \times J_2$, with $J_1 = J_2$,
- $\beta$-connected (respect. unconnected),
- $B_2$ is to the right of $B_1$ (respect. $B_2$ is on the top of $B_1$),

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\begin{itemize}
\item \( \mu(B_1 \cap B_2) \geq (1 - \beta) \max(\mu(B_1), \mu(B_2)) \),
\item \( \mu(W^u(B_1)), \mu((W^u(B_2)) > 0 \) (respect. \( \mu(W^s(B_1)), \mu((W^s(B_2)) > 0) \)),
\end{itemize}
then for sure \( \mu(W^s(B_1) \cap W^s(B_2)) > 0 \) (respect. \( \mu(W^u(B_1) \cap W^u(B_2)) > 0 \)). Simply because \( W^s(B_1) \cap B_2 \subseteq W^s(B_2) \), and \( \mu(W^s(B_1) \cap B_2) > 0 \).

Based on this observation, let us take a box \( \bar{B} \) and cover it by overlapping small boxes. Pick \( \beta \in (0, 1/2) \) and take a grid \( \{ \beta_i \in \bar{B} : i \in \mathbb{Z}^2 \} \) and use the points of this grid as the center of squares of side length \( \frac{1}{n} \). Each such square has area \( \frac{1}{n^2} \), and two adjacent squares overlap on a set of area \( (1 - \beta) \frac{1}{n^2} \). Let us write \( B^\beta_n(\bar{B}) \) for the collection of such overlapping squares. We now state a key result of Sinai regarding the \( \alpha \)-u(s)connected boxes.

**Theorem 6.3** There exists \( \alpha_0 < 1 \) such that for every \( \beta \in (0, \alpha_0) \),

\[
\lim_{n \to \infty} n \mu \left( \bigcup \{ B \in B^\beta_n(\bar{B}) : B \text{ is not either } \beta \text{-uconnected or } \beta \text{-sconnected} \} \right) = 0.
\]

We now demonstrate how Theorem 6.3 can be used to show that \( f^+ \) and \( f^- \) are constant almost everywhere in \( \bar{B} \). We choose \( \beta < \alpha < \alpha_0 \) and would like to show that if \( y, z \in Y' \cap \bar{B} \), then \( f^-(y) = f^+(z) \).

To prove this, we first claim that there exists a full column of boxes in \( B^\beta_n(\bar{B}) \) such that each box \( B \) in this column is \( \alpha \)-uconnected and \( W^u(y) \) reaches two boundary sides of a box in the column provided that \( n \) is sufficiently large. Here \( y \) is fixed and since \( W^u(y) \) is a nontrivial interval, it crosses \( c_1 n \) many columns of total area. If each such column has a box which is not \( \alpha \)-uconnected, then

\[
\mu(\bigcup \{ B \in B^\beta_n(\bar{B}) : B \text{ is not } \alpha \text{-uconnected} \}) \geq c_3 n \cdot \frac{1}{n^2}
\]

for some \( c_3 > 0 \) (note that a point \( x \) belongs to at most \( \left( \frac{1}{23} + 1 \right)^2 \) many boxes). This contradicts Theorem 2.2 for large \( n \). Hence such a column exists. Similarly, we show that there exists a full row of boxes in \( B^\beta_n(\bar{B}) \) such that each box is \( \alpha \)-sconnected and at least one box in this row is fully crossed by \( W^s(z) \). Since \( \beta < \alpha \), we now that \( f^- \) is constant (with the same constant) on \( \cup W^s(B) \) with the union over the boxes \( B \) on that row, and that \( f^+ \) is constant on \( \cup W^u(B) \) with union over the boxes \( B \) on that column. Since the row and the column intersect on a box, we deduce that \( f^+(y) = f^-(z) \). This completes the proof of \( f^+ = f^- = \text{constant a.e. in } \bar{B} \). We now turn to the proof of Theorem 6.3.
**Proof of Theorem** 6.3. First we define a sector

\[ C = \{(a, b) \in \mathbb{R}^2 : |a| \leq \gamma |b|\} \]

which is symmetric about the unstable line \( v_2 \) and contains the two directions of sides of \( \Gamma^- \). We use the explicit value of the slope of \( v_2 \) to see that in fact \( \gamma \) can be chosen in \((0, 1)\). We now argue that all the line segments in \( \bigcup_0^\infty T^i(S^-) \) have directions in the sector \( C \). This is because \( C \) already has the directions of \( S^- \). On the other hand, since the sides of \( S^- \) are not parallel to \( v_1 \), \( T^i \) pushes these lines toward \( v_2 \).

Now let us measure the set of points not in \( W^u(B) \) for a box in \( \mathcal{B}_n^2(B) \). Note that if a point \( x \in B \) is not in \( W^u(B) \), it means that \( W^u(x) \) is cut by one of \( T^i(S^-), i \in \mathbb{N}^* \) inside \( B \). Let us first consider the case when \( B \) is intersected by precisely one line segment of \( \bigcup_i T^i(S^-) \). Since this line segment is in sector \( C \), we learn that \( \mu(B - W^u(B)) \leq \frac{2}{\mu_x} \). This means

\[ \mu(W^u(B)) \geq (1 - \gamma)\mu(B). \]

Let us choose \( \alpha_0 = \frac{1}{2}(1 - \gamma) \) so that if \( \beta < \alpha_0 \) and \( B \) is not \( \beta \)-unconnected, then \( B \) must intersect at least two segments in \( \bigcup_i T^i(S^-) \). (This would be true even when \( \beta < 1 - \gamma \) but we need a smaller \( \beta \) later in the proof.) We now look at \( R_L = \bigcup_{i=0}^{L-1} T^i(S^-) \) and study those boxes which intersect at least two line segments in \( R_L \). Note that each box \( B \) is of length \( 1/n \) and the line segments in \( R_L \) are distinct. So, a box \( B \in \mathcal{B}_n^2 \) intersects at least two lines in \( R_L \) only if it is sufficiently close to an intersection point of two lines in \( R_L \). More precisely, we can find a constant \( c_1(L) \) such that such a box is in a \( c_1(L)/n \) neighborhood of an intersection point. (In fact \( c_1(L) \) can be chosen to be a constant multiple of \( L^2e^{c_0L} \) because there are at most \( 4L(4L - 1) \) intersection points and the smallest possible angle between two line segment in \( R_L \) is bounded below by \( e^{-c_0L} \) for some constant \( c_0 \).) Hence the total area of such boxes is \( c_1(L)n^{-2} \). Now we turn to those boxes which intersect at most one line in \( R_L \) and at least one line in \( R'_L = \bigcup_{i=0}^\infty T^i(S^-) \). Let us write \( \mathcal{D}_L \) for the set of such boxes. Let us write \( B - W^u(B) = B'_L \cup B''_L \), where

\[ B'_L = \{ x \in B : W^u(x) \cap B \cap R_L \neq \emptyset \} \]
\[ B''_L = \{ x \in B : W^u(x) \cap B \cap R' \neq \emptyset \}. \]

If \( B \in \mathcal{D}_L \), then \( B \) can intersect at most one line segment in \( R_L \). Hence \( \mu(B'_L) \leq \gamma \mu(B) \leq (1 - 2\beta)\mu(B) \). If \( B \in \mathcal{D}_L \) is not \( \beta \)-unconnected, then

\[ (1 - \beta)\mu(B) \leq \mu(B - W^u(B)) \leq (1 - 2\beta)\mu(B) + \mu(B''_L). \]

From this we deduce

\[
\mu \left( \bigcup \{ B \in \mathcal{D}_L : \text{B is not } \beta\text{-unconnected} \} \right) \leq \sum \{ \mu(B) \in \mathcal{D}_L : \text{B is not } \beta\text{-unconnected} \}
\]
\[
\leq \beta^{-1} \sum \{ \mu(B''_L) \in \mathcal{D}_L : \text{B is not } \beta\text{-unconnected} \}
\]
\[
\leq \beta^{-1} c\mu \left( \bigcup \{ B''_L \in \mathcal{D}_L : \text{B is not } \beta\text{-unconnected} \} \right),
\]

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where for the last inequity we have used the fact that each point belongs to at most $c(\beta) = (1/(2\beta) + 1)^2$ many boxes in $B^\beta_n$. Let $x \in B'_n$ for some $B \in D_L$. This means that $W^u(x) \cap B$ intersects $T^i(S^-)$ for some $i \geq L$. Hence $T^{-i}(W^u(x) \cap B) \cap S^- \neq \emptyset$. Note that $T^{-i}(W^u(x) \cap B)$ is a line segment of length at most $\lambda^{-1}n^{-1}$. As a result, $T^{-i}(x)$ must be within $\lambda n^{-1}$-distance of $S^-$. That is, $x \in T^i(S^-_{\lambda n^{-1}})$. So,

$$\mu \left( \bigcup \{B''_n : B \in D_L \} \right) \leq \mu \left( \bigcup_{i=L}^{\infty} T^i(S^-_{\lambda n^{-1}}) \right) \leq \sum_{i=L}^{\infty} \mu \left( T^i(S^-_{\lambda n^{-1}}) \right) = \sum_{i=L}^{\infty} \mu(S^-_{\lambda n^{-1}}) \leq c_2 \sum_{i=L}^{\infty} n^{-1} \lambda^i \leq c_3 n^{-1} \lambda^L.$$

This yields

$$\mu \left( \bigcup \{ B \in B^\beta_n(\bar{B}) : B \text{ is not } \alpha \text{-usconnected} \} \right) \leq c_1(L)n^{-2} + c_4(\beta)n^{-1} \lambda^L$$

for every $n$ and $L$. By choosing $L = \eta \log n$ for $\eta = (c_0 - \log \lambda)^{-1}$, we get

$$c_5 n^{-1+\eta \log \lambda (\log n)^2},$$

for the right-hand side. This completes the proof of Theorem 6.3. \hfill \Box

**References**


[R] F. Rezakhanlou, “Lectures on Large Deviation Principle”.

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