Homework 8 and 9

Parts of problems 8.1, 8.5, 9.3, 9.4

$\mathbb{R}^2$: Note that normal vector to $\mathbb{R}^2$ is constant. This implies that $L_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Therefore $H = 0 = K$ and $R_{iljk} = 0$ for all i, j, k, l

$S^1 \times (0,1)$: Problem 10.3 of homework 9 shows that $\mathbb{R}^2$ is locally isometric to $S^1 \times (-\infty, \infty)$. Note $S^1 \times (-\infty, \infty)$ is locally isometric to $S^1 \times (0,1)$. Thus $H = 0 = K$ and $R_{iljk} = 0$ for all i, j, k, l.

Surface of revolution: Take parameterization

$$X(t, \theta) = (r(t)\cos\theta, r(t)\sin\theta, z(t)) \text{ where}$$

$$(r')^2 + (z')^2 \equiv 1. \text{ Note}$$

$$X_t = (r'\cos\theta, r'\sin\theta, z')$$

$$X_\theta = (-r\sin\theta, r\cos\theta, 0)$$

implies

$$n = \frac{X_t \times X_\theta}{|X_t \times X_\theta|} = \frac{1}{r\left[(z')^2 + (r')^2\right]}(-rz'\cos\theta, -rz'\sin\theta, rr')$$

$$= (-z' \cos\theta, -z' \sin\theta, r')$$

Note

$$g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} \text{ implies}$$

$$g^{ij} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{bmatrix}.$$ Therefore

$$L_{ij} = \begin{bmatrix} n \cdot x_{tt} & n \cdot x_{t\theta} \\ n \cdot x_{t\theta} & n \cdot x_{\theta\theta} \end{bmatrix}$$

$$= \begin{bmatrix} r'z'' - r''z' & 0 \\ 0 & rz' \end{bmatrix}$$

and

$$L^i{}_j = \begin{bmatrix} r'z'' - r''z' & 0 \\ 0 & \frac{z'}{r} \end{bmatrix}$$

This was shown in problem 7.5 of homework 7.

We conclude that

$$H = \frac{1}{2}\left(\frac{z'}{r} + r'z'' - r''z'\right)$$

$$K = \frac{z'}{r}\left(r'z'' - r''z'\right)$$

Since L_{ij} is diagonal, the lines of curvature are $\{t = \text{constant}\}$ and $\{\theta = \text{constant}\}$. This shows problem 8.5 of homework 8

Note

$$L_{ik}L_{\ell j} = \begin{cases} L_{11}L_{11} \\ L_{11}L_{22} \\ L_{22}L_{22} \\ 0 & \text{otherwise} \end{cases}$$

This implies

$$R_{1221} = L_{11}L_{22} = rz'(r'z'' - r''z')$$

By problem 9.2 of homework 9 we have

$$R_{1221} = R_{2112} = -R_{2121} = -R_{1212}$$

Therefore

$$R_{i\ell jk} = \begin{cases} rz'(r'z'' - r''z') & (i,\ell,j,k) = (1,2,2,1) \\ & (i,\ell,j,k) = (2,1,1,2) \\ -rz'(r'z'' - r''z') & (i,\ell,j,k) = (2,1,2,1) \\ & (i,\ell,j,k) = (1,2,1,2) \\ 0 & \text{otherwise} \end{cases}$$

S^2: We have $r(t) = \cos t$ and $z(t) = \sin t$.

Therefore

$$L^i_j = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$H = 1$$

$$K = 1$$

$$R_{1221} = R_{2112} = -R_{2121} = -R_{1212} = \cos^2 t$$

T^2: We have $r(t) = 2 + \cos t$ and $z(t) = \sin t$

Therefore

$$L^i{}_j = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{\cos t}{2+\cos t} \end{bmatrix}$$

$$H = \frac{1+\cos t}{2+\cos t}$$

$$K = \frac{\cos t}{2+\cos t}$$

$$R_{1221} = R_{2112} = -R_{2121} = -R_{1212}$$

$$= (1+\cos t)^2 - 1$$