**Math 126**, Springer 2010, F.Rezakhanlou

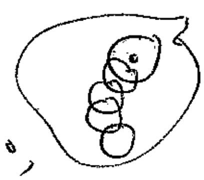
**Your Name:**

1. (a) (2 points) State the mean value property for harmonic functions.

$$\Delta u = 0 \Rightarrow \frac{1}{\mathrm{vol}(B(a,r))}\int_{B(a,r)} u = u(a).$$

(b)(5 points) State and prove the strong form of maximum principle for harmonic functions.

$u$ harmonic in $\Omega$, $\Omega$ bounded connected, $u$ continuous up to $\partial\Omega$, and if the max of $u$ is attained at $a \in \Omega$, then $u$ is constant: If $B(a,r) \subseteq \Omega$, then



2 $\frac{1}{|B(a,r)|}\int_{B(a,r)} u = u(a) \Rightarrow \frac{1}{|B(a,r)|}\int_{B(a,r)} (u - u(a)) = 0$, but $u - u(a) \le 0$,

So $u - u(a) \equiv 0$ in $B(a,r)$. Take any $b \in B(a,r)$ and $B(b,r') \subseteq \Omega$

2 and repeat to deduce $u \equiv u(a)$ in $B(b,r')$ also. Now any $b$ in $\Omega$ can be connected to $a$ by a chain of overlapping balls in $\Omega$. So $u \equiv u(a)$.

(c) (5 points) State and prove uniqueness for the Neumann boundary problem for the PDE $\Delta u = f$ in a bounded domain $D$. Suppose $\begin{cases} \Delta u_i = f & \Omega \text{ for } i = 1,2 \\ \frac{\partial u_i}{\partial n} = g & \partial\Omega \end{cases}$ 1

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If $w = u_2 - u_1$, then $\begin{cases} \Delta w = 0 & \text{in } \Omega \\ \frac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$

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Now $\int_\Omega w \Delta w = 0 = -\int_\Omega |\nabla w|^2 + \int_{\partial\Omega} w \frac{\partial w}{\partial n} = -\int_\Omega |\nabla w|^2 \Rightarrow w$ const.

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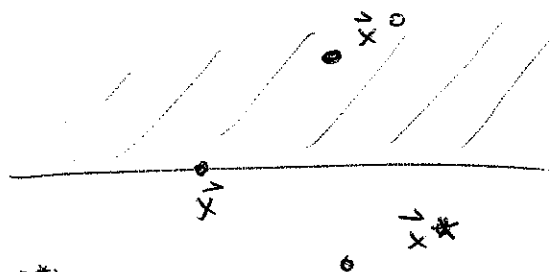
2. (a) (4 points) What is the definition of Green's function of a bounded domain? Prove that the Green's function of a bounded domain is unique.

2 (i) $\Delta(G-\Phi) = 0, \quad G = 0$ on $\partial\Omega$

(ii) Uniqueness: If $G_1, G_2(x^0, x)$ are two Green's function, then $G = G_1 - G_2$ would satisfy $\Delta G \equiv 0$ in $\Omega$, $G = 0$ on $\partial\Omega$ but by max principle $G \equiv 0$.

(b)(6 points) What is the Green's function of the upper half plane $\{(x,y) : y > 0\}$? Derive a formula for a harmonic function in the upper half plane with a prescribed boundary condition.




If $\vec{x}^0 = (x_0, y_0)$, then set

$\vec{x}^* = (x_0, -y_0)$. Now

2 $$G(\vec{x}, \vec{x}^0) = \frac{1}{2\pi}\log|\vec{x} - \vec{x}^0| - \frac{1}{2\pi}\log|\vec{x} - \vec{x}^*|.$$

For Laplace problem, we get

2 $$u(\vec{x}^0) = \int_{y=0} g(\vec{x}) \frac{\partial G}{\partial \vec{n}}(\vec{x}, \vec{x}^0)\, dx$$

1 But $$\nabla G = \frac{1}{2\pi}\left(\frac{\vec{x} - \vec{x}^0}{|\vec{x} - \vec{x}^0|^2} - \frac{\vec{x} - \vec{x}^*}{|\vec{x} - \vec{x}^*|^2}\right) = \frac{\vec{x}^* - \vec{x}^0}{2\pi|\vec{x} - \vec{x}^0|^2}$$ if $\vec{x}$ is on the boundary

$$= \frac{-y}{\pi|\vec{x} - \vec{x}^0|^2}.$$

So $$\frac{\partial G}{\partial n} = \frac{y}{\pi|\vec{x} - \vec{x}^0|^2}$$ and

1 $$u(\vec{x}^0) = \int \frac{g(x)}{\pi[(x - x_0)^2 + y_0^2]}\, dx.$$

3. (6 points) Solve $2u_{tt} + 5u_{xt} + 2u_{xx} = 0$,

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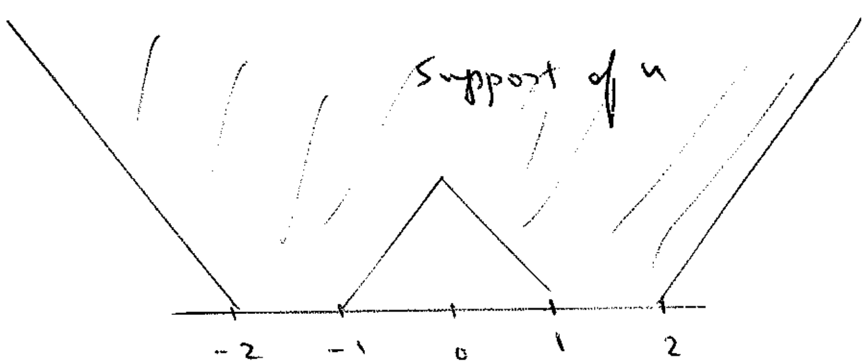
$$2\frac{\partial^2}{\partial t^2} + 5\frac{\partial}{\partial t}\frac{\partial}{\partial x} + 2\frac{\partial^2}{\partial x^2} = \left(2\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + 2\frac{\partial}{\partial x}\right)$$

2 Since $\left(2\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) w = 0$ means that $w = F(x - \frac{t}{2})$

2 and $\left(\frac{\partial}{\partial t} + 2\frac{\partial}{\partial x}\right) w = 0$ means that $w = G(x-2t)$,

1 we get $u(x,t) = F(x - \frac{t}{2}) + G(x-2t)$.

4. (2 points) Suppose that $u$ is a solution to $u_{tt} = u_{xx}$. Suppose that the support of $u$ initially is contained in the intervals $[-2,-1]$ and $[1,2]$. That is, $u(x,0) = 0$ if either $x \leq -2$, or $-1 \leq x \leq 1$ or $x \geq 1$. What can be said about the support of $u$ at later times? Explain your answer.




Use Causality principle.

5. (5 points) (a) Find a series expansion for $u_t = u_{xx} + 2u$ in the interval $[0,1]$ with boundery conditions $u(0,t) = u(1,t) = 0$. Try $w(x,t) = X(x)\,T(t)$

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to get $T'X = X''T + 2XT$, so

$X'' + 2X = \lambda X$

1 $\dfrac{T'}{T} = \dfrac{X'' + 2X}{X} = -\lambda$. To solve

with $X(0) = X(1) = 0$, observe $X'' = (\lambda - 2)X$ and

to have solution, we need $\lambda - 2 < 0$ with

1 $\lambda - 2 = -(k\pi)^2$ for some positive integer $k$.

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$X$ would look like $\sin k\pi x$. The corresponding

$T$ is $e^{(-(k\pi)^2+2)t}$. So the general solution

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is $u(x,t) = \sum_{n=1}^{\infty} a_n e^{(2-(k\pi)^2)t} \sin k\pi x$.

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(b) (2 points) Let $u$ be a solution to $u_t = (u^3)_{xx}$ and $u(x,t) = 0$ for large $x$ and $t \in [0,1]$. Show that $\int u^4(x,t)dx$ is non-increasing in $t$ for $t < 1$.

$$\frac{d}{dt}\int u^4 \underset{1}{=} 4\int u^3 (u^3)_{xx} \underset{1}{=} -4\int\left[(u^3)_x\right]^2 \le 0.$$