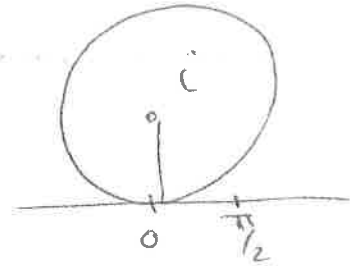


Math 121a, Fall 2021, F.Rezakhanlou

1. (5 points) Define
$$f(z)=\frac{\sin z - z}{z^5}\sin\left(\frac{1}{2z-\pi}\right).$$
Find all the isolated singularitis of f and identify each as a removable, a pole (give the order) or an essential singularity. What is the radius of convergence of the Taylor expansion of f at $z=i$?

1

The singularities are at $z=0$ and $z=\pi/2$.

Since $\sin a = a - \frac{a^3}{3!} + \frac{a^5}{5!} - \cdots$, and $a = \frac{1}{2z-\pi}$, we deduce that $z=\pi/2$ is essential. On the other hand at $z=0$ we have
$$\frac{\sin z - z}{z^5} = -\frac{z^3}{3!\,z^5} + \frac{1}{5!} - \cdots,$$
so this has a pole of order 2. Since $\sin\left(\frac{1}{2z-\pi}\right)$ is analytic near 0, pole is of order 2

2. (5 point) Find the Laurent expansion of $f(z)=(1+z^3)^{-1}+(z+4i)^{-1}$ in the set $\{z : 1<|z|<4\}$.

When $1<|z|<4$, we have
$$\frac{1}{z+4i} = \frac{1}{4i}\,\frac{1}{\frac{z}{4i}+1} = \frac{1}{4i}\left(1-\frac{z}{4i}+\left(\frac{z}{4i}\right)^2 - \cdots\right),$$
$$\frac{1}{1+z^3} = \frac{1}{z^3}\left(\frac{1}{1+\frac{1}{z^3}}\right) = \frac{1}{z^3}\left(1-\frac{1}{z^3}+\frac{1}{z^6}-\cdots\right)$$

So
$$f(z) = \frac{1}{4i} - \frac{z}{(4i)^2} + \cdots + (-1)^{n+1}\frac{z^n}{(4i)^{n+1}} + \cdots$$
$$+ \frac{1}{z^3} - \frac{1}{z^6} + \cdots + (-1)^n \frac{1}{z^{3(n+1)}} - \cdots$$

3. (10 points) Evaluate $\int_0^{2\pi} \frac{1}{3i-sin\theta}d\theta$ and $\int_{-\infty}^{\infty} \frac{x\sin x}{16x^2-\pi^2}dx$.

(1) $$\int_0^{2\pi} \frac{d\theta}{3i-\sin\theta} = \oint_C \frac{1}{3i-\frac{z-z^{-1}}{2i}}\,\frac{dz}{iz} = -\oint \frac{2\,dz}{6z+z^2-1}$$

$z^2+6z-1=0$, $z=-3\pm\sqrt{10}$, $\begin{cases} z_0=-3+\sqrt{10} \\ z_1=-3-\sqrt{10} \end{cases}$ inside the unit circle. Hence

$$\int_0^{2\pi} \frac{d\theta}{\sin\theta+3i} = -2\pi i\, R(-3+\sqrt{10}) \quad \text{for}$$

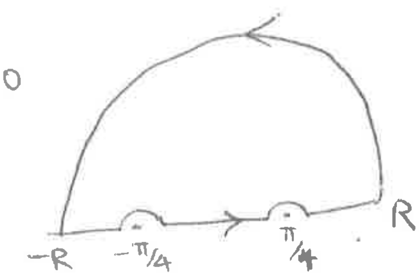
$$f(z)=\frac{2}{(z-z_0)(z-z_1)}, \quad R(F)=\frac{2}{z_0-z_1}=\frac{1}{\sqrt{10}}.$$

Hence integral $= \frac{-2\pi i}{\sqrt{10}}$.

(2) $\int_{-\infty}^{\infty} \frac{xe^{ix}}{16x^2-\pi^2}dx$ Use $\int_\gamma \frac{ze^{iz}}{16z^2-\pi^2}dz=0$ for γ ↓

$$\left|\int_{\frown} \frac{ze^{iz}}{16z^2-\pi^2}dz\right| \le \int_0^{\pi} \frac{R^2}{16R^2-\pi^2}e^{-R\sin\theta}d\theta \to 0$$

as $R\to\infty$ because $e^{-R\sin\theta}\to 0$ for $\theta\in(0,\pi)$.



$$\int_{\frown\,\pi/4} \frac{ze^{iz}\,dz}{16(z-\pi/4)(z+\pi/4)} = \pi i\, R(\pi/4) = \pi i \left.\frac{ze^{iz}}{16(z+\pi/4)}\right|_{z=\pi/4} = \pi i\,\frac{e^{i\pi/4}}{32}$$

$$\int_{\frown\,-\pi/4} = \pi i\, R(-\pi/4) = \pi i\,\frac{e^{-i\pi/4}}{32}.$$ Hence

$$\int_{-\infty}^{\infty} \frac{xe^{ix}}{16x^2-\pi^2}dx = \frac{\pi i}{16}\,\frac{\sqrt{2}}{2}, \qquad \int_{-\infty}^{\infty} \frac{x\sin x}{16x^2-\pi^2}dx = \frac{\pi\sqrt{2}}{32}.$$

4. (4 points) Find the disc of the convergence of the series

$$\sum_{n=1}^{\infty} 3^n \ln(n+\sqrt{n})\ (iz+i+1)^n := \sum_{1}^{\infty} 3^n\, i^n \ln(n+\sqrt{n})\,(z+1-i)$$

$\{ z : |z+1-i| < R \}$ where $R = \frac{1}{\rho} = \frac{1}{3}$ because

$$\rho = \lim_{n\to\infty} \frac{3^{n+1} \ln(n+1+\sqrt{n+1})}{3^n \ln(n+\sqrt{n})} = 3.$$

5. (4 points) Find

$$\left(\frac{1+i}{1-i}\right)^{1/5}.$$

$$\frac{1+i}{1-i} = i = e^{i\pi/2}$$

$$e^{i\frac{\pi}{10} + i\frac{2\pi}{5}k}$$

for $k = 0, 1, 2, 3, 4$.

6. (4 points) Let f be an analytic function. Given a complex number a, define

$$g(z) = \frac{f(z)}{(z-a)^2}.$$

Find the residue of g at a. This this to evaluate the contour integral

$$\int_C \frac{f(z)}{(z-a)^2}\, dz,$$

where C is a closed simple curve that is oriented positively.

Since f is analytic, $f(z) = f(a) + f'(a)(z-a) + \cdots$

Hence $\dfrac{f(z)}{(z-a)^2} = \dfrac{f(a)}{(z-a)^2} + \dfrac{f'(a)}{z-a} + \cdots$

$R(g) = f'(a)$,

$$\int_C \frac{f(z)}{(z-a)^2} = 2\pi i\, f'(a).$$