HW 8

$\checkmark$

8.3 (6) $y' + \dfrac{y}{\sqrt{x^2+1}} = \dfrac{1}{x+\sqrt{x^2+1}}$.

Here $P = \dfrac{1}{\sqrt{x^2+1}}$, its anti derivative is $\ln(x+\sqrt{x^2+1})$. So $e^{-\int P} = \dfrac{1}{x+\sqrt{x^2+1}}$

and $\dfrac{c}{x+\sqrt{x^2+1}}$ is the general solution of $y' + \dfrac{y}{\sqrt{x^2+1}} = 0$.

For a particular solution, observe if y_p solves the ODE, then $(y_p e^{\int P})' = y_p' + P y_p = Q y_p = 1$.

Hence $y_p = x e^{-\int P} = \dfrac{x}{x+\sqrt{x^2+1}}$.

(9) $y' - \dfrac{x}{1-x^2} y = \dfrac{2x}{\sqrt{1-x^2}}$ with $P = -\dfrac{x}{1-x^2}$, $Q = \dfrac{2x}{\sqrt{1-x^2}}$.

$e^{-\int P} = e^{\int \frac{x}{1-x^2}} = e^{-\frac{1}{2}\ln(1-x^2)} = \dfrac{1}{\sqrt{1-x^2}}$.

Homogeneous ODE : $\dfrac{c}{\sqrt{1-x^2}}$.

Particular Solution : $\dfrac{1}{\sqrt{1-x^2}} \int Q\sqrt{1-x^2} = \dfrac{x^2}{\sqrt{1-x^2}}$.

General : $\dfrac{c+x^2}{\sqrt{1-x^2}}$

(14) $\dfrac{dx}{dy} = \dfrac{3y^{\frac{2}{3}} - x}{3y} = y^{-\frac{1}{3}} - \dfrac{1}{3y} x$.

So $P(y) = \dfrac{1}{3y}$, $Q(y) = y^{-1/3}$.

Homogeneous Solution: $c\, e^{-\int \frac{dy}{3y}} = c\, e^{-\frac{1}{3}\ln y} = \dfrac{c}{y^{1/3}}$.

Particular Solution: $\dfrac{1}{y^{1/3}} \int Q\, y^{1/3}\, dy = \dfrac{y}{y^{1/3}} = y^{2/3}$.

General Solution: $\dfrac{c}{y^{1/3}} + y^{2/3}$.

8.5 (4) $y'' + 2y' + 2y = 0$

$P(D) = D^2 + 2D + 2 = (D - (-1-i))(D - (-1+i))$.

Solution: $e^{-x}(c_1 \cos x + c_2 \sin x)$

(9) $D^2 - 4D + 13 = (D-2)^2 + 9 = (D - 2 + 3i)(D - 2 - 3i)$

Solutions $y(x) = e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$

(11) $4D^2 + 12D + 9 = (2D + 3)^2$

$y(x) = e^{-3/2\, x}(c_1 x + c_2)$

8.5 (18) Note that if $v = e^{ax}(1, x, x^2)$,

then $v' = av + e^{ax}(0,1,2x) =: av + w$,

$v'' = av' + e^{ax}(0,0,2) = av' + 2e^{ax} e_3$.

So,

$$\begin{vmatrix} v \\ v' \\ v'' \end{vmatrix} = \begin{vmatrix} v \\ av+w \\ av'+2e^{ax}e_3 \end{vmatrix} = \begin{vmatrix} v \\ av+w \\ 2e^{ax}e_3 \end{vmatrix} = \begin{vmatrix} v \\ w \\ 2e^{ax}e_3 \end{vmatrix}$$

$$= e^{ax} \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2e^{ax}.$$

(19) $D^2 + (1+2i)D + i - 1 = \left(D + \frac{1}{2} + i\right)^2 - \frac{1}{4}$

Roots $= -\left(\frac{1}{2} + i\right) \pm \frac{1}{2} = \begin{cases} -(1+i) \\ -i \end{cases}$

Solutions : $c_1 e^{-ix} + c_2 e^{-(1+i)x}$

(24) $y''' + y = 0$: $D^3 + 1 = 0$:

$y(x) = c_1 e^{-x} + c_2 e^{-x/2} \cos\frac{\sqrt{3}}{2}x + c_3 e^{-x/2} \sin\frac{\sqrt{3}}{2}x$

(25) $(D^4 - 1)^2 y = 0$; roots $1, -1, i, -i$ double roots

$(c_1 + c_2 x)e^{x} + (c_3 + c_4 x)e^{-x} + (c_5 + c_6 x)\cos x + (c_7 + c_8 x)\sin x$.

8.6 (6) $P(D) = D^2+6D+9 = (D+3)^2, \quad P(-1)=4.$

$P(D)y = 12e^{-x} \implies y = (c_1+c_2x)e^{-3x} + 3e^{-x}$

(13) $\underbrace{(D^2-2D+1)}_{P(D)} y = (D-1)^2 y = 2\cos x = 2\,\mathrm{Re}(e^{ix}).$

$P(i) = (i-1)^2 = -2i. \quad y_p(x) = 2\,\mathrm{Re}\left(\frac{1}{P(i)}e^{ix}\right) = \sin x.$

(20) $\underbrace{y'' + 4y' + 8y}_{P(D)y} = 30\,\mathrm{Re}\left(e^{(-\frac{1}{2}+i\frac{5}{2})x}\right).$

Roots: $-2\pm 2i$, $\quad y_p(x) = 30\,\mathrm{Re}\left(\frac{1}{P(-\frac{1}{2}+i\frac{5}{2})}e^{(-\frac{1}{2}+i\frac{5}{2})x}\right)$

$$= 4e^{-x/2}\sin\tfrac{5}{2}x.$$

(25) $(D-3)(D-1)y = 16x^2e^{-x}, \quad P(r) = (r-3)(r-1).$

$$\frac{2}{P(r)} = \frac{1}{r-3} - \frac{1}{r-1},$$

$$\frac{d}{dr}\left(\frac{2}{P(r)}e^{rx}\right) = \left[\frac{1}{(r-1)^2} - \frac{1}{(r-3)^2} + \frac{x}{r-3} - \frac{x}{r-1}\right]e^{rx}$$

$$\frac{d^2}{dr^2}\left(\frac{2}{P(r)}e^{rx}\right) = \left[\frac{1}{(r-3)^3} - \frac{1}{(r-1)^3} + (2x)\left(\frac{1}{(r-1)^2} - \frac{1}{(r-3)^2}\right) + x^2\left(\frac{1}{r-3} - \frac{1}{r-1}\right)\right]e^{rx}$$

At $r=-1$ we get $8\left[\frac{-1}{4^3} + \frac{1}{2^3} + 2x\left(\frac{1}{4} - \frac{1}{16}\right) + x^2\left(\frac{1}{2} - \frac{1}{4}\right)\right]e^{-x}$

$$y_p(x) = \left(2x^2 + 3x + \tfrac{7}{8}\right)e^{-x}$$

8.6 (26) $P(D) = D^2+1 = \overbrace{(D+i)}^{Q(D)}(D-i)$

Consider $P(D)\, y = 8x\, e^{ix}$

$$\left(\frac{d}{dr}\right)^2\left(\frac{1}{2}\,\frac{i}{Q(r)}\,e^{rx}\right) = \left(\frac{d}{dr}\right)^2\left[\frac{1}{2}\,\frac{1}{r+i}\,e^{rx}\right]$$

$$= \frac{d}{dr}\,\frac{1}{2}\left[\frac{-1}{(r+i)^2}\,e^{rx} + \frac{x}{r+i}\,e^{rx}\right]$$

$$= \frac{1}{2}\left[\frac{2}{(r+i)^3} + \frac{-2x}{(r+i)^2} + \frac{x^2}{r+i}\right]e^{rx}$$

Set $r = i$ and multiply by 8 :

$$y(x) = \left(i + 2x - 2ix^2\right)e^{ix}$$

$$\operatorname{Im}(y(x)) = \cos x + 2x\sin x - 2x^2\cos x .$$

General Solution: $c_1 \sin x + c_2 \cos x + 2x \sin x - 2x^2 \cos x$

8.6 (29) Note

$(D-a)(D-b) = D^2 - (a+b)D + ab$. If

$\deg Q_n = n$, the $\deg (D^2-(a+b)D + ab)Q_n = n$

only if $ab \neq 0$. Hence matching coefficients

of x^k for $k \leq n$ yields $(n+1)$ equations for $(n+1)$ unknowns.

If $ab = 0$ but $a+b \neq 0$, then since

$$D(xQ_n(x)) = Q_n(x) + x(DQ_n)(x)$$

is a polynomial of degree n, we can match

it with the right-hand side P_n in the

equation $(D^2-(a+b)D)(xQ_n(x)) = P_n(x)$

and have again $(n+1)$ equations of $(n+1)$ unknowns.

Finally,

$$D^2(x^2Q_n(x)) = D\left(2x\, Q_n(x) + x^2 (DQ_n)(x)\right)$$

$$= 2\, Q_n(x) + 4x\, DQ_n(x) + x^2 D^2Q_n(x)$$

is a polynomial of degree n, so the matching

is possible.