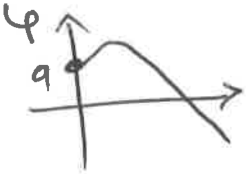
HW 7

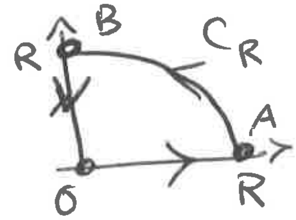
14.7

(43) Consider $f(z) = z^3 + z^2 + 9$.

Observe that if $\varphi(t) = (-t)^3 + (-t)^2 + 9 = -t^3 + t^2 + 9$, then



φ intersects the t-axis once. This means f has one root on $(-\infty, 0]$. We now claim that f has a root a in the first quadrant (which yields a root $\bar{a}$ in the fourth quadrant). To see this consider C_R as in the figure

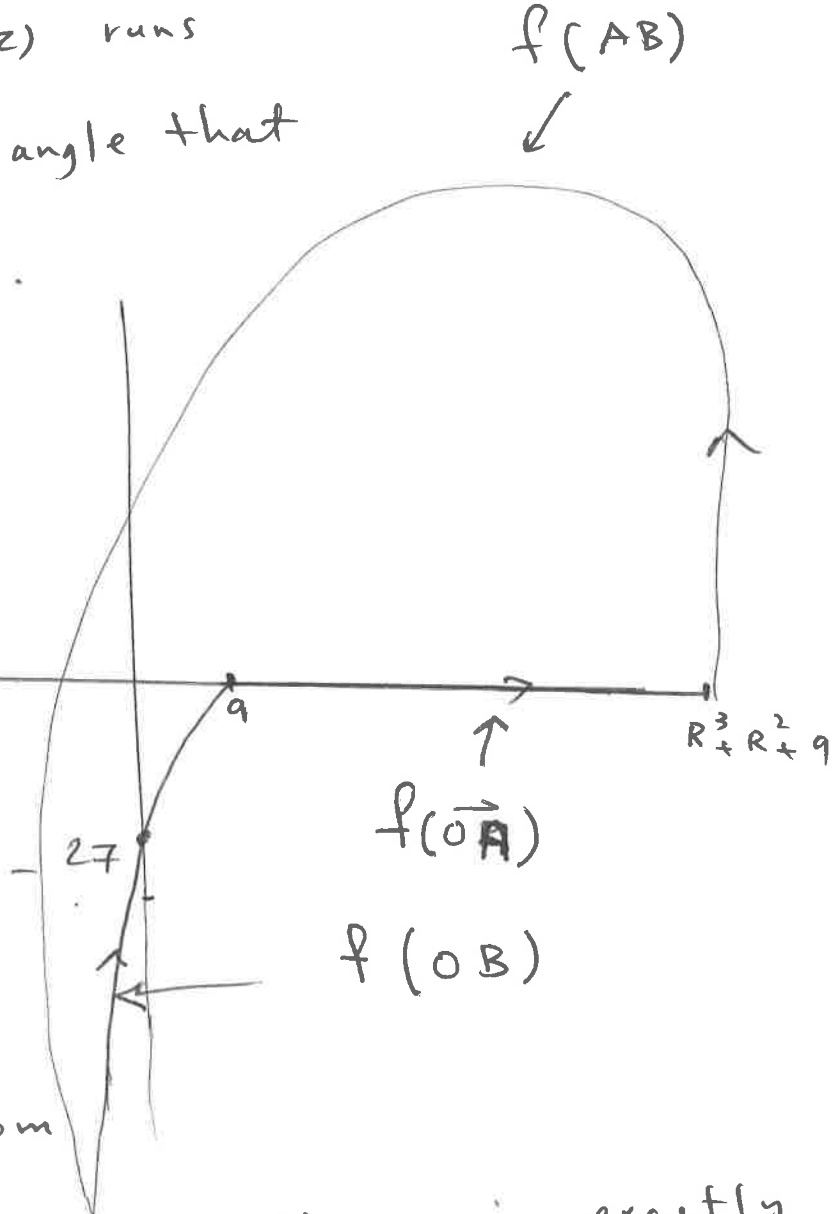


with R very large. We study $f(C_R)$.

First $f(OR) = (9, R^3 + R^2 + 9) \subseteq (0, \infty)$ which is an interval on the x-axis. On the other hand if $\psi(\theta) = f(i\theta)$ for $\theta \in [0, R]$, then $\psi(\theta) = (9 - \theta^2) + i(-\theta^3)$ which makes an angle $\text{Arctan}\,\frac{-\theta^3}{9-\theta^2} = \text{Arctan}\,\frac{\theta^3}{\theta^2 - 9} \approx \text{Arctan}\,\theta$ for large θ which is rough $\frac{3\pi}{2}$. Moreover when $|z| = R$ is large $z^3 + z^2 + 9 \approx z^3$ and as $\arg z$ runs from

0 to $\pi/2$, $f(z)$ runs from 0 to an angle that is roughly $\frac{3\pi}{2}$.

With $f(BO)$ going from $\approx \frac{3\pi}{2}$ to 2π, we have that the argument of f changes from 0 to 2π. Hence there is exactly one root inside C_R when R is large.



(44) Since $\lim_{|z|\to\infty} \left| \frac{f(z)}{a_n z^n} - 1 \right| = 0$,

we learn that $f(z) \approx a_n z^n$ on the circle $C_R = \{z : |z| = R\}$. Hence the change in the angle of $f(z)$ around C_R is the same as the change in the angle of $a_n z^n$ around C_R for large R.

The latter is exactly $2\pi n$. Hence there must be exactly n roots inside C_R for f by (7.8) because f has no pole.

(51) $\oint_{|z|=\frac{1}{2}} \frac{f'(z)}{f(z)} = 2\pi i (N - P$ inside $|z| = \frac{1}{2})$

$= 8\pi i$ because f has a zero of order 4 at 0.

Similarly

$$\oint_{|z|=2} \frac{f'(z)}{f(z)} = 2\pi i \Big(\underset{\substack{\uparrow \\ 0}}{4} + \underset{\substack{\uparrow \\ -1}}{2} - \underset{\substack{\uparrow \\ i}}{2} - \underset{\substack{\uparrow \\ -i}}{2} \Big) = 4\pi i$$

(62) $$f(t) = \text{sum of residues of } \frac{(z-1)^2}{z(z+1)^2}\, e^{zt}$$

$$= R(0) + R(-1).$$

$$R(0) = 1,$$

$$R(-1) = \frac{d}{dz}\, \frac{(z-1)^2 e^{zt}}{z} \Bigg|_{z=-1} = -4te^{-t}$$

Hence $f(t) = 1 - 4te^{-t}$.

(64) $$f(t) = \text{sum of residues of } \frac{z^2}{(z^2-1)(z^2-4)}\, e^{zt}$$

$$= R(1) + R(-1) + R(2) + R(-2)$$

$$= -\frac{1}{6}e^{t} + \frac{1}{6}e^{-t} + \frac{1}{3}e^{2t} - \frac{1}{3}e^{-2t}.$$