14.7

(3) $\int_0^{2\pi}\frac{d\theta}{5-4\sin\theta} = \oint_C \frac{\frac{dz}{iz}}{5-4\,\frac{z-\frac{1}{z}}{2i}} = \oint_C \frac{dz}{5iz-2z^2+2}$

$= \oint_C \frac{dz}{-2(z-2i)(z-i/2)} = \frac{2\pi i}{-2}\left(-\frac{3}{2}i\right)^{-1} = \frac{2\pi}{3}.$

(8) $\int_0^{2\pi}\frac{\sin^2\theta\, d\theta}{13-12\cos\theta} = \oint_C \frac{\left(\frac{z-\frac{1}{z}}{2i}\right)^2\frac{dz}{iz}}{13-12\,\frac{z+\frac{1}{z}}{2}} = \frac{1}{-4i}\oint_C \frac{(z^2-1)^2\,dz}{z^2(13z-6z^2-6)}$

$= \frac{1}{-4i}\oint_C\frac{(z^2-1)^2\,dz}{z^2(-6)\,(z-3/2)(z-\frac{2}{3})} = \frac{2\pi i}{-4i}\left[R(\tfrac{2}{3})+R(0)\right]$

$= \frac{\pi}{12}\,\frac{((\frac{2}{3})^2-1)^2}{(\frac{2}{3})^2\,\frac{(-5)}{6}} + \frac{\pi}{2}\,\frac{13}{36} = -\frac{5\pi}{2\cdot 36}+\frac{\pi\,13}{2\cdot 36} = \frac{\pi}{9}.$

(12) $\int_0^\infty \frac{x^2}{x^4+16} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{x^2}{x^4+16}\,dx = \lim \int \frac{z^2}{z^4+16}\,dz$

$z^4+16=0 \leftrightarrow z=\pm\underbrace{2e^{i\pi/4}}_{a},\ \pm\underbrace{2ie^{i\pi/4}}_{b}.$ Hence

$\frac{1}{2}\int_{-\infty}^{\infty}\frac{x^2}{x^4+16}\,dx = \pi i\,\left(R(2e^{i\pi/4})+R(2ie^{i\pi/4})\right)$

$R(2e^{i\pi/4}) = R(a) = \frac{a^2}{(a^2-b^2)(2a)} = \frac{a}{2(a^2-b^2)}$

$R(2ie^{i\frac{\pi}{4}}) = R(b) = \frac{b^2}{(b^2-a^2)\,2b} = \frac{b}{2(b^2-a^2)}$

$\int_0^\infty \frac{x^2}{x^2+16} = \frac{\pi i}{2}\,\frac{1}{a+b} = \frac{\pi}{2\sqrt{2}}.$

2/

(14) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4x+5}\,dx$, $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+4x+5}\,dx = \lim \int \frac{e^{iz}}{z^2+4z+5}\,dz$

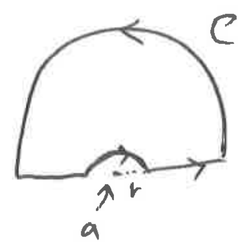
$$= \lim \int \frac{e^{iz}}{(z-a)(z-b)}\,dz \quad \text{for } \begin{cases} a = -2-i \\ b = -2+i \end{cases}$$

Hence $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+4x+5}\,dx = 2\pi i \,\frac{e^{ib}}{b-a} = (2\pi i)\,\frac{e^{ib}}{2i} = \pi e^{ib}$

$$= \pi e^{-2}(\cos 1 + i \sin 1).$$

(16) $\displaystyle\int_0^{\infty} \frac{x \sin x}{9x^2+4}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x \sin x}{9x^2+4} = \frac{1}{2}\,\mathrm{Im} \int_{-\infty}^{\infty} \frac{x e^{ix}}{9x^2+4}$

$$\int_{-\infty}^{\infty} \frac{x e^{ix}}{9x^2+4}\,dx = \int \frac{z e^{iz}}{9z^2+4}\,dz$$



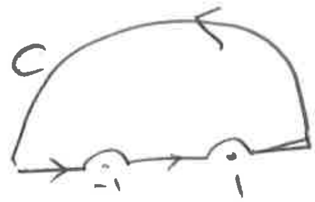
(21) $\displaystyle\int_C \frac{f(z)}{z-a}\,dz = 0$. On the other hand $\displaystyle\lim_{r\to 0} \int \frac{f(z)}{z-a}\,dz = \int \frac{f(z)}{z-a}\,dz$, and

$$\int \frac{f(z)}{z-a}\,dz = \int_0^{\pi} f(z)\, i\, d\theta = \int_0^{\pi} (f(z)-f(a))\, i\, d\theta + i\pi f(a)$$

But $z = a + re^{i\theta}$, and $\displaystyle\lim_{r\to 0} f(z) - f(a) = 0$. Hence

$$\int \frac{f(z)}{z-a}\,dz = \pi i\, f(a)$$

(24) $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin \pi x}{1-x^2}\,dx = \operatorname{Im} \int_{-\infty}^{\infty} \frac{x e^{i\pi x}}{1-x^2}\,dx$.

We have 

$\displaystyle\int_C \frac{z e^{i\pi z}}{1-z^2}\,dz = 0$

which implies $\displaystyle\int_{-\infty}^{\infty} \frac{x e^{i\pi x}}{1-x^2}\,dx = \pi i \left(R(1) + R(-1) \right)$

where Residue is for $f(z) = \dfrac{z e^{i\pi z}}{1-z^2}$. Hence

$$R(1) = \left.\frac{-z e^{i\pi z}}{z+1}\right|_{z=1} = \frac{1}{2}, \quad R(-1) = \left.\frac{-z e^{i\pi z}}{z-1}\right|_{z=-1} = \frac{1}{2}.$$

Hence $\displaystyle\int_{-\infty}^{\infty} \frac{x e^{i\pi x}}{1-x^2} = \pi i$, $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin \pi x}{1-x^2}\,dx = \pi$.

(27) $\displaystyle\int_{-\infty}^{\infty} \frac{e^{i\pi x}}{1-4x^2} = \pi i \left(R(\tfrac{1}{2}) + R(-\tfrac{1}{2}) \right)$ where the residue is for $f(z) = \dfrac{e^{i\pi z}}{1-4z^2}$. Hence

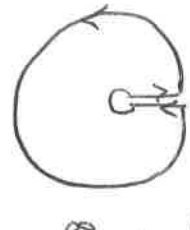
$$R(\tfrac{1}{2}) = \left.\frac{-e^{i\pi z}}{4(z+\frac{1}{2})}\right|_{z=\frac{1}{2}} = \frac{-e^{i\frac{\pi}{2}}}{4} = \frac{-i}{4},$$

$$R(-\tfrac{1}{2}) = \left.\frac{-e^{i\pi z}}{4(z-\frac{1}{2})}\right|_{z=-\frac{1}{2}} = \frac{-e^{-i\pi/2}}{-4} = \frac{-i}{4}.$$

Hence $\displaystyle\int_{-\infty}^{\infty} \frac{e^{i\pi x}}{1-4x^2}\,dx = \pi/2$, $\displaystyle\int_{0}^{\infty} \frac{\cos \pi x}{1-4x^2}\,dx = \frac{\pi}{4}$.

(32) $$\int_0^\infty \frac{dx}{(1+x^4)^2} = \int_0^\infty \frac{u^{\frac{1}{4}-1}\,du}{4(1+u)^2}.$$ Consider

$$\int_C \frac{z^{p-1}\,dz}{(1+z)^2} = \int_{\rightarrow} + \int_{\leftarrow} + \int_{\circlearrowleft} + \int_{\bigcirc}$$



$$\lim_{\substack{r\to 0\\ R\to\infty}} \int_{\rightarrow} \doteq \int_0^\infty \frac{u^{p-1}}{(1+u)^2}, \qquad \lim_{\substack{r\to 0\\ R\to\infty}} \int_{\leftarrow} \frac{z^{p-1}}{(1+z)^2}\,dz = -\int_0^\infty \frac{u^{p-1}e^{i2\pi p}}{(1+u)^2}\,du.$$

$$\lim_{r\to 0} \int_{\circlearrowleft} = -\lim_{r\to 0}\int_0^{2\pi} \frac{(re^{i\theta})^{p-1}\, ire^{i\theta}}{(1+re^{i\theta})^2} = -\lim_{r\to 0} r^p \int_0^{2\pi} \frac{e^{i\theta p}\,d\theta}{(1+re^{i\theta})^2} = 0$$

$$\lim_{R\to\infty} \int_{\bigcirc} = \lim_{R\to\infty} R^p \int_0^{2\pi} \frac{e^{i\theta p}}{(1+Re^{i\theta})^2}\,d\theta \to 0 \quad \text{because } \frac{R^p}{R^2}\to 0.$$

Hence
$$(1-e^{2\pi i p})\int_0^\infty \frac{u^{p-1}}{(1+u)^2}\,du = 2\pi i\, R(-1)$$

The residue is for $\frac{z^{p-1}}{(z+1)^2}$. If $f(z)=z^{p-1}$,

then $f(z) = f(-1) + f'(-1)(z+1) + \cdots$.

So the residue is $f'(-1) = (p-1)\,z^{p-2}\big|_{z=-1} = (p-1)(-1)^{p-2}$

$$= (p-1)\left(e^{i\pi}\right)^{p-2} = (p-1)\,e^{ip\pi}.$$

As a result,
$$\int_0^\infty \frac{u^{p-1}}{(1+u)^2}\,du = 2\pi i\,\frac{(p-1)\,e^{ip\pi}}{1-e^{2\pi i p}} = 2\pi i\,\frac{p-1}{e^{-ip\pi}-e^{\pi p i}} = \frac{\pi(1-p)}{\sin \pi p}.$$

(34) This is similar to (32):

$$\lim_{r\to 0}\int_{\circlearrowright} = -\lim_{r\to 0} r^{1+\frac{1}{2}}\int_0^{2\pi}\frac{e^{i3\theta/2}\,d\theta}{(1+re^{i\theta})^2} = 0$$

$$\lim_{R\to\infty}\int_{\circlearrowleft} = \lim_{R\to\infty} R^{1+\frac{1}{3}}\int_0^{2\pi}\frac{e^{i3\theta/2}\,d\theta}{(1+Re^{i\theta})^2} = 0$$

because $\dfrac{R^{1+\frac{1}{3}}}{R^2}\to 0$.

Also

$$\lim_{\substack{r\to 0\\ R\to\infty}}\int_{\leftarrow}\frac{\sqrt{z}}{(1+z)^2}\,dz = -\int_0^\infty\frac{\sqrt{x}\,e^{i\pi}}{(1+x)^2}\,dx = \int_0^\infty\frac{\sqrt{x}}{(1+x)^2}\,dx.$$

Hence

$$\int_{\rightarrow} + \int_{\leftarrow} = 2\int_0^\infty\frac{\sqrt{x}}{(1+x)^2}\,dx.$$

In summary

$$2\int_0^\infty\frac{\sqrt{x}}{(1+x)^2}\,dx = 2\pi i\, R(-1),$$

the residue of $\dfrac{\sqrt{z}}{(1+z)^2}$ is $f'(-1)$ for $f(z)=\sqrt{z}$.

So $f'(-1) = \dfrac{1}{2\sqrt{-1}} = \dfrac{1}{2i}$ and

$$\int_0^\infty\frac{\sqrt{x}}{(1+x)^2}\,dx = \frac{\pi}{2}.$$

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$$\oint \frac{\ln z}{z^{3/4}(z+1)}\,dz = 2\pi i\ R(-1) = 2\pi i \left.\frac{\ln z}{z^{3/4}}\right|_{z=-1}$$

$$= 2\pi i\,\frac{\ln(-1)}{(-1)^{3/4}} = 2\pi i\,\frac{i\pi}{-e^{-i\pi/4}} = 2\pi^2\,e^{i\pi/4}$$

$$\lim \int_{\rightarrow} = \int_0^\infty \frac{\ln x}{x^{3/4}(1+x)}\,dx$$

$$\lim \int_{\leftarrow} = -\int_0^\infty \frac{\ln(x\,e^{2\pi i})}{x^{3/4}(1+x)\,(e^{2\pi i})^{3/4}}\,dx$$

$$= \frac{1}{i}\int_0^\infty \frac{\ln x + 2\pi i}{x^{3/4}(1+x)}\,dx$$

$$= -i\left[\int_0^\infty \frac{\ln x}{x^{3/4}(1+x)}\,dx + 2\pi i\int_0^\infty \frac{dx}{x^{3/4}(1+x)}\right]$$

Hence

$$(1-i)\int_0^\infty \frac{\ln x}{x^{3/4}(1+x)}\,dx = 2\pi^2\,e^{i\pi/4} - 2\pi\int_0^\infty \frac{dx}{x^{3/4}(1+x)}$$

$$= 2\pi^2\,e^{i\pi/4} - 2\pi\,\frac{\pi}{\sin \pi/4}$$

$$= \pi^2\sqrt{2}(1+i) - 2\pi^2\sqrt{2}$$

$$= \pi^2\sqrt{2}\,(i-1).$$

Thus $\displaystyle\int_0^\infty \frac{\ln x}{x^{3/4}(1+x)}\,dx = -\pi^2\sqrt{2}.$

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$$\lim_{A\to\infty} \int_{\rightarrow} = \int_{-\infty}^{\infty} \frac{e^{px}}{1+e^x}$$

$$\lim_{A\to\infty} \int_{\leftarrow} = -\int_{-\infty}^{\infty} \frac{e^{px} e^{2\pi i p}}{1+e^x e^{2\pi i}}\, dx = -e^{2\pi i p} \int_{-\infty}^{\infty} \frac{e^{px}}{1+e^x}$$

$$\lim_{A\to\infty} \int_{\uparrow} = \lim_{A\to\infty} \int_0^{2\pi} \frac{e^{pA} e^{ipy}}{1+e^A e^{iy}}\, dy$$

observe $\left| \dfrac{e^{pA} e^{ipy}}{|e^A e^{iy}|} \right| = \dfrac{e^{pA}}{e^A} = e^{(p-1)A} \to 0$

as $A\to\infty$. Also

$$\lim_{A\to\infty} \int_{\downarrow} = \lim_{A\to\infty} -\int_0^{2\pi} \frac{e^{-pA} e^{ipy}}{1+e^{-A} e^{ipy}}\, dy = 0$$

because $e^{-A}, e^{-pA} \to 0$. Hence

$$(1-e^{2\pi i p}) \int_{-\infty}^{\infty} \frac{e^{px}}{1+e^x}\, dx = 2\pi i\, R(i\pi)$$

$$= 2\pi i \lim_{z\to i\pi} \frac{e^{pz}}{1+e^z}(z-i\pi)$$

$$= \frac{2\pi i\, e^{ip\pi}}{e^{i\pi}} = -2\pi i\, e^{ip\pi}.$$

So $\displaystyle\int_{-\infty}^{\infty} = -2\pi i \frac{e^{ip\pi}}{1-e^{2\pi i p}} = \frac{\pi}{\sin p\pi}.$