HW 5

(14.3)

(16) $$\int_0^{2\pi} e^{2i\theta} f(e^{i\theta})\, d\theta = \oint_{|z|=1} z^2 f(z) \frac{dz}{iz}$$

$$= \oint_{|z|=1} z f(z)\, dz = 0$$

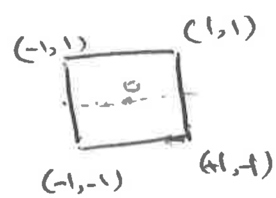
because $z f(z)$ is analytic

(17) $$\oint_{|z|=1} \frac{\sin z}{2z-\pi}\, dz = 0$$ because $\pi/2$ is outside the disk $\{z : |z| \le 1\}$

$$\oint_{|z|=2} \frac{\sin z}{2z-\pi}\, dz = \frac{1}{2} \oint_{|z|=2} \frac{\sin z}{z-\pi/2}\, dz = \pi i \left. \sin z \right|_{z=\pi/2}$$

$$= \pi i$$

(19) Since $\ln 2$ is inside the square



$$\oint \frac{e^{3z}}{z-\ln 2}\, dz = 2\pi i\, e^{3\ln 2} = 16\pi i$$

(24) $$\oint_{|z|=2} \frac{\cosh z}{(2\ln 2 - z)^5}\, dz = \frac{-4!}{2\pi i} \left. \frac{d^4}{dz^4} \cosh z \right|_{z=2\ln 2}$$

$$= \frac{-4!}{2\pi i} \left. \cosh z \right|_{z=2\ln 2} = \frac{30}{\pi}.$$

(14.4) 8 $\dfrac{30}{(z+1)(z-2)(z+3)} = \dfrac{-5}{z+1} + \dfrac{2}{z-2} + \dfrac{3}{z+3}$

$|z|<1$: $(-5)(1-z+z^2-\cdots) - (1+\frac{z}{2}+(\frac{z}{2})^2+\cdots) + (1-\frac{z}{3}+(\frac{z}{3})^2-\cdots)$

$= -5 + \frac{25}{6}z - \frac{175}{36}z^2 + \cdots$

Residue $= 0$

$1<|z|<2$: $-\frac{5}{z}(1-\frac{1}{z}+\frac{1}{z^2}-\cdots) - (1+\frac{z}{2}+(\frac{z}{2})^2+\cdots) + (1-\frac{z}{3}+(\frac{z}{3})^2-\cdots)$

$2<|z|<3$: $-\frac{5}{z}(1-\frac{1}{z}+\frac{1}{z^2}-\cdots) + \frac{2}{z}(1+\frac{2}{z}+(\frac{2}{z})^2+\cdots)$

$+ (1-\frac{z}{3}+(\frac{z}{3})^2-\cdots)$

$3<|z|$: $-\frac{5}{z}(1-\frac{1}{z}+\frac{1}{z^2}-\cdots) + \frac{2}{z}(1+\frac{2}{z}+(\frac{2}{z})^2+\cdots)$

$+ \frac{3}{z}(1-\frac{3}{z}+(\frac{3}{z})^2-\cdots)$

10 (a) $\dfrac{e^z-1}{z^2+4} = \dfrac{e^z-1}{(z+2i)}\ \dfrac{1}{z-2i}$ pole of order 1 at $2i$

(b) $\tan^2 z = \dfrac{\sin^2 z}{\cos^2 z} = \dfrac{1}{\cos^2 z} - 1 = \dfrac{1}{\sin^2(z-\pi/2)} - 1$

Since $\sin(z-\pi/2) = (z-\pi/2)(1-(z-\pi/2)^2+\cdots)$,

we have a pole of order 2 at $\pi/2$.

(c) $\dfrac{1-\cos z}{z^4} = \dfrac{z^2/2 - \frac{z^4}{4!}+\cdots}{z^4} = \dfrac{1}{2z^2} - \dfrac{1}{4!} + \cdots$ pole of order 2

(d) $\cos\dfrac{\pi}{z-\pi} = \dfrac{1}{2}\left(e^{\frac{i\pi}{z-\pi}} + e^{\frac{-i\pi}{z-\pi}}\right)$ essential at π.

(14.5) ③ Note $g(z) = \frac{f(z)}{z-a}$ has a simple pole at a because $\lim_{z\to a} (z-a)\, g(z) = f(a)$. Hence

$$\oint_C g = 2\pi i \operatorname{Res}(a) = 2\pi i\, f(a).$$

(14.6) ⑥ $\sin\frac{1}{z} = \frac{1}{z} - \frac{1}{3!\,z^3} + \frac{1}{5!}\frac{1}{z^5} - \cdots$

$R(0) = 1$

⑧ $$\frac{1+\cos z}{(z-\pi)^2} = \frac{1-\cos(z-\pi)}{(z-\pi)^2} = \frac{\frac{1}{2}(z-\pi)^2 - (z-\pi)^4\frac{1}{4!} + \cdots}{(z-\pi)^2}$$

$$= \frac{1}{2} - \frac{1}{4!}(z-\pi)^2 + \frac{1}{6!}(z-\pi)^4 - \cdots$$

$R(\pi) = 0$.

㉔ $$\frac{1-\cos 2z}{z^3} = \frac{(2z)^2 - (2z)^4/4! + \cdots}{z^3} = \frac{4}{z} - \frac{2}{3}z + \cdots$$

$R(0) = 4$

㊲ Use $\Gamma(z) = \frac{1}{z}\Gamma(z+1) = \frac{1}{z(z+1)}\Gamma(z+2) = \cdots$

$$R(0) = \lim_{z\to 0} z\,\Gamma(z) = \Gamma(1) = 1.$$

$$R(1) = \lim_{z\to -1} (z+1)\,\Gamma(z) = \frac{\Gamma(1)}{-1} = -1,$$

$\vdots$

$$R(n) = \lim_{z\to -n} \frac{(z+n)\,\Gamma(z+n+1)}{z(z+1)\cdots(z+n-1)(z+n)} = \frac{\Gamma(1)}{(-n)(-n+1)\cdots(-1)}$$

$$= \frac{(-1)^n}{n!}.$$