HW 4

2.10

$$\frac{-1-i\sqrt{3}}{2} = e^{i\left(\frac{\pi}{3}+\pi\right)}$$

$$\left(\frac{-1-i\sqrt{3}}{2}\right)^{\frac{1}{8}} = e^{i\left(\frac{\pi}{3}+\pi\right)\frac{1}{8} + i\frac{2\pi k}{8}}$$

for $k = 0, 1, 2, \dots, 7$.

14.1

$$z^2 - \bar{z}^2 = (z-\bar{z})(z+\bar{z}) = 4ixy$$

$$\ln z = \ln(\rho e^{i\theta}) = \ln\rho + i\theta$$

$$= \frac{1}{2}\ln(x^2+y^2) + i\arctan\frac{y}{x}.$$

$$e^{i(x+iy)} = e^{ix-y} = e^{-y}(\cos x + i\sin x)$$

14.2 (24) $u = \dfrac{y}{x^2+y^2}$, $v = \dfrac{-x}{x^2+y^2}$

CR does not hold

$$u_x = \frac{-2xy}{(x^2+y^2)^2}, \quad v_y = \frac{2xy}{(x^2+y^2)^2}$$

(28) $$h^{-1}\left(f(g(z+h)) - f(g(z))\right) = \frac{f(g(z+h)) - f(g(z))}{g(z+h)-g(z)} \frac{g(z+h)-g(z)}{h}$$

If $g(z) = w$, $g(z+h)-g(z) = k$, then we have

$$\frac{f(w+k)-f(w)}{k} \to f'(w)$$ because $k \to 0$ as $h \to 0$.

(31) Let $\ln z = w$, $\ln(z+h) = w'$ with $|h|$ small.

$$e^{w'} = z+h, \quad e^{w} = z, \quad e^{w'-w} = \frac{z+h}{z} = 1 + \frac{h}{z}.$$

So $e^{w'-w} = 1 + w' - w + R$ with $|R| = O(|w'-w|^2)$

Hence $w' - w + R = \frac{h}{z}$, $|w'-w| = O(|h|)$,

$$|R| = O(|h|^2), \qquad \frac{w'-w}{h} + \frac{R}{h} = \frac{1}{z}.$$

Hence $\lim\limits_{h \to 0} \frac{w'-w}{h} = \frac{1}{z}$.

(33)

$$\frac{d}{dz} \cos z = \frac{d}{dz}\left(\frac{e^{iz} + e^{-iz}}{2}\right) = \frac{i e^{iz} - i e^{-iz}}{2} = -\frac{e^{iz} - e^{-iz}}{2i} = -\sin z.$$

$$\frac{d}{dz} \cot z = \frac{d}{dz} \frac{\cos z}{\sin z} = -\frac{\cos^2 z}{\sin^2 z} - 1 = -\frac{1}{\sin^2 z}$$

(39)

$$\frac{1}{2i+z} = \frac{1}{2i} \frac{1}{1+\frac{z}{2i}} = -\frac{i}{2}\left(\sum_{n=0}^{\infty} \left(\frac{z}{2i}\right)^n (-1)^n\right),$$

$$\left|\frac{z}{2i}\right| = \left|\frac{z}{2}\right| < 1 \quad \text{or} \quad |z| < 2.$$

(46) $f(z) = (u+iv)(r, \theta)$, $z = re^{i\theta}$

$$\begin{cases} \dfrac{\partial f}{\partial r} = \dfrac{df}{dz} \dfrac{\partial z}{\partial r} = e^{i\theta} f'(z) \\[2ex] \dfrac{\partial f}{\partial \theta} = \dfrac{df}{dz} \dfrac{\partial z}{\partial \theta} = i r e^{i\theta} f'(z) \end{cases}$$

Hence $i r \frac{\partial f}{\partial r} = \frac{\partial f}{\partial \theta}$

$$i r (u_r + i v_r) = (u_\theta + i v_\theta)$$

$$\begin{cases} u_\theta = -r v_r \\ v_\theta = r u_r \end{cases}$$

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(53) $f = |z|^{1/2} e^{i\theta/2} = r^{1/2} e^{i\frac{\theta}{2}}$

$f_\theta = \frac{i}{2} f$, $f_r = \frac{1}{2} r^{-1/2} e^{i\theta/2} = \frac{1}{2r} f$.

So $\quad i r f_r = f_\theta$.

(61) $u = \frac{x}{x^2+y^2}$, $\quad u_y = \frac{-2xy}{(x^2+y^2)^2}$.

Find v so that $v_x = -u_y = \frac{2xy}{(x^2+y^2)^2}$.

$v = \frac{-y}{x^2+y^2}$ does the job. So

$f = \frac{x-iy}{x^2+y^2} = \frac{1}{z}$.

14.3

(3) $$\int z^2\,dz = \int_{-1}^{1} x^2\,dx - \int_{-1}^{1} (x+i)^2\,dx + \int_0^1 (1+iy)^2\, i\,dy - \int_0^1 (-1+iy)^2\, i\,dy$$

$$= 2 - 2i \int_{-1}^{1} x\,dx + 4i \int_0^1 iy\,dy = 2 - 2 = 0$$

(3) $$\int z^2\,dz = \int_{-1}^{1} x^2\,dx + \int_0^{\pi} e^{2i\theta}\, i e^{i\theta}\,d\theta$$

$$= \frac{2}{3} + \frac{e^{i3\pi}-1}{3} = 0.$$

(10) $$\int_2^{2+i\infty} z e^{iz}\,dz = \int_0^{\infty} (2+iy)\, e^{i(2+iy)}\, i\,dy$$

$$= i e^{2i} \int_0^{\infty} (2+iy)\, e^{-y}\,dy$$

$$= 2i e^{2i} - e^{2i} \int_0^{\infty} y e^{-y}\,dy$$

$$= 2i e^{2i} - e^{2i}.$$