HW 11

1/

8.11

(15) (a) $\int_0^{\pi} \sin x \; \delta(x-\pi/2)\, dx = \sin\frac{\pi}{2} = 1$

(b) $\int_0^{\pi} \sin x \; \delta(x+\pi/2)\, dx = 0$

(c) $\int_{-1}^{1} e^{3x}\, \delta'(x)\, dx = -3\int_{-1}^{1} e^{3x}\, \delta(x)\, dx = -3$

(d) $\int_0^{\pi} \cosh x \; \delta''(x-1) = \int_0^{\pi} \frac{d^2}{dx^2} \cosh x \; \delta(x-1)\, dx$

$= \int_0^{\pi} \cosh x \; \delta(x-1)\, dx = \cosh(1)$

$= \frac{1}{2}(e + 1/e)$.

(16) Let ϕ be a continuously differentiable function s.t. $\phi'(\pm\infty) = \phi(\pm\infty) = 0$. Then

$$\int_{-\infty}^{\infty} \left(\frac{d}{dx}\operatorname{sgn} x\right) \phi(x)\, dx = -\int_{-\infty}^{\infty} \operatorname{sgn}(x)\, \phi'(x)\, dx$$

$$= -\int_{-\infty}^{0} (-1)\, \phi'(x)\, dx - \int_0^{\infty} \phi'(x)\, dx$$

$$= \phi(0) - (-\phi(0)) = 2\phi(0).$$

Hence $\frac{d}{dx} \operatorname{sgn}(x) = 2\delta(x)$.

2/

8.12 ⑥ We wish to solve $y''(t) - a y(t) = \delta(t-t_0)$ with $y(0)=y'(0)=0$. Writing $g(t)$ (in fact $g(t,t_0)$ but we do not display the dependence on t_0 for now), we get for the Laplace transform of the solution

$$(z^2-a^2)\, g(z) = e^{-t_0 z} \quad \text{or} \quad g(z) = (z^2-a^2)^{-1} e^{-t_0 z}.$$

Hence $G(t,t_0) =$ sum of residues of $(z^2-a^2)^{-1} e^{(t-t_0)z}$

Since $(z^2-a^2)^{-1}$ has simple poles at $\pm a$, we have

$$G(t,t_0) = \frac{1}{2a} e^{(t-t_0)a} - \frac{1}{2a} e^{(t-t_0)(-a)}$$

$$= \frac{1}{a} \sinh a(t-t_0)$$

for $t > t_0$; otherwise 0. Hence if the right-hand side is f, we have

$$y(t) = \int_0^t \frac{1}{a} \sinh a(t-t_0)\, f(t_0)\, dt_0$$

In particular, if $f(t_0) = \begin{cases} 0 & t<0, \\ 1 & t>0, \end{cases}$

$$y(t) = \int_{-\infty}^{\infty} \frac{1}{a} \sinh a(t-t_0)\, \mathbb{1}(t>t_0)\, f(t_0)\, dt_0$$

$$= \begin{cases} 0 & \text{if } t<0 \\ \int_0^t \frac{1}{a} \sinh a(t-t_0)\, dt_0 & \text{if } t \geq 0. \end{cases}$$

The latter equals

$$\int_0^t \frac{1}{a} \sinh a t_0 \, dt_0 = \frac{1}{a^2}(\cosh a t, -1)$$

(after change of variable $t - t_0 \to t$).

(15) If we assume that $y = c_1 y_1 + c_2 y_2$

with $y_1 = \sinh$, $y_2 = \cosh$, then the

additional requirement $c_1' y_1 + c_2' y_2 = 0$

leads to $y' = c_1 y_1' + c_2 y_2'$,

$$y'' = c_1 y_1'' + c_2 y_2'' + c_1' y_1' + c_2' y_2'$$

$$= c_1 y_1 + c_2 y_2 + c_1' y_1' + c_2' y_2' = y + c_1' y_1' + c_2' y_2'.$$

Since $y'' - y = \operatorname{sech}$, we deduce

$$\begin{cases} c_1' y_1 + c_2' y_2 = 0 \\ c_1' y_1' + c_2' y_2' = \operatorname{sech} = (\cosh)^{-1} \end{cases}$$

Hence $\begin{bmatrix} 0 \\ \frac{1}{\cosh} \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} \sinh & \cosh \\ \cosh & \sinh \end{bmatrix} \begin{bmatrix} c_1' \\ c_2' \end{bmatrix}$.

Or $\begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = - \begin{bmatrix} \sinh & -\cosh \\ -\cosh & \sinh \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\cosh} \end{bmatrix} = \begin{bmatrix} +1 \\ -\frac{\sinh}{\cosh} \end{bmatrix}$

For example, we may choose $c_1(x) = -x$, $c_2(x) = -\ln(\cosh x)$

For a special solution, $y(x) = x \sinh x - \ln(\cosh x)\cosh x$.

(18) $\begin{cases} c_1' y_1 + c_2' y_2 = 0 \\ c_1' y_1' + c_2' y_2' = (x^2+1)^2 \end{cases}$, $y_1 = x$, $y_2 = 1-x^2$

$$\begin{bmatrix} x & 1-x^2 \\ 1 & -2x \end{bmatrix} \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} 0 \\ (x^2+1)^2 \end{bmatrix},$$

$$\begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \frac{-1}{1+x^2} \begin{bmatrix} -2x & x^2-1 \\ -1 & x \end{bmatrix} \begin{bmatrix} 0 \\ (x^2+1)^2 \end{bmatrix}$$

$$= \frac{-1}{1+x^2} \begin{bmatrix} (x^2-1)(x^2+1)^2 \\ x(x^2+1)^2 \end{bmatrix} = \begin{bmatrix} 1-x^2 \\ -x \end{bmatrix}.$$

$c_1 = x - \frac{x^3}{3}$, $c_2 = -x^2/2$. Hence

$$y(x) = x\left(x - \frac{x^3}{3}\right) - \frac{x^2}{2}(1-x^2) = \frac{x^2}{2} + \frac{x^4}{6}.$$

9.3 (6) $\int_{x_1}^{x_2} \frac{y y'^2}{1+yy'}\, dx$. Here

$F = F(y, y') = \frac{y y'^2}{1+yy'}$. It does not depend on x.

This suggests inverting y i.e we express the integral in terms of $y \mapsto x(y)$, the inverse of $x \mapsto y(x)$.

Note $y' = \frac{1}{x'}$, $dx = x'\, dy$. So we have

$$\int_{y_1}^{y_2} \frac{y \left(\frac{1}{x'}\right)^2}{1+y\frac{1}{x'}}\, x'\, dy = \int_{y_1}^{y_2} \frac{y\, dy}{x'+y} = \int_{y_1}^{y_2} G(y, x')\, dy.$$

The Euler equation is $\frac{d}{dy}\frac{\partial G}{\partial x'} = \frac{\partial G}{\partial x} = 0$, or

$$\frac{\partial G}{\partial x'} = -\frac{y}{(x'+y)^2} = \text{Constant} = -c_0^{-2}.$$ So

$x'+y = c_0\sqrt{y}$, or $x'(y) = c_0\sqrt{y} - y$.

Hence $x(y) = c_1\, y^{3/2} - \frac{1}{2}y^2 + c_2$ for

constants c_1 and c_2.

6/

(7) $$\int_{x_1}^{x_2} (y'^2 + y^2)\, dx \longrightarrow \int_{y_1}^{y_2} \left(\frac{1}{x'^2} + y^2\right) x'\, dy = \int_{y_1}^{y_2} G$$

with $G(y, x') = \frac{1}{x'} + x' y^2$, $\frac{\partial G}{\partial x'} = -\frac{1}{x'^2} + \frac{x}{2}'^2 y^2$

From $\frac{\partial G}{\partial x'} = \frac{1}{2} c_0$ we learn

$$-\frac{2}{x'^2} + x'^2 y^2 = c_0 \quad \text{or} \quad x'^4 y^2 - 2 - c_0 x'^2 = 0$$

Hence $x'^2 = \frac{c_0 \pm \sqrt{c_0^2 - 8y^2}}{y^2}$, $x' = \pm \frac{\sqrt{c_0 \pm \sqrt{c_0^2 - 8y^2}}}{y} + c_1$

and $x(y)$ would be an antiderivative of above.

(9) $$\int_{\phi_1}^{\phi_2} \sqrt{\theta'^2 + \sin^2\theta}\, d\phi = \int_{\theta_1}^{\theta_2} \sqrt{\frac{1}{\phi'^2} + \sin^2\theta}\ \phi'\, d\theta$$

$$= \int_{\theta_1}^{\theta_2} \sqrt{1 + \sin^2\theta\ \phi'^2}\ d\theta = \int_{\theta_1}^{\theta_2} G(\theta, \phi'(\theta))\, d\theta$$

From $\frac{d}{d\theta} \frac{\partial G}{\partial \phi'} = \frac{\partial G}{\partial \phi} = 0$ we deduce

$$\frac{\sin^2\theta\ \phi'(\theta)}{\sqrt{1 + \sin^2\theta\ \phi'^2}} = c_0, \quad \sin^4\theta\ \phi'(\theta)^2 = c_0^2 + c_0^2 \sin^2\theta\ \phi'^2$$

$$(\sin^4\theta - c_0^2 \sin^2\theta)\ \phi'(\theta)^2 = c_0^2, \quad \phi'(\theta) = \frac{c_0}{\sin\theta \sqrt{\sin^2\theta - c_0^2}}$$