# HW 10

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8.9 (26) $\begin{cases} y'' + 2y' + 10y = -6e^{-t}\sin 3t = \operatorname{Im}(-6e^{(-1+3i)t}) \\ y(0) = 0, \quad y'(0) = 1. \end{cases}$

If $Y(p) = \int_0^\infty e^{-pt} y(t)\,dt$, then

$$\int_0^\infty e^{-pt} y'(t)\,dt = p\,Y(p), \quad \int_0^\infty e^{-pt} y''(t)\,dt = p\int_0^\infty e^{-pt} y'(t)\,dt$$

$$-1 = p^2 Y(p) - 1.$$

On the other hand

$$L(\operatorname{Im}(-6e^{(-1+3i)t})) = \operatorname{Im} L(-6e^{(-1+3i)t})$$

$$= \operatorname{Im}\left(\frac{-6}{p-(-1+3i)}\right) = \operatorname{Im}\left(\frac{-6\,(p+1+3i)}{(p+1)^2+9}\right)$$

$$= \frac{-18}{(p+1)^2+9}.$$

Hence

$$(p^2+2p+10)\,Y - 1 = \frac{-18}{(p+1)^2+9},$$

Hence $$Y(p) = \frac{1}{(p+1)^2+9} - \frac{18}{((p+1)^2+9)^2}$$

Now apply the inverse transform using Bromwich:

$y(t) =$ sum of residues of $Y(p)\, e^{pt}$.

$$= \frac{1}{3} e^{-t} \sin 3t - \frac{18}{2\cdot 3^3}\, e^{-t}\,(\sin 3t - 3t \cos 3t)$$

$$= \; te^{-t} \cos 3t$$

(33) $\begin{cases} y' - z' - y = \cos t = \mathrm{Re}(e^{it}) \\ y' + y - 2z = 0 \end{cases}$ $\quad y(0) = -1$, $\quad z(0) = 0$.

$Y =$ Laplace transform of $y$, $\quad Z =$ Laplace transform of $z$

$$L(y')(p) = \int_0^\infty y'(t)\, e^{-pt}\, dt = p \int_0^\infty y(t)\, e^{-pt}\, dt + 1$$

Hence $\begin{cases} p\,Y(p) + 1 - p\,Z(p) - Y(p) = \dfrac{1}{p^2+1} \\ p\,Y(p) + 1 + Y(p) - 2\,Z(p) = 0 \end{cases}$

$$\begin{cases} (p-1)\,Y(p) - p\,Z(p) = \dfrac{1}{p^2+1} - 1 \\ (p+1)\,Y(p) - 2\,Z(p) = -1 \end{cases}$$

$$\begin{bmatrix} p-1 & -p \\ p+1 & -2 \end{bmatrix} \begin{bmatrix} Y \\ Z \end{bmatrix} = \begin{bmatrix} \dfrac{1}{p^2+1} - 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} y \\ z \end{bmatrix} = \frac{1}{p^2-p-1} \begin{bmatrix} -2 & p \\ -p-1 & p-1 \end{bmatrix} \begin{bmatrix} \frac{1}{p^2+1} - 1 \\ -1 \end{bmatrix}$$

$$= \frac{1}{p^2-p-1} \begin{bmatrix} -\frac{2}{p^2+1} + 2 - p \\ -\frac{p+1}{p^2+1} + p+1 - p + 1 \end{bmatrix}$$

$$\begin{cases} y(p) = \dfrac{1}{p^2-p+1}\left(-\dfrac{2}{p^2+1} + 2 - p\right) \\ z(p) = \dfrac{1}{p^2-p+1}\left(-\dfrac{p+1}{p^2+1} + 2\right). \end{cases}$$

It is straightforward to use Bromwich to find $y$ and $z$.

8.10

(8) $L(e^{-at}) = \frac{1}{p+1}$. So for the inverse

Laplace transform of $\frac{1}{(p+a)(p+b)(p+c)}$, we

get $(e^{-at}) * (e^{-bt}) * (e^{-ct})$. Call $e^{-at} =: f_a(t)$.

Note

$$(f_a * f_b)(t) = \int_0^t f_a(s)\, f_b(t-s)\, ds$$

$$= \int_0^t e^{-as} e^{-b(t-s)} ds$$

$$= e^{-bt} \int_0^t e^{-(a-b)s} ds$$

$$= e^{-bt} \frac{1 - e^{-(a-b)t}}{a-b} = \frac{e^{-bt} - e^{-at}}{a-b}$$

provided that $a \neq b$; otherwise we get

$te^{-bt}$. In summary

$$f_a * f_b = \begin{cases} \frac{f_b - f_a}{a-b} & \text{if } a \neq b \\ t\, f_b & \text{if } a = b. \end{cases}$$

In the same fashion if $a, b, c$ are distinct,

$$f_a * f_b * f_c = \frac{1}{a-b}\left(f_b * f_c - f_a * f_c\right)$$

$$= \frac{1}{a-b}\,\frac{1}{b-c}\left(f_c - f_b\right)$$

$$- \frac{1}{a-b}\,\frac{1}{a-c}\left(f_c - f_a\right).$$

(10) We need to calculate the inverse transform of $\dfrac{1}{p\,(p+ia)^2\,(p-ia)^2}$. Hence we have $f_0 * f_{ia} * f_{ia} * f_{-ia} * f_{-ia}$.

We can treat this as in (9).

(17) $y'' - a^2 y = f \Rightarrow y = \dfrac{1}{p^2-a^2}\,L(f)(p)$

$L\left(\frac{1}{a}\sinh at\right) = \dfrac{1}{p^2-a^2}$. Hence for $t > 0$

$$y(t) = \int_0^t \frac{1}{a}\sinh a(t-t_0)\; f(t_0)\,dt_0$$

$$= \int_0^t \frac{1}{a}\sinh a(t-t_0)\,dt_0 = \int_0^t \frac{1}{a}\sinh(a\theta)\,d\theta$$

$$= \frac{1}{a^2}\left(\cosh at - 1\right).$$

# 8.11

(9) $y'' + 2y' + 10y = \delta(t-t_0)$

We have $(p^2+2p+10)\, Y(p) = e^{-t_0 p}$ by taking the Laplace Transform of both sides. So

$$Y(p) = \frac{1}{(p+1)^2+9}\, e^{-t_0 p}.$$

By Bromwich's formula,

$$y(t) = \frac{1}{3} e^{-(t-t_0)} \sin 3(t-t_0)\ \mathbb{1}(t>t_0)$$

(11) $y^{(4)} - y = \delta(t-t_0)$. So $(p^4-1)\, Y(p) = e^{-t_0 p}$,

or $Y(p) = \dfrac{e^{-t_0 p}}{p^4-1}$. For $t > t_0$,

$y(t) =$ sum of residues of $\dfrac{e^{(t-t_0)z}}{z^4-1}$

$$= \frac{1}{4}\left(e^{t-t_0} - e^{-(t-t_0)}\right) + \frac{-1}{4i}\left(e^{-i(t-t_0)} - e^{+i(t-t_0)}\right)$$

$$= \frac{1}{2}\left(\sinh(t-t_0) - \sin(t-t_0)\right).$$