## Kinetic Description of Hamilton-Jacobi PDE II

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PDE/Probability Student Seminar

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**Motivation** 

**Convex Duality** 

**Tessellation and Triangulation** 

Second Polytope

Minkowski-Alexandrov Problem and Optimal Transport

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Hamilton-Jacobi Dynamics

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Hamilton-Jacobi Dynamics

Voronoi tessellations are used to model/study various phenomena in nature:





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Given *n* distinct points  $\rho_1, \ldots, \rho_n$  (in general position), consider the optimization problem

$$w(x) = \min_i |x - \rho_i|.$$

For each *i*, set

$$X(\rho_i) = \{ x : w(x) = |x - \rho_i| \}.$$



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Write  $\hat{P}$  for the convex hull of P.

- 1. If  $\rho$  is an extreme point of  $\hat{P}$ , then  $X(\rho)$  is unbounded.
- 2. If  $\rho$  is not an extreme point of  $\hat{P}$ , then  $X(\rho)$  is bounded.

3. Each  $X(\rho)$  is a polyhedron/polytope.

- 4. We say *P* is generic (points in *P* are in general position) if no k points of *P* lie on a k 1 affine set (for  $k \in \{2, ..., d + 1\}$ ), and no set of d + 2 points in *P* lie on the boundary of a ball whose interior does not intersect *P*.
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## **Delaunay triangulation**



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Write  $\hat{P}$  for the convex hull of P.

- 1. If  $\rho$  is an extreme point of  $\hat{P}$ , then  $X(\rho)$  is unbounded.
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- 3. Each  $X(\rho)$  is a polyhedron/polytope.
- 4. We say *P* is generic (points in *P* are in general position) if no k points of *P* lie on a k 1 affine set (for  $k \in \{2, ..., d + 1\}$ ), and for any set of d + 2 points  $m_1, ..., m_{d+1} \in P$ , we have

- 5. For generic P, we have a graph of degree d + 1; Its dual is a triangulation (weighted Delaunay triangulation).
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Given a set P and a map  $f : P \to \mathbb{R}$ , we define a (marked) tessellation

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for  $P(x) = \partial u(x)$ ? What we have is simply the Laguerre tessellation associated with  $u^* = f^o$ . This is the dual tessellation. If *f* is generic, then cells of this dual tessellation are simplices (triangles when d = 2). They are also dual in graph theoretical sense. Write *X* for the set of vertices in the original tessellation:

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for  $P(x) = \partial u(x)$ ? What we have is simply the Laguerre tessellation associated with  $u^* = f^o$ . This is the dual tessellation. If *f* is generic, then cells of this dual tessellation are simplices (triangles when d = 2). They are also dual in graph theoretical sense. Write *X* for the set of vertices in the original tessellation:

$$X = \{X(\rho) : \rho \in \mathbb{R}^d, \sharp X(\rho) = 1\}.$$

Then

$$u^*(\rho) = \sup_{x} (x \cdot \rho - u(x)) = \sup_{x \in X} (x \cdot \rho - u(x)).$$

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