# Kinetic Description of Hamilton-Jacobi PDE II 

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## Outline

Motivation
Convex Duality
Tessellation and Triangulation
Second Polytope
Minkowski-Alexandrov Problem and Optimal Transport
Hamilton-Jacobi Dynamics
Poisson-Laguerre Point Process

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## Voronoi Tessellation

Voronoi tessellations are used to model/study various phenomena in nature:


Voronoi Tessellation
Given $n$ distinct points $\rho_{1}, \ldots, \rho_{n}$ (in general position), consider the optimization problem

$$
w(x)=\min _{i}\left|x-\rho_{i}\right|
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For each $i$, set

$$
X\left(\rho_{i}\right)=\left\{x: w(x)=\left|x-\rho_{i}\right|\right\}
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## Voronoi Tessellation (alternative formulation)

For $f(\rho)=|\rho|^{2} / 2$, consider

$$
u(x)=\sup \left(x \cdot \rho_{i}-f\left(\rho_{i}\right)\right)
$$

$$
X\left(\rho_{i}\right)=\left\{x: u(x)=x \cdot \rho_{i}-f\left(\rho_{i}\right)\right\}
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Set $P=\left\{\rho_{1}, \ldots, \rho_{n}\right\}, h(\rho)=f(\rho)+\infty \Uparrow 1(\rho \notin P)$, then $u=h^{*}$.


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## Write $\hat{P}$ for the convex hull of $P$.

1. If $\rho$ is an extreme point of $\hat{P}$, then $X(\rho)$ is unbounded.
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4. We say $P$ is generic (points in $P$ are in general position) if no $k$ points of $P$ lie on a $k-1$ affine set (for $k \in\{2, \ldots, d+1\}$ ), and no set of $d+2$ points in $P$ lie on the boundary of a ball whose interior does not intersect $P$.
5. For generic $P$, we have a graph of degree $d+1$; Its dual is a triangulation (Delaunay triangulation).


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Delaunay triangulation


## Laguerre Tessellation

Given a set $n$ distinct points $P=\left\{\rho_{1}, \ldots, \rho_{n}\right\}$, and $c: P \rightarrow \mathbb{R}$, consider the optimization problem

$$
w^{\prime}(x)=\min _{\rho \in P}\left\{\mid x-\rho^{\prime 2} / 2-c(p)\right\} .
$$

(When $c=0$, we are back to Voronoi scenario) For each $\rho$, set

$$
x(\rho)=\left\{x: w(x)=|x-\rho|^{2} / 2-c(\rho)\right\} .
$$

Observe that $f(\rho)=|\rho|^{2} / 2-c(\rho)$,

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\begin{aligned}
u(x) & :=\sup _{\rho \in P}(x \cdot \rho-f(\rho))=\frac{1}{2}|x|^{2}- \\
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\left\{x: x \cdot\left(m_{i}-m_{j}\right)=f\left(m_{i}\right)-f\left(m_{j}\right), \text { for all } i, j\right\}=\emptyset
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5. For generic $P$, we have a graph of degree $d+1$;

Its dual is a triangulation (weighted Delaunay triangulation).
6. $X(\rho)$ could be empty for some $\rho$ if $P$ is not minimal
(can be replaced with a proper subset of $P$ in the definition).
This has to do that $f$ may not be strictly convex.
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## Dual Tessellation=Legendre Transform

Given a set $P$ and a map $f: P \rightarrow \mathbb{R}$, we define a (marked) tessellation

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\{(\rho, X(\rho)): \rho \in P\}
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This is nothing other than

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In fact $u^{*}=f^{0}$ is the convex hull of $f$. On $X(\rho)$, we have $u(x)=x \cdot \rho-f(\rho)=x \cdot \rho-u^{*}(\rho)$. It is more convenient to consider

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X=\left\{X(\rho): \rho \in \mathbb{R}^{d}, \sharp X(\rho)=1\right\} .
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