POINCARÉ-BIRKHOFF THEOREMS IN RANDOM DYNAMICS

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To Alan Weinstein on his 70th birthday, with admiration.

Abstract. We propose a generalization of the Poincaré-Birkhoff Theorem on area-preserving twist maps to area-preserving twist maps F that are random with respect to an ergodic probability measure. In this direction, we will prove several theorems concerning existence, density, and type of the fixed points. To this end first we introduce a randomized version of generalized generating functions, and verify the correspondence between its critical points and the fixed points of F, a fact which we successively exploit in order to prove the theorems. The study we carry out needs to combine probabilistic techniques with methods from nonlinear PDE, and from differential geometry, notably Moser's method and Conley-Zehnder theory. Our stochastic model in the periodic case coincides with the classical setting of the Poincaré-Birkhoff Theorem.

1. Introduction

Poincaré understood that preserving area has global implications for a dynamical system. We give instances when this connection persists in a random setting. We do it by using random generating functions to reduce the proofs to finding critical points of random maps. This paper proposes an extension of the classical theory of area-preserving twist maps to the random setting.

In his work in celestial mechanics [Po93] Poincaré showed the study of the dynamics of certain cases of the restricted 3-Body Problem may be reduced to investigating area-preserving maps (see Le Calvez [Le91] and Mather [Ma86] for an introduction to area-preserving maps). He concluded that there is no reasonable way to solve the problem explicitly in the sense of finding formulae for the trajectories. New insights appear regularly (eg. Albers et al. [AFFHO12], Bruno [Br94], Galante et al. [GK11], and Weinstein [We86]). Instead of aiming at finding the trajectories, in dynamical systems one aims at describing their analytical and topological behavior. Of a particular interest are the constant ones, i.e., the fixed points. The development of the modern field of dynamical systems was markedly influenced by Poincaré's work in mechanics, which led him to state (1912) the Poincaré-Birkhoff Theorem [Po12, Bi13]. It was proved in full by Birkhoff in 1925. The result says that an area-preserving periodic twist map

$$F\colon \mathcal{S} \to \mathcal{S}$$

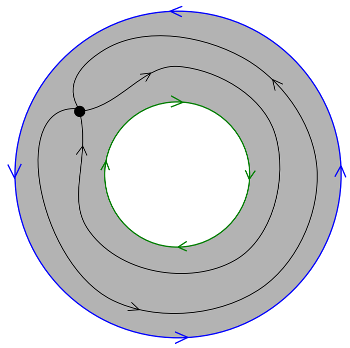

FIGURE 1.1. Fixed point of an area-preserving twist defined by a flow.

of

$$\mathcal{S} := \mathbb{R} \times [-1,\, 1]$$

has two geometrically distinct fixed points. More precisely:

Definition 1.1. A diffeomorphism $F\colon \mathcal{S} \to \mathcal{S}$,

$$F(q,p) = (Q(q,p), P(q,p)),$$

is an *area-preserving periodic twist* if the following conditions are satisfied:

(1) *area preservation*: it preserves area;
(2) *boundary invariance*: it preserves $\ell_{\pm} := \mathbb{R} \times \{\pm 1\}$, i.e.

$$P(q, \pm 1) := \pm 1;$$

(3) *boundary twisting*: F is orientation preserving and

$$\pm Q(q, \pm 1) > \pm q$$

for all q;
(4) *periodicity*: $F(q+1,p) = (1,0) + F(q,p)$ for all p, q.

We may alternatively replace (3) and (4) by

(3') $q \mapsto Q(q, \pm 1)$ is increasing and $\pm Q(q, \pm 1) > \pm q$ for all q, and (4) by
(4') $F(q,\, p) = (q + \bar{Q}(q,\, p), \bar{P}(q,\, p))$ for a map

$$\bar{F} := (\bar{Q}, \bar{P})\colon \mathcal{S} \to \mathcal{S}$$

such that

$$\bar{F}(q+1,\, p) = \bar{F}(q,\, p)$$

for all $(q,\, p)$.

Now we are ready to state the famous result of Poincaré and Birkhoff on area-preserving twist maps.

Theorem A (Poincaré-Birkhoff)**.** *An area-preserving periodic twist $F\colon \mathcal{S} \to \mathcal{S}$ has at least two geometrically distinct fixed points.*

For the purpose of our article, the most useful proof of Theorem A follows Chaperon's viewpoint [Ch84, Ch84b, Ch89] and the so called theory of "generating functions". Generalizations including a number of new ideas have been obtained by several authors, eg. see Carter [Ca82], Ding [Di83], Franks [Fr88, Fr88b, Fr06], Le Calvez-Wang [Le10], Neumann [Ne77], and Jacobowitz [Ja76, Ja77]. Theorem A was proved[1] in certain cases by Poincaré [Po12]. Later Birkhoff gave a full proof and presented generalizations [Bi13, Bi26]; in [Bi66] he explored its applications to dynamics. See [BG97, Section 7.4] and [BN77] for an expository account.

There is no unique way to generalize Theorem A, but at least one should hope that a generalization to the random setting recovers it as a particular case. In our paper we prove a parallel generalization of Theorem A to twist maps that are random with respect to a given probability measure. Our stochastic model in the so called periodic case *coincides* with the classical setting of Poincaré-Birkhoff theorem. While random dynamics has been explored quite throughly, eg. Brownian motions [Ei56, Ne67], the implications of the area preservation assumption remain relatively unknown. We recommend [AA68, KH95, Ko57, Mo73, Sm67] for modern accounts of dynamics, and [BH12, HZ94, MS98, Pol01] for treatments emphasizing symplectic techniques.

2. The space of all twist maps and Main Theorems

The primary goal of this article is the study of the set of fixed points of an area preserving twist map that is not necessarily periodic. To describe our results, let us write $\mathcal{T}$ for the space of area preserving twist maps. That is, the set of diffeomorphism $F : \mathcal{S} \to \mathcal{S}$, such that the Axioms $(1) - (3)$ of Introduction are valid. For our purposes, let us also write $\overline{\mathcal{T}}$ for the space of maps $\bar{F} : \mathcal{S} \to \mathcal{S}$ such that if

$$\ell(\bar{F})(q,p) := (q,0) + \bar{F}(q,p),$$

then $\ell(\bar{F}) \in \mathcal{T}$. We think of the operator $\ell : \overline{\mathcal{T}} \to \mathcal{T}$ as sending $\bar{F}$ to its *lift* $F = \ell(\bar{F})$. We have a natural family of shifts $\big\{\tau_a : \overline{\mathcal{T}} \to \overline{\mathcal{T}} : \ a \in \mathbb{R}\big\}$, that are defined by

$$\tau_a \bar{F}(q,p) = \bar{F}(q+a,p).$$

Given $F \in \mathcal{T}$, we write

$$\mathrm{Fix}(F) = \big\{x \in \mathcal{S} : \ F(x) = x\big\} \subset \mathcal{S},$$

for the set of its fixed points. If we abuse the notation and write τ_a for

$$\tau_a A = \{x : \ x + a \in A\},$$

then we have the trivial commutative relationship

$$\tau_a \mathrm{Fix}(\ell(\bar{F})) = \mathrm{Fix}(\ell(\tau_a \bar{F})). \tag{2.1}$$

[1] One can use symplectic dynamics to study area-preserving maps, see [BH12]. Section 3.4 of Bramham et al. [BH12] proves Theorem A using the important tool of finite energy foliations [HWZ03].

The type of results we will establish in this paper have the following flavor: For generic members of $\mathcal{T}$ we will show that the set $\mathrm{Fix}(F)$ is infinite, and in some cases of *positive density*. We use probabilist means to decide on genericness; we explore quantitative properties of $\mathrm{Fix}(F)$ that are valid with probability one with respect to a probability measure that is defined on the set $\mathcal{T}$, or equivalently on $\overline{\mathcal{T}}$. More precisely, we take a probability measure $\mathbb{Q}$ on $(\overline{\mathcal{T}}, \mathcal{F})$, where $\mathcal{F}$ denotes the Borel σ-algebra associated with the topology of uniform norm, and show that the probability of those $\bar{F}$ for which the set $\mathrm{Fix}(\ell(\bar{F}))$ enjoys certain quantitative properties is one. The type of quantitative properties we have in mind, would not be affected if the set $\mathrm{Fix}(F)$ is translated. In view of the translation property (2.4), it is natural to assume that the probability measure $\mathbb{Q}$ is translation invariant (or in the probability theory jargon *stationary*). This property also allows us to apply the Ergodic Theorem to guarantee that density of $\mathrm{Fix}(\ell(\bar{F}))$ is well-defined for $\mathbb{Q}$-almost all choices of $\bar{F}$. We also assume that the probability measure $\mathbb{Q}$ is *ergodic* with respect to the translations $\{\tau_a : \ a \in \mathbb{R}\}$. This is simply an irreducibility assumption; if $\mathbb{Q}$ is not ergodic, then we can express it as an average over its *ergodic components* and our results would be valid for each component. The advantage/raison d'etre of the ergodicity is that the $\mathbb{Q}$-almost sure density of $\mathrm{Fix}(F)$ is a constant independent of F.

We note that in the periodic case, the fact that we have at least two geometrically distinct fixed points, translates to asserting that in the interval $[0, \ell]$, $\ell \in \mathbb{N}$, there are at least 2ℓ fixed points. How much of this is true in the random case? On account of Theorem A, let us formulate a wish-list that we would like to have for the set of fixed points $\mathrm{Fix}(F)$ of a randomly selected area-preserving twist map $F(\cdot, \cdot)$:

(i) At the very least, we would like to show that the set $\mathrm{Fix}(F)$ is nonempty (and in fact unbounded from both sides by ergodicity) almost surely.

(ii) Better yet, we would like to show that the set $\mathrm{Fix}(F)$ has a positive *density*. That is, the large ℓ limit of $\ell^{-1}\#\big(\mathrm{Fix}(F) \cap [0, \ell]\big)$ exists and is positive almost surely. (For a set B, $\#B$ denote its cardinality.)

(iii) Ideally, we can come up with two distinct properties (formulated in terms of the derivative dF at the fixed point) such that if the set of fixed points with those properties are denoted by $\mathrm{Fix}_1(F)$ and $\mathrm{Fix}_2(F)$, then both $\mathrm{Fix}_1(F)$ and $\mathrm{Fix}_2(F)$ are of positive density almost surely. This would be our analog of "two geometrically distinct" aspect of Poincaré-Birkhoff Theorem in the random setting.

2.1. **Monotone Twist Maps.**

As we will demonstrate in our first theorem, we have a rather satisfactory result in the case that $\mathbb{Q}$ is concentrated on the space of *monotone twists*.

Definition 2.1. **(1)** We write $\overline{\mathcal{MT}_+}$ for the space of maps $\bar{F} = (\bar{Q}, \bar{P}) \in \overline{\mathcal{T}}$ such that for every q, the function $f\colon [-1,\, 1] \to \mathbb{R}$, given by $f(p) := \bar{Q}(q,\, p)$ is increasing. We also write $\mathcal{MT}_+$ for the set of $\ell(\bar{F})$, with $\bar{F} \in \overline{\mathcal{MT}_+}$. We refer to the members of $\mathcal{MT}_+$ as the (positive) monotone twist maps. Similarly, we write $\mathcal{MT}_-$ for the space of maps F such that $F^{-1} \in \mathcal{MT}_+$. We refer to the members of $\mathcal{MT}_-$ as the negative monotone twist maps.

(2) A fixed point $x = (q,\, p)$ of $F(\cdot,\, \cdot)\colon \mathcal{S} \to \mathcal{S}$ is of $+$ (respectively -) type if the eigenvalues of $\mathrm{d}F(q,\, p)$ are positive (respectively negative). We write

$$\mathrm{Fix}_{\pm}(F) = \big\{ x \in \mathrm{Fix}(F) :\ x \text{ is of } \pm \text{ type } \big\}.$$

Our interest in $\mathcal{MT}_\pm$ stems from the fact that all monotone twist maps possess *generating functions*. That is, for $F = \ell(\bar{F}) \in \mathcal{MT}_+$, we can find a scalar-valued function $\mathcal{G}(q,Q) = \mathcal{G}(q,Q;\bar{F})$ such that

$$F(q, -\mathcal{G}_q(q,Q)) = (Q, \mathcal{G}_Q(q,Q)). \tag{2.2}$$

Because of the boundary conditions of $F(q,p) = (P(q,p), Q(q,p))$, we only need to define $\mathcal{G}(q,Q)$ for the pair (q,Q), such that

$$Q(q,-1) \leqslant Q \leqslant Q(q,+1). \tag{2.3}$$

In particular

$$\psi(q;\bar{F}) := \mathcal{G}(q,q;\bar{F}),$$

is well defined. As we will see later (see (2.4) below)

$$\psi(\cdot;\tau_a\bar{F}) = \tau_a\psi(\cdot;\bar{F}).$$

This implies that the law of $\psi(q;\bar{F})$ is translation invariant (hence independent of q), by the stationarity of $\mathbb{Q}$.

As is demonstrated in our Theorem B below, **(iii)** of our wish-list is (almost) materialized for randomly selected monotone twist maps:

Theorem B. *Let $\mathbb{Q}$ be a translation invariant ergodic probability measure on $\big(\overline{\mathcal{T}}, \mathcal{F}\big)$, such that $\mathbb{Q}\big(\overline{\mathcal{MT}_+}\big) = 1$. Then the following statements are true with probability one with respect to $\mathbb{Q}$:*

(1) *The sets $\mathrm{Fix}_\pm\big(\ell(\bar{F})\big)$ are nonempty.*

(2) *If the random pair*

$$\left(\frac{\mathrm{d}}{\mathrm{d}q}\psi(q;\bar{F}), \frac{\mathrm{d}^2}{\mathrm{d}q^2}\psi(q;\bar{F})\right),$$

has a probability density $\rho(a,b)$ (which is independent of q by translation invariance), then the sets $\mathrm{Fix}_\pm\big(\ell(\bar{F})\big)$ have positive density $\lambda_\pm$, given by

$$\lambda_\pm = \int_{-\infty}^{\infty} b^\pm \rho(0,b)\ db.$$

As we mentioned earlier, each monotone twist map has a generating function that is unique modulo a constant. It may appear that it would be hard to come up with examples of probability measures that are concentrated on monotone twist maps because the set $\mathcal{MT}_+$ is rather complicated; after all a function $F \in \mathcal{MT}_+$ must satisfy (1) – (3) of the Introduction, and the monotonicity condition. However, it would be easier if we start from a randomly selected generating function $\mathcal{G}$, and construct F from it with the aid of (2.2). For this to be useful, we need to figure out what conditions must be satisfied by $\mathcal{G}$ in order to be in the range of the map $\bar{F} \mapsto \mathcal{G}(\cdot,\cdot;\bar{F})$. It is not hard to see

$$\mathcal{G}(q+a, Q+a; \bar{F}) = \mathcal{G}(q, Q; \tau_a \bar{F}). \tag{2.4}$$

(See Section 5.) This suggests defining $\mathcal{L}$ by

$$\mathcal{G}(q, Q; \bar{F}) = \mathcal{L}(q, Q-q; \bar{F}),$$

and try to find the range of the map $\mathcal{A}(\bar{F}) := \mathcal{L}(\cdot,\cdot;\bar{F})$. In some sense, $\mathcal{L}(q,v) = \mathcal{L}(q,v;\bar{F})$ is the *Lagrangian* of F, with v playing the role of the velocity. We also note that in terms of the Lagrangian,

$$\psi(q;\bar{F}) = \mathcal{L}(q,0;\bar{F}).$$

For our second result, we determine the range of the map $\mathcal{A}$ and give a precise recipe for constructing $\mathcal{A}^{-1}$, and an ergodic stationary law on $\overline{\mathcal{MT}_+}$. We note that in view of (2.3), the domain of definition for the function $\mathcal{L}(q,v)$ is

$$\bar{Q}(q,-1) \leqslant v \leqslant \bar{Q}(q,1).$$

However, it is more convenient to start with an extension of $\mathcal{L}$ to $\mathbb{R}^2$, and figure out what the domain of $\mathcal{L}$ is from this extension. To have a simpler description for the operator $\mathcal{A}^{-1}$, we would even start from a suitable translate of the extended Lagrangian function, denoted by ω, and build $\mathcal{G}$ from this translation. In the following Definition, we give the necessary axioms for ω and make preparations for the construction of $\mathcal{G}$ from ω.

Definition 2.2. Let us write Ω_0 for the space of functions

$$\omega : \mathbb{R}^2 \to \mathbb{R}$$

such that $\omega(q,a) > 0$ for $a > 0$, $\omega(q,0) = 0$, and,

$$\eta(q;\omega) = \inf\{a \mid \omega(q,a) = 2\} < \infty,$$

for every q. We then set

$$Q^-(q;\omega) = \frac{1}{2}\int_0^{\eta(q;\omega)} \omega(q,a)\mathrm{d}a - \eta(q;\omega),$$

$$G(q,Q;\omega) = \omega(q, Q-q-Q^-(q;\omega)).$$

We write Ω_1 for the space of $\omega \in \Omega_0$ such that $G_q(q,Q;\omega) < 0$ for all (q,Q).

Theorem C. *For every $\omega \in \Omega_1$, there exists a unique function $\bar{F}(\cdot,\cdot;\omega)$ such that if $F(\cdot,\cdot;\omega) = \ell\big(\bar{F}(\cdot,\cdot;\omega)\big)$, then the equation (2.2) is valid for the function $\mathcal{G}(q,Q) = \mathcal{G}(q,Q;\omega)$, that is given by*

$$\mathcal{G}(q,Q;\omega) = \int_{q+Q^-(q;\omega)}^{Q} \omega(q,a)\mathrm{d}a - (Q-q).$$

Moreover, if $\tau_a\omega(q,v) = \omega(q+a,v)$, then

$$\bar{F}(\cdot,\cdot;\tau_a\omega) = \tau_a\bar{F}(\cdot,\cdot;\omega). \tag{2.5}$$

Theorem C provides a recipe for constructing examples of random monotone twist maps for which Theorem B is applicable.

Recipe for Monotone Twist Map: Take any probability measure $\mathbb{P}$ on Ω_0 that is stationary and ergodic with respect to $\{\tau_a : \ a \in \mathbb{R}\}$. If $\mathbb{P}(\Omega_1) = 1$, then the push forward $\mathbb{Q} := \mathcal{B}_\sharp\mathbb{P}$ of $\mathbb{P}$ with respect to the map

$$\mathcal{B} : \Omega_1 \to \overline{\mathcal{MT}_+}, \qquad \mathcal{B}(\omega)(\cdot,\cdot) = \bar{F}(\cdot,\cdot;\omega),$$

gives a stationary and ergodic probability measure on $\overline{\mathcal{MT}_+}$.

It is straightforward to construct stationary and ergodic probability measures on Ω_0. The only nontrivial condition to worry about is $\mathbb{P}(\Omega_1) = 1$. To see how in practice such a condition is verified, let us consider a concrete example.

Example 2.3 If we assume that ω is linear in a, and write $\omega(q, a) = R(q)a$, for a function $R : \mathbb{R} \to \mathbb{R}$, then $\omega \in \Omega_0$ means that $R > 0$. On the other hand, if the derivative of R satisfies $2|R'| \leqslant R^2$, $\mathbb{P}$-almost surely, then $\mathbb{P}(\Omega_1) = 1$. For example, we can start from an arbitrary uniformly positive stationary process R_0 with bounded derivative, and build ω via $\omega(q, a) = cR_0(q)a$. This would satisfy $\mathbb{P}(\Omega_1) = 1$ provided that c is sufficiently large. See Example 5.7 for details.

2.2. **Hamiltonian Systems.** We now describe an important class of twist maps that may include non-monotone examples. Let Ω_2 denote the set C^2 (Hamiltonian) functions $\omega(q, p, t)$ with uniformly bounded second derivatives such that

$$\pm\omega_p(q, \pm 1, t) > 0, \qquad \omega_q(q, \pm 1, t) = 0.$$

(Note that even though $\omega(q, \pm 1, t)$ is independent of q, the function $\omega_p(q, \pm 1, t)$ may depend on q.) Given $\omega \in \Omega_2$, set

$$\tau_a\omega(q, p, t) = \omega(q + a, p, t),$$

as before, and write $\phi^\omega_t(q, p)$ for the flow of the corresponding Hamiltonian system

$$\dot{q} = \omega_p(q, p, t), \qquad \dot{p} = -\omega_q(q, p, t).$$

It is not hard to show that if

$$F^t(q, p; \omega) = \phi^\omega_t(q, p), \qquad \bar{F}^t(q, p; \omega) = \phi^\omega_t(q, p) - (q, 0),$$

then $F^t(\cdot, \cdot; \omega) \in \mathcal{T}$, and

$$\bar{F}^t\big(q, p; \tau_a\omega\big) = \tau_a\bar{F}^t\big(q, p; \omega\big).$$

This means that if we start with a τ-invariant ergodic probability measure $\mathbb{P}$ on Ω_2, then its push-forward $\mathbb{Q}^t$ under the map $\omega \mapsto \bar{F}^t(\cdot, \cdot; \omega)$ is a τ-invariant ergodic probability measure on $\overline{\mathcal{MT}_+}$.

Theorem D. *Let $\mathbb{P}$ be a τ-invariant ergodic probability measure $\mathbb{P}$ on Ω_2. Then the probability that $F^t(\cdot,\cdot;\omega)$ has infinitely many fixed points is one for every $t \geq 0$, i.e.*

$$\mathbb{P}\Big(\#\mathrm{Fix}\big(F^t(\cdot,\,\cdot,\,;\omega)\big) = \infty\Big) = 1.$$

Example 2.4 In contrast with Theorem C, it is not hard to come up with examples of Q^t because any q-stationary process $\omega(q,p,t)$ does the job provided that the boundary conditions are satisfied. Here are two examples:

(i) Let $A(q,t)$ be a positive C^2, q-stationary random process with bounded second q-derivative. Take any deterministic C^2 function $B(p,t)$ with bounded second p-derivative such that $\pm B_p(\pm 1,t) > 0$, and $B(\pm 1,t) = 0$. Then the Hamiltonian function $\omega(q,p,t) = A(q,t)B(p,t)$ will be in Ω_2.

(ii) One classical way of constructing a stationary Hamiltonian function is starting from a fixed deterministic Hamiltonian function $H^0(q,p,t)$ of compact support in q-variable that satisfies the boundary conditions, and set

$$\omega(q,p,t;\alpha) = \sum_i H^0(q - q_i, p, t),$$

where

$$\alpha = \{q_i \ : \ i \in \mathbb{Z}\},$$

is a *stationary point process*. By this we mean that the set α is a discrete subset $\mathbb{R}$ that is selected randomly with a law that is invariant with respect to the translations

$$\tau_a\alpha = \{q - a : \ q \in \alpha\}.$$

A Poisson point process of constant intensity is an example of a stationary point process.

Theorem C raises two questions:

- **(i)** Let $\mathbb{Q}$ be a stationary ergodic measure on $\overline{\mathcal{T}}$. Can we find a stationary ergodic measure $\mathbb{P}$ on Ω_2 such that $\mathbb{Q} = \mathbb{Q}^1$? In other words, $\mathbb{Q}$ is the push-forward of $\mathbb{P}$ with respect to the map $\omega \mapsto \bar{F}^1(\cdot,\cdot;\omega)$?
- **(ii)** What can be said about the type of the fixed points we have in Theorem D?

To rephrase the first question, let us write $\mathcal{C}([0,1];\overline{\mathcal{T}})$ for the space of C^1 maps $\gamma : [0,1] \to \overline{\mathcal{T}}$, such that $\ell(\gamma(0))$ is identity. The shift operator τ on $\overline{\mathcal{T}}$ induces a shift operator (again denoted by τ) on $\mathcal{C}([0,1];\overline{\mathcal{T}})$ by $(\tau_a\gamma)(t) = \tau_a(\gamma(t))$. Note that if we have a stationary ergodic measure $\mathbb{P}$ on Ω_2, then the map

$$\omega \mapsto \big(\bar{F}^t(q,p;\omega) = \phi_t^\omega(q,p) - (q,0) : \ t \in [0,1]\big),$$

pushes forward $\mathbb{P}$ onto a stationary ergodic probability measure $\mathcal{Q}$ on $\mathcal{C}([0,1];\overline{\mathcal{T}})$. It is not hard to show that the converse is also true; a stationary ergodic probability measure $\mathcal{Q}$ on $\mathcal{C}([0,1];\overline{\mathcal{T}})$ is necessarily comes from a unique a stationary ergodic measure $\mathbb{P}$ on Ω_2. As a result, we may rephrase the first question as

(i') Let $\mathbb{Q}$ be a stationary ergodic measure on $\overline{\mathcal{T}}$. Can we find a stationary ergodic measure $\mathcal{Q}$ on $\mathcal{C}([0,1];\overline{\mathcal{T}})$, such that $\mathbb{Q}$ is the push forward of $\mathcal{Q}$ under the time-1 map $\pi_1 : \mathcal{C}([0,1];\overline{\mathcal{T}}) \to \overline{\mathcal{T}}$? (By time-1 map we mean $\pi_1\gamma := \gamma(1)$.)

We conjecture that the answer to this question is affirmative. In Theorem E below, we partially resolve this conjecture by showing that if there is a path γ such that $\ell(\gamma)$ is area preserving only in the average sense, then it can be deformed to a path of area preserving maps. To state this carefully, we make a definition.

Definition 2.5. Let $\mathcal{D}$ denotes the space of diffeomorphism $F : \mathcal{S} \to \mathcal{S}$. We write $\overline{\mathcal{D}}$ for the space of functions $\bar{F}$ such that $\ell(\bar{F}) \in \mathcal{D}$. Let $\mathcal{Q}$ be a stationary ergodic measure on $\mathcal{C}([0,1];\overline{\mathcal{D}})$. We say that $\mathcal{Q}$ is *regular* if the following conditions are true:

(a)
$$\int \sup_{t\in[0,1]} \left[\|\dot{\gamma}(t)\|_\infty + \|d\gamma(t)\|_\infty + \left\|d\gamma(t)^{-1}\right\|_\infty\right] \, \mathcal{Q}(d\gamma) < \infty,$$
with $\|\cdot\|_\infty$ denoting the L^∞ norm, and $\dot{\gamma}(t)$ and $d\gamma(t)$ denoting the derivatives of $\gamma(t)$ with respect to t and $x = (q,p)$.

(b)
$$\frac{1}{2}\int \left[\int_{-1}^{1} \det(\mathrm{d}\gamma(t)(q,p))\, \mathrm{d}p\right] \, \mathcal{Q}(d\gamma) = 1;$$
for every $t \in [0,1]$. (This expression is independent of q by the stationarity of $\mathcal{Q}$.)

We note that the condition (b) is trivially satisfied if $\mathcal{Q}$ is concentrated on $\mathcal{C}([0,1];\overline{\mathcal{T}})$, because for an area preserving $\ell(\gamma(t))$, we simply have
$$\det(\mathrm{d}\gamma(t)(q,p)) = 1.$$

Theorem E. *Let $\mathbb{Q}$ be a stationary ergodic measure on $\overline{\mathcal{T}}$. If there exists a regular stationary ergodic measure $\mathcal{Q}$ on $\mathcal{C}([0,1];\overline{\mathcal{D}})$, such that $\mathbb{Q}$ is the push forward of $\mathcal{Q}$ under the time-1 map $\pi_1\gamma := \gamma(1)$, then there exits another stationary ergodic measure $\mathcal{Q}'$ on $\mathcal{C}([0,1];\overline{\mathcal{T}})$, such that $\mathbb{Q}$ is also the push forward of $\mathcal{Q}$ under the time-1 map π_1.*

2.3. **The Complexity of an Isotopy.** We now would like to address the second question we asked in Subsection 2.2:

(ii) What can be said about the type of the fixed points we have in Theorem D?

To address this question, we first need to explain what role the measure $\mathcal{Q}$ on $\mathcal{C}([0,1];\overline{\mathcal{T}})$ plays in the proof of Theorem D. When the measure $\mathbb{Q}$ on $\overline{\mathcal{T}}$ comes from a measure $\mathbb{P}$ on Ω_2, or when the assumptions of Theorem D are satisfied, we have an *isotopy* of area preserving twists $F^t(\cdot,\cdot;\omega)$ that connects the identity to $F(\cdot,\cdot;\omega) := F^1(\cdot,\cdot;\omega)$, with F^1 distributed according to $\mathbb{Q}$.

Following the Chaperon's strategy for proving Conley-Zehnder's Theorem, we may use this isotopy to express F as a finite composition of monotone twist maps. More precisely,

Theorem F. *Let $\mathbb{P}$ be a τ-invariant ergodic probability measure $\mathbb{P}$ on Ω_2. Let $F = F^1$ be as in Theorem D. Then there exists a deterministic integer $N \geqslant 0$ and area-preserving random twists F_j, $0 \leqslant j \leqslant N$, such that for $\mathbb{P}$ almost all $\omega \in \Omega_2$, we have a decomposition:*

$$F(\cdot, \cdot; \omega) = F_N(\cdot, \cdot; \omega) \circ \ldots \circ F_2(\cdot, \cdot; \omega) \circ F_1(\cdot, \cdot; \omega) \circ F_0(\cdot, \cdot; \omega), \tag{2.6}$$

where

- *F_j is negative monotone if j is even;*
- *F_j is positive monotone if j is odd;*
- *$\bar{F}_j(q, p; \omega) := F_j(q, p; \omega) - (q, 0)$ is stationary i.e. $\bar{F}_j(q, p; \tau_a\omega) = \tau_a\bar{F}_j(q, p; \omega)$, for every j.*

The integer N in (2.6) is the *complexity of* F. We may use an L^∞ bound on the first derivative of ω to get an upper bound on N. Statements [Le91, Propositions 2.6 & 2.7, Lemma 2.16] have the flavor to Theorem F for classical twists (see also [MS98, Section 9.2]).

When $N = 1$ or 2, more can be said about the set of fixed points and the nature of dF at the fixed points. We refer to Theorems 6.5 and 6.6 when $N = 1$, and Theorem 7.3 when $N = 2$.

2.4. **Almost periodic twists.** In fact our main results do cover the classical Poicare-Birkhoff Theorem A in some cases. For example, Theorem D implies that the flow of any deterministic Hamiltonian function $H^0(q, p, t)$ on strip $\mathcal{S}$, that is 1-periodic in q, and satisfies the twist boundary conditions $(H^0_q(q, \pm 1, t) = 0, \pm H^0_p(q, \pm, t) > 0)$, possesses 1-periodic orbits (or its time one map has fixed points). In other words, we may recast the deterministic periodic model as an example of a random twist model. The interpretation we have in mind is also applicable if H^0 is *almost periodic* in q. We explain this by three models of random area-preserving twist maps:

Example 2.6 *(Periodic Twists)* As the simplest example, take any $\bar{F}_0 \in \overline{\mathcal{T}}$, that is 1-periodic in q-variable, and set

$$\Gamma(\bar{F}_0) = \big\{\tau_a \bar{F}_0 \ : \ a \in \mathbb{R}\big\} \subset \overline{\mathcal{T}}. \tag{2.7}$$

We now take a τ-invariant probability measure $\mathbb{Q}$ on $\overline{\mathcal{T}}$ that is concentrated on $\Gamma(\bar{F}_0)$. Since $\bar{F}_0$ is a 1-periodic function in q-variable, the set $\Gamma(\bar{F})$ is homeomorphic to the circle. (Here were are thinking of circle as the interval $[0, 1]$ with $0 = 1$.) Since $\mathbb{Q}$ is τ-invariant, it can only be the push forward of the Lebesgue measure under the map $a \mapsto \tau_a\bar{F}$. Now any almost sure statement for the fixed points of the lifts of maps in the support of $\mathbb{Q}$ is equivalent to an analogous statement for the map $\bar{F}_0$. This is because if (q_0, p_0) is a fixed point for

$$F(q, p) := \ell\big(\tau_a\bar{F}\big) = (q, 0) + \tau_a\bar{F}_0(q, p),$$

then $(q_0 + a, p_0)$ is a fixed point for $F_0 = \ell(\bar{F}_0)$. For example, the statement that for $\mathbb{Q}$-almost choices of $\bar{F}$, its lift $\ell(\bar{F})$ has a fixed point is equivalent to asserting that F_0 has a fixed point. In summary, our stochastic model *coincides* with the classical setting of Poincaré-Birkhoff in this case.

Example 2.7 *(Quasi-periodic twists)* Pick a function

$$\bar{F}_1 : \mathbb{T}^k \times [-1, -1] \to \mathbb{R} \times [-1, 1],$$

where $\mathbb{T}^k$ denotes the k-dimensional torus. Pick a vector $v \in \mathbb{R}^k$ that satisfies the following condition:

$$\langle v,\, n\rangle = 0, \quad n \in \mathbb{Z}^k \quad \Rightarrow \quad n = 0. \tag{2.8}$$

Let

$$\bar{F}_0(q, p) = \bar{F}_1(qv, p)$$

and define $\Gamma(\bar{F}_0)$ as in (2.7). Note that if $k > 1$, the set $\Gamma(\bar{F}_0)$ is not closed. However, the condition (2.8) guarantees that its topological closure $\Gamma'(\bar{F}_0)$ consists of functions of the form

$$\bar{F}(q, p; b) = \bar{F}_0(qv + b, p),$$

with $b \in \mathbb{T}^k$. (Here we regard $\mathbb{T}^k$ as $[0, 1]^k$ with $0 = 1$, and $qv - b$ is a *Mod* 1 summation.) Assume that $\mathbb{Q}$ is concentrated on the set $\Gamma'(\bar{F}_0)$. Again, since $\mathbb{Q}$ is τ-invariant, the pull-back of $\mathbb{Q}$ with respect to the transformation $b \in \mathbb{T}^k \mapsto \bar{F}(\cdot, \cdot; b)$ can only be the uniform measure on $\mathbb{T}^k$. Hence, a $\mathbb{Q}$-almost sure statement regarding the fixed points of $F = \ell(\bar{F})$, is equivalent to an analogous statement for the map

$$F(\cdot, \cdot; b) = \ell\big(\bar{F}(\cdot, \cdot; b)\big),$$

for almost all $b \in \mathbb{T}^k$. In other words, our main result does not guarantee the existence of fixed points for a given quasi-periodic map $F_0 = \ell(\bar{F}_0)$. Instead our main results say that for almost all choices of b, the map $F(\cdot, \cdot; b)$ possesses fixed points.

Example 2.8 *(Almost periodic-twists)* Given a function $\bar{F}_0 \in \overline{\mathfrak{T}}$, let us assume that the corresponding $\Gamma(\bar{F}_0)$ is precompact with respect to the topology of uniform convergence. We write $\Gamma'(\bar{F}_0)$ for the topological closure of $\Gamma(\bar{F}_0)$. By the classical theory of almost periodic functions, the set $\Gamma'(\bar{F}_0)$ can be turned to a compact topological group and for $\mathbb{Q}$, we may choose a normalized *Haar* measure on $\Gamma'(\bar{F}_0)$. Again, our main results only guarantee the existence of fixed points for $F(q, p) = \ell(\bar{F})$, for $\mathbb{Q}$-almost all choices of $\bar{F}$.

In the random stationary setting, we may start with a function $\bar{F}_0$ such that the corresponding $\Gamma'(\bar{F}_0)$ is not compact and may not have a group structure. Instead we may insist on the existence of an ergodic translation invariant measure that is concentrated on $\Gamma'(\bar{F}_0)$. Even the last requirement can be relaxed and our measure $\mathbb{Q}$ may not be concentrated on $\Gamma'(\bar{F}_0)$ for some $\bar{F}_0$. The measure $\mathbb{Q}$ in some sense plays the role of the normalized Haar measure in our third example above.

2.5. **Abstract Formulation.** So far we have stated several results for the set of fixed point of $\ell(\bar{F})$ where $\bar{F}$ is selected according to a suitable ergodic stationary probability measure $\mathbb{Q}$ on $\overline{\mathcal{T}}$. In all the models we have discussed in the preceding subsections, the measure $\mathbb{Q}$ is expressed as a push forward of another ergodic stationary probability measure $\mathbb{P}$ that is now defined on a probability space Ω. In other words, our $\mathbb{Q}$-selected function can be expressed as $\bar{F}(\cdot,\cdot;\omega)$ with ω selected according to a suitable stationary ergodic measure $\mathbb{P}$, and the map $\omega \mapsto \bar{F}(\cdot,\cdot;\omega)$ satisfies

$$\tau_a \bar{F}(\cdot,\cdot;\omega) = \bar{F}(\cdot,\cdot;\tau_a\omega).$$

This is equivalent to

$$\bar{F}(q,p;\omega) = \bar{F}(0,p;\tau_q\omega) := \bar{F}_0(\tau_q\omega, p). \tag{2.9}$$

The space Ω takes different forms depending on the type of result we have in mind. For example

- $\Omega = \Omega_1$ where Ω_1 is the space of functions in Ω_0 that satisfies a suitable nondegeneracy condition as was discussed in Subsection 2.1.
- $\Omega = \Omega_2$ is the space of certain Hamiltonian functions as we described in Subsection 2.3.
- $\Omega = \Gamma'(\bar{F}_0)$ is the topological closure of the translates of a fixed $\bar{F}_0 \in \overline{\mathcal{T}}$ as we discussed in Subsection 2.4.

For a more flexible formulation, we set up a framework for random area-preserving twist map that is defined on a general and unspecified Ω. With this goal in mind, we propose the following abstract setting to study area-preserving dynamics: a *probability space*, that is, a quadruple:

$$\hat{\Omega} := (\Omega,\, \mathcal{F},\, \mathbb{P},\, \tau). \tag{2.10}$$

Here Ω is a separable metric space, $\mathcal{F}$ is the Borel sigma-algebra on Ω, $\tau\colon \mathbb{R}\times\Omega \to \Omega$ is a continuous $\mathbb{R}$-action, and $\mathbb{P}$ is a τ-invariant ergodic probability measure on $(\Omega, \mathcal{F})$. Denote $\tau_a := \tau(a,\cdot)\colon \Omega \to \Omega$. In addition, we assume:

(i) $\mathbb{P}$-*positivity*: if $U \in \mathcal{F}$ is a nonempty open set, then $\mathbb{P}(U) > 0$.
(ii) $\mathbb{P}$-*preservation by* τ: $\mathbb{P}(\tau_a A) = \mathbb{P}(A)$ for every $a \in \mathbb{R}$, and every $A \in \mathcal{F}$.
(iii) *Ergodicity*: for every $A \in \mathcal{F}$, if $\tau_a A = A$ for all $a \in \mathbb{R}$, then $\mathbb{P}(A) = 1$ or $\mathbb{P}(A) = 0$.

If (i), (ii), and (iii) hold we say that $\mathbb{P}$ is a τ-*invariant ergodic* probability measure. For instance, take a smooth manifold Ω which admits a smooth global flow $\phi : \mathbb{R}\times\Omega \to \Omega$ with an ergodic invariant probability measure $\mathbb{P}$ that is positive on nonempty open subsets of Ω (it is non-trivial to find ϕ with these properties), $\mathcal{F}$ the Borel sigma-algebra of Ω, and $\tau_a := \phi(a,\cdot)$.

In what follows, let $\hat{\Omega}$ be a probability space as in (2.10). Let

$$\bar{F}_0\colon \Omega \times [-1,\, 1] \to \mathcal{S}$$

be a measurable map with respect to the product measure of $\mathbb{P}$ and the Lebesgue measure on $[-1,\, 1]$. Write

$$\bar{F}_0(\omega,p) = (\bar{Q}(\omega,p), \bar{P}(\omega,p))$$

and suppose that $F\colon \mathcal{S}\times\Omega \to \mathcal{S}$ is of the form $F(q,p;\omega) = (Q(q,\, p;\, \omega), P(q,\, p;\, \omega))$, with

$$\begin{cases} Q(q,\, p;\, \omega) \;=\; q + \bar{Q}(\tau_q\omega,\, p) \\ P(q,\, p;\, \omega) \;=\; \bar{P}(\tau_q\omega,\, p). \end{cases} \tag{2.11}$$

Write $\mathbb{E}$ for the expected value with respect to the probability measure $\mathbb{P}$.

Definition 2.9. We say that F in (2.11) is an *area-preserving random twist* if the following hold for $\mathbb{P}$-almost all ω:

(1) *area-preservation*: $F(\cdot\,,\,\cdot\,;\,\omega)\colon \mathcal{S} \to \mathcal{S}$ is an area-preserving diffeomorphism;
(2) *boundary invariance*: $P(q,\pm1;\omega) = \pm1$;
(3) *boundary twisting*: $q \mapsto Q(q,\pm1;\omega)$ is increasing, and $\pm\bar{Q}(\omega,\pm1) > 0$.

If additionally $\bar{Q}(\omega,p)$ is increasing in p, we refer to F as a *(positive) monotone area-preserving random twist*. We call F a *negative monotone area-preserving random twist*, if F^{-1} is a positive area-preserving random twist. When there is no danger of confusion, we may simply call such map as a positive/negative twist.

It is this abstract formulation that will be used for the rest of the paper. By a slight change of notion, we can readily restate our main results Theorems B-F in terms of an abstract area-preserving random twist.

2.6. **Arnol'd Conjecture.** Arnol'd formulated the higher dimensional analogue of Theorem A: the Arnol'd Conjecture [Ar78] (see also [Au13], [Ho12], [HZ94], [Ze86]). The first breakthrough on the conjecture was by Conley and Zehnder [CZ83], who proved it for the $2n$-torus (a proof using generating functions was later given by Chaperon [Ch84]). The second breakthrough was by Floer [Fl88, Fl89, Fl89b, Fl91]. Related results were proven eg. by Hofer-Salamon, Liu-Tian, Ono, Weinstein [Ho85, HS95, LT98, On95, We83].

The first breakthrough on Arnold's Conjecture was achieved by Conley and Zehnder [CZ83]. According to their theorem, any smooth symplectic map $F : \mathbb{T}^{2d} \to \mathbb{T}^{2d}$ that is isotopic to identity has at least $2d+1$ many fixed points. For the stochastic analog of [CZ83], we take a $2d$-dimensional stationary process

$$X(x;\omega) = \bar{X}(\tau_x\omega)$$

with

$$\bar{X} : \Omega \to \mathbb{R}^{2d};\; x \in \mathbb{R}^{2d},$$

and assume that its lift

$$F(x;\omega) = x + X(x;\omega)$$

is symplectic with probability one. Our strategy of proof is also applicable to such random symplectic maps. The main ingredients for proving results analogous to our main theorems are Morse Theory and Spectral Theorem for multi-dimensional stationary processes. In a subsequent paper, we will work out a generalization of Conley and Zehnder's Theorem in the stochastic setting.

2.7. **Main Strategy and Outline of the Paper.** As we mentioned earlier, we have adopted Chaperon's approach to establish our main results. Here is an outline of what follows:

(i) In Section 3 we give a definition for the *generalized generating function* of an area-preserving twist map F in our random setting and show that there is a one-to-one corresponding between fixed points F and the critical points of its generalized generating function (Proposition 3.3).

(ii) The purpose of Section 4 is threefold:

- We show that if (2.6) is valid for F, then F has a generalized generating function. On account of Proposition 3.3, we may prove our results about the fixed points of F by proving analogous results for the critical points of its generalized generating function. (See Lemma 4.4.)
- We establish Theorem F so that (2.6) is valid whenever our area-preserving random twist map comes from a Hamiltonian ODE.
- We establish Theorem E so that any regular $\mathcal{Q}$ comes from a Hamiltonian ODE.

(iii) By Theorem F, we can associate a nonnegative integer N to our twist map that measures its complexity. The case $N = 0$ corresponds to the monotone twist maps and they are studied in Section 5. In particular, we establish Theorems B and C in this Section.

(iv) In Sections 6 and 7 we study the case of an area-preserving twist map with complexity $N \leq 2$. Most notably we show that if $N = 1$, then the corresponding F possesses infinitely many fixed points of different types (Theorems 6.5, 6.6). In the case of $N = 2$, we have a similar result for the critical points of its generating function. Though we have not been able to prove an analogous result for the fixed points of F because of a complicated formula that relates the eigenvalues of the first derivative of F to the the second derivatives of its generating function.

(v) Section 8 is devoted to the proof of Theorem D.

3. Calculus of random generating functions

We construct the principal novelty of the paper, random generating functions, and explain how to use them to find fixed points. Recall that Ω is as in (2.10).

Definition 3.1. We say that a measurable function $G : \Omega \to \mathbb{R}$ is *ω-differentiable* if the limit

$$\nabla G(\omega) := \lim_{t \downarrow 0} t^{-1} \left(G(\tau_t \omega) - G(\omega) \right)$$

exists for $\mathbb{P}$-almost all ω. For a measurable map $K \colon \Omega \times [0,\, 1] \to \mathbb{R}$ we write $K_p = \frac{\partial K}{\partial p}$ and $K_\omega = \frac{\partial K}{\partial \omega}$ for the partial derivatives of K. We say that K *is* C^1 if the partial derivatives of K exist and are continuous for $\mathbb{P}$-almost all ω.

Given an area-preserving random twist as in (2.11), consider the sets (see Figure 3.1):

$$
\begin{cases} \bar{A} := \{(\omega;\, v) \mid \bar{Q}(\omega,\, -1) \leqslant v \leqslant \bar{Q}(\omega,\, 1)\} \subseteq \Omega \times \mathbb{R} \\ \bar{A}_\omega := \{v \mid (\omega;\, v) \in \bar{A}\} \subseteq \mathbb{R} \\ \bar{A}^v := \{\omega \mid (\omega;\, v) \in \bar{A}\} \subseteq \Omega \\ A_\omega := \{(q, Q) \mid (\tau_q\omega;\, Q - q) \in \bar{A}\}. \end{cases} \tag{3.1}
$$

We write $F^{-1}(P, Q) = (q(Q, P), p(Q, P))$.

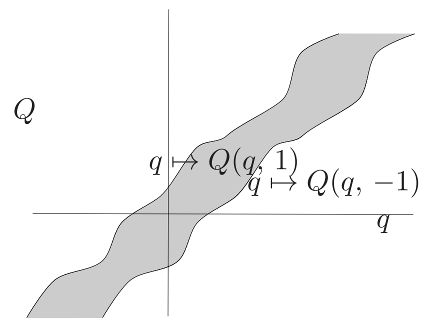

FIGURE 3.1. A_ω in (3.1) bounded by the graphs of $q \mapsto Q(q,\, 1)$, $q \mapsto Q(q,\, -1)$, respectively.

Definition 3.2. Given an area-preserving random twist map (2.11), we say that $\mathcal{L} : \bar{A} \times \mathbb{R}^N \to \mathbb{R}$ is a *generalized generating function of complexity* N if $\mathcal{L}$ is C^1 and the function

$$
\mathcal{G}(q, Q; \xi) = \mathcal{G}(q, Q; \xi, \omega) := \mathcal{L}(\tau_q\omega, Q - q, \xi_1 - q, \ldots, \xi_N - q),
$$

with, $\xi = (\xi_1, \ldots, \xi_N)$, satisfies:

$$
\mathcal{G}_\xi(q, Q; \xi, \omega) = 0 \Rightarrow F\left(q, -\mathcal{G}_q(q, Q; \xi, \omega); \omega\right) = \left(Q, \mathcal{G}_Q(q, Q; \xi, \omega)\right). \tag{3.2}
$$

Our interest in generalized generating functions is due to the following.

Proposition 3.3. *Let $\mathcal{L}$ be a generalized generating function for F. Set*

$$
\mathcal{I}(q, \xi; \omega) = \mathcal{L}(\tau_q\omega, 0, \xi_1 - q, \ldots, \xi_N - q).
$$

If $(\bar{q}, \bar{\xi})$ is a critical point for $\mathcal{I}(\cdot, \cdot; \omega)$, then $\vec{x} := (\bar{q},\, -\mathcal{G}_q(\bar{q},\, \bar{q};\, \bar{\xi}))$ is a fixed point of $F(\cdot, \cdot; \omega)$.

Proof. Observe that if $(\bar{q}, \bar{\xi})$ is a critical point of $\mathcal{I}$, then by the definition of $\mathcal{G}$, $\mathcal{G}_Q(\bar{q},\, \bar{q};\, \bar{\xi}) = -\mathcal{G}_q(\bar{q},\, \bar{q};\, \bar{\xi})$ and $\mathcal{G}_\xi(\bar{q},\, \bar{q};\, \bar{\xi}) = 0$. Since $\mathcal{L}$ is a generating function, $\mathcal{G}_\xi = 0$ gives $F(\vec{x}) = \vec{x}$. □

The underlying theme of the paper is to show that fixed points of F are in correspondence with critical points of the associated random generating function $\mathcal{G}$, and then prove existence of critical points of $\mathcal{G}$. Viterbo has used generating functions with great success [Vi11]. Golé [Go01] describes several results in this direction.

4. Proofs of Theorems E and F

We begin by introducing stationary lifts.

Definition 4.1. A function $f(q,\omega)$ is *stationary* if $f(q,\omega) = \bar{f}(\tau_q\omega)$ for a continuous $\bar{f}\colon \Omega \to \mathbb{R}$. We say that f is a *stationary lift* if $f(q,\omega) = q + \bar{f}(\tau_q\omega)$ for a continuous $\bar{f}\colon \Omega \to \mathbb{R}$.

Definition 4.2. A vector-valued map $f(q,p;\omega)$ with $f(\cdot,\omega)\colon \mathbb{R}^2 \to \mathbb{R}^2$ is *q-stationary* if $f(q,p,\omega) = \bar{f}(\tau_q\omega, p)$ for some $\bar{f}\colon \Omega\times\mathbb{R} \to \mathbb{R}^2$. A similar definition is given for $f(\cdot,\omega)\colon \mathcal{S} \to \mathbb{R}^2$. We say that such f is a *q-stationary lift* if f can be expressed as $f(q,p,\omega) = (q,0) + \bar{f}(\tau_q\omega, p)$.

Proposition 4.3. *The following properties hold:*

(P.1) *If $f(q,\omega)$ is an increasing stationary lift in C^1, then f^{-1} is an increasing lift. The same holds for q-stationary diffeomorphism lifts $f(q,p,\omega)$.*

(P.2) *The composition of q-stationary lifts is a q-stationary lift. If f is a q-stationary lift and g is q-stationary, $g\circ f$ is q-stationary.*

(P.3) *For every differentiable $\bar{f}:\Omega\to\mathbb{R}$ we have that $\mathbb{E}\nabla\bar{f} = 0$.*

Proof. The proof of (P.2) is trivial. We only prove (P.1) for a stationary lift $f(q,p,\omega)$ because the case of $f(q,\omega)$ is done in the same way. Assume that $f(q,p,\omega)$ is a q-stationary lift so that for every $a\in\mathbb{R}$, $f(q+a,p,\omega) = (a,0)+f(q,p,\tau_a\omega)$, and write $g(q,p,\omega)$ for its inverse. To show that $g(q,p,\omega)$ is a q-stationary lift it suffices to check that $g(q+a,p,\omega) = (a,0)+g(q,p,\tau_a\omega)$. In order to do this, let us fix a and write $\tilde{g}(q,p,\omega)$ for the right-hand side $(a,0)+g(q,p,\tau_a\omega)$. Observe that since f is a q-stationary lift,

$$f(\tilde{g}(q,p,\omega),\omega) = (a,0) + f(g(q,p,\tau_a\omega),\tau_a\omega) = (a,0)+(q,p) = (q+a,p).$$

By uniqueness, $\tilde{g}(q,p,\omega) = g(q+a,p,\omega)$, which concludes the proof of (P.2). As for (P.3), write $f(x,\omega) = \bar{f}(\tau_x\omega)$ and observe that for any smooth $J\colon\mathbb{R}\to\mathbb{R}$ of compact support, with $\int_{\mathbb{R}} J(x)\mathrm{d}x = 1$,

$$\begin{aligned}
\mathbb{E}\nabla\bar{f} &= \int_{\mathbb{R}} J(x)\left(\mathbb{E}f_x(x,\omega)\right)\,\mathrm{d}x\\
&= -\mathbb{E}\int_{\mathbb{R}} J'(x) f(x,\omega)\,\mathrm{d}x\\
&= -\left(\int_{\mathbb{R}} J'(x)\,\mathrm{d}x\right)\left(\mathbb{E}\bar{f}\right),
\end{aligned} \tag{4.1}$$

so $\mathbb{E}\nabla\bar{f} = 0$. □

The proof of Theorem E draws on spectral theory for random processes. To this end, let us recall the statement of the Spectral Theorem for random processes. The Spectral Theorem allows us to represent a random process in terms of an auxiliary process with randomly orthogonal increments. Such a representation reduces to a Fourier series expansion if the stationary process is periodic. In order to apply the Spectral Theorem to a stationary process $a(q) = \bar{a}(\tau_q\omega)$, one follows the steps:

(i) Assume that $a(q)$ is *centered* in the sense that $\mathbb{E}\bar{a}(\omega) = 0$. We define the *correlation* $R(z) = \mathbb{E}\bar{a}(\omega)\bar{a}(\tau_z\omega)$.

(ii) There always exists a nonnegative measure G such that

$$R(z) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}z\xi}\, G(\mathrm{d}\xi).$$

(iii) One can construct an auxiliary process $(Y(\xi) : \xi \in \mathbb{R})$ or alternatively the random measure $Y(\mathrm{d}\xi) = Y(\mathrm{d}\xi, \omega)$ that are related by $Y(I) = Y(b) - Y(a)$, where $I = [a, b]$. The process Y has orthogonal increments in the following sense:

$$I \cap J = \varnothing \Longrightarrow \mathbb{E}Y(I)Y(J) = 0. \tag{4.2}$$

The relationship between the measure $G(\mathrm{d}\xi)$ or its associated nondecreasing function $G(\xi)$ is given by $\mathbb{E}Y(I)^2 = G(b) - G(a) = G(I)$.

The *Spectral Theorem* ([Do53]) says that for any stationary process a for which $\mathbb{E}\bar{a}^2 < \infty$, we may find a process Y satisfying (4.2) such that $\bar{a}(\tau_q\omega) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}q\xi} Y(\mathrm{d}\xi)$. Note that

$$\mathbb{E}\bar{a}(\tau_q\omega)\bar{a}(\omega) = \mathbb{E}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}q\xi}\, Y(\mathrm{d}\xi) \int_{-\infty}^{\infty} Y(\mathrm{d}\xi') = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}q\xi}\, G(\mathrm{d}\xi). \tag{4.3}$$

Also, $\bar{a}(\omega) = \int_{-\infty}^{\infty} Y(\mathrm{d}\xi, \omega)$, and the stationarity of $a(q, \omega)$ means

$$Y(\mathrm{d}\xi, \tau_q\omega) = \mathrm{e}^{\mathrm{i}q\xi} Y(\mathrm{d}\xi, \omega).$$

For our application below, we will have a family of random maps $(a(q, t) \mid t \in [0, 1])$ that varies smoothly with t. In this case we can guarantee that the associated measures $Y(\mathrm{d}\xi, t)$ depend smoothly in t.

The main difficulties of the proof are due to the fact that the "random and area-preservation properties" do not integrate well, for instance when arguing about t-dependent deformations which must preserve both properties. The proof consists of four steps.

Proof of Theorem E. Write $x = (q, p)$. Since F is random isotopic to the identity, there is a path $\mathcal{F} = (F^t \mid t \in [0, 1])$ of diffeomorphisms that connects F to the identity map, F^t is a stationary lift for each $t \in [0, 1]$, we have the normalization $\frac{1}{2}\int_{-1}^{1} \mathbb{E}\det(\mathrm{d}F^t)\mathrm{d}p = 1$ for every $t \in [0, 1]$, and F^t is regular for a constant independent of t. There are four steps to the proof:

Step 1. (*General strategy to turn* $\mathcal{F}$ *into a path of area-preserving random twists*). Write

$$\rho^t(x) = \rho^t(q, p) = \bar{\rho}^t(\tau_q\omega, p) = \det(\mathrm{d}F^t(x)),$$

so that $(F^t)^*\,\mathrm{d}x = \rho^t\,\mathrm{d}x$, where $\mathrm{d}x = \mathrm{d}q \wedge \mathrm{d}p$, and by assumption,

$$\frac{1}{2}\int_{-1}^{1} \mathbb{E}\rho^t \mathrm{d}p = \frac{1}{2}\int_{-1}^{1} \mathbb{E}\bar{\rho}^t \mathrm{d}p = 1, \;\; \rho^0 = \rho^1 = 1.$$

Since F^t is regular uniformly on t, the function ρ^t is bounded and bounded away from 0 by a constant that is independent of t. That is, there exists a constant $C_0 > 0$ such that $C_0^{-1} \leqslant \rho^t(x;\omega) \leqslant C_0$, almost surely. We now construct, out of F^t, an area-preserving path Λ^t which is a stationary lift for every t. We achieve this by using Moser's deformation trick, namely we construct a path G^t such that $\Lambda^t = F^t \circ G^t$ is an area-preserving stationary lift for all t. As it will be clear from the construction of G^t below, G^0 and G^1 are both the identity and, as a result, Λ^t is a path of area-preserving maps that connects F to identity. We need

$$(G^t)^*(\rho^t \mathrm{d}x) = \mathrm{d}x,$$

and G^t is constructed as a 1-flow map of a vector field

$$\mathcal{X}(x,\theta) = \mathcal{X}(x,\theta;t).$$

So we wish to find some vector field $\mathcal{X}$ such that $G^t = \phi^1$ where ϕ^θ, $\theta \in [0,1]$, denotes the flow of $\mathcal{X}$. In fact, we also have to make sure that the vector field $\pm\mathcal{X}$ is parallel to the q-axis at $p = \pm 1$. This guarantees that the strip $\mathcal{S}$ is invariant under the flow of $\mathcal{X}$.

Let

$$m(\theta, x) := \theta\rho^t(x) + (1-\theta),$$

so that $m(\theta,x)\,\mathrm{d}x$ is connecting the area form $\mathrm{d}x$ to $\rho^t\,\mathrm{d}x$. We need to find a vector field $\mathcal{X}$ such that its flow ϕ^θ satisfies $(\phi^\theta)^*\mathrm{d}x = m(\theta,x)\,\mathrm{d}x$. Equivalently, m must satisfies the Liouville's equation

$$m_\theta + \nabla\cdot(\mathcal{X}m) = \rho^t - 1 + \nabla\cdot(\mathcal{X}m) = 0. \tag{4.4}$$

The strategy to solve equation (4.4) for $\mathcal{X}$ is as follows. Search for a solution $\mathcal{X}$ such that $m\mathcal{X} = \nabla_x u$ is a gradient. Of course we insist that u is q-stationary so that $\mathcal{X}$ is also q-stationary;

$$\begin{aligned}
&u(q,p,\theta) = \bar{u}(\tau_q\omega, p, \theta),\\
&(m\mathcal{X})(q,p,\theta) = (m\mathcal{X})(q,p,\theta;\omega) = (\bar{m}\bar{\mathcal{X}})(\tau_q\omega,p,\theta)\\
&= (\bar{u}_\omega(\tau_q\omega,p,\theta), \bar{u}_p(\tau_q\omega,p,\theta)).
\end{aligned}$$

Since t is fixed, we drop t from our notations and write $\rho^t = \rho$. The equation (4.4) in terms of u is an elliptic partial differential equation of the form

$$\Delta u = 1 - \rho =: \eta, \tag{4.5}$$

with $\eta(q,p) = \bar{\eta}(\tau_q\omega,p)$ and $\int_{-1}^1 \mathbb{E}\eta(\omega,p)\mathrm{d}p = 0$. This concludes Step 1.

Step 2. (*Applying Spectral Theorem to solve* (4.5)). To apply the Spectral Theorem for each p, set

$$\hat{\eta}(\omega,p) = \bar{\eta}(\omega,p) - k(p)$$

for $k(p) = \mathbb{E}\bar{\eta}(\omega,p)$, and write

$$R(q;p) := \mathbb{E}\hat{\eta}(\omega,p)\hat{\eta}(\tau_q\omega,p) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}qz}\, G(\mathrm{d}z,p).$$

Note that $\mathbb{E}\hat{\eta}(\omega,p)=0$ for every p and $\int_{-1}^{1}k(p)\mathrm{d}p=0$. We have the representation

$$\eta(q,p)=k(p)+\bar{\eta}(\tau_q\omega,p)=k(p)+\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}qz}\,Y(\mathrm{d}z,p), \tag{4.6}$$

where $Y(\mathrm{d}z,p)=Y(\mathrm{d}z,p;\omega)$ satisfies

$$Y(\mathrm{d}z,p;\tau_q\omega)=\mathrm{e}^{\mathrm{i}qz}Y(\mathrm{d}z,p;\omega). \tag{4.7}$$

We want to find a solution to the partial differential equation

$$\Delta u(q,p)=\eta(q,p),$$

which is still stationary in the q variable. First choose $h_0(p)$ such that $h_0''(p)=k(p)$ and satisfy the boundary conditions

$$h_0(\pm 1)=0. \tag{4.8}$$

We write $u=h_0+v$ and search for a random v satisfying

$$\Delta v(q,p)=\hat{\eta}(q,p):=\hat{\eta}(\tau_q\omega,p).$$

Since $\gamma(q,p)=\mathrm{e}^{(\mathrm{i}q\pm p)z}$ is harmonic for each $z\in\mathbb{R}$, the function h given by

$$h(q,p):=\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}qz}\left(\mathrm{e}^{zp}\,\Gamma_1(\mathrm{d}z)+\mathrm{e}^{-zp}\,\Gamma_2(\mathrm{d}z)\right), \tag{4.9}$$

is harmonic for any measures Γ_1 and Γ_2. We will find a solution of the form $v=w+h$ where $\Delta w=\eta$ and h will be selected to satisfy the boundary conditions $v_p(q,\pm 1)=0$. Indeed w given by

$$\begin{aligned} w(q,p) &:= \int_{-1}^{p}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}qz}}{z}\sinh((p-a)z)Y(\mathrm{d}z,a)\mathrm{d}a \\ &= \int_{-1}^{p}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}qz}\,\frac{\mathrm{e}^{(p-a)z}-\mathrm{e}^{(a-p)z}}{2z}Y(\mathrm{d}z,a)\mathrm{d}a, \end{aligned}$$

satisfies all of the required properties. In order to verify this observe that

$$\begin{aligned} w_{qq}(q,p) &= -\frac{1}{2}\int_{-1}^{p}\int_{-\infty}^{\infty}z\mathrm{e}^{\mathrm{i}qz}\left(\mathrm{e}^{(p-a)z}-\mathrm{e}^{(a-p)z}\right)Y(\mathrm{d}z,a)\mathrm{d}a, \\ w_p(q,p) &= \frac{1}{2}\int_{-1}^{p}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}qz}\,\left(\mathrm{e}^{(p-a)z}+\mathrm{e}^{(a-p)z}\right)Y(\mathrm{d}z,a)\mathrm{d}a, \\ w_{pp}(q,p) &= \frac{1}{2}\int_{-1}^{p}\int_{-\infty}^{\infty}z\mathrm{e}^{\mathrm{i}qz}\,\left(\mathrm{e}^{(p-a)z}-\mathrm{e}^{(a-p)z}\right)Y(\mathrm{d}z,a)da+\hat{\eta}(q,p). \end{aligned}$$

This clearly implies that $\Delta w=\eta$.

On the other hand, the process w is q-stationary. In other words

$$w(q,p)=w(q,p;\omega)=\bar{w}(\tau_q\omega,p),$$

for a process $\bar{w}$. This can be verified by checking that

$$w(q+b,p;\omega)=w(q,p;\tau_b\omega),$$

which is an immediate consequence of (4.7):

$$w(q+b,p;\omega)=\int_{-\infty}^{\infty}\int_{-1}^{p}\frac{\mathrm{e}^{\mathrm{i}qz}}{z}\sinh((p-a)z)\mathrm{e}^{\mathrm{i}bz}Y(\mathrm{d}z,a;\omega)\mathrm{d}a=w(q,p;\tau_b\omega).$$

This concludes Step 2.

Step 3. (*Checking that* Γ_1 *and* Γ_2 *in* (4.9) *can be chosen to satisfy the boundary conditions* (4.8)). At $p=\pm1$, $\pm\nabla u$ should point in the direction of the q-axis, we need to have that

$$u_p(q,\pm1)=v_p(q,\pm1)=0,$$

because $h_0(\pm1)=0$. First, the condition $v_p(q,1)=0$, means

$$\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}qz}z(\mathrm{e}^{z}\Gamma_1(\mathrm{d}z)-\mathrm{e}^{-z}\Gamma_2(\mathrm{d}z)) \tag{4.10}$$

$$+\frac{1}{2}\int_{-1}^{1}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}qz}(\mathrm{e}^{(1-a)z}+\mathrm{e}^{(a-1)z})Y(\mathrm{d}z,a)\mathrm{d}a=0,$$

and the condition $v_p(q,-1)=0$, means

$$\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}qz}z(\mathrm{e}^{-z}\Gamma_1(\mathrm{d}z)-\mathrm{e}^{z}\Gamma_2(\mathrm{d}z))=0.$$

Since we need to verify the above conditions for all q, we must have that $\Gamma_1=\mathrm{e}^{2z}\Gamma_2$, and

$$z\mathrm{e}^{z}(\mathrm{e}^{2z}-\mathrm{e}^{-2z})\Gamma_2(\mathrm{d}z)+Y'(\mathrm{d}z)\;=\;0,$$

where

$$Y'(\mathrm{d}z)=\frac{1}{2}\int_{-1}^{1}(\mathrm{e}^{(1-a)z}+\mathrm{e}^{(a-1)z})Y(\mathrm{d}z,a)\mathrm{d}a.$$

In summary,

$$\Gamma_2(\mathrm{d}z)=-z^{-1}\mathrm{e}^{-z}(\mathrm{e}^{2z}-\mathrm{e}^{-2z})^{-1}Y'(\mathrm{d}z),\quad \Gamma_1=\mathrm{e}^{2z}\Gamma_2. \tag{4.11}$$

Since Y satisfies (4.7), the same property holds true for both Γ_1 and Γ_2. From this it follows that the process h (and hence u) is q-stationary; this is proven in the same way we established the stationarity of w. The q-stationarity of u implies that $\mathcal{X}$ is q-stationary. This in turn implies that the flow ϕ^θ is a q-stationary lift for each θ. To see this, observe that since both $\phi^\theta(q+a,p;\omega)$ and $(a,0)+\phi^\theta(q,p;\tau_a\omega)$ satisfy the ordinary differential equation $y'(\theta)=\mathcal{X}(y(\theta),\theta;\omega)$ for the same initial data $(q+a,p)$, we deduce $\phi^\theta(q+a,p;\omega)=(a,0)+\phi^\theta(q,p;\tau_a\omega)$, which concludes this step.

Step 4. (*Producing a twist decomposition for F from the path* Λ). We claim that there exists a q-stationary process $H(q,p,t;\omega) = \bar{H}(\tau_q\omega, p, t)$ such that

$$\frac{\mathrm{d}\Lambda^t}{\mathrm{d}t} = J\,\nabla H \circ \Lambda^t$$

holds. Indeed, since Λ^t is a q-stationary lift, $\frac{\mathrm{d}\Lambda^t}{\mathrm{d}t}$ is q-stationary. Hence by Proposition 4.3, the composite $\frac{\mathrm{d}}{\mathrm{d}t}\Lambda^t \circ (\Lambda^t)^{-1}$ is q-stationary. Set

$$A(t,q,p;\omega) = \frac{\mathrm{d}\Lambda^t}{\mathrm{d}t} \circ (\Lambda^t)^{-1}(q,p,\omega).$$

We need to express A as $J\,\nabla H$. Observe that since Λ^t is area preserving, A is divergence free. Write $A(t,q,p;\omega) = (a(\tau_q\omega,p,t), b(\tau_q\omega,p,t))$. We have $a_\omega + b_p = 0$. Set

$$H(q,p,t;\omega) = \int_0^p a(\tau_q\omega, p', t)\mathrm{d}p' - b(\tau_q\omega, 0, t).$$

Clearly $H_q = -b$, $H_p = a$, and H is stationary. Note that since $\frac{\mathrm{d}\Lambda^t}{\mathrm{d}t}$ and $(\Lambda^t)^{-1}$ are bounded in C^1, A is bounded in C^1. $\square$

Proof of Theorem F. Set $A(\omega) = (\omega_p, -\omega_q)$,

$$\Omega_2(k) = \{\omega :\ \|A(\omega)\|, \|DA(\omega)\| \le k\}.$$

Let us write $(\Lambda^{s,t} \mid s \le t)$ for the flow of the vector field A so that $\Lambda^{0,t} = \Lambda^t$ and $\Lambda^{s,s} = \mathrm{id}$. On the other hand

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda^{s,t} = A \circ \Lambda^{s,t},$$

implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}\Lambda^{s,t} = \mathrm{D}A \circ \Lambda^{s,t}\ \mathrm{D}\Lambda^{s,t}.$$

Hence for $\omega \in \Omega_2(k)$,

$$\|\mathrm{D}\Lambda^{s,t}\| \leqslant \mathrm{e}^{k(t-s)},$$

and $\|\mathrm{D}\Lambda^{s,t} - \mathrm{id}\| \leqslant (t-s)\mathrm{e}^{k(t-s)}$. It follows that

$$\sup_{0\le s\le t\le 1} \|\Lambda^{s,t} - \mathrm{id}\|_{\mathrm{C}^1} \leqslant c_0\,(t-s),$$

for a constant c_0. So we may write

$$F = \Lambda^1 = \psi^1 \circ \psi^2 \circ \ldots \circ \psi^n \quad \text{ with } \quad \psi^j = \Lambda^{\frac{jt}{n},\frac{(j-1)t}{n}}$$

satisfying $\|\psi^j - \mathrm{id}\| \leqslant c_0\,n^{-1}$. Hence, for large n, we can arrange

$$\max_{1\leqslant j\leqslant n} \|\psi^j - \mathrm{id}\|_{\mathrm{C}^1} \ \leqslant\ \delta.$$

Let $\varphi^0(q,p) = (q+p,p)$. Then

$$\|\psi^j \circ \varphi^0 - \varphi^0\|_{\mathrm{C}^1} \leqslant \delta.$$

The map φ^0 is a positive monotone twist map and we can readily show that $\psi^j \circ \varphi^0$ is positive monotone twist if $\delta < 1$. Hence $\psi^j = \eta^j \circ (\varphi^0)^{-1}$ where η^j is a positive monotone twist and $(\varphi^0)^{-1}$ is a negative monotone twist. In summary, we have established (2.6) for $\omega \in \Omega_2(k)$, for a positive integer N that depends only on k. However, if we define $N(\omega)$ for the smallest nonnegative integer such that (2.6) is true, then we can readily check that $N(\omega) = N(\tau_a\omega)$. Since $\mathbb{P}$ is ergodic, we deduce that $N(\omega)$ is constant $\mathbb{P}$-almost surely. This concludes the proof of Theorem F. □ □

Next we give an application to random generating functions of complexity N. For the following, recall the definition of $\bar{A}$ in (3.1).

Lemma 4.4. *Let F be a area-preserving random twist map of the form $F = F_N \circ \ldots \circ F_0$, where each F_i is a monotone area-preserving random twist with generating function of the form $\mathcal{G}^i(q, Q; \omega) := \mathcal{L}^i(\tau_q\omega, Q - q)$. Then F has a generalized generating function $\mathcal{L} \colon \bar{A} \times \mathbb{R}^N \to \mathbb{R}$ of complexity N, $\mathcal{L}(\omega, v; \xi)$, that is given by*

$$\mathcal{L}^0(\omega,\, \xi_1) + \sum_{j=1}^{N-1} \mathcal{L}^j(\tau_{\xi_j}\omega,\, \xi_{j+1} - \xi_j) + \mathcal{L}^N(\tau_{\xi_N}\omega,\, v - \xi_N),$$

or equivalently

$$\mathcal{G}(q,\, Q;\, \xi) = \mathcal{G}^0(q,\, \xi_1) + \sum_{j=1}^{N-1} \mathcal{G}^j(\xi_j,\, \xi_{j+1}) + \mathcal{G}^N(\xi_N,\, Q)$$

where $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$.

Proof. We write $\xi_0 = q, \xi_{N+1} = Q$. To verify (3.2), observe that $\mathcal{G}_\xi = 0$ means that $\mathcal{G}^i_Q(\xi_i, \xi_{i+1}) = -\mathcal{G}^{i+1}_q(\xi_{i+1}, \xi_{i+2})$ for $i = 0, \ldots, N-1$. We have that $F_i(q_i,\, p_i) = (Q_i(q_i,\, p_i),\, P_i(q_i,\, p_i))$, with $\mathcal{G}^i_Q(q_i,\, Q_i) = P_i$, $\mathcal{G}^i_q(q_i,\, Q_i) = -p_i$. By definition we have that $F_0(q_1, -\mathcal{G}^0_q(q,\, \xi_1)) = (\xi_1,\, \mathcal{G}^0_Q(q,\, \xi_1))$. Since $\mathcal{G}^0_Q(q,\, \xi_1) = -\mathcal{G}^1_q(\xi_1,\, \xi_2)$ we have that

$$F_1(\xi_1,\, -\mathcal{G}^1_q(\xi_1,\, \xi_2)) = (\xi_2,\, \mathcal{G}^1_Q(\xi_1,\, \xi_2)).$$

Iterating N times we get

$$F_N(\xi_N,\, -\mathcal{G}^N_q(\xi_N,\, Q)) = (Q,\, \mathcal{G}^N_Q(\xi_N,\, Q)),$$

so $F(q,\, -\mathcal{G}_q(q,\, Q;\, \xi)) = (Q,\, \mathcal{G}_Q(q,\, Q;\, \xi))$. □

5. Area-preserving random monotone twists

5.1. **Existence of random generating functions.** The map $v \mapsto \bar{p}(\omega, v)$ is defined to be the inverse of the map $p \mapsto \bar{Q}(\omega, p)$. This means that

$$Q \mapsto p(q, Q) = \bar{p}(\tau_q\omega, Q - q)$$

is the inverse of

$$p \mapsto Q(q, p) = q + \bar{Q}(\tau_q\omega, p).$$

Note that the map $\bar{p}$ is defined on the set $\bar{A}$ so that $v \in [\bar{Q}(\omega, -1), \bar{Q}(\omega, 1)]$. The following explicit description is needed in upcoming proofs.

Proposition 5.1. *Write $Q^{\pm}(\omega) = \bar{Q}(\omega, \pm 1)$ and set*

$$\mathcal{L}(\omega, v) := \int_{Q^-(\omega)}^{v} \bar{P}(\omega, \bar{p}(\omega, a)) \, \mathrm{d}a - Q^-(\omega). \tag{5.1}$$

Then $\mathcal{L}(\omega, v)$ is a generating function of F of complexity 0.

Proof. We prove it if F is positive monotone; the negative monotone case is similar. From (5.1) we deduce that the corresponding $\mathcal{G}(q, Q; \omega) = \mathcal{L}(\tau_q\omega, Q - q)$ is equal to

$$\int_{q+Q^-(\tau_q\omega)}^{Q} P(q,\, p(q, \tilde{Q}))\, \mathrm{d}\tilde{Q} - Q^-(\tau_q\omega) =: \mathcal{G}'(q, Q; \omega) - (Q - q)$$

which is equal to

$$\begin{aligned} &\int_{q+Q^-(\tau_q\omega)}^{Q} \Big(P(q,\, p(q, \tilde{Q})) + 1\Big)\, \mathrm{d}\tilde{Q} - (Q - q) \\ &= \int_{q}^{Q+q^-(\tau_Q\omega)} (p(\tilde{q}, Q) + 1)\, \mathrm{d}\tilde{q} - (Q - q) \\ &= \int_{q}^{Q+q^-(\tau_Q\omega)} p(\tilde{q}, Q) \mathrm{d}\tilde{q} + q^-(\tau_Q\omega). \end{aligned} \tag{5.2}$$

For the first equality in (5.2), we used that F is area-preserving. Here $F^{-1}(Q, P) = (q(Q, P), p(Q, P))$ and $q^{\pm}$ is defined by $q(Q, \pm 1) = Q + q^{\pm}(\tau_Q\omega)$ so that $Q \mapsto Q + q^{\pm}(\tau_Q\omega)$ is the inverse of the map $q \mapsto q + Q^{\pm}(\tau_q\omega)$. Applying the Fundamental Theorem of Calculus to (5.2) we obtain that $\mathcal{G}_Q(q,\, Q) = P(q,\, p)$ and $-\mathcal{G}_q(q,\, Q) = p$. Then (3.2) follows. ☐

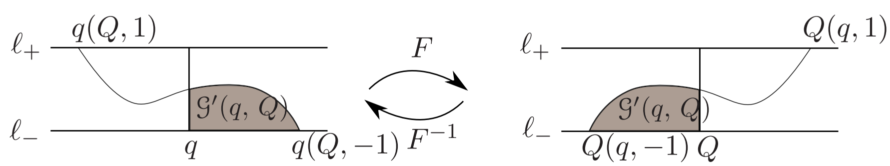

FIGURE 5.1. Area-preserving random twist $F\colon \mathcal{S} \to \mathcal{S}$ and inverse. The area of the shaded regions is $\mathcal{G}'(q,\, Q)$ in (5.2).

5.2. **Fixed points.**

Proposition 5.2. *Let $F\colon \mathcal{S} \times \Omega \to \mathcal{S}$ be an area-preserving random montone twist with generating function $\mathcal{L}\colon \bar{A} \to \mathbb{R}$. Then $\psi\colon \bar{A}_0 \to \mathbb{R}$ given by*

$$\psi(a,\, \omega) = \bar{\psi}(\tau_a\omega) := \mathcal{L}(\tau_a\omega,\, 0)$$

has infinitely many critical points. Furthermore, except for degenerate cases, ψ has maximum and minimum critical points. In degenerate cases ψ has a continuum of critical points. If ψ is bounded and non-constant, it oscillates infinitely many times, so it has maximums and minimums.

Proof. We prove the last statement by contradiction. Suppose that $\psi(a,\,\omega)$ is monotone for large a. Then $\lim_{a\to\infty}\psi(a,\,\omega)=\psi(\infty,\,\omega)$ is well-defined. By ergodicity $\psi(\infty,\,\omega)=\psi(\infty)$ is independent of ω. On the other hand, for any bounded continuous function $J\colon\mathbb{R}\to\mathbb{R}$ we have that $\mathbb{E}\,J(\psi(a,\,\omega))=\mathbb{E}\,J(\bar\psi(\omega))$ for every a, and therefore $J(\psi(\infty))=\mathbb{E}\,J(\bar\psi(\omega))$. Thus $\bar\psi(\omega)=\psi(\infty)$ a.s. In other words, if $\psi(a,\,\omega)$ doesn't oscillate, then $\psi(a,\,\omega)$ is constant. □

5.3. **Construction of random monotone twists and spectral nature of fixed points.** As we argued in Proposition 5.1, a monotone twist map may be determined in terms of its generating function. We now explain how we can start from a scalar-valued function $H(\omega,v)$ and construct a monotone twist map from it. To explain this construction, let us derive a useful property of generating functions. Recall $Q^{\pm}(\omega)=\bar Q(\omega,\pm1)$.

Proposition 5.3. *Let $\mathcal{L}(\omega,v)$ be as in Proposition 5.1. Then the function*

$$\mathcal{L}(\omega,Q^{+}(\omega))-Q^{+}(\omega), \tag{5.3}$$

is constant and $\mathcal{L}(\omega,Q^{-}(\omega))=-Q^{-}(\omega)$.

Proof. From $F(q,-\mathcal{G}_q(q,Q;\omega))=(Q,\mathcal{G}_Q(Q,q;\omega))$, we deduce

$$\bar F(\omega,\mathcal{L}_v(\omega,v)-\mathcal{L}_\omega(\omega,v))=(v,\mathcal{L}_v(\omega,v)).$$

Since $P=\pm1$ if and only if $p=\pm1$, we obtain $\mathcal{L}_\omega(\omega,Q^{\pm}(\omega))=0$ and $\mathcal{L}_v(\omega,Q^{\pm}(\omega))=\pm1$. But

$$\nabla_\omega\left(\mathcal{L}(\omega,Q^{\pm}(\omega))\right)=\mathcal{L}_\omega(\omega,Q^{\pm}(\omega))+\mathcal{L}_v(\omega,Q^{\pm}(\omega))Q^{\pm}_\omega(\omega)=\pm Q^{\pm}_\omega(\omega),$$

which means that the function $\mathcal{L}(\omega,Q^{\pm}(\omega))\mp Q^{\pm}(\omega)$ is constant by the ergodicity of $\mathbb{P}$. On the other hand, by the definition of $\mathcal{L}$ (see (5.1)) we know that $\mathcal{L}(\omega,Q^{-}(\omega))=-Q^{-}(\omega)$. □

We are ready to give a recipe for constructing a monotone twist map from a C^2 function $H:\Omega\times\mathbb{R}\to\mathbb{R}$, which satisfy the following conditions

$$\begin{cases} H(\omega,0)=0, \quad H(\omega,a)>0 \text{ for } a>0, \\ \\ \eta(\omega)=\inf\{a>0\mid H(\omega,a)=2\}<+\infty, \end{cases} \tag{5.4}$$

almost surely. For such a function H, we set

$$\sigma(\omega)=\eta(\omega)-\frac{1}{2}\int_0^{\eta(\omega)}H(\omega,a)\mathrm{d}a$$

and

$$\begin{aligned} &(5.5) & Q^{-}(\omega) &= -\sigma(\omega), \quad Q^{+}(\omega)=(\eta-\sigma)(\omega);\\ &(5.6) & \bar G(\omega,v) &= H(\omega,v+\sigma(\omega)), \quad G(q,Q;\omega)=\bar G(\tau_q\omega,Q-q);\\ && \mathcal{L}(\omega,v) &= \int_0^{v+\sigma(\omega)}H(\omega,a)\mathrm{d}a-v; \quad \mathcal{G}(q,Q;\omega)=\mathcal{L}(\tau_q\omega,Q-q). \end{aligned}$$

Theorem 5.4. *Assume that $H\colon \Omega\times\mathbb{R}\to\mathbb{R}$ satisfies (5.4) and the condition $G_q<0$ with G defined as in (5.6). Then there exists a unique monotone twist map F such that $F(q,-\mathcal{G}_q(q,Q))=(Q,\mathcal{G}_Q(q,Q))$, and $F(q,\pm1)=(q+Q^{\pm}(\tau_q\omega),\pm1)$ with $Q^{\pm}$ defined by (5.5). Moreover, if $\bar q$ is a local maximum (respectively minimum) for $q\mapsto\psi(q)=\mathcal{G}(q,q)$, then $\mathrm{D}F$ at the F-fixed point $(\bar q,-\mathcal{G}_q(\bar q,\bar q))$ has negative (respectively positive) eigenvalues.*

Proof. By the definition,

$$\mathcal{G}(q,Q)=\int_{q+Q^-(\tau_q\omega)}^{Q} G(q,Q')\ \mathrm{d}Q'-(Q-q),$$

which implies

$$\mathcal{G}_Q=G-1,\quad \mathcal{G}_{Qq}=G_q<0. \tag{5.7}$$

From (5.7) we learn that the map $Q\mapsto\mathcal{G}_q(q,Q)$ is decreasing and, as a result, the equation

$$\mathcal{G}_q(q,Q)=-p \tag{5.8}$$

may be solved for Q, to yield a p-increasing function $Q=Q(q,p)$. We set

$$P(q,p)=\mathcal{G}_Q(q,Q(q,p))=G(q,Q(q,p))-1,$$

so that

$$F(q,p)=(Q(q,p),P(q,p)).$$

Note that the monotonicity condition is satisfied because Q is increasing in p. We need to show that the boundary conditions are satisfied and that F is area-preserving. For the latter, observe that by differentiating both sides of the relationship (5.8), we obtain $\mathcal{G}_{qq}+\mathcal{G}_{Qq}Q_q=0$, $\mathcal{G}_{qQ}Q_p=-1$, $P_q=\mathcal{G}_{Qq}+\mathcal{G}_{QQ}Q_q$, and $P_p=\mathcal{G}_{QQ}Q_p$. It follows that

$$\mathrm{D}F=-\mathcal{G}_{Qq}^{-1}\begin{bmatrix}\mathcal{G}_{qq} & 1\\ \mathcal{G}_{qq}\mathcal{G}_{QQ}-\mathcal{G}_{qQ}^2 & \mathcal{G}_{QQ}\end{bmatrix}. \tag{5.9}$$

It follows from (5.9) that if the eigenvalues of $\mathrm{D}F$ are λ and λ^{-1}, then $\lambda>0$ if and only if

$$\mathrm{Trace}(\mathrm{D}F)=\frac{\mathcal{G}_{qq}+\mathcal{G}_{QQ}}{-\mathcal{G}_{qQ}}=\lambda+\lambda^{-1}\geqslant 2.$$

Equivalently $\mathrm{D}F$ has positive eigenvalues if and only if

$$\psi''(q)=(\mathcal{G}_{qq}+\mathcal{G}_{QQ}+2\mathcal{G}_{qQ})(q,q)>0.$$

The case of negative eigenvalues may be treated in the same way.

For the boundary conditions, we first establish

$$\mathcal{L}_\omega(\omega,Q^{\pm}(\omega))=0,\quad \mathcal{L}_v(\omega,Q^{\pm}(\omega))=\pm1. \tag{5.10}$$

For the second equality in (5.10), observe that $\mathcal{L}_v=\bar G-1$, and by definition $\bar G(\omega,Q^-(\omega))=H(\omega,0)=0$, and $\bar G(\omega,Q^+(\omega))=H(\omega,Q^+(\omega)-Q^-(\omega))=$

$H(\omega,\eta(\omega))=2$. As for the first equality in (5.10), observe that by the definition of σ, G and $\mathcal{L}$,

$$\begin{aligned}
&\mathcal{L}(\omega,Q^-(\omega))+Q^-(\omega)=0,\\
&\mathcal{L}(\omega,Q^+(\omega))-Q^+(\omega)=\int_0^{Q^+(\omega)+\sigma(\omega)}H(\omega,a)\mathrm{d}a-2Q^+(\omega)\\
&\qquad\qquad\qquad\qquad\qquad\;=\int_0^{\eta(\omega)}H(\omega,a)\mathrm{d}a-2(\eta-\sigma)(\omega)=0.
\end{aligned}$$

As a result

$$\mathcal{L}(\omega,Q^\pm(\omega))\mp Q^\pm(\omega)=0. \tag{5.11}$$

Differentiating (5.11) with respect to ω yields

$$0=\mathcal{L}_\omega(\omega,Q^\pm(\omega))+\mathcal{L}_v(\omega,Q^\pm(\omega))Q^\pm_\omega(\omega)\mp Q^\pm_\omega(\omega)=\mathcal{L}_\omega(\omega,Q^\pm(\omega)),$$

which is precisely the first equality in (5.10).

We are now ready to verify the boundary conditions. We wish to show that $Q(q,\pm1)=q+Q^\pm(\tau_q\omega)$, or equivalently

$$\pm1=-\mathcal{G}_q(q,q+Q^\pm(\tau_q\omega))=(\mathcal{L}_v-\mathcal{L}_\omega)(\tau_q\omega,Q^\pm(\tau_q\omega)).$$

This is an immediate consequence of (5.10). It remains to verify $P(q,\pm1)=\pm1$. We certainly have

$$P(q,\pm1)=\mathcal{G}_Q(q,q+Q^\pm(\tau_q\omega))=G(q,q+Q^\pm(\tau_q\omega))-1=\bar G(\tau_q\omega,Q^\pm(\tau_q\omega))-1$$

This and (5.10) imply $P(q,\pm1)=\pm1$, because $\bar G-1=\mathcal{L}_v$. □

Remark 5.5. σ in (5.5) is motivated by (5.3). It is chosen so that $\mathcal{L}(\omega,Q^+(\omega))=Q^+(\omega)$.

Remark 5.6. The monotonicity condition $G_q=\mathcal{G}_{Qq}<0$ may be expressed as $H_\omega(\omega,a)<H_a(\omega,a)(1-\sigma'(\omega))$. The derivative of σ may be calculated with the aid of (5.5):

$$\begin{aligned}
\sigma'(\omega)&=\eta'(\omega)-\frac12H(\omega,\eta(\omega))\eta'(\omega)-\frac12\int_0^{\eta(\omega)}H_\omega(\omega,a)\mathrm{d}a\\
&=-\frac12\int_0^{\eta(\omega)}H_\omega(\omega,a)\mathrm{d}a.
\end{aligned}$$

Example 5.7 We now give a concrete example of H that satisfies the assumptions of Theorem 8.4. Consider $H(a,\omega)=\bar R(\omega)a$ for $R(q,\omega)=\bar R(\tau_q\omega)$ a positive C^1 stationary process. For such H, we have $\eta=2\sigma=2\bar R^{-1}$, and $Q^\pm=\pm R^{-1}$. The condition $G_q<0$ is equivalent to

$$R'(Q-q)-R<0. \tag{5.12}$$

for $q,Q\in[Q^-,Q^+]$. Eqivalently,

$$\begin{aligned}
R'>0 \quad&\Rightarrow\quad 2R'R^{-1}\le R,\\
R'<0 \quad&\Rightarrow\quad -2R'R\le R.
\end{aligned}$$

In summary, we need $2|R'| \le R^2$ to hold.

5.4. **The density of fixed points.** When F is a positive twist map, it has a generating function $\mathcal{G}(q,Q,\omega) = \mathcal{L}(\tau_q\omega, Q-q)$ and any fixed point of F is of the form $(q_0, \mathcal{L}_v(\tau_{q_0}\omega, 0))$ where q_0 is a critical point of the random process $\psi(q,\omega) = \bar\psi(\tau_q\omega)$ (Propositions 3.3 and 5.1). We have also learned that any random process ψ has infinitely many local maximums and minimums. In this section we give sufficient conditions to ensure that such a random process has a positive density of critical points, which in turn yields a positive density for fixed points of a monotone twist map. Let $\sharp B$ be the cardinality of a set B.

Definition 5.8. The *density* of $A \subset \mathbb{R}$ is $\mathrm{den}(A) := \lim_{\ell\to\infty} (2\ell)^{-1}\sharp(A \cap [-\ell,\ell])$.

Let us state a set of assumptions for the random process $\psi(q,\omega) = \bar\psi(\tau_q\omega)$ that would guarantee the existence of a density for the set

$$Z(\omega) := \{q \mid \psi'(q,\omega) = 0\}.$$

Hypothesis 5.9. (i) *$\psi(q,\omega)$ is twice differentiable almost surely and if*

$$\phi_\ell(\delta;\omega) = \sup\Big\{|\psi''(q,\omega) - \psi''(\hat q,\omega)| \;\mid\; q,\hat q \in [-\ell,\ell],\ |q-\hat q| \leqslant \delta\Big\},$$

then $\lim_{\delta\to 0}\mathbb{E}\ \phi_\ell(\delta;\omega) = 0$ for every $\ell > 0$.

(ii) *The random pair $(\bar\psi_\omega(\omega), \bar\psi_{\omega\omega}(\omega))$ has a probability density $\rho(x,y)$. In other words, for any bounded continuous function $J(x,y)$,*

$$\mathbb{E}J(\psi'(q,\omega),\psi''(q,\omega)) = \int_{\mathbb{R}} J(x,y)\,\rho(x,y)\,\mathrm{d}x\mathrm{d}y.$$

(iii) *There exists $\varepsilon > 0$ such that $\rho(x,y)$ is jointly continuous for x satisfying $|x| \geqslant \varepsilon$.*

We define $\bar Z^{\pm}(\omega) := \{q \mid \psi'(q,\omega) = 0,\ \pm\psi''(q,\omega) > 0\}$ and $N_\ell^{\pm}(\omega) := \bar Z^{\pm}(\omega) \cap [-\ell,\ell]$. It is well known that if we assume Hypothesis 5.9, then

$$\mathbb{E}\ N_\ell(\omega) = 2\ell\int_{\mathbb{R}} \rho(0,y)y^{\pm}\ \mathrm{d}y. \tag{5.13}$$

This is the celebrated *Rice Formula* and its proof can be found in [Ad00, Az09]. Next we state a direct consequence of Rice Formula and the Ergodic Theorem.

Theorem 5.10. *If ψ satisfies Hypothesis 5.9 then $\bar Z(\omega) = Z(\omega)$ almost surely and*

$$\lim_{\ell\to\infty}\mathbb{E}\left|\frac{1}{2\ell}N_\ell^{\pm}(\omega) - \int_{\mathbb{R}}\rho(0,y)y^{\pm}\ \mathrm{d}y\right| = 0. \tag{5.14}$$

Proof. Pick a smooth function $\zeta : \mathbb{R} \to [0,\infty)$ such that its support is contained in the interval $[-1,1]$, $\zeta(-a) = \zeta(a)$, and $\int_{\mathbb{R}}\zeta(q)\mathrm{d}q = 1$. Set $\zeta_\varepsilon(q) := \varepsilon^{-1}\zeta(q/\varepsilon)$. It is not hard to show

$$\frac{1}{2\ell}N_\ell^{\pm}(\omega) \geqslant \frac{1}{2\ell}\int_{-\ell+\varepsilon}^{\ell-\varepsilon}\left|\zeta_\varepsilon' * \hat\psi(q,\omega)\right|\mathrm{d}q =: X_\varepsilon^{\pm}(\ell,\omega), \tag{5.15}$$

where $\hat{\psi}(q,\omega) = 1\!\!1(\psi'(q,\omega) > 0)$ (this is [Az09, Lemma 3.2]). We note that if $\eta_\varepsilon(\omega) = \left| \int_{\mathbb{R}} \zeta'_\varepsilon(a)\hat{\psi}(a,\omega)\ \mathrm{d}a \right|$, then

$$\begin{aligned}\eta_\varepsilon(\tau_q\omega) &= \left| \int_{\mathbb{R}} \zeta'_\varepsilon(a)\hat{\psi}(a,\tau_q\omega)\ \mathrm{d}a \right| = \left| \int_{\mathbb{R}} \zeta'_\varepsilon(a)\hat{\psi}(a+q,\omega)\ \mathrm{d}a \right| \\ &= \left| \zeta'_\varepsilon * \hat{\psi}(q) \right|.\end{aligned}$$

From this and the Ergodic Theorem we deduce

$$\lim_{\ell\to\infty} \frac{1}{2\ell} \int_{-\ell}^{\ell} \left| \zeta'_\varepsilon * \hat{\psi}(q,\omega) \right| \mathrm{d}q = \mathbb{E}\eta_\varepsilon, \tag{5.16}$$

almost surely and in the $\mathrm{L}^1(\mathbb{P})$ sense.

On the other hand,

$$\lim_{\varepsilon\to 0} \mathbb{E}\eta_\varepsilon = \int_{\mathbb{R}} \rho(0,y)y^{\pm}\ \mathrm{d}y =: \bar{X}^{\pm}. \tag{5.17}$$

This follows the proof of Rice Formula, see [Az09, proof of Theorem 3.4].

Again by Rice Formula,

$$0 = \mathbb{E}\left[\frac{1}{2\ell}N_\ell^{\pm}(\omega) - \bar{X}^{\pm}\right] = \mathbb{E}\left[\frac{1}{2\ell}N_\ell^{\pm}(\omega) - X_\varepsilon^{\pm}(\ell,\omega)\right] - \mathbb{E}\left[X_\varepsilon^{\pm}(\ell,\omega) - \bar{X}^{\pm}\right],$$

which implies

$$\lim_{\varepsilon\to 0}\limsup_{\ell\to\infty} \mathbb{E}\left[\frac{1}{2\ell}N_\ell^{\pm}(\omega) - X_\varepsilon^{\pm}(\ell,\omega)\right] = 0, \tag{5.18}$$

because by (5.16) and (5.17),

$$\lim_{\varepsilon\to 0}\limsup_{\ell\to\infty} \mathbb{E}\left|X_\varepsilon^{\pm}(\ell,\omega) - \bar{X}^{\pm}\right| = 0. \tag{5.19}$$

From (5.15) and (5.18) we deduce

$$\lim_{\varepsilon\to 0}\limsup_{\ell\to\infty} \mathbb{E}\left|\frac{1}{2\ell}N_\ell^{\pm}(\omega) - X_\varepsilon^{\pm}(\ell,\omega)\right| = 0. \tag{5.20}$$

Then (5.20) and (5.19) imply (5.14). □

6. Complexity $N = 1$ area-preserving random twists

6.1. Domain of random generating functions. We begin by describing the domain the random generating function of a complexity one twist.

Lemma 6.1. *Let F be an area-preserving random twist of complexity one with decomposition $F = F_1 \circ F_0$, where F_1 is a positive monotone area-preserving random twist and F_0 is negative monotone area-preserving random twist. Let $\mathcal{G}^0$, $\mathcal{G}^1$ be the generating functions, respectively, of the monotone twists F_0, F_1. Then $G_1 := F_1^{-1}$ is a negative area-preserving random twist with generating function given by $\hat{\mathcal{G}}^1(q,\,\xi) := -\mathcal{G}^1(\xi,\,q)$, and if*

$$D_0 := \mathrm{Domain}(\mathcal{G}^0) \ \text{ and } \ D_1 := \mathrm{Domain}(\hat{\mathcal{G}}^1), \tag{6.1}$$

then we have a proper inclusion of sets $D_0 \subsetneq D_1$ (see Figure 6.1).

Proof. Note that $G_1(a, \pm 1) = (Q_1^{\pm}(a), \pm 1)$ and $F_0(a, \pm 1) = (Q_0^{\pm}(a), \pm 1)$, with $\pm(Q_i^{\pm}(a) - a) < 0$ and $Q_i^{\pm}$ increasing. Since F is an area-preserving random twist map, we may write $F^{-1}(q, \pm 1) = (\hat{Q}^{\pm}(q), \pm 1)$ with $\hat{Q}^{\pm}$ increasing and such that $\pm(\hat{Q}^{\pm}(q) - q) < 0$ for all q. For $i = 0, 1$ let $\partial^{\pm} D_i = \{(a, Q_i^{\mp}(a)) \mid a \in \mathbb{R}\}$ denote the boundary curves of D_i. From $G_1 = F_0 \circ F^{-1}$, we deduce $Q_0^{\pm}(\hat{Q}^{\pm}(q)) = Q_1^{\pm}(q)$, and therefore

$$Q_0^{-}(q) < Q_1^{-}(q) \tag{6.2}$$

and

$$Q_0^{+}(q) > Q_1^{+}(q). \tag{6.3}$$

Then (6.2) (respectively (6.3)) implies that the upper (respectively lower) boundary of D_1 is strictly above (respectively below) D_0. It follows that $D_0 \subsetneq D_1$, as desired. $\square$

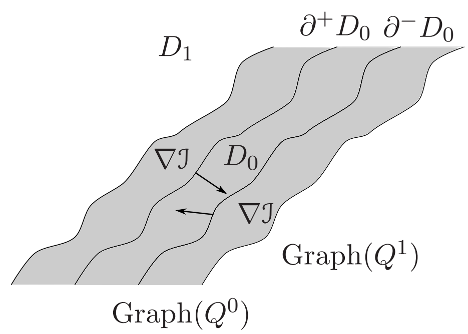

FIGURE 6.1. The domains D_0 and D_1 and the gradient $\nabla \mathcal{J}$.

6.2. **Gradients and geometry of domains.** Let D_0 be defined by (6.1).

Corollary 6.2. *The map*

$$\mathcal{J}(q, \xi) := \mathcal{G}^0(q, \xi) + \mathcal{G}^1(\xi, q) \tag{6.4}$$

is well-defined on the set D_0, cf. (6.1).

Proof. If $(\xi, q) \in D_0 \cap D_1$ then the sum $\mathcal{G}^0(q, \xi) + \mathcal{G}^1(\xi, q)$ is well defined. The corollary follows from Lemma 6.1. $\square$

Lemma 6.3. *The gradient $\nabla \mathcal{J}$ of $\mathcal{J}\colon D_0 \to \mathbb{R}$ is inward on $\partial^{\pm} D_0$ and*

$$\mp \mathcal{J}_{\xi}, \pm \mathcal{J}_q > 0$$

on $\partial^{\pm} D_0$.

Proof. If $F_0(q, p) = (\xi, \eta)$ and $F_1(\xi, \eta') = (q, P)$, then $\mathcal{I}_q(q, \xi) = P - p$ and $\mathcal{I}_\xi(q, \xi) = \eta - \eta'$ hold. We express the domain D_0 of $\mathcal{I}$ given by (6.1) as $\{(\xi, q) \mid p = p(q, \xi) = -\mathcal{G}^0_q(q, \xi) \in [-1, 1]\}$. On $\partial^- D_0$, $\eta = p = 1$ and $P, \eta' < 1$ (because $D_0 \subsetneq D_1$). So on $\partial^- D_0$ we have $\mathcal{I}_\xi(q, \xi) > 0$ and $\mathcal{I}_q(q, \xi) < 0$. On $\partial^+ D_0$ we have $\eta = p = -1$ and $\eta', P < 1$. So on $\partial^+ D_0$ we have $\mathcal{I}_\xi(q, \xi) < 0$ and $\mathcal{I}_q(q, \xi) > 0$. The lower boundary $\partial^- D_0$ is the graph of an increasing function $q \mapsto h(q)$, and of course $h'(q) > 0$. So, the tangent to $\partial^- D_0$ is $(1, h'(q))$ and the inward normal is $(-h'(q), 1)$. On $\partial^- D_0$ we have $\mathcal{I}_\xi(q, \xi) > 0$ and $\mathcal{I}_q(q, \xi) < 0$. So we have that the dot product $\langle(\mathcal{I}_q(q, \xi), \mathcal{I}_\xi(q, \xi)), (-h'(q), 1)\rangle = -h'(q)\mathcal{I}_q(q, \xi) + \mathcal{I}_\xi(q, \xi) > 0$. That is, on the lower boundary $\nabla\mathcal{I}$ is inward.

The case of the upper boundary is analogous. □

6.3. **Fixed points.** If we set $\hat{D} := \{(q, a) \mid (q, q+a) \in D_0\}$, we have that, for a pair of random processes $B^-(\tau_q\omega), B^+(\tau_q\omega) > 0$, $\hat{D} = \{(q, a) \mid -B^-(\tau_q\omega) < a < B^+(\tau_q\omega)\}$. We then use the notation of Lemma 4.4 to set $\bar{\mathcal{I}}(\tau_q\omega, a) := \mathcal{L}^0(\tau_q\omega, a) + \mathcal{L}^1(\tau_a\tau_q\omega, -a) = \mathcal{I}(q, q+a)$. Define the map $\bar{K}\colon \Omega \times [-1, 1] \to \mathbb{R}$ by

$$K(q, p; \omega) = \bar{K}(\tau_q\omega, p) = \bar{\mathcal{I}}\left(\tau_q\omega, B(\tau_q\omega, p)\right),$$

where

$$B(\tau_q\omega, p) = \frac{p+1}{2}B^+(\tau_q\omega) + \frac{p-1}{2}\,B^-(\tau_q\omega).$$

Note that

$$\begin{aligned} K_p(q, p; \omega) =& \frac{1}{2}\bar{\mathcal{I}}_a\left(\tau_q\omega, B(\tau_q\omega, p)\right)\left(B^+(\tau_q\omega) + B^-(\tau_q\omega)\right), \\ K_q(q, p; \omega) =& \bar{\mathcal{I}}_\omega\left(\tau_q\omega, B(\tau_q\omega, p)\right) + \bar{\mathcal{I}}_a\left(\tau_q\omega, B(\tau_q\omega, p)\right)B_\omega(\tau_q\omega, p). \end{aligned} \tag{6.5}$$

Hence there is a one-one correspondence between the critical points of K and $\mathcal{I}$. From (6.5) and Lemma 6.3 we conclude the following.

Lemma 6.4. *The gradient ∇K of $K\colon \mathcal{S} \times \Omega \to \mathbb{R}$ is inward on the boundary of $\mathcal{S}$.*

Theorem 6.5. *Let $\bar{K}\colon \Omega \times [-1, 1] \to \mathbb{R}$ be a C^1-map such that $\mp\bar{K}_p(\cdot, \pm 1) > 0$. Let $K(q, p; \omega) := \bar{K}(\tau_q\omega, p)$.*

(a) *K has infinitely many critical points;*
(b) *Furthermore, the critical points of K occur as follows:*
  (1) *Either K has a continuum of critical points;*
  (2) *Or K has both infinitely many local maximums, and infinitely many saddle points or local minimums.*

Proof. We prove (b). If $\hat{K}(\omega) := \max_{a\in[-1,1]} \bar{K}(\omega, a)$, then either $\hat{K}$ is constant or $\hat{K}(\tau_q\omega)$ oscillates almost surely. In the former case for almost all ω, there exists $a(\omega)$ such that $\bar{K}(\omega, a(\omega))$ is a maximum and (of course) $a(\omega) \notin \{-1, 1\}$ by the assumption $\mp\bar{K}_p(\cdot, \pm 1) > 0$. More concretely, we set

$$a(\omega) = \max\{p \in [-1, 1] \mid \bar{K}(\omega, p) = \hat{K}(\omega)\}.$$

Hence K has a continuum of critical points of the form $\{(q, a(\tau_q\omega)) \mid q \in \mathbb{R}\}$. In the latter case, there are infinitely many local maximums. Choose $\bar{q}$ so that $\hat{K}(\tau_{\bar{q}}\omega)$ is a local maximum. For such $(\bar{q}, \omega)$ choose $a(\tau_{\bar{q}}\omega)$ so that $\bar{K}(\tau_{\bar{q}}\omega, a(\tau_{\bar{q}}\omega)) = \hat{K}(\tau_{\bar{q}}\omega)$. Therefore K has infinitely many local maximums by Proposition 5.2.

Note that if

$$\Omega_0 := \Big\{\omega \;\mid\; \{\tau_a\omega \,|\, a > a_0\} \text{ is dense for every } a_0\Big\},$$

then $\mathbb{P}(\Omega_0) = 1$. This is true because the family $\{\tau_a : a \in \mathbb{R}\}$ is ergodic and by assumption $\mathbb{P}(U) > 0$ for every open set U. Given $\omega \in \Omega_0$, consider the ordinary differential equation with initial value condition

$$\begin{cases} q'(t) \;=\; \bar{K}_\omega(\tau_{q(t)}\omega,\, p(t)) \\ p'(t) \;=\; \bar{K}_p(\tau_{q(t)}\omega,\, p(t)) \\ q(0) \;=\; 0, \qquad p(0) = a. \end{cases} \tag{6.6}$$

There are two possibilities; the first possibility is that for some a, we have that $q(t)$ is unbounded as $t \to \infty$, and in this case we claim that there is a continuum of critical points. The second possibility is that $q(t)$ is *always* bounded as $t \to \infty$, and in this case we claim that K has either infinitely many saddle points or local minimums. We proceed with case by case.

Case 1. (*The map $q(t)$ is unbounded as $t \to \infty$ for some $\omega \in \Omega_0$*). We want to prove that K has a continuum of critical points. Define $\omega(t) := \tau_{q(t)}\omega$, and let ϕ^r be the flow of (6.6). Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{K}(\omega(t),\, p(t)) = |\nabla \bar{K}(\omega(t),\, p(t))|^2 \geqslant 0.$$

Since $q(t)$ is unbounded, $\omega(t)$ can approach almost any point in Ω. Moreover if $\tau_{q(t_n)}\omega \to \overline{\omega}$ and $p(t_n) \to \overline{p}$, then we claim that $\nabla \bar{K}(\overline{\omega}, \overline{p}) = 0$. Indeed, if $\lambda := \sup_{t>t_0} \bar{K}(\omega(t),\, p(t))$, we have $\lambda = \bar{K}(\overline{\omega},\, \overline{p})$, and since

$$\lambda = \sup_{t>t_0} \bar{K}(\omega(t+r),\, p(t+r)),$$

we have, for any $r > 0$, that $\lambda = \bar{K}(\overline{\omega},\, \overline{p}) = \bar{K}(\phi^r(\overline{\omega},\, \overline{p}))$. Hence $\nabla \bar{K}(\overline{\omega},\, \overline{p}) = 0$; otherwise

$$\frac{\mathrm{d}}{\mathrm{d}r}\bar{K}(\phi^r(\overline{\omega},\, \overline{p}))|_{r=0} > 0,$$

which is impossible. Note that $\overline{\omega}$ could be any point in Ω and therefore for such $\overline{\omega}$ there exists $\overline{p} = \overline{p}(\overline{\omega})$ such that $\nabla K(\overline{\omega}, \overline{p}(\overline{\omega})) = 0$, i.e. we have a continuum of critical points. This concludes Case 1.

Case 2. (*The map $q(t) = q(t,\, \omega)$ is bounded for every $\omega \in \Omega_0$*). We claim that if $\bar{K}$ does not have a continuum of fixed points, then K has infinitely many critical points which are local minimums or saddle points. Suppose that this is not the case, then we want to arrive at a contradiction. In order to do this let $\bar{x} = (\overline{q}, \overline{p})$ be a local maximum, which we know it always exists by the paragraphs preceding Case 1. In fact we may take a $\delta > 0$ such that $K(x) \leqslant$

$K(\bar{x})$ for every $x = (q, p)$ with $q \in (\bar{q} - \delta, \bar{q} + \delta)$. Now take a closed curve γ such that $(\overline{q}, \overline{p})$ is inside γ and if $a \in \gamma$, then $\lim_{t\to\infty} \phi^t(a) = (\overline{q}, \overline{p}) = \overline{a}$. For example, we may take γ to be part of level set of the function $(q, p) \mapsto K(q, p)$ with value $c < K(\bar{x})$ very close to $K(\bar{x})$. Since K does not have a continuum of critical points, we may choose such level set γ such that K has no critical point on γ. From this latter property we deduce that γ is homeomorphic to a circle. Let $a \in \gamma$. If there is no other type of critical points, then the curve

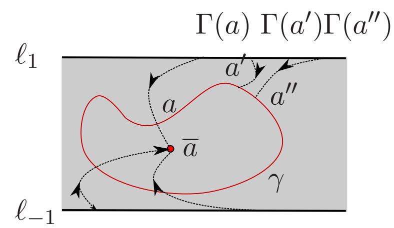

FIGURE 6.2. Note that $a \in \gamma$ while $\overline{a}$ is enclosed by γ.

$t \mapsto \phi^t(a)$, where $t \leqslant 0$, must reach the boundary for some $t_a < 0$, because $\frac{\mathrm{d}}{\mathrm{d}t} K(\phi^t(a)) \geqslant 0$. This defines a map $\Gamma\colon \gamma \to (\mathbb{R} \times \{-1\}) \cup (\mathbb{R} \times \{1\})$, $\Gamma(a) := \phi_{t_a}(a)$. We now argue that in fact Γ is continuous. To show the continuity of Γ at $a \in \gamma$, extend K continuously near $\Gamma(a)$, choose $\varepsilon > 0$ and set

$$\eta = (\phi^\theta(a) \mid \theta \in [t_a - \varepsilon, \varepsilon]).$$

Choose ε sufficiently small so that $\phi^\theta(a)$ is inside γ for $\theta \in (0, \varepsilon]$, and $\phi^t(a)$ is outside the strip for $t \in (t_a - \varepsilon, t_a)$. Choose $\hat{a} \in \gamma$ close to a so that $\eta' = (\phi^\theta(b) \mid \theta \in [t_a - \varepsilon, \varepsilon])$ is uniformly close to η. Since $\phi^{t_a}(\hat{a})$ is near $\Gamma(a)$, we can choose $\hat{a}$ close enough to a to guarantee that $\Gamma(\hat{a})$ is close to $\Gamma(a)$. Moreover, we can easily show that $\Gamma(c)$ is between $\Gamma(a)$ and $\Gamma(\hat{a})$ for any c between a and $\hat{a}$ on γ. Hence Γ is a homeomorphism from a neighborhood of a onto its image. Since γ is homeomorphic to S^1, its homeomorphic image $\Gamma(\gamma)$ cannot be fully contained inside of $\mathbb{R} \times \{-1\} \cup \mathbb{R} \times \{+1\}$. Therefore there exists $a \in \gamma$ such that any limit point z of $\phi^t(a)$ as $t \to -\infty$ is a critical point inside the strip that is not a local maximum. Clearly $z \notin (\bar{q} - \delta, \bar{q} + \delta)$. Let us assume for example that $z = (q_1, p_1)$ with $q_1 > \bar{q} + \delta$. Take another local maximum $\hat{x} = (\hat{q}, \hat{p})$ to the right of $\bar{x}$ and assume that $K(\hat{x}) \geqslant K(x)$ for all $x \in (\hat{q} - \hat{\delta}, \hat{q} + \hat{\delta}) \times [-1, 1]$. Since $\phi^t(a)$ cannot enter $(\hat{q} - \hat{\delta}, \hat{q} + \hat{\delta}) \times [-1, 1]$ we deduce that $q_1 \in (\bar{q} + \delta, \hat{q} - \hat{d})$.

Repeating the above argument for other local maximums, we deduce that there exist infinitely critical points in between local maximums that are not local maximums. $\square$

6.4. **Nature of the fixed points in terms of generating function.** A result similar to Theorem 5.4 holds for complexity $N = 1$ twist maps.

Theorem 6.6. *Let F and $\mathfrak{I}$ be as in Lemma 6.1 and Corollary 6.2.*

Let $(\bar{q},\bar{\xi})$ be a critical point of $\mathcal{I}$ and $\vec{x}$ be the corresponding fixed point of F as in Proposition 3.3. Assume that $\mathcal{I}_{\xi\xi}(\bar{q},\bar{\xi}) \neq 0$. Then $\mathrm{D}F(\vec{x})$ has positive (respectively negative) eigenvalues if and only if $\det \mathcal{I}(\bar{q},\bar{\xi}) \geqslant 0$ (respectively $\leqslant 0$).

Proof. Recall that $\mathcal{S}(q,Q;\xi) = \mathcal{S}^0(q,\xi) + \mathcal{S}^1(\xi,Q)$ and:

$$\mathcal{G}_\xi(q,Q;\xi) = 0 \Rightarrow F(q,-\mathcal{G}_q(q,Q;\xi)) = (Q,\mathcal{G}_Q(q,Q;\xi)).$$

Observe that if $cI(\bar{q},\bar{\xi}) = cG_{\xi\xi}(\bar{q},\bar{q};\bar{\xi}) \neq 0$, then near $(\bar{q},\bar{q},\bar{\xi})$, we can solve $\mathcal{G}_\xi(q,Q;\xi) = 0$ as $\xi = \xi(q,Q)$. Write $\mathcal{T}(q,Q) = \mathcal{G}(q,Q;\xi(q,Q))$. Then $\mathcal{T}_q = \mathcal{G}_q,\ \ \mathcal{T}_Q = \mathcal{G}_Q$, and $F(q,-\mathcal{T}_q(q,Q)) = (Q,\mathcal{T}_Q(q,Q))$. As a result, we can show

$$\mathrm{D}F = \frac{1}{-\mathcal{T}_{qQ}} \begin{bmatrix} \mathcal{T}_{qq} & 1 \\ \mathcal{T}_{qq}T_{QQ} - \mathcal{T}_{qQ}^2 & \mathcal{T}_{QQ} \end{bmatrix},$$

in the same way we derived (5.9). Observe that $\mathrm{Trace}(\mathrm{D}F) = \frac{\mathcal{T}_{qq}+\mathcal{T}_{QQ}}{-\mathcal{T}_{qQ}}$. Since $\mathcal{T}_{qq} = \mathcal{G}_{qq} + \mathcal{G}_{q\xi}\xi_q$, $\mathcal{T}_{QQ} = \mathcal{G}_{QQ} + \mathcal{G}_{Q\xi}\xi_Q$, $\mathcal{T}_{qQ} = \mathcal{G}_{qQ} + \mathcal{G}_{q\xi}\xi_Q$, and $\mathcal{T}_{Qq} = \mathcal{G}_{Qq} + \mathcal{G}_{Q\xi}\xi_q$, we have that

$$\mathcal{T}_{qq} + \mathcal{T}_{QQ} + 2\mathcal{T}_{qQ} = \mathcal{G}_{qq} + \mathcal{G}_{QQ} + 2\mathcal{G}_{qQ} + (\mathcal{G}_{q\xi} + \mathcal{G}_{Q\xi})(\xi_q + \xi_Q).$$

On the other hand, by differentiating the relationship $\mathcal{G}_\xi(q,Q;\xi(q,Q)) = 0$, we have $\mathcal{G}_{\xi q} + \mathcal{G}_{\xi\xi}\xi_q = 0$ and $\mathcal{G}_{\xi Q} + \mathcal{G}_{\xi\xi}\xi_Q = 0$, or equivalently, $\xi_q = -\frac{\mathcal{G}_{\xi q}}{\mathcal{G}_{\xi\xi}}$, $\xi_Q = -\frac{\mathcal{G}_{\xi Q}}{\mathcal{G}_{\xi\xi}}$. In particular, $\mathcal{G}_{\xi q} + \mathcal{G}_{\xi_Q} + \mathcal{G}_{\xi\xi}(\xi_q + \xi_Q) = 0$, which in turn implies

$$\mathcal{T}_{qq} + \mathcal{T}_{QQ} + 2\mathcal{T}_{qQ} = \mathcal{G}_{qq} + \mathcal{G}_{QQ} + 2\mathcal{G}_{qQ} - \frac{1}{\mathcal{G}_{\xi\xi}}(\mathcal{G}_{q\xi} + \mathcal{G}_{Q\xi})^2.$$

Furthermore, if $\mathcal{I}(q,\xi) = \mathcal{G}(q,q;\xi)$, then $\mathcal{I}_q = \mathcal{G}_q + \mathcal{G}_Q$, $\mathcal{I}_\xi = \mathcal{G}_\xi$, and

$$\mathrm{D}^2\mathcal{I} = \begin{bmatrix} \mathcal{G}_{qq} + \mathcal{G}_{QQ} + 2\mathcal{G}_{qQ} & \mathcal{G}_{\xi Q} + \mathcal{G}_{\xi q} \\ \mathcal{G}_{\xi Q} + \mathcal{G}_{\xi q} & \mathcal{G}_{\xi\xi} \end{bmatrix}.$$

So $\mathcal{T}_{qq} + \mathcal{T}_{QQ} + 2\mathcal{T}_{qQ} = \frac{\det(\mathrm{D}^2\mathcal{I})}{\mathcal{G}_{\xi\xi}}$. Also, $\mathcal{T}_{qQ} = \mathcal{G}_{qQ} - \frac{\mathcal{G}_{q\xi}\mathcal{G}_{Q\xi}}{\mathcal{G}_{\xi\xi}}$. Now

$$\mathrm{Trace}(\mathrm{D}F) - 2 = \frac{\mathcal{T}_{qq} + \mathcal{T}_{QQ} + 2\mathcal{T}_{qQ}}{-\mathcal{T}_{qQ}} = \frac{\det(\mathrm{D}^2\mathcal{I})}{\mathcal{G}_{qQ}\mathcal{G}_{\xi\xi} - \mathcal{G}_{q\xi}\mathcal{G}_{Q\xi}}. \tag{6.7}$$

Recall $\mathcal{G}(q,Q;\xi) = \mathcal{G}^0(q,\xi) + \mathcal{G}^1(\xi,Q)$ with $\mathcal{G}^0_{q\xi} > 0$, and $\mathcal{G}^1_{Q\xi} < 0$ because F^0 is a negative monotone twist and F^1 is a positive monotone twist. Hence we obtain $-\mathcal{G}_{q\xi}\mathcal{G}_{Q\xi} > 0$. On the other hand $\mathcal{G}_{qQ} = 0$, which simplifies (6.7) to

$$\mathrm{Trace}(\mathrm{D}F) - 2 = \frac{\mathcal{T}_{qq} + \mathcal{T}_{QQ} + 2\mathcal{T}_{qQ}}{-\mathcal{T}_{qQ}} = \frac{\det(\mathrm{D}^2\mathcal{I})}{-\mathcal{G}_{q\xi}\mathcal{G}_{Q\xi}}.$$

This expression has the same sign as $\det(\mathrm{D}^2\mathcal{I})$. Finally $\mathrm{D}F$ has positive eigenvalues if and only if $\mathrm{Trace}(\mathrm{D}F) \geqslant 2$, if and only if $\det(\mathrm{D}^2\mathcal{I}) \geq 0$, which concludes the proof. ∎

7. Complexity $N = 2$ area-preserving random twists

7.1. Domain of random generating functions. Next we describe the domain of a random generating function associated to a complexity $N = 2$ twist.

Lemma 7.1. *Let F be an area-preserving random twist of complexity $N = 2$. Suppose that F decomposes as $F = F_2 \circ F_1 \circ F_0$, where F_1 is a positive monotone area-preserving random twist and F_j is negative monotone area-preserving random twist for $j = 0, 2$. Let $\mathcal{G}^0, \mathcal{G}^1, \mathcal{G}^N$ be the corresponding generating functions. Write $G_i = F_i^{-1}$ and define $Q_i^{\pm}$ and $\hat{Q}_i^{\pm}$ by $F_i(q, \pm 1) = (Q_i^{\pm}(q), \pm 1)$ and $G_i(q, \pm 1) = (\hat{Q}_i^{\pm}(q), \pm 1)$. Then the function $\mathcal{I}(q, \xi_1, \xi_2) := \mathcal{G}^0(q, \xi_1) + \mathcal{G}^1(\xi_1, \xi_2) + \mathcal{G}^2(\xi_2, q)$, is well-defined on the set*

$$D = \left\{ (q, \xi_1, \xi_2) \ \mid \ Q_0^+(q) \leqslant \xi_1 \leqslant Q_0^-(q), \quad \hat{Q}_2^-(q) \leqslant \xi_2 \leqslant \hat{Q}_2^+(q) \right\},$$

Moreover, if $(q, \xi_1, \xi_2) \in D$, then $Q_1^-(\xi_1) < \xi_2 < Q_1^+(\xi_1)$.

Proof. Since $F_1 = G_2 \circ F \circ G_0$, we have

$$\hat{Q}_2^{\pm} \circ Q^{\pm} \circ \hat{Q}_0^{\pm} = Q_1^{\pm}, \tag{7.1}$$

where $Q^{\pm}$ are defined by the relationship $F(q, \pm 1) = (Q^{\pm}(q), \pm 1)$. On the set D, $\mathcal{G}^0(q, \xi_1)$ and $\mathcal{G}^2(\xi_2, q)$ are well defined. It is sufficient to check that if $(q, \xi_1, \xi_2) \in D$, then $\mathcal{G}^1(\xi_1, \xi_2)$ is well-defined. That is, $Q_1^-(\xi_1) < \xi_2 < Q_1^+(\xi_1)$. To see this observe that by (7.1),

$$\pm Q_1^{\pm}(\xi_1) = \pm \left(\hat{Q}_2^{\pm} \circ Q^{\pm} \circ \hat{Q}_0^{\pm} \right)(\xi_1) \geqslant \pm \left(\hat{Q}_2^{\pm} \circ Q^{\pm} \right)(q) > \pm \hat{Q}_2^{\pm}(q) \geqslant \pm \xi_2,$$

as desired. Here for the first inequality we used the fact that $Q^{\pm}$ and $\hat{Q}_2^{\pm}$ are increasing and that in D, we have $\hat{Q}_0^-(\xi_1) \leqslant q \leqslant \hat{Q}_0^+(\xi_1)$; for the second inequality we used $\pm Q^{\pm}(q) > \pm q$, which concludes the proof. □

We define $B_0^{\pm}(\omega), B_2^{\pm}(\omega) > 0$, by $Q_0^{\pm}(q) = q \mp B_0^{\pm}(\tau_q \omega)$ and $\hat{Q}_2^{\pm}(q) = q \pm B_2^{\pm}(\tau_q \omega)$. Let

$$K(q, p; \omega) = \bar{K}(\tau_q \omega, p) = \mathcal{I}(q, \xi(q, p)) = \bar{\mathcal{I}}(\tau_q \omega, q + \bar{\xi}(\tau_q \omega, p)), \tag{7.2}$$

where $p = (p_1, p_2)$, $\bar{\xi}(\omega, p) = (\bar{\xi}_1(\omega, p_1), \bar{\xi}_2(\omega, p_2))$, $\xi(q, p) = (q + \bar{\xi}_1(\tau_q \omega, p_1), q + \bar{\xi}_2(\tau_q \omega, p_2))$, and $\bar{\xi}_1$ and $\bar{\xi}_2$ are defined by $\bar{\xi}_1(\omega, p_1) := \frac{p_1 + 1}{2} B_0^-(\omega) + \frac{p_1 - 1}{2} B_0^+(\omega)$ and $\bar{\xi}_2(\omega, p_2) := \frac{p_2 + 1}{2} B_2^+(\omega) + \frac{p_2 - 1}{2} B_2^-(\omega)$.

Lemma 7.2. *Let $K : \mathbb{R} \times [-1, 1]^2 \times \Omega \to \mathbb{R}$ be as in (7.2). The following hold:*

(i) *There exists a one-to-one correspondence between critical points of $\mathcal{I}$ and K.*
(ii) *The vector ∇K is pointing inward on the boundary of $\mathbb{R} \times [-1, 1]^2$.*

Proof. Evidently $K(q, p_1, p_2) = K(q, p; \omega)$ satisfies

$$(7.3) \begin{cases} K_{p_1}(q, p_1, p_2) = \frac{1}{2}\mathcal{I}_{\xi_1}(q, \xi(q,p))\left(B_0^+ + B_0^-\right)(\tau_q\omega), \\ K_{p_2}(q, p_1, p_2) = \frac{1}{2}\mathcal{I}_{\xi_2}(q, \xi(q,p))\left(B_2^+ + B_2^-\right)(\tau_q\omega), \\ K_q(q, p_1, p_2) = \mathcal{I}_q(q, \xi(q,p)) + \mathcal{I}_{\xi_1}(q, \xi(q,p)) + \mathcal{I}_{\xi_2}(q, \xi(q,p)) \\ \quad +\mathcal{I}_{\xi_1}(q, \xi(q,p))\left(\frac{p_1+1}{2}\nabla B_0^- + \frac{p_1-1}{2}\nabla B_0^+\right)(\tau_q\omega) \\ \quad +\mathcal{I}_{\xi_2}(q, \xi(q,p))\left(\frac{p_1+1}{2}\nabla B_2^- + \frac{p_1-1}{2}\nabla B_2^+\right)(\tau_q\omega). \end{cases}$$

It follows from (7.3) that there exists a one-to-one correspondence between the critical points of $\mathcal{I}$ and K because $B_i^\pm > 0$ for $i = 0, 2$. This proves (i).

We now examine the behavior of K across the boundary. Observe that the functions K_{p_1} and $\mathcal{I}_{\xi_1}$ (respectively K_{p_2} and $\mathcal{I}_{\xi_2}$) have the same sign. Moreover,

$$\begin{aligned} p_1 &= \pm 1 \Leftrightarrow \xi_1 = Q_0^\mp(q), \\ p_2 &= \pm 1 \Leftrightarrow q = Q_2^\pm(\xi_2). \end{aligned}$$

It remains to verify

$$\begin{aligned} \xi_1 = Q_0^\mp(q) &\Rightarrow \pm\mathcal{I}_{\xi_1} < 0, \\ q = Q_2^\pm(\xi_2) &\Rightarrow \pm\mathcal{I}_{\xi_2} < 0. \end{aligned}$$

Let us write ξ_0 for q and ξ_3 for Q. We define functions $p^i(\xi_i, \xi_{i+1})$ and $P^i(\xi_i, \xi_{i+1})$ by $F^i\left(\xi_i, p^i(\xi_i, \xi_{i+1})\right) = \left(\xi_{i+1}, P^i(\xi_i, \xi_{i+1})\right)$. We then have $\mathcal{I}_{\xi_1} = \mathcal{G}_Q^0 + \mathcal{G}_q^1 = P^0 - p^1$ and $\mathcal{I}_{\xi_2} = \mathcal{G}_Q^1 + \mathcal{G}_q^2 = P^1 - p^2$. Finally we assert,

$$\begin{aligned} p_1 = \pm 1 \Rightarrow \xi_1 = Q_0^\mp(q) \Rightarrow p^0 = P^0 = \mp 1 \Rightarrow \pm\mathcal{I}_{\xi_1} < 0, \\ p_2 = \pm 1 \Rightarrow \xi_2 = \hat{Q}_2^\pm(q) \Rightarrow p^2 = P^2 = \pm 1 \Rightarrow \pm\mathcal{I}_{\xi_2} < 0, \end{aligned}$$

as desired. Here we are using the fact that if $p^0 = P^0 = \mp 1$ or $p^2 = P^2 = \pm 1$, then $Q_1^-(\xi_1) < \xi_2 < Q_1^+(\xi_1)$ or equivalently $p^1, P^1 \notin \{-1, 1\}$. □

7.2. **Fixed points.** The following proof is sketched because it is similar to that of Theorem 6.5.

Theorem 7.3. *Let $K : \mathbb{R} \times [-1,1]^2 \times \Omega \to \mathbb{R}$, and $K(q,\, p; \omega) := \bar{K}(\tau_q\omega,\, p)$ be C^1 up to the boundary with ∇K pointing inwards on the boundary. Then*

(a) *K has infinitely many critical points.*
(b) *The critical points of K occur as follows:*
  (1) *Either K has a continuum of critical points;*
  (2) *Or K has both infinitely many local maximums, and infinitely many saddle points or local minimums.*

Proof. We prove (b). As in the proof of Theorem 6.5, we assume that K does not have a continuum of critical points and deduce that K has infinitely many isolated local maximums. The q component of the flow remains bounded almost surely. We take a local maximum a and a connected component γ of a level set of K associated with a regular value c of K, very close to the value $K(a)$. The surface γ is an oriented closed manifold and if K has no other type of critical point, then $\Gamma : \gamma \to \mathbb{R} \times \partial[-1,1]^2$, is a homeomorphism from γ onto its image. Since the set $\mathbb{R} \times \partial[-1,1]^2$ cannot contain a homeomorphic image of γ, we arrive at a contradiction. From this we deduce the conclusion of the theorem as in the proof of Theorem 6.5. □

8. Complexity $N \geqslant 3$ area-preserving random twists

8.1. **Geometry of the domain of the generating function.** Let F be an area-preserving random twist of complexity N. As in Theorem F, we assume that N is an odd number and that F decomposes as in (2.6). Recall that $\mathcal{G}^0, \dots, \mathcal{G}^N$ denote the generating functions, respectively, of the monotone twists $F_0, \dots, F_N$. Set

$$\mathcal{I}(q,\xi) = \mathcal{G}(q,q;\xi) = \mathcal{L}(\tau_q\omega, 0; \xi - q), \quad \mathcal{I}'(q,\eta) = \mathcal{I}(q, \eta + q) =: \bar{\mathcal{I}}(\tau_q\omega, \eta),$$

where $\mathcal{G}$ and $\mathcal{L}$ are defined by Lemma 4.4, and $\eta + q = (\eta_1 + q, \dots, \eta_N + q)$. Given a realization ω, we write $D = D(\omega)$ for the domain of the definition of $\mathcal{I}$. We also set $D'(\omega) = \{\eta \in \mathbb{R}^N \mid (0,\eta) \in D(\omega)\}$ so that the domain of the function $\mathcal{I}'$ is exactly $\{(q,\eta) \mid \eta \in D'(\tau_q\omega)\}$. To simplify the notation, we write ξ_0 for q and ξ_{N+1} for Q. In this way, we can write

$$F^i(\xi_i, p^i) = (\xi_{i+1}, P^i),$$

where

$$p^i = p^i(\xi_i, \xi_{i+1}) = -\mathcal{G}^i_q(\xi_i, \xi_{i+1})$$

and

$$P^i = P^i(\xi_i, \xi_{i+1}) = \mathcal{G}^i_Q(\xi_i, \xi_{i+1}).$$

Here by $\mathcal{G}^i_q$ and $\mathcal{G}^i_Q$ we mean the partial derivatives of $\mathcal{G}^i$ with respect to its first and second arguments respectively. As before, we write G^i for the inverse of F^i and define increasing functions $Q^\pm_i$ and $\hat{Q}^\pm_i$ by $F^i(q,\pm 1) = (Q^\pm_i(q), \pm 1)$ and $G^i(q,\pm 1) = (\hat{Q}^\pm_i(q), \pm 1)$. Let

$$E(\xi_1,\xi_N) = \bigcap_{i=1}^{N-1} \Big\{(\xi_2,\dots,\xi_{N-1}) \,|\, (-1)^{i+1}Q^-_i(\xi_i) \leqslant (-1)^{i+1}\xi_{i+1} \leqslant (-1)^{i+1}Q^+_i(\xi_i)\Big\}.$$

Then the set D consists of points (q,ξ) such that $\xi_1 \in [Q^+_0(q), Q^-_0(q)]$, $\xi_N \in [\hat{Q}^+_N(q), \hat{Q}^-_N(q)]$ and $(\xi_2,\dots,\xi_{N-1}) \in E(\xi_1,\xi_N)$. Alternatively, we can write

$$E(\xi_1,\xi_N) = \bigcap_{i=1}^{N-1} \Big\{(\xi_2,\dots,\xi_{N-1}) \,|\, -1 \leqslant p^i(\xi_i,\xi_{i+1}),\ \ P^i(\xi_i,\xi_{i+1}) \leqslant 1\Big\}.$$

We write $\partial D = \partial^+ D \cup \partial^- D$, where $\partial^+ D$ and $\partial^- D$ represent the upper and lower boundaries of D. Then $\partial^+ D = \bigcup_{i=0}^N \partial_i^+ D$ and $\partial^- D = \bigcup_{i=0}^N \partial_i^- D$, where

$$\partial_0^{\pm} D = \left\{(q,\xi) \in D \,|\, \xi_1 = Q_0^{\mp}(q)\right\} = \left\{(q,\xi) \in D \,|\, p^0(q,\xi_1) = P^0(q,\xi_1) = \mp 1\right\},$$
$$\partial_N^{\pm} D = \left\{(q,\xi) \in D \,|\, \xi_N = \hat{Q}_N^{\mp}(q)\right\} = \left\{(q,\xi) \in D \,|\, p^N(\xi_N, q) = P^N(\xi_1, q) = \mp 1\right\},$$
$$\partial_i^{\pm} D = \left\{(q,\xi) \in D \,|\, \xi_{i+1} = Q_i^{\pm}(\xi_i)\right\} \quad \text{for } i \text{ odd and } 1 < i < N,$$
$$\partial_i^{\pm} D = \left\{(q,\xi) \in D \,|\, \xi_i = \hat{Q}_i^{\pm}(\xi_{i+1})\right\} \quad \text{for } i \text{ even and } 1 < i < N.$$

We also write $\partial_i D = \partial_i^+ D \cup \partial_i^- D$, $\bar{\partial}_i^{\pm} D = \partial_i^{\pm} D \setminus \left(\partial_{i-1}^{\pm} D \cup \partial_i^{\pm} D\right)$, $\bar{\partial}_0^{\pm} D = \partial_0^{\pm} D \setminus \left(\partial_1^{\pm} D \cup \partial_N^{\pm} D\right)$, $\bar{\partial}_N^{\pm} D = \partial_N^{\pm} D \setminus \left(\partial_0^{\pm} D \cup \partial_{N-1}^{\pm} D\right)$.

8.2. **The gradient.** Next examine the behavior of $\nabla \mathcal{I}$ across the boundary. The randomness of $D(\omega)$ and $\mathcal{I}$ play no role and the proof is analogous in the periodic case ([Go01]).

Proposition 8.1. *Let $F = F_N \circ \cdots \circ F_1 \circ F_0$ be an area-preserving random twist decomposition as in (2.6). Then the following properties hold.*

(P.i) *If $1 < i < N$ is even, $\nabla \mathcal{I}$ is inward along $\bar{\partial}_i^{\pm} D$;*
(P.ii) *If $1 < i < N$ is odd, $\nabla \mathcal{I}$ is outward along $\bar{\partial}_i^{\pm} D$;*
(P.iii) *$\nabla \mathcal{I}$ is outward along $\bar{\partial}_N^{\pm} D$;*
(P.iv) *$\nabla \mathcal{I}$ is inward along $\bar{\partial}_0^{\pm} D$.*

Proof. Evidently, $\mathcal{I}_q(q,\xi) = P^N - p^0$ and $\mathcal{I}_{\xi_i}(q,\xi) = P^{i-1} - p^i$, for $i = 1, \ldots, N$. We wish to study the behavior of the function $\mathcal{I}$ across the boundary of D. On $\partial_0^{\pm} D$, we have $p_0 = P_0 = \mp 1$. Since $\mathcal{I}_{\xi_1} = P^0 - p^1$, we deduce

$$\pm\, \mathcal{I}_q > 0, \quad \pm \mathcal{I}_{\xi_1} < 0 \quad \text{on} \quad \bar{\partial}_0^{\pm} D. \tag{8.1}$$

On $\partial_N^{\pm} D$, we have $p_N = P_N = \mp 1$. Since $\mathcal{I}_{\xi_N} = P^{N-1} - p^N$, we deduce

$$\pm\, \mathcal{I}_{\xi_{N-1}} < 0, \quad \pm \mathcal{I}_{\xi_N} > 0 \quad \text{on} \quad \bar{\partial}_N^{\pm} D. \tag{8.2}$$

On $\partial_i^{\pm} D$ we have $P^i = p^i = \pm(-1)^{i+1}$; hence $\pm(-1)^i \mathcal{I}_{\xi_i}(q,\xi) = \pm(-1)^i (P^{i-1} - p^i) \geqslant 0$ and $\pm(-1)^i \mathcal{I}_{\xi_{i+1}}(q,\xi) = \pm(-1)^i (P^i - p^{i+1}) \leqslant 0$ if $1 < i < N$. The inequalities are strict on $\bar{\partial}_i^{\pm} D$.

The boundary $\partial_0^{\pm} D$ is the set of points (q,ξ) such that $\xi_1 = Q_0^{\mp}(q)$ with $q \mapsto Q_0^{\mp}(q)$ increasing. So, if we write $\dot{Q}_0^{\mp}(q)$ for the derivative of $Q_0^{\mp}(q)$, then any vector that has $(1, \dot{Q}_0^{\mp}(q))$ for its projection onto (q,ξ_1)-space would be tangent to $\partial_0^{\pm} D$. Hence a vector n_0 that has $\pm(\dot{Q}_0^{\mp}(q), -1)$ for the first two components and 0 for the other components, is an inward normal vector to the $\partial_0^{\pm} D$ part of boundary. As a result, we have that on $\partial_0^{\pm} D$

$$\langle \nabla \mathcal{I}, n_0 \rangle = \pm \left(\dot{Q}_0^{\mp}(q) \mathcal{I}_q - \mathcal{I}_{\xi_1} \right) > 0,$$

by (8.1). Here $\langle \cdot, \cdot \rangle$ denotes the dot product. That is, on $\bar{\partial}_0^{\pm} D$, the gradient $\nabla \mathcal{I}$ is inward, proving (P.iv). Similarly we use (8.2) to establish (P.iii).

Assume that i is odd. The boundary $\partial_i^{\pm} D$ is the set of points (q,ξ) such that the components ξ_i and ξ_{i+1} lie on the graph $\xi_{i+1} = Q_i^{\pm}(\xi_i)$. Again, if we write $\dot{Q}_i^{\pm}$ for the derivative of $Q_i^{\pm}$, then any vector that has $(1,\, \dot{Q}_i^{\pm}(\xi_i))$ for its projection onto (ξ_i, ξ_{i+1})-space would be tangent to $\partial_i^{\pm} D$. As a result, the vector n_i that has $\pm(\dot{Q}_i^{\pm}(\xi_i),\, -1)$ for $(i, i+1)$ components and 0 for the other components, is an inward normal to the $\partial_i^{\pm} D$ portion of the boundary. Hence on $\dot{Q}_i^{\pm}$,

$$\langle \nabla \mathcal{I},\, n_i \rangle = \pm \left(\dot{Q}_i^{\pm} \mathcal{I}_{\xi_i} - \mathcal{I}_{\xi_{i+1}} \right) < 0,$$

proving (P.ii). (P.i) is established similarly. □

Define $\partial_{\mathrm{in}} D := \{x \in \partial D \,|\, \nabla \mathcal{I}(x) \text{ is inward}\}$, and similarly define

$$\partial_{\mathrm{out}} D := \{x \in \partial D \,|\, \nabla \mathcal{I}(x) \text{ is outward}\}\,.$$

We write $\mathbb{D}^k$ for the k-dimensional unit ball.

With the same proof as Gole [Go01], Proposition 8.1 implies the following lemma.

Lemma 8.2. *Suppose that $N = 2k+1$ with $k \geq 1$. Then the sets $\partial_{\mathrm{out}} D$ and $\partial_{\mathrm{in}} D$ are homeomorphic to $\mathbb{R} \times \mathbb{D}^{k+1} \times \partial \mathbb{D}^k$ and $\mathbb{R} \times \partial \mathbb{D}^{k+1} \times \mathbb{D}^k$ respectively.*

8.3. **Fixed points.**

Theorem 8.3. *Let $Z(\omega)$ be the set of critical points of $\mathcal{I}$ and*

$$\hat{Z} := \{q \,|\, (q,\xi) \in Z(\omega)\}.$$

Then:

(a) $\sup \hat{Z} = +\infty$ *and* $\inf \hat{Z} = -\infty$ *with probability* 1*;*
(b) $\mathcal{I}$ *has infinitely many critical points in D almost surely.*

Proof. (b) follows from (a). Consider the ordinary differential equation

$$\begin{cases} q'(t) \;=\; \mathcal{I}_q(q(t),\, \xi(t);\omega) \;=\; \bar{\mathcal{I}}_\omega(\tau_{q(t)}\omega,\, \xi(t)), \\ \xi'(t) \;=\; \mathcal{I}_\xi(q(t),\, \xi(t);\omega) \;=\; \bar{\mathcal{I}}_\xi(\tau_{q(t)}\omega,\, \xi(t)). \end{cases}$$

Now we distinguish two cases (in analogy with the proof of Theorem 6.5).
Case 1. (*The map $q(t)$ is unbounded either as $t \to \infty$ or $t \to -\infty$*). Analogously to Case 1 in Theorem 6.5, we are assuming that for a realization $\omega \in \Omega_0$, either

$$(x(t) = (q(t), \xi(t)) : t \geq 0)$$

or

$$(x(t) = (q(t), \xi(t)) : t \leq 0)$$

remains inside the domain $D(\omega)$ and the q-component is unbounded. As in the proof of Case 1 in Theorem 6.5, we can show that for all $\omega \in \Omega$ there exists $\xi(\omega)$ such that $(\omega,\, \xi(\omega))$ is a critical point for $\bar{\mathcal{I}}$. In particular $\mathcal{I}$ has a continuum of critical points.

Case 2. (*The map $q(t)$ is always bounded as $t \to \pm\infty$*). We want to show that $\mathcal{J}$ has critical points strictly inside of $D = D(\omega)$. Let us first assume by contradiction that $\mathcal{J}$ has no critical point inside of $D(\omega)$ for a realization of ω. Consider the flow

$$\phi^t(q,\,\xi) := (q(t),\,\xi(t)) = x(t),$$

which starts at the point $x = (q,\xi) \in \partial_{in} D$. Since $q(t)$ stays bounded and we are assuming that there is no critical point inside, the flow must exit at some positive time $e(x)$. Write $\hat{\phi}(x) = \phi^{e(x)}(x)$. Note that the sets $\partial_{\rm in} D$ and $\partial_{\rm out} D$ are open relative to ∂D. We now argue that the function $\hat{\phi}(x)$ is continuous. For example, $\hat{\phi}$ is continuous at x Simply because we may extend $\mathcal{J}$ near $\hat{\phi}(x)$ across the boundary so that for some small $\varepsilon > 0$, the flow $\phi^t(x)$ is well-defined and lies outside D for $t \in (e(x), e(x)+\varepsilon)$. We can then guarantee that $\phi^t(y)$ is close to $\phi^t(x)$ for $t \in [0, e(x)+\varepsilon)$ and y sufficiently close to x. As a result, for y sufficiently close to x, the point $\phi^{e(y)}(y)$ is close to $\phi^{e(x)}(x)$, concluding the continuity of $\hat{\phi}$. In fact by interchanging $\partial_{\rm out} D$ with $\partial_{\rm in} D$, we can show that $\hat{\phi}^{-1}$ is continuous. As a result $\hat{\phi}$ is a homeomorphism from $\partial_{\rm in} D$ onto $\partial_{\rm out} D$. This is impossible because $\partial_{\rm in} D$ is not homeomorphic to $\partial_{\rm out} D$by Lemma 8.2. Hence $\mathcal{J}$ has at least one critical point in $\mathrm{Int}(D)$ and $Z(\omega) \neq \emptyset$.

It remains to show that the set $Z(\omega)$ is unbounded on both sides. We only verify the unboundedness from above as the boundedness from below can be established in the same way. Suppose to the contrary that $Z(\omega)$ is bounded above with positive probability. Since

$$Z(\tau_q\omega) = \tau_{-q} Z(\omega) = \{(a-q,\xi) \,|\, (a,\xi) \in Z(\omega)\}, \tag{8.3}$$

by stationarity, we learn that the set $Z(\omega)$ is bounded above almost surely. Define $\bar{x}(\omega) = (\bar{q}(\omega), \bar{\xi}(\omega))$ by

$$\bar{q}(\omega) = \max\{q \,|\, (q,\xi) \in Z(\omega)\}$$

and

$$\bar{\xi}(\omega) = \max\{\xi \,|\, (\bar{q}(\omega),\xi) \in Z(\omega)\}.$$

Again by (8.3), $\bar{q}(\tau_a\omega) + a = \bar{q}(\omega)$ and $\bar{\xi}(\tau_a\omega) = \bar{\xi}(\omega)$, for every $a \in \mathbb{R}$. As a result,

$$\mathbb{P}\big(\bar{q}(\omega) \geqslant \ell\big) = \mathbb{P}\big(\bar{q}(\tau_a\omega) + a \geqslant \ell\big) = \mathbb{P}\big(\bar{q}(\tau_a\omega) \geqslant \ell - a\big) = \mathbb{P}\big(\bar{q}(\omega) \geqslant \ell - a\big),$$

for every a and ℓ. Since this is impossible unless $\bar{q} = \infty$, we deduce that the set $Z(\omega)$ is unbounded from above. □

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Dedication. The authors dedicate this article to Alan Weinstein, whose fundamental and deep insights in so many areas of geometry are a continuous source of inspiration.

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