Lectures on Dynamical Systems

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Introduction

The main goal of the theory of dynamical system is the study of the global orbit structure of maps and flows. In these notes, we review some fundamental concepts and results in the theory of dynamical systems with an emphasis on differentiable dynamics.

Several important notions in the theory of dynamical systems have their roots in the work of Maxwell, Boltzmann and Gibbs who tried to explain the macroscopic behavior of fluids and gases on the basic of the classical dynamics of many particle systems. The notion of *ergodicity* was introduced by Boltzmann as a property satisfied by a Hamiltonian flow on its constant energy surfaces. Boltzmann also initiated a mathematical expression for the *entropy* and the *entropy production* to derive Maxwell's description for the equilibrium states. Gibbs introduced the notion of *mixing systems* to explain how reversible mechanical systems could approach equilibrium states. The ergodicity and mixing are only two possible properties in the hierarchy of stochastic behavior of a dynamical system. Hopf invented a versatile method for proving the ergodicity of geodesic flows. The key role in Hopf's approach is played by the hyperbolicity. Lyapunov exponents and Kolmogorov–Sinai entropy are used to measure the hyperbolicity of a system.

Dynamical Systems come in two flavors: discrete and continuous:

Discrete Systems. We have a set X of possible *states/configurations*. X is often is equipped with a metric. There exists a map $f : X \to X$ that is often continuous (or even more regular). We set $x_n = f^n(x)$ and call the sequence $(x_n : n \in \mathbb{N}^*)$ the orbit starting from the initial state $x_0 = x$:

$$x,\ f(x),\ f(f(x)) = f^2(x), \dots,\ f^n(x), \dots$$

Continuous Systems. X is now a nice manifold, and we have a flow on X. That is, a family of homeomorphisms/diffeomorphisms $\phi_t : X \to X, \quad t \in \mathbb{R}$ such that

$$\phi_0(x) = x, \qquad \phi_{t+s}(x) = \phi_t(\phi_s(x)).$$

The path $(\phi_t(a) :\ t \in \mathbb{R})$ is an orbit starting from the initial state $\phi_0(a) = a$. For example, $x(t) = \phi_t(a)$ solves an ODE: $\dot{x} = b(x)$ where b is a vector field on the manifold X. Ideally we wish to have a complete (explicit) description of orbits.

If this can be achieved, we have a *completely integrable/exactly solvable* model. This is rarely the case for models we encounter in nature. Failing this, we may wish to find some qualitative information about some/most/all orbits. This was originated in the work of Poincare 1890-1899 [Po1-2]; the birth of the theory of dynamical systems.

What qualitative descriptions do we have in mind? Many of our models in dynamic systems have their roots in celestial mechanics and statistical physics. We already mentioned that the work of Poincare in celestial mechanics led to many fundamental concepts in the

the theory of dynamical systems. This include the notion of *symplectic maps* and the birth of *symplectic geometry* (the flow maps ϕ_t in celestial mechanics are examples of symplectic maps). Moreover, several important notions in the theory of dynamical systems can be traced back to the work of Maxwell, Boltzmann and Gibbs who tried to explain the macroscopic behavior of fluids and gases on the basic of the classical dynamics of many particle systems.

Boltzmann's Ergodicity. In the microscopic description of a solid or a fluid/gas, we are dealing with a huge number of particles: (Avogadro number) 10^{23} for a fluid, and 10^{19} for a dilute gas. It is not practical or even useful to analyze the exact locations/ velocities of all particles in the system. A more realistic question is that what a generic particle does in average. Boltzmann formulated the following question: If A is a set of states (subset of X), then what fraction of time the orbit $\phi_t(x)$ spends in the set A? Boltzmann formulated the following *ansatz* to answer the above question for models that are governed by Newton's law: For generic initial state x,

$$\frac{1}{\ell}\ \{t \in [0,\ell]:\ \phi_t(x) \in A\} \approx \text{ volume of } \ A.$$

as $\ell \to \infty$. Here we have an example of an *ergodic* dynamical system. The above ansatz is not true in general and requires some polishing. We now have more realistic reformulation of the above ansatz in the form of a conjecture that is still wide open. Sinai made a breakthrough in 1960s when he established the above conjecture for a planar *billiard* with two balls (elastic collision).

Entropy. The entropy comes in two flavors: metric (measure theoretical) and topological. The rough idea goes back to Boltzmann: In microscopic model the number of states N is exponentially large. The entropy is proportional to $\log N$. How this can be formulated for a dynamical system associated with $f : X \to X$? Introduce a resolution $\delta > 0$. When two states are within distance δ, regard them the same. In this way we replace our infinite state space with a finite set!

$$\text{Number of orbits up to time } n \ \approx \ e^{nh_{top}(f)},$$

for large n and small δ. Metric (Kolmogorov-Sinai) entropy was defined by Kolomogorov as an invariance of a dynamical system: He wanted to associate a number to a dynamical system that does not change if we make a change of variable: In other words if we have two dynamical systems $T : X \to X$, $\hat{T} : \hat{X} \to \hat{X}$, and a homeomorphism $h : X \to \hat{X}$, such that $\hat{T} = h \circ T \circ h^{-1}$, then we would like to have $entropy(T) = entropy(\hat{T})$. Motivated by the work of Boltzmann (Statistical Mechanics) and Shannon (Information Theory), Kolmogorov define the entropy as the rate of gain in information as we observe more and more of our system: Introduce a (measure theoretical) resolution. That is, a finite partition of X, so that if all points in a member of the partition is regarded as one. In this way we are dealing with a finite set. Suppose the n-orbit $(x, f(x), \ldots, f^n(x))$ of a point with respect to a partition is known. How accurately we can locate x? In chaotic dynamical systems the accuracy improves exponentially fast. The exponential rate of the improvement/gain of information is the entropy : $h_\mu(T)$. We need a measure μ to measure the size of the set of possible location of x based on the information available.

1 Invariant Measures and Ergodic Theorem

By a *discrete dynamical system* we mean a pair (X, T), where $X = (X, d)$ is a complete separable metric space (in short *Polish space*) with metric d, and $T : X \to X$ is a continuous map. By an orbit of (X, T) we mean sequences of the form $O(x) = \big(x_n = T^n(x) :\ n \in \mathbb{N}^*\big)$, where $\mathbb{N}^*$ denotes the set of nonnegative integers. Here are some examples of dynamical systems that should be kept in mind for understanding various notions that will be developed in this Chapter.

Example 1.1(i) *(Rotation)* $X = \mathbb{T}^d$ is the d-dimensional torus. We may regard $\mathbb{T}$ as the interval $[0, 1]$ with $0 = 1$. Given a vector $\alpha = (\alpha_1, \dots, \alpha_d) \in [0, 1)^d$, we define $T : \mathbb{T}^d \to \mathbb{T}^d$ by $T(x) = x + \alpha \pmod 1$. In other words, when $x \in [0, 1]^d$, then $x + \alpha$ is understood as the sum of points x and α in $\mathbb{R}^d$. Though if $x + \alpha \notin [0, 1)^d$, then by dropping its integer parts of its coordinates, we obtain a point in $[0, 1)^d$. Alternatively, if we regard the circle $\mathbb{T}$ as the set of complex numbers $z = e^{2\pi i\theta}$ such that $|z| = 1$, and set $\beta = (\beta_1, \dots, \beta_d)$, with $\beta_j = e^{2\pi i\alpha_j}$, then $T(z_1, \dots, z_d) = (\beta_1 z_1, \dots, \beta_d z_d)$.

(ii) *(Expansion)* Given an integer $m \geq 2$, we define $T = T_m : \mathbb{T} \to \mathbb{T}$, by $T(x) = mx \pmod 1$. Alternatively, if we regard the circle $\mathbb{T}$ as the set of complex numbers $z = e^{2\pi i\theta}$ such that $|z| = 1$, then $T(x) = z^m$.

(iii) *(Shift)* Given a Polish space E, set $X = E^{\mathbb{N}}$ (respectively $E^{\mathbb{Z}}$) for the space of sequences $\omega = (\omega_n :\ n \in \mathbb{N})$ (respectively $\omega = (\omega_n :\ n \in \mathbb{Z})$) in E. Consider the *shift map* $\tau : X \to X$ that is defined by $(\tau\omega)_n = \omega_{n+1}$.

(iv) *(Contraction)* Let (X, T) be a discrete dynamical system, ans assume that there exists $\lambda \in (0, 1)$ such that $d\big(T(x), T(y)\big) \leq \lambda d(x, y)$. Then there exists a unique $a \in X$ such that $T(a) = a$ and

$$d\big(T^n(x), a\big) \leq \lambda^n d(x, a).$$

As a consequence $d\big(T^n(x), a\big) \to 0$, as $n \to \infty$, for every $x \in X$. □

Given a dynamical system (X, T), we may wonder how often a subset of X is visited by an orbit of T. For example, in the dynamical systems described in Example **1.1**, most orbits (for "most" α in part **(i)**) are dense and every nonempty open set is visited infinitely often for any such orbit. To measure the asymptotic fraction of times a set is visited, we may look at the limit points of the sequence

$$\frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_A(T^j(x)) \tag{1.1}$$

as $n \to \infty$. More generally, we may wonder whether or not the limit

$$\lim_{n\to\infty} \Phi_n(f)(x) := \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} f(T^j(x)) \tag{1.2}$$

exists for a function $f : X \to \mathbb{R}$. Let us write $C_b(X)$ for the space of bounded continuous functions $f : X \to \mathbb{R}$. Given $x \in X$, if the limit of (1.2) exists for every $f \in C_b(X)$, then the limit $\Phi^x(f)$ enjoys some obvious properties:

(i) $f \geq 0 \Rightarrow \Phi^x(f) \geq 0$, $\Phi^x(1\!\!1) = 1$.

(ii) $\Phi^x(f)$ is linear in f.

(iii) $|\Phi^x(f)| \leq \sup_{y \in X} |f(y)|$.

(iv) $\Phi^x(f \circ T) = \Phi_x(f)$.

If X is also locally compact, then we can use Riesz Representation Theorem to assert that there exists a unique (Radon) probability measure μ such that $\Phi_x(f) = \int f \, d\mu$. Evidently, such a measure $\mu(A)$ measures how often a set A is visited by the orbit $O(x)$. Motivated by (iv), we make the following definition:

Definition 1.1(i) Given a Polish space X, with a metric d, we write $\mathcal{B}(X)$ for the Borel σ-algebra of (X, d), and $\mathcal{M}(X)$ for the set of Borel Radon probability measures on X.

(ii) We write $\mathcal{I}_T$ for the set of Radon probability measures μ such that

$$\int f \circ T \, d\mu = \int f \, d\mu, \tag{1.3}$$

for every $f \in C_b(X)$. Any such measure μ is called an *invariant measure* of T. Equivalently, $\mu \in \mathcal{I}_T$ iff $\mu(A) = \mu\big(T^{-1}(A)\big)$ for every $B \in \mathcal{B}(X)$. □

It seems natural that for analyzing the limit points of (1.1), we should first try to understand the space $\mathcal{I}_T$ of invariant measures. Note that in (1.2), what we have is $\int f \, d\mu_n^x$ where $\mu_n^x = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{T^j(x)}$. We also learned that if (1.2) exists for every f, then μ_n^x has a limit and its limit is an invariant measure. Of course there is a danger that the limit (1.2) does not exist in general. This is very plausible if the orbit is unbounded and some of the mass of the measure μ_n^x is lost as $n \to \infty$ because $T^j(x)$ goes off to infinity. This would not happen if we assume X is compact. To this end, let us review the notion of weak convergence for measures.

Definition 1.2 We say a sequence $\{\mu_n\}_{n \in \mathbb{N}}$ in $\mathcal{M}(X)$ converges weakly to $\mu \in \mathcal{M}(X)$ (in short $\mu_n \Rightarrow \mu$), if

$$\int f \, d\mu_n \to \int f \, d\mu, \tag{1.4}$$

for every $f \in C_b(X)$. □

It turns out that for the weak convergence, we only need to verify (1.4) for $f \in U_b(X)$ where $U_b(X)$ denotes the space of bounded uniformly continuous functions. Since $U_b(X)$ is separable, we can metrize the space of probability measures $\mathcal{M}(X)$. (See for example [P].)

Theorem 1.1 *Suppose X is a compact metric space.*

- **(i)** (Krylov–Bogobulov) *Let $\{x_n\}$ be a sequence in X. Then any limit point of the sequence $\{\mu_n^{x_n}\}$ is in $\mathcal{I}_T$. In particular, $\mathcal{I}_T \neq \emptyset$.*

- **(ii)** *If $\mathcal{I}_T = \{\bar{\mu}\}$ is singleton, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j(x)) = \int f \; d\bar{\mu}$$

  uniformly for every $f \in C(X)$. In fact $D(\mu_n^x, \bar{\mu}) \to 0$ uniformly in x.

- **(iii)** *If $\{\Phi_n(f)\}$ converges pointwise to a constant for functions f in a dense subset of $C(X)$, then $\mathcal{I}_T$ is a singleton.*

Proof(i) This is an immediate consequence of Exercise (ii) below at the end of this chapter, and what we have seen in the beginning of this chapter.

(ii) Let $\{x_n\}$ be any sequence in X and put $\nu_n = \mu_n^{x_n}$. Since any limit point of $\{\nu_n\}$ is in $\mathcal{I}_T = \{\mu\}$, we deduce that $\nu_n \Rightarrow \bar{\mu}$. From this we can readily deduce that in fact $\mu_n^x \Rightarrow \mu$ uniformly.

(iii) We are assuming that $\Phi_n(f)$ converges to a constant $\hat{f}$ for f in a dense set $\mathcal{T} \subseteq C(X)$. Since the sequence $\{\Phi_n(f)\}$ is uniformly bounded for $f \in C(X)$, and

$$\int \Phi_n(f) \; d\mu = \int f \; d\mu,$$

for every $\mu \in \mathcal{I}_T$, we deduce that the constant $\hat{f}$ can only be $\int f \; d\mu$. As a consequence, if $\mu, \nu \in \mathcal{I}_T$, then $\int f \; d\mu = \int f \; d\nu$ for a dense set of functions f. This implies that $\mu = \nu$ and we conclude that $\mathcal{I}_T$ is a singleton. □

From Theorem 1.1 we learn that when $\mathcal{I}_T$ is a singleton, the statistics of the orbits are very simple. However, this is a rather rare situation and when it happens, we say that the transformation T is *uniquely ergodic.*

Example 1.2(i) Consider the dynamical system of Example 1.1**(i)**, when $d = 1$, and $\alpha = p/q$ a rational number with p and q coprime. Note that every orbit is periodic of period q. Moreover, for every $x \in \mathbb{T}$, the measure

$$\mu^x = \frac{1}{q} \sum_{j=0}^{q-1} \delta_{T^j(x)},$$

is invariant for T. One can show that any $\mu \in \mathcal{I}_T$ can be expressed as

$$\mu = \int_{\mathbb{T}} \mu^x \ \theta(dx),$$

where θ is a probability measure on $\mathbb{T}$. To avoid repetition, we only need to take a probability measure that is concentrated on the interval $[0, q^{-1})$, or the interval $[0, q^{-1}]$, with $0 = q^{-1}$.

(ii) Again, consider the dynamical system of Example 1.1**(i)**, but now in any dimension and for any α. We wish to find the necessary and sufficient for T to be uniquely ergodic. We note that the Lebesgue measure ℓ on $\mathbb{T}^d$ is always invariant for T. To apply Theorem 1.1**(iii)**, let us take $\mathcal{A}$ to be the set of trigonometric polynomials

$$\sum_{j\in A} c_j e^{2\pi i j\cdot x},$$

with A any finite subset of $\mathbb{Z}^d$. For calculating the limit of $\Phi_n(f)$ as $n \to \infty$, it suffice to consider the case $f(x) = f_j(x) = e^{2\pi i j\cdot x}$. Indeed since

$$\Phi_n(f) = \frac{1}{n}\sum_{\ell=0}^{n-1} e^{2\pi i j\cdot(x+\ell\alpha)} = e^{2\pi i j\cdot x}\left(\frac{1}{n}\sum_{\ell=0}^{n-1} e^{2\pi i \ell j\cdot\alpha}\right) = \frac{1}{n}\ e^{2\pi i j\cdot x}\left(\frac{1-e^{2\pi i n j\cdot\alpha}}{1-e^{2\pi i j\cdot\alpha}}\right),$$

whenever $j \cdot \alpha \notin \mathbb{Z}$, we have

$$\lim_{n\to\infty} \Phi_n(f) = \begin{cases} 0 & \text{if} \quad j\cdot\alpha \notin \mathbb{Z}, \\ e^{2\pi i j\cdot x} & \text{if} \quad j\cdot\alpha \in \mathbb{Z}. \end{cases}$$

From this and Theorem 1.1**(iii)** we deduce that T is uniquely ergodic iff the following condition is true:

$$j \in \mathbb{Z}^d \setminus \{0\} \quad \Rightarrow \quad j\cdot\alpha \notin \mathbb{Z}. \tag{1.5}$$

We note that the ergodicity of the Lebesgue measure also implies the denseness of the sequence $\{x + n\alpha\}$. This latter property is known as the *topological transitivity.*

(iii) Consider the dynamical system of Example 1.1**(i)**, when $d = 2$, and $\alpha = (\alpha_1, 0)$ with $\alpha_1 \notin \mathbb{Q}$. Let μ^{x_2} denotes the one-dimensional Lebesgue measure that is concentrated on the circle

$$\mathbb{T}_{x_2} = \big\{(x_1, x_2) \in \mathbb{T}^2 : \ x_1 \in \mathbb{T}\big\}.$$

Clearly this measure is invariant. In fact all invariant measures can be expressed as

$$\mu = \int_{\mathbb{T}} \mu^{x_2} \ \tau(dx_2),$$

where τ is a probability measure on $\mathbb{T}$. We also note

$$\lim_{n\to\infty} \Phi_n(f)(x_1, x_2) = \int_{\mathbb{T}} f(y_1, x_2)\ dy_1.$$

(iii) Consider the dynamical system of Example 1.1**(iv)**. Then $\mathcal{I}_T = \{\delta_a\}$. □

Remark 1.1(i) When $d = 1$, the condition (1.6) is equivalent to $\alpha \notin \mathbb{Q}$. The fact that Lebesgue measure is the only invariant measure when α is irrational is equivalent to that fact that the sequence $\{n\alpha\}$ is dense on the circle $\mathbb{T}$. To see this, observe that if $\mu \in \mathcal{I}_T$, then

$$\int f(x + n\alpha)\ \mu(dx) = \int f(x)\ \mu(dx)$$

for every continuous f and any $n \in \mathbb{N}$. Since $\{n\alpha\}$ is dense, we deduce that μ is translation invariant. As is well known, the only translation invariant finite measure on $\mathbb{T}$ is the Lebesgue measure.

(ii) According to a classical result of Poincaré, if an orientation preserving homeomorphism $T : \mathbb{T} \to \mathbb{T}$ has a dense orbit, then it is isomorphic to a rotation (i.e. there exists a change of coordinates $h : \mathbb{T} \to \mathbb{T}$ such that $h^{-1} \circ T \circ h$ is a rotation). □

As we mentioned earlier, in most cases $\mathcal{I}_T$ is not a singleton. There are some obvious properties of the set $\mathcal{I}_T$ which we now state. Note that $\mathcal{I}_T$ is always a convex and closed subset of $\mathcal{M}(X)$. Also, $\mathcal{I}_T$ is compact when X is compact because $\mathcal{M}(X)$ is compact. Let us recall a theorem of Choquet that can be used to get a picture of the set $\mathcal{I}_T$. Recall that if $\mathcal{C}$ is a compact convex set then a point $a \in \mathcal{C}$ is *extreme* if $a = \theta b + (1 - \theta)c$ for some $\theta \in [0, 1]$ and $b, c \in \mathcal{C}$ implies that either $a = b$ or $a = c$. According to *Choquet's theorem*, if $\mathcal{C}$ is convex and compact, then any $\mu \in \mathcal{C}$ can be expressed as an average of the extreme points. More precisely, we can find a probability measure θ on the set of extreme points of $\mathcal{C}$ such that

$$\mu = \int_{C^{ex}} \alpha\ \theta(d\alpha). \tag{1.6}$$

Motivated by (1.6) and Example 1.2, we formulate two natural concepts:

Definition 1.3(i) We write $\mathcal{I}_T^{ex}$ for the set of extreme points of $\mathcal{I}_T$.

(ii) Given $\mu \in \mathcal{I}_T^{ex}$, we set

$$X_\mu = \{x : \mu_n^x \Rightarrow \mu \text{ as } n \to \infty\}. \tag{1.7}$$

□

Example 1.3 In Example 1.1**(i)**, we have $\mathcal{I}_T^{ex} = \big\{\mu^x:\ x \in [0,q^{-1})\big\}$. Example 1.2**(iii)**, we have $\mathcal{I}_T^{ex} = \big\{\mu^{x_2}:\ x_2 \in \mathbb{T}\big\}$. □

Given $\mu \in \mathcal{I}_T^{ex}$, clearly the set X_μ is invariant under T. That is, if $x \in X_\mu$, then $T(x) \in X_\mu$. Also, if $\mu_1 \neq \mu_2 \in \mathcal{I}_T^{ex}$, then $X_{\mu_1} \cap X_{\mu_2} = \emptyset$. Our second *Ergodic Theorem* below implies that $\mu(X_\mu) = 1$. This confirms the importance of extreme measures among the invariant measures. Later we find more a practical criterion for the extremity in terms of the invariant sets and functions.

One way to study the large n limit of the sequence $\Phi_n(f)$ is by examining the convergence of the empirical measures $\big\{\mu_n^x\big\}_{n\in\mathbb{N}}$. Alternatively, we may fix an invariant measure μ and examine the convergence of the sequence $\Phi_n(f)$ in $L^p(\mu)$. Observe that if $\Phi_n f \to \bar f$, then $\bar f$ must be invariant with respect to the dynamics. This suggests studying the set of invariant functions. Moreover, the pairing $(f,\mu) \mapsto \int f\ d\mu$ suggests considering functions that are *orthogonal* to invariant functions, namely functions of the form $f = g\circ T - g$.

Definition 1.4(i) Let $\mu \in \mathcal{M}(X)$. We write $\mathcal{F}_T$ (respectively $\mathcal{F}_T^\mu$) for the set of bounded measurable functions $f : X \to \mathbb{R}$ such that $f\circ T = f$ (respectively $f \circ T = f$, μ-a.e.). Also, set

$$L_T^p(\mu) = \{f \in L^p(\mu):\ f\circ T = f \quad \mu - \text{a.e.}\}\,. \tag{1.8}$$

We refer to functions in $\mathcal{F}_T$ as *T-conserved or invariant* functions. With a slight abuse of notation, by $A \in \mathcal{F}_T$ (respectively $A \in \mathcal{F}_T^\mu$) we mean that $1\!\!1_A \in \mathcal{F}_T$ (respectively $1\!\!1_A \in \mathcal{F}_T^\mu$). Note that $A \in \mathcal{F}_T$ iff $A \in \mathcal{B}(X)$ with $T^{-1}(A) = A$. Similarly, $A \in \mathcal{F}_T^\mu$ iff $A \in \mathcal{B}(X)$ with

$$\mu\big(T^{-1}(A)\Delta A\big) = 0.$$

(ii) We define

$$\mathcal{H}_T(\mu) = \big\{g\circ T - g:\ g \in L^2(\mu)\big\}.$$

□

Theorem 1.2 *(von Neumann) Let $T : X \to X$ be a Borel measurable transformation and let $\mu \in \mathcal{I}_T$. If $f \in L^2(\mu)$, then $\Phi_n(f) = \frac{1}{n}\sum_0^{n-1} f\circ T^j$ converges in L^2-sense to Pf, where Pf is the projection of f onto $L_T^2(mu)$.*

Proof. Observe that if $f \in L_T^2(\mu)$, then $\mu(A_1) = 1$, where

$$A_n = \big\{x \in X:\ f(x) = f(T(x)) = \cdots = f(T^n(x))\big\}.$$

Since $\mu \in \mathcal{I}_T$, we deduce that $\mu\big(T^n(A_1)\big) = 1$. Hence $\mu(A_n) = 1$. This implies that for such f, we have that $\Phi_n(f) = f$ μ-a.e.

We note that $\Phi_n : L^2(\mu) \to L^2(\mu)$ is a bounded linear operator with

$$\left\|\Phi_n(f)\right\|_{L^2} \le \|f\|_{L^2},$$

because $\|f_k \circ T^j\|_{L^2} = \|f_k\|_{L^2}$ by invariance. Also observe that if $f = g \circ T - g$ for some $g \in L^2$, then $\Phi_n(f) \to 0$ as $n \to \infty$. If $f \in \bar{\mathcal{H}}$, then we still have $\Phi_n(f) \to 0$ as $n \to \infty$. Indeed if $f_k \in \mathcal{H}$ converges to f in L^2, then

$$\|\Phi_n(f)\|_{L^2} \le \|\Phi_n(f_k)\|_{L^2} + \|f - f_k\|_{L^2},$$

Since $\|\Phi_n(f_k)\|_{L^2} \to 0$ as $n \to \infty$ and $\|f - f_k\|_{L^2} \to 0$ as $k \to \infty$, we deduce that $\Phi_n(f) \to 0$ as $n \to \infty$.

Given any $f \in L^2(\mu)$, write $f = g + h$ with $g \in \bar{\mathcal{H}}$ and $h \perp \mathcal{H}$. If $h \perp \mathcal{H}$, then $\int h\ \varphi \circ T\ d\mu = \int h\varphi\ d\mu$, for every $\varphi \in L^2(\mu)$. Hence $\int (h \circ T - h)^2\ d\mu = 0$. This means that $h \circ T = h$. As a result, $h \in L^2_T(\mu)$, and $\Phi_n(f) = \Phi_n(g) + \Phi_n(h) = \Phi_n(g) + h$. Since $\Phi_n(g) \to 0$, we deduce that $\Phi_n(f) \to h$ with $h = Pf$. □

Theorem 1.2 is also true in $L^1(\mu)$ setting. To explain this, let us first make sense of Pf for $f \in L^1(\mu)$. For example, we may approximate any $f \in L^1(\mu)$ by the sequence $f_k = f 1\!\!1(|f| \le k)$, and define

$$Pf = \lim_{k \to \infty} Pf_k.$$

The limit exists because the sequence $\{Pf_k\}$ is Cauchy in $L^1(\mu)$ (this is an immediate consequence of Exercise **(ii)** below). We are now ready to state and prove the Ergodic Theorem for L^1-functions.

Corollary 1.1 *Suppose $\mu \in \mathcal{I}_T$ and $f \in L^1(\mu)$. Let Pf be as above. Then $\Phi_n(f)$ converges to Pf in L^1 sense.*

Proof Clearly,

$$\left\|\Phi_n(f)\right\|_{L^1(\mu)} \le \left\|f\right\|_{L^1(\mu)},$$

by the invariance of μ. From this and $\int |Pg|\ d\mu \le \int |g|\ d\mu$ (see Exercise 1.2**(ii)** below) we learn

$$\begin{aligned} \|\Phi_n(f) - Pf\|_{L^1(\mu)} &\le \|\Phi_n(f_k) - Pf_k\|_{L^1(\mu)} + \|\Phi_n(f - f_k)\|_{L^1(\mu)} + \|P(f - f_k)\|_{L^1(\mu)} \\ &\le \|\Phi_n(f_k) - Pf_k\|_{L^1(\mu)} + 2\|f - f_k\|_{L^1(\mu)}, \end{aligned}$$

where $k \in \mathbb{N}$, $f_k = f 1\!\!1(|f| \le k)$. The proof follows because by Theorem 1.2 the first term goes to 0 as $n \to \infty$, and by approximation, the second term goes to 0, as $k \to \infty$. □

Remark 1.2 We note that if $\mu \in \mathcal{I}_T$, then the operator $Uf = f \circ T$ is an isometry of $L^2(\mu)$ and the subspace $L^2_T(\mu)$ is the eigenspace associated with the eigenvalue one. Hence von

Neumann's theorem simply says that $\frac{1}{n}(I + U + \cdots + U^{n-1}) \to P$, with P representing the projection onto the eigenspace associated with the eigenvalue 1. Note that if $\lambda = e^{i\theta}$ is an eigenvalue of U and if $\lambda \neq 1$, then $\frac{1}{n}(1 + \lambda + \cdots + \lambda^{n-1}) = \frac{\lambda^n - 1}{n(\lambda - 1)} \to 0$ as $n \to \infty$. This suggests that Theorem 1.2 may be proved using the Spectral Theorem for unitary operators. The above theorem also suggests studying the spectrum of the operator U for a given T. Later we will encounter the notion of mixing dynamical systems. It turns out that the mixing condition implies that discrete spectrum of the operator U consists of the point 1 only. □

From Theorem 1.2 we learn the relevance of the invariant (conserved) functions for a dynamical system. One possibility is that the only invariant function in the support of μ is the constant function. In fact if there are non constant functions in $L^2(\mu)$, we may use them to decompose μ into invariant measures with smaller support. The lack of nontrivial conserved functions is an indication of the *irreducibility* of our invariant measure. We may check such irreducibility by evaluating μ at T-invariant subsets of X. More precisely, we have the following definition.

Definition 1.5 An invariant measure μ is called *ergodic* if $\mu(A) \in \{0, 1\}$ for every $A \in \mathcal{F}_T$. The set of ergodic invariant measures is denoted by $\mathcal{I}_T^{er}$. □

We will see in Exercise **(vi)** below that the sets in $\mathcal{F}_T^{\mu}$ differ from sets in $\mathcal{F}_T$ only in a set of measure 0. Also, we will see later that $\mathcal{I}^{er} = \mathcal{I}^{ex}$. In view of (1.6), any $\mu \in \mathcal{I}_T$ can be expressed as an average of ergodic ones.

Remark 1.3 By Theorem 1.2, we know that if $f \in L^2(\mu)$, then Pf is the projection of f onto the space of invariant functions. For $f \in L^1(\mu)$, we may define Pf as the unique $\mathcal{F}_T^{\mu}$-measurable function such that

$$\int_A Pf\ d\mu = \int_A f\ d\mu \tag{1.9}$$

for every $A \in \mathcal{F}_T$. Note that since Pf is $\mathcal{F}_T^{\mu}$-measurable, we have

$$Pf \circ T = Pf,$$

μ-almost everywhere. Alternatively, Pf is uniquely defined as the Radon–Nikodym derivative of $f\mu$ with respect to μ, if we restrict it to $\mathcal{F}_T - \sigma$-algebra. More precisely

$$Pf = \frac{d(f\mu|_{\mathcal{F}_T})}{d\mu|_{\mathcal{F}_T}}.$$

□

As our next goal, we consider an almost everywhere mode of convergence.

Theorem 1.3 *(Birkhoff Ergodic Theorem) Suppose $\mu \in \mathcal{I}_T$ and $f \in L^1(\mu)$. Then*

$$\mu\left\{x : \lim_{n\to\infty} \Phi_n(f)(x) = Pf(x)\right\} = 1.$$

Proof Set $g = f - Pf - \epsilon$ for a fixed $\epsilon > 0$. Evidently $Pg \equiv -\epsilon < 0$ and $\Phi_n(f - Pf - \epsilon) = \Phi_n(f) - Pf - \epsilon$. Hence, it suffices to show

$$\limsup_{n\to\infty} \Phi_n(g) \leq 0 \quad \mu - \text{a.e.}$$

We expect to have

$$g + g \circ T + \cdots + g \circ T^{n-1} = -\varepsilon n + o(n).$$

From this, it is reasonable to expect that the expression $g + \cdots + g \circ T^{n-1}$ to be bounded above μ-a.e. Because of this, let us define $G_n = \max_{j \leq n} \sum_0^{j-1} g \circ T^i$. Set $A = \{x : \lim_{n\to\infty} G_n(x) = +\infty\}$. Without loss of generality, we may assume that g is finite everywhere. Clearly $A \in \mathcal{F}_T$ because $G_{n+1} = g + \max(0, G_n \circ T)$. Note also that if $x \notin A$, then $\limsup_{n\to\infty} \Phi_n(g) \leq 0$. To complete the proof, it remains to show that $\mu(A) = 0$. To see this, observe

$$\begin{aligned} 0 &\leq \int_A (G_{n+1} - G_n) d\mu = \int_A (G_{n+1} - G_n \circ T) d\mu \\ &= \int_A [g + \max(0, G_n \circ T) - G_n \circ T] d\mu = \int_A (g - \min(0, G_n \circ T)) d\mu. \end{aligned}$$

On the set A, $-\min(0, G_n \circ T) \downarrow 0$. On the other hand, if

$$h_n = g - \min(0, G_n \circ T),$$

then $g \leq h_n \leq h_1 = g + (g \circ T)^-$. Hence by the Dominated Convergence Theorem, $0 \leq \int_A g d\mu = \int_A Pg d\mu \leq -\epsilon\mu(A)$. Thus we must have $\mu(A) = 0$. □

Remark 1.4(i) If μ is ergodic, then the σ-algebra $\mathcal{F}_T$ is trivial and any $\mathcal{F}_T$ measurable function is constant. Hence Pf is constant and this constant can only be $\int f \, d\mu$.

(ii) Since $\mu_n^x \Rightarrow \mu$ iff $\int f \, d\mu_n^x \to \int f \, d\mu$, for f in a countable dense set of continuous functions, we learn from Part (i) that $\mu(X_\mu) = 1$, where X_μ was defined by (1.6). However, if μ is not ergodic, then Pf is not constant in general and if $\mu_T(x, dy)$ denotes the conditional distribution of μ given $\mathcal{F}_T$, then

$$Pf(x) = \int f(y) \mu_T(x, dy).$$

From this we deduce that in this case,

$$\mu\left\{x : \lim_{n\to\infty} \mu_n^x = \mu_T(x, \cdot)\right\} = 1,$$

Moreover, $\mu_T(x,\cdot) \in \mathcal{I}_T^{er}$, for μ-almost all x.

(iii) If T is invertible, then we can have an ergodic theorem for T^{-1} as well. Since $\mathcal{F}_T = \mathcal{F}_{T^{-1}}$, it is clear that $P_T f = P_{T^{-1}} f$. As a consequence we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_0^{n-1} f \circ T^j = \lim_{n\to\infty} \frac{1}{n} \sum_0^{n-1} f \circ T^{-j} = Pf.$$

$\mu -$ a.e. □

Our proof of Theorem 1.2 implies that any $f \in L^2(\mu)$ can be written as

$$f = Pf + g_k \circ T - g_k + h_k,$$

such that $g_k, h_k \in L^2(\mu)$ with $\|h_k\|_{L^2(\mu)} \to 0$, as $k \to \infty$. A similar decomposition is also valid for $f \in L^1(\mu)$ with $g_k \in L^\infty(\mu), h_k \in L^1(\mu)$, $\|h_k\|_{L^1(\mu)} \to 0$, as $k \to \infty$. We note that for every $g \in L^\infty(\mu)$,

$$\lim_{n\to\infty} \Phi_n\big(g \circ T - g\big) = 0,$$

μ-a.e. Because of this, we may wonder whether or not Theorem 1.3 can be established with the aid of such a decomposition. For this however we need to show that the error term $\Phi_n(h_k)$ does not contribute to the pointwise limit. This can be done by a maximal type inequality. Put

$$M_n(h) = \sup_{1\le j\le n} \Phi_j(h), \qquad M(h) = \sup_{1\le j} \Phi_j(h).$$

Theorem 1.4 *If , then*

$$\mu\left(\big\{x:\ M(h)(x) > t\big\}\right) \le t^{-1}\|h\|_{L^1(\mu)}.$$

First Proof If $g = h - t$, and

$$E = \left\{\max_{n\ge1} \Phi_n(g) > 0\right\} = \left\{\max_{n\ge1}(g + \cdots + g \circ T^{n-1}) > 0\right\},$$

then it suffices to show

$$\int_E g \, d\mu \ge 0.$$

Note that if

$$G_n = \max_{1\le i\le n}(g + \cdots + g \circ T^{i-1}), \qquad F_n = \max(0, G_n),$$

then $E = \cup_n E_n$, where $E_n = \{F_n > 0\}$. Since $E_n \subseteq E_{n+1}$, it suffices to show that $\int_{E_n} g \, d\mu \ge 0$, for every n. Observe that on the set E_n, we have $F_n = G_n > 0$, and

$$g + F_n \circ T = \max(g, g + g \circ T, \ldots, g + \cdots + g \circ T^n) \ge F_n.$$

Hence,

$$\int_{E_n} g\, d\mu \geq \int_{E_n} (F_n - F_n \circ T)\, d\mu = \int_X F_n\, d\mu - \int_{E_n} F_n \circ T\, d\mu \geq \int_X (F_n - F_n \circ T)\, d\mu = 0,$$

as desired. Here we have used $F_n \geq 0$, and that $F_n = 0$ on E_n^c.

Second Proof We now offer a proof that is based on a *discrete Hardy-Littlewood maximal inequality.* To motivate our strategy, let us consider examine our inequality when $X = \mathbb{Z}$ and $T = \tau$ is the shift $\tau(i) = i + 1$. Note that the counting measure m is the only (σ-finite) invariant measure. For $F : \mathbb{Z} \to \mathbb{R}$,

$$\Phi_n(F)(i) = n^{-1} \sum_{j=0}^{n-1} F(i + j), \qquad \widehat{M}(F)(i) = \sup_{n \geq 1} \Phi_n(F)(i),$$

and the analog of our maximal inequality reads as

$$\sharp\big\{i : \ \widehat{M}(F)(i) > t\big\} \leq 3t^{-1} \sum_{j \in \mathbb{Z}} |F(j)|.$$

Accepting this inequality for now, let us take any $f \in L^1(\mu)$ and $(x, n) \in X \times \mathbb{N}$, and define the sequence

$$F^{x,n}(i) = f(T^i x)\ \mathbb{1}(0 \leq i \leq n).$$

Evidently,

$$M_\ell(f)(T^k x) = \sup_{1 \leq j \leq \ell} j^{-1} \sum_{i=0}^{j-1} f(T^{k+i}(x)) = \sup_{1 \leq j \leq \ell} j^{-1} \sum_{i=0}^{j-1} F^{x,n}(k + i) \leq \widehat{M}(F^{x,n})(k),$$

whenever $\ell + k \leq n$. As a result,

$$\begin{aligned}
\mu\big(M_\ell(f) > t\big) &= (n - \ell + 1)^{-1} \sum_{k=0}^{n-\ell} \mu\big(M_\ell(f) \circ T^k > t\big) \\
&\leq (n - \ell + 1)^{-1} \sum_{k=0}^{n-\ell} \mu\big(\widehat{M}(F^{x,n})(k) > t\big) \\
&\leq 3(t(n - \ell + 1))^{-1} \int \sum_k F^{x,n}(k)\ \mu(dx) \\
&= 3n(t(n - \ell + 1))^{-1} \int f\ d\mu.
\end{aligned}$$

We finally send $n \to \infty$ and $\ell \to \infty$ in this order to complete the proof. $\square$

Remark 1.5 Given $\mathbf{a} = \{\mathbf{a}_n : n \in \mathbb{N}\}$, a family of non-negative sequences $\mathbf{a}_n = (a_i^n : i \in \mathbb{N}^*)$, with $\sum_i a_n^i = 1$, we define

$$\Phi_n^{\mathbf{a}}(f) = \sum_{i \in \mathbb{N}^*} a_n^i f \circ T^i.$$

Note that if $\mathbf{a}_n^i = n^{-1}1\!\!1(0 \le i \le n-1)$, then $\Phi_n^{\mathbf{a}} = \Phi_n$. We may wonder under what conditions on $\mathbf{a}$, we have $\Phi_n^{\mathbf{a}}(f) \to f$ in large n limit. It turns out that for $L^2(\mu)$ convergence, the necessary and sufficient condition is the existence of the limit

$$\lim_{n\to\infty} \hat{\mathbf{a}}_n(\alpha) = \lim_{n\to\infty} \sum_{j=0}^{n-1} a_j^n e^{ij\alpha},$$

for every α. The necessity of this condition is immediate because this condition is equivalent to the convergence when T is the rotation of Example 1.1**(i)**. For the analog of Theorem 1.4, we may repeat our Second Proof to argue that the following discrete Hardy-Littlewood maximal inequality

$$\sharp\left\{j : \sup_n \widehat{\Phi}_n^{\mathbf{a}}(h)(j) > t\right\} \le c_0 \sum_j |h(j)|,$$

with

$$\widehat{\Phi}_n^{\mathbf{a}}(h)(j) = \sum_i a_n^i h(i+j),$$

implies

$$\mu\left(\Phi_n^{\mathbf{a}}(f) > t\right) \le c_0 t^{-1} \int f \, d\mu,$$

The converse is also true. According to a result of Bellow and Calderon [BC], the above maximal inequality holds if $\mathbf{a}$ satisfies the following condition: There exists a constant c_1, and $\alpha \in (0,1]$ such that

$$|a_{i+j}^n - a_i^n| \le c_1 \frac{|j|^\alpha}{i^{\alpha+1}},$$

for every n and (i,j), with $2|j| \le i$. □

a As an immediate consequence of Theorem 1.4, we have the following pointwise ergodic theorem with varying function.

Corollary 1.2 *Suppose $\mu \in \mathcal{I}_T$ and $\{f_n\}_{n\in\mathbb{N}^*}$ is a sequence in $L^1(\mu)$ such that $f_n \to f$ as $n \to \infty$, μ-a.e., and in L^1-sense. Then*

$$\mu\left\{x : \lim_{n\to\infty} n^{-1} \sum_{i=0}^{n-1} f_i\big(T^i(x)\big) = Pf(x)\right\} = 1.$$

Proof To ease the notation, let us set $g_i = |f_i - f|$, and

$$h_m = \sup_{n \geq m} \ g_n.$$

On account of Theorem 1.3, it suffices to check

$$\lim_{n\to\infty} n^{-1} \sum_{i=0}^{n-1} g_i\big(T^i(x)\big) = 0, \tag{1.10}$$

for μ-a.e. x. To prove this, observe that for any positive $m \leq n$,

$$\begin{aligned} n^{-1}\sum_{i=0}^{n-1} g_i \circ T^i =& n^{-1}\left[\sum_{i=0}^{m-1} + \sum_{i=m}^{n-1}\right] g_i \circ T^i \leq n^{-1}\sum_{i=0}^{m-1} g_i \circ T^i + n^{-1}\sum_{i=m}^{n-1} h_m \circ T^i \\ \leq& n^{-1}\sum_{i=0}^{m-1} g_i \circ T^i + M(h_m). \end{aligned}$$

Sending $n \to \infty$ yields

$$\limsup_{n\to\infty} n^{-1}\sum_{i=0}^{n-1} g_i \circ T^i \leq M(h_m).$$

This implies (1.10) because

$$\mu\left(\big\{M(h_m) > \delta\big\}\right) \leq 3\delta^{-1}\|h_m\|_{L^1(\mu)},$$

by Theorem 1.4, and

$$\lim_{m\to\infty} \|h_m\|_{L^1(\mu)} = 0.$$

□

We continue with more consequences of Theorem 1.3.

Proposition 1.1 *We have $\mathcal{I}_T^{ex} = \mathcal{I}_T^{er}$. Moreover, if μ_1 and μ_2 are two distinct ergodic measures, then μ_1 and μ_2 are mutually singular.*

Proof Suppose that $\mu \in \mathcal{I}_T$ is not ergodic and choose $A \in \mathcal{F}_T$ such that $\mu(A) \in (0,1)$. If we define

$$\mu_1(B) = \frac{\mu(A \cap B)}{\mu(A)}, \ \mu_2(B) = \frac{\mu(A^c \cap B)}{\mu(A^c)},$$

then $\mu_1, \mu_2 \in \mathcal{I}_T$ and $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$ for $\alpha = \mu(A)$. Hence $\mathcal{I}^{ex} \subseteq \mathcal{I}^{er}$

Conversely, let μ be ergodic and $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$ for some $\mu_1, \mu_2 \in \mathcal{I}_T$ and $\alpha \in (0,1)$. Note that we also have that $\mu_i(A) = 0$ or 1 if $A \in \mathcal{F}_T$ and $i = 1$ or 2. As a result, $\mu_1, \mu_2 \in \mathcal{I}_T^{er}$ and $\mu\big(X_\mu\big) = \mu_1\big(X_{\mu_1}\big) = \mu_2\big(X_{\mu_2}\big) = 1$. But $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$, and $\mu\big(X_\mu\big) = 1$, imply that $\mu_1\big(X_\mu\big) = \mu_2\big(X_\mu\big) = 1$. This is compatible with $\mu_1\big(X_{\mu_1}\big) = \mu_2\big(X_{\mu_2}\big) = 1$, only if $\mu = \mu_1 = \mu_2$.

Finally, if if μ_1 and μ_2 are two distinct ergodic measures, then $X_{\mu_1} \cap X_{\mu_2} = \emptyset$. This implies that $\mu_1 \perp \mu_2$, because $\mu_1\big(X_{\mu_1}\big) = \mu_2\big(X_{\mu_2}\big) = 1$. □

1.1 Mixing

As we mentioned in the introduction, many important ergodic measures enjoy a stronger property known as mixing.

Definition 1.6 A measure $\mu \in \mathcal{I}_T$ is called *mixing* if for any two measurable sets A and B,

$$\lim_{n\to\infty} \mu(T^{-n}(A) \cap B) = \mu(A)\mu(B).$$

The set of mixing invariant measures is denoted by $\mathcal{I}_T^{mix} = \mathcal{I}_T^{mix}(X)$. □

Remark 1.6 Mixing implies the ergodicity because if $A \in \mathcal{F}_T$, then $T^{-n}(A) = A$ and $T^{-n}(A) \cap A^c = \emptyset$. As a result,

$$\mu(A) = \lim_n \mu(T^{-n}(A) \cap A) = \mu(A)\mu(A),$$

which implies that either $\mu(A) = 0$ or $\mu(A) = 1$. Also note that if μ is ergodic, then

$$\mu\left\{x : \frac{1}{n}\sum_0^{n-1} 1\!\!1_A \circ T^j \to \mu(A)\right\} = 1,$$

which in turn implies

$$\lim_{n\to\infty} \int \left(\frac{1}{n}\sum_0^{n-1} 1\!\!1_A \circ T^j\right) 1\!\!1_B \, d\mu = \mu(A)\mu(B).$$

Hence the ergodicity means

$$\lim_{n\to\infty} \frac{1}{n}\sum_0^{n-1} \mu(T^{-j}(A) \cap B) = \mu(A)\mu(B). \tag{1.11}$$

So, the ergodicity is some type of a weak mixing. □

Example 1.4 Let $T : \mathbb{T}^d \to \mathbb{T}^d$ be a translation $T(x) = x + \alpha \pmod 1$ with α satisfying (1.5). We now argue that T is not mixing. To see this, take a set B with $\bar{\mu}(B) > 0$ and assume that B is not dense. Pick $x_0 \notin B$ and let $\delta = \text{dist.}(x_0, B) > 0$. Take any set A open with $\mu(A) > 0$ and $\text{diam}(A) < \delta/2$. By topological transitivity, $x_0 \in T^{-n}(A)$ for infinitely many $n \in \mathbb{N}$. Since $\text{diam}(T^{-n}(A)) = \text{diam}(A)$, we deduce that $T^{-n}(A) \cap B = \emptyset$ for such n's. Clearly $\bar{\mu}(T^{-n}(A) \cap B) = 0$ does not converge to $\bar{\mu}(A)\bar{\mu}(B) \neq 0$ as $n \to \infty$. □

Before discussing examples of mixing systems, let us give an equivalent criterion for mixing.

Proposition 1.2 *A measure μ is mixing iff*

$$\lim_{n \to \infty} \int f \circ T^n \; g \; d\mu = \int f d\mu \int g d\mu \tag{1.12}$$

for f and g in a dense subset of $L^2(\mu)$.

Proof If μ is mixing, then (1.12) is true for $f = 1\!\!1_A$, $g = 1\!\!1_B$. Hence (1.12) is true if both f and g are simple, i.e., $f = \sum_{j=1}^m c_j 1\!\!1_{A_j}$, $g = \sum_{j=1}^m c'_j 1\!\!1_{B_j}$. We now use the fact that the space of simple functions is dense in $L^2(\mu)$ to deduce that (1.12) is true for a dense set of functions.

For the converse, it suffices to show that if (1.12) is true for a dense set of functions, then it is true for every $f \in L^2(\mu)$. Observe that if $\|f - \hat{f}\|_{L^2}$ and $\|g - \hat{g}\|_{L^2}$ are small, then

$$\left| \int f \circ T^n \; g \; d\mu - \int \hat{f} \circ T^n \; \hat{g} \; d\mu \right|,$$

is small. Indeed,

$$\begin{aligned} \left| \int f \circ T^n \; g \; d\mu - \int \hat{f} \circ T^n \; \hat{g} \; d\mu \right| &\leq \left| \int (f \circ T^n - \hat{f} \circ T^n) g \; d\mu \right| + \left| \int \hat{f} \circ T^n \; (g - \hat{g}) \; d\mu \right| \\ &\leq \|f - \hat{f}\| \|g\| + \|\hat{f}\| \|g - \hat{g}\| \end{aligned}$$

by invariance and Schwartz Inequality. □

Remark 1.7 We learn from Proposition 1.2 that μ is mixing iff for every $f \in L^2(\mu)$, we have $f \circ T^n \rightharpoonup \hat{f}$, as $n \to \infty$, where the constant $\hat{f}$ is $\int f \; d\mu$. □

Before working out some examples of mixing invariant measure, we discuss *isomorphic systems* who are dynamically equivalent.

Definition 1.7(i) Let X and Y be two Polish spaces, and let $h : X \to Y$ be a Borel map. Then $h_\sharp : \mathcal{M}(X) \to \mathcal{M}(Y)$ is defined as

$$h_\sharp(\mu)(A) = \mu\big(h^{-1}(A)\big).$$

(ii) Let (X, T) and (Y, S) be two dynamical systems. A continuous surjective transformation $h : X \to Y$ is called a *factor map* if $S \circ h = h \circ T$. If such a map exists, then we refer to (Y, S) as a factor of (X, T). If the map h is also injective, then we say (X, T) and (Y, S) are *isomorphic.* □

Note that $\mu \in \mathcal{I}_T = \mathcal{I}(X)$ iff $T_\sharp(\mu) = \mu$. The following result is straightforward and its proof is left as an exercise.

Proposition 1.3 *If h is a factor map between (X, T) and (Y, S), then*

$$\begin{aligned} &h^{-1}\big(\mathcal{F}_S\big) \subseteq \mathcal{F}_T, && h_\sharp\big(\mathcal{I}_T(X)\big) \subseteq \mathcal{I}_S(Y), \\ &h_\sharp\big(\mathcal{I}_T^{er}(X)\big) \subseteq \mathcal{I}_S^{er}(Y), && h_\sharp\big(\mathcal{I}_T^{mix}(X)\big) \subseteq \mathcal{I}_S^{mix}(Y). \end{aligned}$$

Example 1.5(i) Let (X, τ) be as in Example 1.1**(iii)** and assume that the corresponding is a compact metric space. Given $\beta \in \mathcal{M}(E)$, let us write $\mu^\beta \in \mathcal{M}(X)$ for the product measure with marginals β. Clearly $\mu^\beta \in \mathcal{I}_\tau$. We now argue that indeed μ^β is mixing. To see this, write $\mathcal{A}_{loc}$ for the space of $L^2\big(\mu^\beta\big)$ functions that depend on finitely many coordinates. We claim that $\mathcal{A}_{loc}$ is dense in $L^2\big(\mu^\beta\big)$. This can be shown in two ways:

(1) Use Stone-Weirstrass Theorem to show that the space of $\mathcal{A}_{loc} \cap C(X)$ is dense in $C(X)$, and then apply Lusin's Theorem to deduce that $\mathcal{A}_{loc}$ is dense in $L^2\big(\mu^\beta\big)$.

(2) Write $\mathcal{F}_k$ for the σ-algebra of sets that depend on the first k coordinates. Now, given $f \in L^2\big(\mu^\beta\big)$, write $f_k = \mu^\beta\big(f|\mathcal{F}_k\big)$ for the conditional expectation of f, given $\mathcal{F}_k$. Note that $f_k \in \mathcal{A}_{loc}$. On the other hand $f_k \to f$ in $L^2\big(\mu^\beta\big)$ as $k \to \infty$ by the celebrated *Martingale Convergence Theorem.*

In view of Proposition 1.2, μ^β is mixing if we can show

$$\lim_{n\to\infty} \int f \circ \tau^n \ g \ d\mu^\beta = \int f \ d\mu^\beta \int g \ d\mu^\beta,$$

for every $f, g \in \mathcal{A}_{loc}$. Indeed if f and g depend on the first k variables, then we simply have

$$\int f \circ \tau^n \ g \ d\mu^\beta = \int f \ d\mu^\beta \int g \ d\mu^\beta,$$

whenever $n > k$.

Also note that if $\bar{\omega} \in X$ is a periodic sequence of period exactly k, then

$$\mu^{\bar{\omega}} = \frac{1}{k} \sum_{i=0}^{k-1} \delta_{\tau^i(\bar{\omega})} \in \mathcal{I}_\tau^{er}.$$

Though $\mu^{\bar{\omega}}$ is not mixing unless $k = 1$.

(ii) We next consider $T = T_m : \mathbb{T} \to \mathbb{T}$ of Example **1.1(ii)**. Observe that if (X, τ) is as in Part **(i)**, for $E = \{0, \dots, m-1\}$, and $F : X \to [0,1]$ is defined by

$$F(\omega_1, \omega_2, \dots, \omega_k, \dots) = \sum_{i=1}^{\infty} \omega_i m^{-i},$$

then F is continuous. In fact, if we equip X with the metric

$$d(\omega, \omega') = \sum_{i=1}^{i} m^{-i} |\omega_i - \omega_i'|,$$

then

$$|F(\omega) - F(\omega')| \le d(\omega, \omega').$$

Given any $p = (p_0, \dots, p_{m-1})$ with $p_j \ge 0$ and $p_0 + \dots + p_{m-1} = 1$, we can construct a unique probability measure μ_p such that

$$\mu_p[\cdot\omega_1 \dots \omega_k, \cdot\omega_1 \dots \omega_k + m^{-k}) = p_{\omega_1} p_{\omega_2} \dots p_{\omega_k},$$

where

$$.\omega_1\omega_2 \dots \omega_k = \omega_1 m^{-1} + \omega_2 m^{-2} + \dots + \omega_k m^{-k},$$

is a base m expansion with $\omega_1, \dots, \omega_k \in \{0, 1, \dots, m-1\}$. Clearly $\mu_p = F_\sharp \mu^p$. From this we learn that $\mu_p \in \mathcal{I}_T^{mix}$ by Proposition 1.3.

Here are some examples of μ_p:

(1) If $p_j = 1$ for some $j \in \{0, 1, \dots, m-1\}$, then $\mu_p = \delta_{y_j}$ with

$$y_j = j/(m-1) = jm/(m-1) \pmod 1.$$

Note that $y_0 = y_{m-1}$; otherwise, $y_0, \dots, y_{m-2}$ correspond to the (distinct) fixed points.

(2) If $p_0 = \dots = p_{m-1} = \frac{1}{m}$, then μ_p is the Lebesgue measure.

(3) If for example $m = 3$ and $p_0 = p_2 = 1/2$, then the support of the measure μ_p is the classical Cantor set of *Hausdorff dimension* $\alpha = \log_3 2$ (solving $3^{-\alpha} = 2^{-1}$).

Note also that if x is a periodic point of period k, then $\mu = \frac{1}{k}\sum_{j=0}^{k-1}\delta_{T^j(x)}$ is an ergodic measure. Such μ is never mixing unless $k = 1$.

(iii) Consider a linear transformation on $\mathbb{R}^2$ associated with a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $a, b, c, d \in \mathbb{Z}$, then $T(x) = Ax \pmod 1$ defines a transformation on the 2-dimensional torus $\mathbb{T}^2$. Here we are using the fact that if $x = y \pmod 1$, then $Ax = Ay \pmod 1$. If we assume $\det A = 1$, then the Lebesgue measure ℓ on $\mathbb{T}^2$ is invariant for the transformation T. To have ℓ mixing, we need to assume that the eigenvalues of T are real and different from 1 and -1. Let us assume that A has eigenvalues α and α^{-1} with $\alpha \in \mathbb{R}$ and $|\alpha| < 1$. By Proposition 1.1, ℓ is mixing if we can show that for every $n, m \in \mathbb{Z}^2$,

$$\lim_{n\to\infty}\int (\varphi_k \circ T^n)\varphi_m \, d\ell = \int \varphi_k \, d\ell \int \varphi_m \, d\ell \tag{1.13}$$

where $\varphi_k(x) = \exp(2\pi i k \cdot x)$. If $m = 0$ or $k = 0$, then (1.13) is obvious. If $k, m \neq 0$, then the right-hand side of (1.13) is zero. We now establish (1.13) for $m, k \neq 0$ by showing that the left-hand side is zero for sufficiently large n. Clearly

$$\int \varphi_k \circ T^n \varphi_m \, d\ell = \int \varphi_{(A^T)^n k + m} \, d\ell, \tag{1.14}$$

where A^T denotes the transpose of A. To show that (1.14) is zero for large n, it suffices to show that $(A^T)^n k + m \neq 0$ for large n. For this, it suffices to show that $\lim_{n\to\infty}(A^T)^n k$ is either ∞ or 0. Write v_1 and v_2 for eigenvectors associated with eigenvalues α and α^{-1}. We have $\lim_{n\to\infty}(A^T)^n k = \infty$ if k is not parallel to v_1 and $\lim_{n\to\infty}(A^T)^n k = 0$ if k is parallel to v_1. □

1.2 Continuous dynamical systems

We now turn our attention to the notion of the ergodicity of continuous dynamical system.

Definition 1.8(i) Let X be a Polish space. By a *continuous flow* we mean a continuous map $\phi : X \times \mathbb{R} \to X$, such that if $\phi_t(x) = \phi(x, t)$, then the family $\{\phi_t : \ t \in \mathbb{R}\}$, satisfies the following conditions: $\phi_0 = id$, and $\phi_s \circ \phi_t = \phi_{s+t}$ for all $s, t \in \mathbb{R}$.

(ii) Given a continuous flow ϕ, we set

$$\begin{aligned}
\mathcal{F}_\phi &= \{f \in \mathcal{B}(X) : \ f \circ \phi_t = f \quad \forall t \in \mathbb{R}\} \\
\mathcal{F}_\phi^\mu &= \{f \in \mathcal{B}(X) : \ f \circ \phi_t = f \quad \mu \ a.e. \quad \forall t \in \mathbb{R}\} \\
\mathcal{I}_\phi &= \left\{\mu \in \mathcal{M}(X) : \ \int f \circ \phi_t \, d\mu = \int f \, d\mu \quad \forall \ (f, t) \in C_b(X) \times \mathbb{R}\right\} \\
\mathcal{I}_\phi^{er} &= \{\mu \in \mathcal{I}_\phi : \ \mu(A) \in \{0, 1\} \quad \text{for every} \quad A \in \mathcal{F}_\phi\}.
\end{aligned}$$

□

Theorem 1.5 *Assume that* $\mu \in \mathcal{I}_\phi$ *and* $f \in L^1(\mu)$. *Then*

$$\mu\left(\left\{x : \lim_{t\to\infty} \frac{1}{t}\int_0^t f\circ\phi_\theta(x)\ d\theta = Pf\right\}\right) = 1,$$

where $Pf = \mu\big(f|\mathcal{F}_\phi\big)$.

Proof *(Step 1)* We first claim

$$\mu\left(\left\{x :\ \hat{P}f := \lim_{t\to\infty} \frac{1}{t}\int_0^t f\circ\phi_\theta(x)\ d\theta\ \text{ exists}\right\}\right) = 1 \tag{1.15}$$

$$\lim_{t\to\infty} \frac{1}{t}\int \left|\int_0^t f\circ\phi_\theta - \hat{P}f\right|\ d\mu = 0. \tag{1.16}$$

To reduce this to the discrete case, let us define $\Omega = \prod_{j\in\mathbb{Z}}\mathbb{R}$ and $\Gamma : X \to \Omega$ by

$$\Gamma(x) = (\omega_j(x) : j \in \mathbb{Z}) = \left(\int_j^{j+1} f\circ\phi_\theta(x)\ d\theta : j\in\mathbb{Z}\right).$$

Clearly $\Gamma\circ\phi_1 = \tau\circ\Gamma$. Also, if $\mu \in \mathcal{I}_\phi$, then $\tilde{\mu}$ defined by $\tilde{\mu}(A) = \mu(\Gamma^{-1}(A))$ belongs to $\mathcal{I}_T$. Indeed,

$$\int g\circ\tau\ d\tilde{\mu} = \int g\circ\tau\circ\Gamma\ d\mu = \int g\circ\Gamma\circ\phi_1\ d\mu = \int g\circ\Gamma\ d\mu = \int g\ d\tilde{\mu}.$$

We now apply Theorem 1.6 to assert

$$\tilde{\mu}\left(\left\{\omega :\ A(\omega) := \lim_{n\to\infty}\frac{1}{n}\sum_0^{n-1}\omega_j\ \text{ exists}\right\}\right) = 1,$$

$$\lim_{n\to\infty}\int\left|\frac{1}{n}\sum_0^{n-1}\omega_j - A(\omega)\right|\ \tilde{\mu}(d\omega) = 0.$$

Hence, (1.15) and (1.16) are true for $\hat{P}f = A\circ\Gamma$, if the convergence occurs along $n\in\mathbb{N}$ in place $t\in\mathbb{R}$. To complete the proof of (1.15) and (1.16), observe

$$\frac{1}{t}\int_0^t f\circ\phi_\theta\ d\theta = \frac{[t]}{t}\frac{1}{[t]}\int_0^{[t]} f\circ\phi_\theta\ d\theta + \frac{1}{t}\int_{[t]}^t f\circ\phi_\theta\ d\theta.$$

Hence it suffices to show

$$\lim_{n\to\infty}\frac{1}{n}\int_n^{n+1}|f\circ\phi_\theta|\;d\theta=0\quad\mu-\text{a.e. and in }\;L^1(\mu) \tag{1.17}$$

To prove this, observe

$$\lim_{n\to\infty}\frac{1}{n}\int_0^{n}|f\circ\phi_\theta|\;d\theta\text{ exists }\mu-\text{a.e. and in }\;L^1(\mu),$$

and this implies

$$\begin{aligned}\frac{1}{n}\int_n^{n+1}|f\circ\phi_\theta|\;d\theta &= \frac{1}{n}\int_0^{n+1}|f\circ\phi_\theta|\;d\theta-\frac{1}{n}\int_0^{n}|f\circ\phi_\theta|\;d\theta\\ &= \frac{n+1}{n}\frac{1}{n+1}\int_0^{n+1}|f\circ\phi_\theta|\;d\theta-\frac{1}{n}\int_0^{n}|f\circ\phi_\theta|\;d\theta\end{aligned}$$

converges to 0 $\mu-$ a.e., proving (1.17). This in turn implies (1.15) and (1.16).

(Step 2) It remains to check that $\hat{P}=P$. To see this, take any bounded $g\in\mathcal{F}_\phi^\mu$ and observe that by (1.16)

$$\lim_{t\to\infty}\int\left[\frac{1}{t}\int_0^t f\circ\phi_\theta\;d\theta\right]\;g\;d\mu=\int\hat{P}f\;g\;d\mu. \tag{1.18}$$

On the other hand, since g is invariant, we have

$$\int f\circ\phi_t\;g\;d\mu=\int f\;g\circ\phi_{-t}\;d\mu=\int fg\;d\mu$$

Hence the left-hand of side (1.18) is $\int fg\;d\mu$. This means that $\hat{P}f=Pf$, as desired. □

Example 1.7 A prime example for a continuous dynamical system is a flow associated with and ODE $\dot{x}=b(x)$ on a Riemannian manifold where b is a Lipschitz vector field. More precisely we write $\phi_t(a)$ for a solution with initial condition $x(0)=a$. Here are some examples for the invariant measures:

(i) If $b(a)=0$, then $\mu=\delta_a$ is an invariant measure. If $\phi_T(a)=a$ for some $T>0$, then

$$\mu(dx)=T^{-1}\int_0^T\delta_{\phi_t(a)}(dx)\;dt,$$

with support $\gamma=\big\{\phi_t(a):\;t\in[0,T)\big\}$. In fact since $b\big(\phi_t(a)\big)\;dt=d\ell$, is the length element, alternatively we may represents $d\mu$ as the measure $|b|^{-1}\;d\ell$ on γ.

(ii) When *div* $b=0$, the normalized volume measure is invariant. □

1.3 Subadditive ergodic theorem

The following generalization of Theorem 1.3 has many applications in probability theory and dynamical system. We will see several of its applications in Chapter 4 where we discuss *Lyapunov exponents.*

Theorem 1.6 *(Kingman) Let $\mu \in \mathcal{I}_T^{er}$ and suppose that $\{S_n(\cdot) : n = 0, 1, \dots\}$ is a sequence of $L^1(\mu)$ functions satisfying*

(1.19) $$S_{n+m}(x) \le S_n(x) + S_m(T^n(x)),$$

Then

$$S(x) := \lim_{n\to\infty} \frac{1}{n} S_n(x),$$

exists for μ-almost all x. Moreover S is T-invariant, and

$$\int S(x)\ \mu(dx) = \lambda := \inf_m \left\{ \frac{1}{m} \int S_m\ d\mu \right\} \in [-\infty, +\infty).$$

The following short proof of Theorem 1.6 is due Avila and Bochi [AB]. This proof also provides us with a new proof of the ergodic theorem (Theorem 1.3). The main ingredient for the proof is the following Lemma.

Lemma 1.1 *Let S_n be sequence that satisfies the assumptions of Theorem 1.6, and set*

$$\underline{S}(x) := \liminf_{n\to\infty} n^{-1} S_n(x).$$

Then $\underline{S} \circ T = \underline{S}$, μ-a.e., and $\int \underline{S}\ d\mu = \lambda$.

Proof From $S_{n+1} \le S_1 + S_n \circ T$, we learn that $\underline{S}(x) \le \underline{S}(T(x))$. This means that for any $a \in \mathbb{R}$,

$$\big\{x :\ \underline{S}(x) \ge a\big\} \subseteq T^{-1}\big(\big\{x :\ \underline{S}(x) \ge a\big\}\big).$$

Since these two sets have equal μ-measure, we learn that their difference is μ-null. From this we deduce that $\underline{S} = \underline{S} \circ T$, μ-a.e. As a consequence, if

$$X_0 = \big\{x :\ \underline{S}(T^j(x)) = \underline{S}(x) \ \text{ for all } j \in \mathbb{N}\big\},$$

then $\mu(X_0) = 1$.

Let us first assume that $S_n \ge c_0 n$, for a finite constant c_0, so that $\bar{S} \ge c_0$. Given $\varepsilon > 0$, and $k \in \mathbb{N}$, we define

$$X_k = \big\{x :\ m^{-1} S_m(x) \le \underline{S}(x) + \varepsilon \ \text{ for some } m \in \{1, \dots, k\}\big\},$$

so that $\cup_k X_k = X$. Given $k \in \mathbb{N}$, define a (possibly finite) sequence

$$m_0 = 0 \le n_1(x) < m_1(x) \le \dots \le n_i(x) < m_i(x) \le \dots,$$

inductively in the following manner:

- Given m_{i-1}, we choose n_i as the smallest $n \geq m_{i-1}$ such that $T^n(x) \in X_k$. If $T^n(x) \notin E_k$ for all $n \geq m_{i-1}$, we set $n_i = \infty$, and our sequence ends.
- Since $y = T^{n_i}(x) \in X_k$, we can find $r_i = r_i(x) \in \{1, \dots, k\}$ such that $r_i^{-1} S_{r_i}\big(y\big) \leq \underline{S}(y) + \varepsilon$. We then set $m_i = n_i + r_i$, for a choice of r_i.

Given $n \geq k$, let us write ℓ for the largest integer such that $m_\ell \leq n$. Note that we always have $m_{\ell+1} > n$, though $n_{\ell+1} > n$ or $n_{\ell+1} \leq n$ are both possible ($n_{\ell+1} = \infty$ is alsso a possibility). We then use subadditivity to write

$$S_n(x) \leq \sum_{j \in A_n(x)} S_1\big(T^j(x)\big) + \sum_{i=1}^{\ell(x)} S_{r_i(x)}\big(T^{n_i(x)}(x)\big),$$

where A_n consists of those integers j that are in the set

$$\cup_{i=1}^{\ell}[m_{i-1}, n_i) \cup [m_\ell, n).$$

Note that if A'_n consists of those integers in the set

$$\cup_{i=1}^{\ell}[m_{i-1}, n_i) \cup [m_\ell, n \wedge n_{\ell+1}).$$

then $T^j(x) \notin E_k$, whenever $j \in A'_n$. Because of this, we define

$$f_k(x) = \max\{S_1(x), \underline{S}(x) + \varepsilon\} 1\!\!1\big(x \notin E_k\big) + (\underline{S}(x) + \varepsilon) 1\!\!1\big(x \in E_k\big),$$

to assert that if $x \in X_0$, then

$$\begin{aligned} S_n(x) &\leq \sum_{j \in A_n(x) \setminus A'_n(x)} S_1\big(T^j(x)\big) 1\!\!1\big(T^j(x) \in E_k\big) + \sum_{j \in A_n(x)} f_k\big(T^j(x)\big) + (\underline{S}(x) + \varepsilon) \sum_{i=0}^{\ell(x)} r_i(x) \\ &\leq \sum_{j \in A_n(x) \setminus A'_n(x)} S_1\big(T^j(x)\big) 1\!\!1\big(T^j(x) \in E_k\big) + \sum_{j=0}^{n-1} f_k\big(T^j(x)\big). \end{aligned}$$

Since $n - (n \wedge n_{\ell+1}) \leq m_{\ell+1} - n_{\ell+1} \leq k$, we learn that $\sharp\big(A_n(x) \setminus A'_n(x)\big) \leq k$. As a result,

$$n \geq k \quad \implies \quad \int S_n \, d\mu \leq k \int |S_1| \, d\mu + n \int f_k \, d\mu,$$

which in turn implies

$$\lambda \leq \int f_k \, d\mu.$$

Since $\mu(X_k) \to 1$ as $k \to \infty$, we deduce that $\lambda \le \int \underline{S}\ d\mu + \varepsilon$. We then send $\varepsilon \to 0$ to arrive at $\lambda \le \int \underline{S}\ d\mu$. The reverse inequality is a consequence of Fatou's lemma.

For general case, pick any $c \in \mathbb{R}$, and set $S_n^c = \max\{S_n, cn\}$. Then S_n^c satisfies the subadditivity condition. Note

$$\liminf_{n\to\infty} n^{-1} S_n^c = \max\{\underline{S}, c\},$$

As a result,

$$\begin{aligned}\int \underline{S}\ d\mu &= \inf_c \int \max\{\underline{S}, c\}\ d\mu = \inf_c \inf_n \int n^{-1} S_n^c\ d\mu \\ &= \inf_n \inf_c \int n^{-1} S_n^c\ d\mu = \inf_n \lim_{c\to-\infty} \int n^{-1} S_n^c\ d\mu \\ &= \inf_n \int n^{-1} S_n\ d\mu,\end{aligned}$$

where the Monotone Convergence Theorem is used for the last equality. This completes the proof. □

We note that if we apply Lemma 1.1 to $S_n = \Phi_n(f)$ and $S_n = \Phi_n(-f)$, then we deduce Theorem 1.3.

Proof of Theorem 1.6 *(Step 1)* Set

$$\bar{S}(x) := \limsup_{n\to\infty} n^{-1} S_n(x).$$

We first claim

$$\limsup_{n\to\infty} n^{-1} S_n \le \limsup_{n\to\infty} (nk)^{-1} S_{nk}, \tag{1.20}$$

for every $k \in \mathbb{N}$. To see this, write $n = kq_n + r_n$, with $q_n = [n/k]$, and $r_n \in \{0, 1, \dots, k-1$, and use subadditivity to write

$$S_n \le S_{kq_n} + S_{r_n} \circ T^{kq_n} \le S_{kq_n} + h \circ T^{kq_n},$$

where

$$h = \max\left\{S_1^+, \dots, S_{k-1}^+\right\}.$$

From this, and $(kq_n)/n \to 1$, we learn

$$\limsup_{n\to\infty} n^{-1} S_n \le \limsup_{n\to\infty} (kn)^{-1} S_{kn} + \limsup_{n\to\infty} n^{-1} h \circ T^n.$$

On the other hand, for $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mu\big(n^{-1} h \circ T^n > \varepsilon\big) = \sum_{n=1}^{\infty} \mu\big(\varepsilon^{-1} h > n\big) \le \int \varepsilon^{-1} h \; d\mu < \infty,$$

which implies

$$\limsup_{n\to\infty} n^{-1} h \circ T^n \le 0,$$

by Borel-Cantelli Lemma. This completes the proof of (1.20).

Fix $k \in \mathbb{N}$, and use (1.20) and the subadditivity to assert

$$\bar{S} \le \limsup_{n\to\infty} (kn)^{-1} S_{kn} \le \limsup_{n\to\infty} (kn)^{-1} \sum_{i=0}^{n-1} S_k \circ T^{ik} =: k^{-1} \limsup_{n\to\infty} \Phi_n^k(S_k).$$

The expression $\Phi_n^k(S_k)$ is an additive sequence for the dynamical system (X, T^k). Hence by Theorem 1.3, or even Lemma 1.1,

$$\int \limsup_{n\to\infty} \Phi_n^k(S_k) \; d\mu = \int S_k \; d\mu.$$

As a result,

$$\int \bar{S} \; d\mu \le \int k^{-1} S_k \; d\mu.$$

This and Lemma 1.1 imply

$$\int \bar{S} \; d\mu \le \lambda = \int \underline{S} \; d\mu.$$

This completes the proof. □

Second Proof of Theorem 1.6 Fix $m > 0$. Any $n > m$ can be written as $n = qm + r$ for some $q \in \mathbb{N}^*$ and $r \in \{0, 1, \dots, m-1\}$. As a result, if $k \in \{0, \dots, m-1\}$, then $n = k + q'm + r'$ with $q' = q'(k) = \begin{cases} q & \text{if } r \ge k \\ q-1 & \text{if } r < k \end{cases}$, $r' = r'(k) = \begin{cases} r-k & \text{if } r \ge k \\ r-k+m & \text{if } r < k \end{cases}$. By subadditivity,

$$\begin{aligned} S_n(x) &\le S_k(x) + S_{q'm}(T^k(x)) + S_{r'}(T^{k+q'm}(x)) \\ &\le S_k(x) + \sum_{j=0}^{q'-1} S_m(T^{k+jm}(x)) + S_{r'}(T^{k+q'm}(x)). \end{aligned}$$

We now sum over k to obtain

$$S_n(x) \le \frac{1}{m} \sum_{0}^{m-1} S_k(x) + \sum_{0}^{n-1} \frac{S_m}{m}(T^i(x)) + \frac{1}{m} \sum_{0}^{m-1} S_{r'(k)}(T^{k+q'(k)m}(x)),$$

where $S_0 = 0$. Hence

$$\frac{1}{n} S_n(x) \le \frac{1}{n} \sum_0^{n-1} \frac{S_m}{m}(T^i(x)) + R_{n,m}(x),$$

where $\|R_{n,m}\|_{L^1} \le \text{constant} \times \frac{m}{n}$, because $\int |S_l(T^r)| d\mu = \int |S_l| d\mu$. By the Ergodic Theorem,

$$\limsup_{n\to\infty} \frac{1}{n} S_n(x) \le \int \frac{S_m}{m} d\mu.$$

Since m is arbitrary,

$$\limsup_{n\to\infty} \frac{1}{n} S_n(x) \le \lambda, \tag{1.21}$$

almost everywhere and in L^1-sense.

For the converse, we only need to consider the case $\lambda > -\infty$. Let us take a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ that is nondecreasing in each of its arguments. We certainly have

$$\begin{aligned} \int \varphi(S_1, \dots, S_n) d\mu &= \int \varphi(S_1 \circ T^k, \dots, S_n \circ T^k) d\mu \\ &\ge \int \varphi(S_{k+1} - S_k, S_{k+2} - S_k, \dots, S_{k+n} - S_k) d\mu \end{aligned}$$

for every k. Hence

$$\begin{aligned} \int \varphi(S_1, \dots, S_n) d\mu &\ge \frac{1}{N} \sum_{k=0}^{N-1} \int \varphi(S_{k+1} - S_k, \dots, S_{k+n} - S_k) d\mu \\ &= \int \varphi(S_{k+1} - S_k, \dots, S_{k+n} - S_k) d\mu \; \nu_N(dk) \end{aligned} \tag{1.22}$$

where $\nu_N = \frac{1}{N} \sum_0^{N-1} \delta_k$. We think of k as a random number that is chosen uniformly from 0 to $N-1$. To this end let us define $\Omega = \mathbb{R}^{\mathbb{Z}^+} = \{w : \mathbb{Z}^+ \to \mathbb{R}\}$ and $T : M \times \mathbb{N} \to \Omega$ such that $T(x,k) = w$ with $w(j) = S_{k+j}(x) - S_{k+j-1}(x)$. We then define a measure μ_N on Ω by $\mu_N(A) = (\mu \times \nu_N)(T^{-1}(A))$. Using this we can rewrite (1.22) as

$$\int \varphi(S_1, \dots, S_n) d\mu \ge \int \varphi(w(1), w(1) + w(2), \dots, w(1) + \dots + w(n)) \mu_N(dw). \tag{1.23}$$

We want to pass to the limit $N \to \infty$. Note that Ω is not a compact space. To show that

μ_N has a convergent subsequence, observe

$$\begin{aligned}
\int w(j)^+\mu_N(dw) &= \int (S_{k+j}(x) - S_{k+j-1}(x))^+\mu(dx)\nu_N(dx) \\
&= \frac{1}{N}\sum_0^{N-1}\int (S_{k+j}(x) - S_{k+j-1}(x))^+\mu(dx) \\
&\leq \frac{1}{N}\sum_0^{N-1}\int (S_1(T^{k+j-1}(x)))^+\mu(dx) = \int S_1^+ d\mu, \\
\int w(j)\mu_N(dw) &= \frac{1}{N}\sum_0^{N-1}\int (S_{k+j}(x) - S_{k+j-1}(x))\mu(dx) \\
&= \frac{1}{N}\int (S_{j+N-1} - S_{j-1})d\mu \geq \lambda\frac{j+N-1}{N} - \frac{1}{N}\int S_{j-1}d\mu \\
&> -\infty,
\end{aligned}$$

uniformly in N. As a result $\int w(j)^-\mu_N(dw)$ is uniformly bounded. Hence

$$\sup_N \int |w_j| d\mu_N = \beta_j < \infty$$

for every j. We now define

$$K_\delta = \left\{ w : |w_j| \leq \frac{2^{j+1}\beta_j}{\delta} \right\}.$$

The set K_δ is compact and

$$\mu_N(K_\delta^c) \leq \frac{1}{2}\sum_j 2^{-j}\beta_j^{-1}\delta\beta_j = \delta.$$

From this and Exercise 1.1**(iv)** we deduce that μ_N has a convergent subsequence. Let $\bar{\mu}$ be a limit point and set $\bar{S}_j = w(1) + \cdots + w(j)$. By (1.23),

$$\int \varphi(S_1, \dots, S_n)d\mu \geq \int \varphi(\bar{S}_1, \dots, \bar{S}_n)d\bar{\mu}, \tag{1.24}$$

for every continuous monotonically decreasing φ. We now define $\tau : \Omega \to \Omega$ by $(\tau w)(j) = w(j+1)$. It is not hard to see $\bar{\mu} \in \mathcal{I}_\tau$. By Ergodic Theorem, $\frac{1}{n}\bar{S}_n \to Z$ for almost all w. Moreover, $\int Z d\bar{\mu} = \int w(1)\bar{\mu}(dw) = \lim_{N\to\infty}\int \frac{1}{N}(S_N - S_0)d\mu = \lambda$. We use (1.24) to assert that for every bounded continuous increasing ψ,

$$\int \psi\left(\min_{k\leq n\leq l}\frac{S_n}{n}\right)d\mu \geq \int \psi\left(\min_{k\leq n\leq l}\frac{\bar{S}_n}{n}\right)d\bar{\mu}.$$

We now apply the bounded convergence theorem to deduce

$$\int \psi(\underline{S})d\mu \geq \int \psi(Z)d\bar{\mu}$$

where $\underline{S} = \liminf_{n\to\infty} \frac{S_n}{n}$. Choose $\psi(z) = \psi^{r,l}(z) = (zv(-l)) \wedge r$, $\psi_l(z) = zv(-l)$. We then have

$$\int \psi_l(\underline{S})d\mu \geq \int \psi^{r,l}(\underline{S})d\mu \geq \int \psi^{r,l}(Z)d\bar{\mu}.$$

After sending $r \to \infty$, we deduce

$$\int \psi_l(\underline{S})d\mu \geq \int Z d\bar{\mu} = \lambda, \text{ or} \tag{1.25}$$

$$\int (\psi_l(\underline{S}) - \lambda)d\mu \geq 0.$$

Recall $\underline{S} \leq \limsup \frac{S_n}{n} \leq \lambda$. But (1.25) means

$$\int_{\underline{S}\geq -l} (\underline{S} - \lambda)d\mu + (-l - \lambda)\mu\{\underline{S} \leq -l\} \geq 0.$$

Since $\lambda > -\infty$, we can choose l large enough to have $-l - \lambda < 0$. For such l, $\underline{S} - \lambda = 0$ on the set $\{\underline{S} \geq -l\}$. By sending $l \to +\infty$ we deduce $\underline{S} = \lambda$ almost everywhere, and this completes the proof. □

Exercises

(i) Show that the topology associated with (1.4) is metrizable with the metric given by

$$D(\mu,\nu) = \sum_{n=1}^{\infty} 2^{-n} \frac{\left|\int f_n d\mu - \int f_n d\nu\right|}{1 + \left|\int f_n d\mu - \int f_n d\nu\right|},$$

where $\{f_n : n \in N\}$ is a countable dense subset of $U_b(X)$.

(ii) Let $\mu_n \Rightarrow \mu$ and $\mu(\partial A) = 0$. Deduce that $\mu_n(A) \to \mu(A)$. (*Hint:* For such A, we can approximate the indicator function of A with continuous functions.)

(iii) Show that if X is a compact metric space, then $\mathcal{M}(X)$ is compact.

(iv) Let X be a Polish space. Suppose that $\{\mu_N\}$ is a sequence of probability measures on X. Assume that for every $\delta > 0$ there exists a compact set K_δ such that $\sup_N \mu_N(K_\delta^c) \leq \delta$. Show that $\{\mu_N\}$ has a convergent subsequence.

(v) Show that if X is compact and $f_n : X \to \mathbb{R}$ is a sequence of continuous functions, then $f_n \to f$ uniformly iff

$$x_n \to x \quad \Rightarrow \quad \lim_{n\to\infty} f_n(x_n) = f(x).$$

(vi) Assume that $\mu \in \mathcal{I}_T$. Let A be a measurable set with $\mu(A\Delta T^{-1}(A)) = 0$. Show that there exists a set $B \in \mathcal{F}_T$ such that $\mu(A\Delta B) = 0$.

(vii) Show that $\int |Pf| d\mu \le \int |f| d\mu$.

(viii) Show that the decimal expansion of 2^n may start with any finite combination of digits. (*Hint*: Use $T : \mathbb{T} \to \mathbb{T}$ defined by $T(x) = x + \alpha \pmod 1$ with $\alpha = \log_{10} 2$.)

(ix) In the case of an irrational rotation $T : \mathbb{T} \to \mathbb{T}$, $T(x) = x + \alpha \pmod 1$, show that the operators $\Phi_n = n^{-1}\big(I + U + \cdots + U^{n-1}\big)$ do not converge to the projection operator P strongly. More precisely,

$$\liminf_{n\to\infty} \sup_{\|f\|_{L^2}=1} \|\Phi_n(f) - Pf\|_{L^2} > 0.$$

(x) Show that if $\mu \in \mathcal{I}_T$ and $f \in L^p(\mu)$ for some $p \in [1,\infty)$, then the $\big\|\Phi_n(f) - Pf\big\|_{L^p(\mu)} \to 0$ as $n \to \infty$. (*Hint:* Approximate f by bounded functions and use Theorem 1.3.)

(xi) Let a be a periodic point for T of period ℓ. Show that $\mu = \frac{1}{\ell}\sum_{j=0}^{\ell-1} \delta_{T^j(x)}$ is not mixing if $\ell > 1$.

(xii) Show that if μ is mixing and $f \circ T = \lambda f$, then either $\lambda = 1$ or $f = 0$.

(xiii) Show that the Lebesgue measure ℓ is ergodic for $T(x_1, x_2) = (x_1 + \alpha, x_1 + x_2) \pmod 1$ iff α is irrational. Show that m is never mixing.

(xiv) Let $m > 1$ be a prime number and assume that $\mu \in \mathcal{I}_T^{er} \setminus \mathcal{I}_{T^m}^{er}$. Show that there exists a measurable set A such that $\mu(A) = m^{-1}$, the collection $\xi = \big\{A, T^{-1}(A), \dots, T^{-m+1}(A)\big\}$ is a partition of X, and the set of T^m invariant sets $\mathcal{F}_{T^m}$ is the σ-algebra generated by ξ.

(xv) Assume that $f \in L^2(\mu)$ with $\mu \in \mathcal{I}_T$. Let $\{a_n\}_{n\in\mathbb{N}^*}$ be a sequence of positive numbers with the following properties:

- The sequence $\{a_n\}_{n\in\mathbb{N}^*}$ is either non-decreasing or non-increasing.
- The sequence $\{a_n\}_{n\in\mathbb{N}^*}$ satisfies

$$\lim_{n\to\infty} a_n^{-1}(a_0 + \cdots + a_n) = \infty.$$

Show

$$\lim_{n\to\infty} \widehat{\Phi}_n(f) := \lim_{n\to\infty} \frac{a_0\ f + a_1\ f\circ T + \cdots + a_{n-1}\ f \circ T^{n-1}}{a_0 + a_1 + \cdots + a_{n-1}} = Pf,$$

in $L^2(\mu)$.

(xvi) Assume that $f \in L^2(\mu)$ with $\mu \in \mathcal{I}_T$. Let $\{a_n\}_{n\in\mathbb{N}^*}$ and $\widehat{\Phi}_n(f)$ be as in **(xv)**. prove the analog of Theorem 1.4 for the sequence $\{\widehat{\Phi}_n(f) :\ n \in \mathbb{N}^*\}$ when this sequence satisfies [BC] condition of Remark 1.5. □

2 Transfer Operator, Liouville Equation

In the previous section we encountered several examples of dynamical systems for which it was rather easy to find "nice" ergodic invariant measures. We also observed in the case of expanding map that the space of invariant measures is rather complex. One may say that the Lebesgue measure is the "nicest" invariant measure for an expanding map. Later in Section 3, we show how the Lebesgue measure stands out as the unique invariant measure of maximum entropy.

In general, it is not easy to find some natural invariant measure for our dynamical system. For example, if we have a system on a manifold with a Riemannian structure with a volume form, we may wonder whether or not such a system has an invariant measure that is absolutely continuous with respect to the volume form. To address and study this sort of questions in a systematic fashion, let us introduce an operator on measures that would give the evolutions of measures with respect to our dynamical system. This operator is simply the dual of the operator $Uf = f \circ T$, namely $U^* = T_\sharp$. Even though we have some general results regarding the spectrum of U, the corresponding questions for the operator $\mathcal{T}$ are far more complex. We can now cast the existence of an invariant measure with some properties as the existence of a fixed point of $T_\sharp$ with those properties. The operator $T_\sharp$ is called *Perron–Frobenious*, *Perron–Frobenious–Ruelle* or *Transfer Operator*, once an expression for it is derived when μ is absolutely continuous with respect to the volume form. To get a feel for the operator $T_\sharp$, let us state a Proposition and examine some examples.

Proposition 2.1 *Recall* $\Phi_n^* = n^{-1}\big(I + T_\sharp + \cdots + T_\sharp^{n-1}\big)$.

(i) $\Phi_n^*\delta_x = \mu_n^x$. *Moreover any limit point of* $\Phi_n^*\nu$ *is an invariant measure.*

(ii) *A measure* $\mu \in \mathcal{I}_T^{er}$ *iff* $\Phi_n^*\nu$ *converges to* μ *in high* n *limit, for every* $\nu << \mu$.

(iii) *A measure* μ *is a mixing invariant measure iff* $T_\sharp^n\nu$ *converges to* μ *in high* n *limit, for every* $\nu << \mu$.

The elementary proof of Proposition 2.1 is left as an exercise.

Example 2.1(i) $T : \mathbb{T}^d \to \mathbb{T}^d$, $T(x) = x + \alpha \pmod 1$. The operator $T_\sharp$ simply translates a measure for the amount α. We assume that the numbers $\alpha_1 \dots \alpha_d$, and 1 are rationally independent. We can study the asymptotic behavior of $T_\sharp^n\mu$ for a given μ. The sequence $\{T_\sharp^n\mu\}$ does not converge to any limit as $n \to \infty$. In fact the set of limit points of the sequence $\{T_\sharp^n\mu\}$ consists of all translates of μ. However

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} T_\sharp^j\mu = \ell, \tag{2.1}$$

where ℓ denotes the Lebesgue measure. The proof of (2.1) follows from the unique ergodicity of T that implies

$$\Phi_n(f) \to \int f d\lambda$$

uniformly for every continuous f. This implies

$$\lim_{n\to\infty} \int \Phi_n(f) d\mu = \lim_{n\to\infty} \int f \ d\left(\frac{1}{n}\sum_{j=0}^{n-1} T_\sharp^j \mu\right) = \int f \ d\lambda,$$

proving (2.1).

(ii) Let (X,d) be a complete metric space and suppose $T : X \to X$ is a *contraction*. In other words, there exists a constant $\alpha \in (0,1)$ such that $d(T(x),T(y)) \leq \alpha d(x,y)$. In this case T has a unique fix point $\bar{x}$ and $\lim_{n\to+\infty} T^n(x) = \bar{x}$ for every x (the convergence is locally uniform). As a consequence we learn that $\lim_{n\to\infty} T_\sharp^n \mu = \delta_{\bar{x}}$ for every measure $\mu \in \mathcal{M}(X)$. For example, if $X = \mathbb{R}$ and $T(x) = \alpha x$ with $\alpha \in (0,1)$, then $d\mu = \rho \ dx$ results in a sequence $T_\sharp^n \mu = \rho_n \ dx$ with

$$\rho_n(x) = \alpha^{-n} \rho\left(\frac{x}{\alpha^n}\right).$$

In other words, the measure μ under $T_\sharp$ becomes more concentrated about the origin.

(iii) Let $T : \mathbb{T} \to \mathbb{T}$ be the expansion $T(x) = 2x (\operatorname{mod} 1)$. If $d\mu = \rho \ dx$ and $T_\sharp^n \mu = \rho_n \ dx$, then $\rho_1(x) = \frac{1}{2}\left(\rho\left(\frac{x}{2}\right) + \rho\left(\frac{x+1}{2}\right)\right)$ and

$$\rho_n(x) = \frac{1}{2^n}\sum_{j=0}^{2^n-1} \rho\left(\frac{x}{2^n} + \frac{j}{2^n}\right).$$

From this, it is clear that if ρ is continuous, then $\lim_{n\to\infty} \rho_n(x) \equiv 1$. Indeed

$$\lim_{n\to\infty} \left|\rho_n(x) - \frac{1}{2^n}\sum_{j=0}^{2^n-1} \rho\left(\frac{j}{2^n}\right)\right| = \lim_{n\to\infty} \left|\frac{1}{2^n}\sum_{j=0}^{2^n-1} \left(\rho\left(\frac{x}{2^n} + \frac{j}{2^n}\right) - \rho\left(\frac{j}{2^n}\right)\right)\right| = 0,$$

$$\lim_{n\to\infty} \frac{1}{2^n}\sum_{j=0}^{2^n-1} \rho\left(\frac{j}{2^n}\right) = \int \rho dx = 1.$$

This can also be seen by looking at the Fourier expansion of ρ. For the following argument, we only need to assume that $\rho \in L^2[0,1]$. If

$$\rho(x) = \sum_n a_n e^{2\pi i n x},$$

then $a_0 = 1$ and

$$\rho_1(x) = \sum_k a_{2k} e^{2\pi i k x},$$

and by induction,

$$\rho_n(x) = \sum_k a_{2^n k} e^{2\pi i k x}.$$

As a result,

$$\int_0^1 |\rho_n(x) - 1|^2 dx = \sum_{k \neq 0} a_{2^n k}^2 \to 0.$$

□

There is a couple of things to learn from Example 2.1. First, when there is a contraction, the operator $T_\sharp$ makes measures more concentrated in small regions. Second, if there is an expansion then $T_\sharp$ has some smoothing effect. In hyperbolic systems we have both expansion and contraction. In some sense, if we have more contraction than the expansion, then it is plausible that there is a fractal set that attracts the orbits as $n \to \infty$. If this happens, then there exists no invariant measure that is absolutely continuous with respect to the volume measure. Later in this section, we will see an example of such phenomenon. As a result, to have an absolutely continuous invariant measure, we need to make sure that, in some sense, the expansion rates and the contraction rates are balanced out. Let us first derive a formula for $T_\sharp\mu$ when μ is absolutely continuous with respect to a volume form. As a warm up, first consider a transformation $T : \mathbb{T}^d \to \mathbb{T}^d$ that is smooth. We also assume that T is invertible with a smooth inverse, i.e., T is a diffeomorphism. We then consider $d\mu = \rho dx$. We have

$$\int_{\mathbb{T}^d} f \circ T \ \rho \ dx = \int_{\mathbb{T}^d} f \ \rho \circ T^{-1} |JT^{-1}| \ dy$$

where $JT^{-1} = \det DT^{-1}$. As a result, if $T_\sharp\mu = \hat{\rho} \ dx$, then $\hat{\rho} = |JT^{-1}|\rho \circ T^{-1} = \frac{\rho \circ T^{-1}}{|JT \circ T^{-1}|}$. This suggests defining

$$\mathcal{T}\rho(x) = \frac{\rho \circ T^{-1}}{|JT \circ T^{-1}|}, \tag{2.2}$$

regarding $\mathcal{T}$ as an operator acting on probability density functions. More generally, assume that X is a smooth manifold and T is C^∞. Let ω be a volume form (non-degenerate d-form where d is the dimension of X). Then $T^*\omega$, the pull-back of ω under T, is also a k-form and we define $JT(x)$ to be the unique number such that $T^*\omega_x = JT(x)\omega_{T(x)}$. More precisely, $T^*\omega_x(v_1 \dots v_k) = \omega_{T(x)}(DT(x)v_1, \dots, DT(x)v_k) = JT(x)\omega_{T(x)}(v_1 \dots v_k)$. We then have

$$\int_X (f \circ T)\rho \ \omega = \int_X f(\rho \circ T^{-1})|JT^{-1}| \ \omega.$$

Hence (2.2) holds in general.

If T is not invertible, one can show

$$\mathcal{T}\rho(x) = \sum_{y \in T^{-1}(\{x\})} \frac{\rho(y)}{|JT(y)|}. \tag{2.3}$$

The next proposition demonstrates how the existence of an absolutely continuous invariant measure forces a bound on the Jacobians.

Proposition 2.2 *Let X be a smooth manifold with a volume form ω. Let $T : X \to X$ be a diffeomorphism with $JT > 0$. The following statements are equivalent:*

- **(i)** *There exists $\mu = \rho\omega \in \mathcal{I}_T$ for a bounded uniformly positive ρ.*
- **(ii)** *The set $\{JT^n(x) : x \in X,\ n \in \mathbb{Z}\}$ is uniformly bounded.*

Proof **(i) $\Rightarrow$ (ii)** Observe

$$\mathcal{T}^n\rho = \frac{\rho \circ T^{-n}}{JT^n \circ T^{-n}},\ n \in \mathbb{N}.$$

Also, $\mathcal{T}^{-1}\rho = (\rho \circ T)JT$, and by induction

$$\begin{aligned}\mathcal{T}^{-n}\rho &= (\rho \circ T^n)JT^n \\ &= (\rho \circ T^n)JT^{-n} \circ T^n; \quad n \in \mathbb{N}.\end{aligned}$$

Hence

$$\mathcal{T}^n\rho = \frac{\rho \circ T^{-n}}{JT^n \circ T^{-n}}; \quad n \in \mathbb{Z}.$$

If $\rho\omega$ is invariant, then $\mathcal{T}^n\rho = \rho$ for all $n \in \mathbb{Z}$. As a result, $(JT^n \circ T^{-n})\rho = \rho \circ T^{-n}$, or

$$JT^n = \frac{\rho}{\rho \circ T^n}; \quad n \in \mathbb{Z}.$$

Now it is clear that if ρ is bounded and uniformly positive, then $\{JT^n(x) : n \in \mathbb{Z},\ x \in X\}$ is uniformly bounded.

(ii) $\Rightarrow$ (i) Suppose $\{JT^n(x) : n \in \mathbb{Z}$ and $x \in X\}$ is bounded and define

$$\rho(x) = \sup_{n \in \mathbb{Z}} JT^n(x).$$

We then have

$$\begin{aligned} JT(x)(\rho \circ T)(x) &= \sup_{n \in \mathbb{Z}} (JT^n) \circ T(x) JT(x) \\ &= \sup_{n \in \mathbb{Z}} J(T^n \circ T)(x) = \rho(x). \end{aligned}$$

Hence $\mathcal{T}\rho = \rho$. Evidently ρ is bounded. Moreover

$$1/\rho = \inf_n [1/JT^n(x)] = \inf_n JT^{-n} \circ T^n = \inf_n JT^n \circ T^{-n}$$

is uniformly bounded by assumption. □

Recall that expansions are harmless and have smoothing effect on $\mathcal{T}\rho$. As a test case, let us consider an expansion of $[0, 1]$ given by

$$T(x) = \begin{cases} T_1(x) & x \in [0, \theta_0) = I_1 \\ T_2(x) & x \in [\theta_0, 1] = I_2 \end{cases}$$

with T_1, T_2 smooth functions satisfying $|T_i'(x)| \geq \lambda$ for $x \in I_i$. We assume $\lambda > 1$ and that $T_i(I_i) = [0, 1]$. In this case

$$\mathcal{T}\rho(x) = \frac{\rho_1 \circ T_1^{-1}(x)}{T_1' \circ T_1^{-1}(x)} + \frac{\rho \circ T_2^{-1}(x)}{T_2' \circ T_2^{-1}(x)}. \tag{2.4}$$

Writing $\ell(dx) = dx$ for the Lebesgue measure, clearly $\mathcal{T} : L^1(\ell) \to L^1(\ell)$ is a linear operator such that

$$\rho \geq 0 \quad \Longrightarrow \quad \mathcal{T}\rho \geq 0, \qquad \int \mathcal{T}\rho \; dx = \int \rho \; dx,$$
$$\int \big|\mathcal{T}\rho_1 - \mathcal{T}\rho_2)\big| \; dx \leq \int \big|\rho_1 - \rho_2\big| \; dx.$$

Theorem 2.1 *If $T_1, T_2 \in C^2$, then there exists $\mu \in \mathcal{I}_T$ of the form $d\mu = \rho dx$ with ρ of finite variation.*

Proof Write $S_i = T_i^{-1}$ so that

$$\mathcal{T}\rho = (\rho \circ S_1)S_1' + (\rho \circ S_2)S_2'.$$

Using $S_i' \leq \frac{1}{\lambda}$, we learn

$$\int_0^1 |(\mathcal{T}\rho)'| dx \leq \lambda^{-1} \int_0^1 \mathcal{T}|\rho'| \; dx + \beta_0 \int_0^1 \mathcal{T}\rho \; dx,$$

where

$$\beta_0 = \max_x \max_{i \in \{1,2\}} \frac{|S_i''(x)|}{S_i'(x)}.$$

Hence

$$\int_0^1 |(\mathcal{T}\rho)'|dx \le \lambda^{-1}\int_0^1 |\rho'|dx + \beta_0.$$

By induction,

$$\int_0^1 |(\mathcal{T}^n\rho)'|dx \le \lambda^{-n}\int_0^1 |\rho'|dx + \beta_0\frac{1-\lambda^{-n}}{1-\lambda^{-1}}.$$

From this we learn that

$$\sup_n \|\mathcal{T}^n\rho\|_{BV} < \infty.$$

Hence $\mathcal{T}^n\rho$ has convergent subsequences in $L^1[0,1]$. But a limit point may not be an invariant density. To avoid this, let us observe that we also have

$$\sup_n \left\|\frac{1}{n}\sum_0^{n-1}\mathcal{T}^j\rho\right\|_{BV} < \infty.$$

Hence the sequence $\left\{\rho_n = n^{-1}\sum_0^{n-1}\mathcal{T}^j\rho\right\}_n$ has convergent subsequences by Helley Selection Theorem. If $\bar\rho$ is a limit point, then $\mathcal{T}\bar\rho = \bar\rho$ by Proposition 2.1. Also, for every periodic $\varphi \in C^1$,

$$\left|\int_0^1 \varphi'\bar\rho\ dx\right| = \lim_{n\to\infty}\left|\int_0^1 \varphi'\rho_n\ dx\right| \le \|\varphi\|_{L^\infty}\ \limsup_{n\to\infty}\|\rho_n\|_{BV} \le \frac{\beta_0}{1-\lambda^{-1}}\|J\|_{L^\infty}.$$

Hence $\bar\rho \in BV$. □

A review of the proof of Theorem 2.1 reveals that in fact

$$\mathcal{A}\mathcal{B}_a \subseteq \mathcal{B}_{a\lambda^{-1}+\beta_0},$$

where $\mathcal{B}_a$ denotes the space of probability densities ρ such that $\int_0^1 |\rho'|\ dx \le a$. In particular, if a is sufficiently large, then $a\lambda^{-1}+\beta_0 < a$ and $\mathcal{A}$ maps the set $\mathcal{B}_a$ to a strictly smaller subset of $\mathcal{B}_a$. From this, we may wonder whether $\mathcal{A}$ is a contraction with respect to a suitable metric on $\mathcal{B}_a$. Such a contraction for sure guarantees the existence of a fixed point and the convergence of $\mathcal{A}^n\rho$, and this is exactly what we are looking for. Instead of working on the space $\mathcal{B}_a$, we would rather work on smaller space which yields the convergence of $\mathcal{T}^n\rho$ even with respect to the uniform topology. Let us consider the following function space:

$$\mathcal{C}_a = \{e^g : |g(x)-g(y)| \le a|x-y| \text{ for } x,y\in[0,1]\}. \tag{2.5}$$

We note that $\rho \in \mathcal{C}_a \cup \{0\}$, iff $\rho \ge 0$ and for all $x, y \in [0,1]$,

$$\rho(x) \le \rho(y)e^{a|x-y|}.$$

Recall that $S_i = T_i^{-1}$ and $\beta_0 = \max_{i\in\{1,2\}}\max_x \frac{|S_i''(x)|}{S_i'(x)}$.

Lemma 2.1 *We have that* $\mathcal{T}\mathcal{C}_a \subseteq \mathcal{C}_{a\sigma}$*, whenever* $a \geq \frac{\beta_0}{\sigma - \lambda^{-1}}$ *and* $\sigma > \lambda^{-1}$.

Proof Let $\rho = e^g \in \mathcal{C}_a$. Then

$$\begin{aligned}
\mathcal{T}\rho(x) &= \sum_{i=1}^{2} \rho \circ S_i(x) S_i'(x) \\
&\leq \sum_{i=1}^{2} \rho \circ S_i(y) e^{a|S_i(x)-S_i(y)|} S_i'(x) \\
&= \sum_{i=1}^{2} \rho \circ S_i(y) e^{a|S_i(x)-S_i(y)|} S_i'(y) e^{\log(S_i'(x)) - \log(S_i'(y))} \\
&\leq \sum_{i=1}^{2} \rho \circ S_i(y) |S_i'(y)| e^{a\lambda^{-1}|x-y|} e^{\beta_0|x-y|} \\
&= \mathcal{T}\rho(y) e^{(a\lambda^{-1}+\beta_0)|x-y|}.
\end{aligned}$$

As a result, $\mathcal{T}\mathcal{C}_a \subseteq \mathcal{C}_{a\lambda^{-1}+\beta_0} \subseteq \mathcal{C}_{\sigma a}$. □

What we learn from Lemma 2.1 is that if $\sigma \in (\lambda^{-1}, 1]$, then we can find a function space $\mathcal{C}_a$ that is mapped into itself by $\mathcal{T}$. Note that indeed $\mathcal{C}_a$ is a cone in the sense that

$$\begin{cases} \text{if } \rho \in \mathcal{C}_a, & \text{then } \lambda\rho \in \mathcal{C}_a \text{ for } \lambda > 0, \\ \text{if } \rho_1, \rho_2 \in \mathcal{C}_a, & \text{then } \rho_1 + \rho_2 \in \mathcal{C}_a. \end{cases}$$

Define a partial order

$$\rho_1 \preccurlyeq \rho_2 \text{ iff } \rho_2 - \rho_1 \in \mathcal{C}_a \cup \{0\}. \tag{2.6}$$

In other words, $\rho_1 \preccurlyeq \rho_2$ iff $\rho_1 \leq \rho_2$ and

$$\rho_2(x) - \rho_1(x) \leq (\rho_2(y) - \rho_1(y)) e^{a|x-y|}, \ \ x, y \in [0, 1]. \tag{2.7}$$

Hilbert metric associated with our cone $\mathcal{C}_a$ is defined as

$$d_a(\rho_1, \rho_2) = \log(\beta_a(\rho_1, \rho_2)\beta_a(\rho_2, \rho_1)), \tag{2.8}$$

where $\beta_a(\rho_1, \rho_2) = \inf\{\lambda \geq 0 : \rho_2 \preccurlyeq \lambda\rho_1\}$. By convention, $\beta_a(\rho_1, \rho_2) = \infty$ if there exists no such λ. We certainly have

$$\begin{aligned}
d_a(\rho_1, \rho_2) &= \inf_{\alpha} \inf_{\beta} \left\{ \log \frac{\beta}{\alpha} : \alpha\rho_1 \preccurlyeq \rho_2 \preccurlyeq \beta\rho_1 \right\} \\
&= \inf_{\gamma} \{\gamma : \alpha\rho_1 \preccurlyeq \rho_2 \preccurlyeq e^{\gamma}\alpha\rho_1 \text{ for some } \alpha > 0\} \geq 0.
\end{aligned} \tag{2.9}$$

Lemma 2.2 **(i)** $\beta_a(\rho_1,\rho_2) = \sup\limits_{\substack{x,y\\ x\neq y}} \dfrac{e^{a|x-y|}\rho_2(y)-\rho_2(x)}{e^{a|x-y|}\rho_1(y)-\rho_1(x)} \geq \sup\limits_x \dfrac{\rho_2(x)}{\rho_1(x)}$.

(ii) *d_a is a quasi-metric with $d_a(\rho_1,\rho_2)=0$ iff $\rho_1=\lambda\rho_2$ for some $\lambda>0$.*

(iii) *If $a_1 \leq a_2$ then $d_{a_1}(\rho_1,\rho_2) \geq d_{a_2}(\rho_1,\rho_2)$ for $\rho_1,\rho_2 \in \mathcal{C}_{a_1}$.*

Proof **(i)** If $\rho_2 \preccurlyeq \lambda\rho_1$, then $\rho_2 \leq \lambda\rho_1$ and

$$\begin{aligned} -\rho_2(x)+\lambda\rho_1(x) &\leq e^{a|x-y|}(-\rho_2(y)+\lambda\rho_1(y)), \\ -\rho_2(x)+e^{a|x-y|}\rho_2(y) &\leq \lambda(-\rho_1(x)+e^{a|x-y|}\rho_1(y)). \end{aligned}$$

From this we deduce

$$\beta_a(\rho_1,\rho_2) = \max\left\{\sup_x \frac{\rho_2(x)}{\rho_1(x)},\ \sup_{x\neq y} \frac{e^{a|x-y|}\rho_2(y)-\rho_2(x)}{e^{a|x-y|}\rho_1(y)-\rho_1(x)}\right\}.$$

Note that if $\sup\limits_x \dfrac{\rho_2(x)}{\rho_1(x)} = \dfrac{\rho_2(\bar{x})}{\rho_1(\bar{x})}$, then

$$\frac{e^{a|x-\bar{x}|}\rho_2(\bar{x})-\rho_2(x)}{e^{a|x-\bar{x}|}\rho_1(\bar{x})-\rho_1(x)} = \frac{e^{a|x-\bar{x}|}\rho_1(\bar{x})\frac{\rho_2(\bar{x})}{\rho_1(\bar{x})}-\rho_1(x)\frac{\rho_2(x)}{\rho_1(x)}}{e^{a|x-\bar{x}|}\rho_1(\bar{x})-\rho_1(x)} \geq \frac{\rho_2(x)}{\rho_1(x)}.$$

This completes the proof of **(i)**

(ii) The triangle inequality is a consequence of the fact that if $\rho_2 \preccurlyeq \lambda_1\rho_1$ and $\rho_3 \preccurlyeq \lambda_2\rho_2$, then $\rho_3 \preccurlyeq \lambda_1\lambda_2\rho_1$.

(iii) First observe $\mathcal{C}_{a_1} \subseteq \mathcal{C}_{a_2}$. Hence $\rho_2 \preccurlyeq \lambda\rho_1$ in $\mathcal{C}_{a_1}$ implies the same inequality in $\mathcal{C}_{a_2}$.

□

Recall that we are searching for a fixed point for the operator $\mathcal{T}$. By Lemma 2.1, if $\sigma \in (\lambda^{-1},1)$ and $a > \frac{\beta_0}{\sigma-\lambda^{-1}}$, then $\mathcal{T}(\mathcal{C}_a) \subseteq \mathcal{C}_{a\sigma} \subseteq \mathcal{C}_a$. As our next step,, we show that $\mathcal{T}$ is a contraction on $\mathcal{C}_a$. But first let us demonstrate that in fact that the set $\mathcal{C}_{a\sigma}$ is a bounded subset of $\mathcal{C}_a$.

Lemma 2.3 *diam* $\mathcal{C}_{a\sigma} = \sup\limits_{\rho_1,\rho_2\in\mathcal{C}_{a\sigma}} d_a(\rho_1,\rho_2) \leq b := 2\log\dfrac{1+\sigma}{1-\sigma} + 2a\sigma$.

Proof From $\rho_2(x) \leq \rho_2(y)e^{-a\sigma|x-y|}$ and $\rho_1(x) \leq \rho_1(y)e^{a\sigma|x-y|}$ we deduce

$$\beta_a(\rho_1,\rho_2) \leq \sup_{x,y} \frac{e^{a|x-y|}-e^{-a\sigma|x-y|}}{e^{a|x-y|}-e^{a\sigma|x-y|}}\frac{\rho_2(y)}{\rho_1(y)}.$$

To calculate this, set $z = a|x - y|$. Then $z \geq 0$ and

$$\lim_{z \to 0} \frac{e^z - e^{-\sigma z}}{e^z - e^{\sigma z}} = \frac{1+\sigma}{1-\sigma}.$$

On the other hand,

$$\frac{e^z - e^{-\sigma z}}{e^z - e^{\sigma z}} \leq \frac{1+\sigma}{1-\sigma}$$

which is the consequence of the convexity of the exponential function;

$$e^{\sigma z} \leq \frac{2\sigma}{1+\sigma} e^z + \frac{1-\sigma}{1+\sigma} e^{-\sigma z}.$$

As a result,

$$\beta_a(\rho_1, \rho_2) \leq \frac{1+\sigma}{1-\sigma} \sup_y \frac{\rho_2(y)}{\rho_1(y)} \leq \frac{1+\sigma}{1-\sigma} \frac{\rho_2(y_0) e^{a\sigma/2}}{\rho_1(y_0) e^{-a\sigma/2}} = \frac{\rho_2(y_0)}{\rho_1(y_0)} e^{a\sigma} \frac{1+\sigma}{1-\sigma}$$

for $y_0 = \frac{1}{2}$. Hence

$$\beta_a(\rho_1, \rho_2) \beta_a(\rho_2, \rho_1) \leq \left(\frac{1+\sigma}{1-\sigma} \right)^2 e^{2a\sigma},$$

completing the proof of lemma. □

We are now ready to show that $\mathcal{T}$ is a contraction.

Lemma 2.4 *For every $\rho_1, \rho_2 \in \mathcal{C}_a$,*

$$d_a(\mathcal{T}\rho_1, \mathcal{T}\rho_2) \leq \tanh\left(\frac{b}{4}\right) d_a(\rho_1, \rho_2).$$

Proof By Lemma 2.3, $\operatorname{diam}(\mathcal{T}\mathcal{C}_a) \leq b$. As a consequence if $\beta\rho_1 \succcurlyeq \rho_2 \succcurlyeq \alpha\rho_1$, then

$$d_a(\mathcal{T}(\rho_2 - \alpha\rho_1),\ \mathcal{T}(\beta\rho_1 - \rho_2)) \leq b$$

for every $\rho_1, \rho_2 \in \mathcal{C}_a$ and $\alpha, \beta \geq 0$. This means that we can find two constants $\lambda_1, \lambda_2 \geq 0$ such that $\log \frac{\lambda_2}{\lambda_1} \leq b$ and

$$\frac{\beta + \alpha\lambda_2}{1+\lambda_2} \mathcal{T}\rho_1 \preccurlyeq \mathcal{T}\rho_2 \preccurlyeq \frac{\beta + \alpha\lambda_1}{1+\lambda_1} \mathcal{T}\rho_1.$$

As a result,

$$d_a(\mathcal{T}\rho_1, \mathcal{T}\rho_2) \leq \log \frac{\beta + \alpha\lambda_1}{1+\lambda_1} \frac{1+\lambda_2}{\beta + \alpha\lambda_2} = \log \frac{\frac{\beta}{\alpha} + \lambda_1}{\frac{\beta}{\alpha} + \lambda_2} + \log \frac{1+\lambda_2}{1+\lambda_1}.$$

Minimizing over β/α yields

$$\begin{aligned} d_a(\mathcal{T}\rho_1, \mathcal{T}\rho_2) &\leq \log \frac{e^{d_a(\rho_1,\rho_2)} + \lambda_1}{e^{d_a(\rho_1,\rho_2)} + \lambda_2} - \log \frac{1+\lambda_1}{1+\lambda_2} \\ &= \int_0^{d_a(\rho_1,\rho_2)} \frac{e^\theta(\lambda_2 - \lambda_1)}{(e^\theta + \lambda_1)(e^\theta + \lambda_2)} d\theta \leq d_a(\rho_1,\rho_2) \frac{\sqrt{\lambda_2} - \sqrt{\lambda_1}}{\sqrt{\lambda_2} + \sqrt{\lambda_1}} \end{aligned}$$

because $\max_{x\geq 1} \dfrac{x(\lambda_2 - \lambda_1)}{(x+\lambda_1)(x+\lambda_2)} = \dfrac{\sqrt{\lambda_2} - \sqrt{\lambda_1}}{\sqrt{\lambda_2} + \sqrt{\lambda_1}}$. Finally from $\log \frac{\lambda_2}{\lambda_1} \leq b$ we obtain

$$d_a(\mathcal{T}\rho_1, \mathcal{T}\rho_2) \leq d_a(\rho_1,\rho_2) \frac{e^{\frac{1}{2}b} - 1}{e^{\frac{1}{2}b} + 1} = d_a(\rho_1,\rho_2) \tanh\left(\frac{b}{4}\right).$$

□

This evidently gives us a contraction on $\mathcal{C}_a$ for any $a \geq \frac{\beta_0}{\sigma - \lambda^{-1}}$ provided that $\sigma \in (\lambda^{-1}, 1)$, because $\tanh\left(\frac{b}{4}\right) < 1$ always. We may minimize the rate of contraction $\tanh\left(\frac{b}{4}\right)$ by first choosing the best a, namely $a = \frac{\beta_0}{\sigma - \lambda^{-1}}$, and then minimizing b in σ as σ varies in $(\lambda^{-1}, 1)$. Our goal is to show that $\lim_{n\to\infty} \mathcal{T}^n \rho$ converges to a unique invariant density $\bar{\rho}$. For this, let us establish an inequality connecting $d_a(\rho_1, \rho_2)$ to $\|\rho_1 - \rho_2\|_{L^1}$.

Lemma 2.5 *For every $\rho_1, \rho_2 \in \mathcal{C}_a$, with $\int_0^1 \rho_1 \, dx = \int_0^1 \rho_2 \, dx = 1$, we have*

$$\int_0^1 |\rho_1 - \rho_2| \, dx \leq (e^{d_a(\rho_1,\rho_2)} - 1), \quad |\rho_1 - \rho_2| \leq (e^{d_a(\rho_1,\rho_2)} - 1)\rho_1.$$

Proof Let us write $d_a(\rho_1, \rho_2) = \log \frac{\beta}{\alpha}$ with $\alpha\rho_1 \preccurlyeq \rho_2 \preccurlyeq \beta\rho_1$. This in particular implies that $\alpha\rho_1 \leq \rho_2 \leq \beta\rho_1$. Integrating this over $[0,1]$ yields $\alpha \leq 1 \leq \beta$, which in turn implies that $\alpha\rho_1 \leq \rho_1 \leq \beta\rho_1$. As a result,

$$(\alpha - \beta)\rho_1 \leq \rho_2 - \rho_1 \leq (\beta - \alpha)\rho_1.$$

Thus

$$\begin{aligned} &|\rho_1 - \rho_2| \leq (\beta - \alpha)\rho_1 \leq (\beta/\alpha - 1)\rho_1 \leq \left(e^{d_a(\rho_1,\rho_2)} - 1\right)\rho_1, \\ &\int_0^1 |\rho_2 - \rho_1| \, dx \leq \beta - \alpha \leq \frac{\beta - \alpha}{\alpha} = \frac{\beta}{\alpha} - 1 = e^{d_a(\rho_1,\rho_2)} - 1. \end{aligned}$$

□

We are now ready to state and prove the main result of this section.

Theorem 2.2 *Let* $a = \frac{\beta_0}{\sigma - \lambda^{-1}}$ *and* $\sigma \in (\lambda^{-1}, 1)$. *Then for every* $\rho \in \mathcal{C}_a$ *with* $\int_0^1 \rho = 1$, $\lim_{n\to\infty} \mathcal{T}^n \rho = \bar{\rho}$ *exists uniformly and* $\bar{\rho}\ dx \in \mathcal{I}_T$ *with* $\bar{\rho} \in \mathcal{C}_{a\sigma}$. *Moreover, there exists a constant* $\bar{c}_1$ *such that*

$$\left| \int_0^1 f \circ T^n\ g\ dx - \int_0^1 g\ dx \int_0^1 f\bar{\rho}\ dx \right| \leq \bar{c}_1 \hat{\lambda}^n \|f\|_{L^1} (\|g\|_{L^1} + \|g'\|_{L^\infty}) \tag{2.10}$$

where $\hat{\lambda} = \tanh\left(\frac{b}{4}\right)$, $b = 2\log\frac{1+\sigma}{1-\sigma} + 2a\sigma$, $f \in L^1$, *and* g *is Lipschitz.*

An immediate consequence of Theorem 2.2 is the mixing property of $\bar{\rho}$ because we may choose $g = h\bar{\rho}/\int h\bar{\rho}$ to deduce

$$\lim_{n\to\infty} \int_0^1 f \circ T^n\ h\bar{\rho}\ dx = \int_0^1 f\bar{\rho}dx \int_0^1 h\bar{\rho}dx.$$

Proof of Theorem 2.2 We first show that if $\rho \in \mathcal{C}_a$, then $\mathcal{T}^n\rho$ converges to a function $\bar{\rho} \in \mathcal{C}_a$ in L^1-sense. Indeed

$$\begin{aligned} \|\mathcal{T}^{n+m}\rho - \mathcal{T}^n\rho\|_{L^1} &\leq \exp(d_a(\mathcal{T}^{n+m}\rho, \mathcal{T}^n\rho)) - 1 \\ &\leq \exp(\hat{\lambda}^{n-1} d_a(\mathcal{T}^{m+1}\rho, \mathcal{T}\rho)) - 1 \\ &\leq e^{\hat{\lambda}^{n-1}b} - 1 \leq \hat{\lambda}^{n-1} b e^{\hat{\lambda}^{n-1}b} \leq c_0 \hat{\lambda}^{n-1} \end{aligned}$$

for a constant c_0 that depends on b only. This implies that $\mathcal{T}^n\rho$ is Cauchy in L^1. Let $\bar{\rho} = \lim \rho_n$ where $\rho_n = \mathcal{T}^n\rho$. Since $\rho_n(x) \leq \rho_n(y)e^{a\sigma|x-y|}$ and $\rho_{n_k} \to \bar{\rho}$ a.e. for a subsequence, we deduce that $\bar{\rho}(x) \leq \bar{\rho}(y)e^{a\sigma|x-y|}$ for a.e. x and $y \in [0,1]$. By modifying $\bar{\rho}$ on a set of zero Lebesgue measure if necessary, we deduce that $\bar{\rho} \in \mathcal{C}_{\sigma a}$. Note that $\bar{\rho}$ is never zero, because if $\bar{\rho}(x_0) = 0$ for some x_0, then $\bar{\rho}(x) \leq \bar{\rho}(x_0)e^{a\sigma|x_0-x|}$ implies that $\bar{\rho}(x) = 0$ for every x. But $\int_0^1 \rho dx = 1$ implies that $\int_0^1 \bar{\rho} dx = 1$. So $\bar{\rho} > 0$, completing the proof of $\bar{\rho} \in \mathcal{C}_a$.

We now show that $\mathcal{T}^n\rho \to \bar{\rho}$ uniformly. Indeed from $\mathcal{T}^n\rho \to \bar{\rho}$ in L^1 we deduce that $\int f \circ T^n \rho dx \to \int f\bar{\rho}dx$ for every bounded f, which implies that $\mathcal{T}\bar{\rho} = \bar{\rho}$. Moreover

$$\begin{aligned} |\mathcal{T}^n\rho - \bar{\rho}| &= |\mathcal{T}^n\rho - \mathcal{T}^n\bar{\rho}| \leq (e^{d_a(\mathcal{T}^n\rho, \mathcal{T}^n\bar{\rho})} - 1)\mathcal{T}^n\bar{\rho} \\ &\leq (e^{\hat{\lambda}^{n-1} d_a(\mathcal{T}\rho, \mathcal{T}\bar{\rho})} - 1)\bar{\rho} \leq (e^{\hat{\lambda}^{n-1}b} - 1)\bar{\rho} \\ &\leq \hat{\lambda}^{n-1} b e^{\hat{\lambda}^{n-1}b} \bar{\rho} \leq c_0 \hat{\lambda}^n \bar{\rho} \end{aligned}$$

with c_0 depending on b only. From this we learn that

$$\|\mathcal{T}^n\rho - \bar{\rho}\|_{L^\infty} \leq c_0 \hat{\lambda}^n \|\bar{\rho}\|_{L^\infty},$$

for every $\rho \in \mathcal{C}_a$ with $\int_0^1 \rho dx = 1$.

We now turn to the proof of (2.13). Without loss of generality, we may assume that $g \geq 0$. Given such a function g, find $l > 0$ large enough so that $\rho = g + l\bar{\rho} \in \mathcal{C}_a$. Indeed, for $y > x$, we have that $\rho(y) \leq g(y) + l\bar{\rho}(x)\exp(a\sigma(y-x)) =: \exp(h(y))$. On the other hand

$$h'(y) = \frac{g'(y) + la\sigma\bar{\rho}(x)e^{a\sigma(y-x)}}{g(y) + l\bar{\rho}(x)e^{a\sigma(y-x)}} \leq \frac{\|g'\|_{L^\infty}}{l\bar{\rho}(x)} + \frac{la\sigma\bar{\rho}(x)e^{a\sigma(y-x)}}{l\bar{\rho}(x)e^{a\sigma(y-x)}} \leq \frac{\|g'\|_{L^\infty}}{\inf\bar{\rho}}\frac{1}{l} + a\sigma.$$

This is at most a if we choose

$$l = \frac{\|g'\|_{L^\infty}}{a(1-\sigma)\inf\bar{\rho}}.$$

Hence

$$\left\| \mathcal{T}^n \frac{g + l\bar{\rho}}{Z} - \bar{\rho} \right\|_{L^\infty} \leq c_0 \hat{\lambda}^n \|\bar{\rho}\|_{L^\infty}$$

where $Z = \int_0^1 (g + l\bar{\rho})dx$. Since $\mathcal{T}\bar{\rho} = \bar{\rho}$, we deduce

$$\begin{aligned} \left\| \frac{\mathcal{T}^n g}{Z} + \frac{l}{Z}\bar{\rho} - \bar{\rho} \right\|_{L^\infty} &\leq c_0 \hat{\lambda}^n \|\bar{\rho}\|_{L^\infty}, \\ \|\mathcal{T}^n g - (Z - l)\bar{\rho}\|_{L^\infty} &\leq c_0 \hat{\lambda}^n \|\bar{\rho}\|_{L^\infty} Z. \end{aligned}$$

Hence

$$\left\| \mathcal{A}^n g - \bar{\rho} \int_0^1 g\ dx \right\|_{L^\infty} \leq c_1 \hat{\lambda}^n \left[\int g\ dx + l \right] \leq c_2 \hat{\lambda}^n \left[\int g\ dx + \|g'\|_{L^\infty} \right].$$

From this, we can readily deduce (2.13). □

Example 2.2 Let

$$T(x) = \begin{cases} \frac{x}{1-x} & \text{for } x \in \left[0, \frac{1}{2}\right), \\ 2x - 1 & \text{for } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Note that for this example, the condition $|T'(x)| > 1$ is violated at a single point $x = 0$. It turns out T has no invariant measure which is absolutely continuous with respect to Lebesgue measure. We omit the proof and refer the reader to [LaYo]. □

As our next scenario, let us study an example of a 2-dimensional system that has expanding and contracting direction but there is no absolutely continuous invariant measure. As a toy model for such a phenomenon, we consider a (generalized) *baker's transformation*:

$$T : \mathbb{T}^2 \to \mathbb{T}^2,\ T(x_1, x_2) = \begin{cases} \left(\frac{x_1}{\alpha}, \beta x_2\right) & \text{if } 0 \leq x_1 < \alpha, \\ \left(\frac{x_1 - \alpha}{\beta}, \beta + \alpha x_2\right) & \text{if } \alpha \leq x_1 < 1. \end{cases}$$

with $\alpha, \beta > 0$ and $\alpha + \beta = 1$. Observe that if we project a T orbit onto the x-axis, we obtain an orbit of an expanding dynamical system associated with the map $S : \mathbb{T} \to \mathbb{T}$, that is given by

$$S(x_1) = \begin{cases} \frac{x_1}{\alpha} & \text{if } 0 \le x_1 < \alpha, \\ \frac{x_1 - \alpha}{\beta} & \text{if } \alpha \le x_1 < 1, \end{cases}$$

We can readily show that in fact the Lebesgue measure is an invariant measure for S. The same cannot be said about the projection onto the x_2-axis.

Note

$$|JT(x_1, x_2)| = \begin{cases} \frac{\beta}{\alpha} & \text{if } 0 \le x \le \alpha, \\ \frac{\alpha}{\beta} & \text{if } \alpha < x \le 1. \end{cases}$$

As we will see later, the transformation T does not have an absolutely continuous invariant measure unless $\alpha = \beta = \frac{1}{2}$. To analyze Perron–Frobenious operator, let us define

$$F(x_1, x_2) = F_\mu(x_1, x_2) = \mu([0, x_1] \times [0, x_2]).$$

If $\hat{F} = F_{\mathcal{T}\mu}$, then

$$\hat{F}(x_1, x_2) = \begin{cases} F(\alpha x_1, x_2/\beta) & \text{if } 0 \le x_2 < \beta, \\ F(\alpha x_1, 1) + F\left(\beta x_1 + \alpha, \frac{x_2 - \beta}{\alpha}\right) - F\left(\alpha, \frac{x_2 - \beta}{\alpha}\right) & \text{if } \beta \le x_2 < 1. \end{cases} \tag{2.11}$$

To see this, recall that $\hat{F}(x_1, x_2) = \mu(T^{-1}([0, x_1] \times [0, x_2]))$. Also

$$T^{-1}(x_1, x_2) = \begin{cases} \left(\alpha x_1, \frac{x_2}{\beta}\right) & \text{if } 0 \le x_2 < \beta, \\ \left(\alpha + \beta x_1, \frac{x_2 - \beta}{\alpha}\right) & \text{if } \beta \le x_2 < 1. \end{cases} \tag{2.12}$$

Note that T is discontinuous on $x_1 = \alpha$, and T^{-1}. Now if $0 \le x_2 \le \beta$, then $T^{-1}([0, x_1] \times [0, x_2]) = [0, \alpha x_1] \times \left[0, \frac{x_2}{\beta}\right]$ which implies that $\hat{F}(x_1, x_2) = F\left(\alpha x_1, \frac{x_2}{\beta}\right)$ in this case. On the other hand, if $\beta < x_2 \le 1$, then

$$\begin{aligned} T^{-1}([0, x_1] \times [0, x_2]) &= T^{-1}([0, x_1] \times [0, \beta]) \cup T^{-1}([0, x_1] \times [\beta, x_2]), \\ T^{-1}([0, x_1) \times [0, \beta]) &= [0, \alpha x_1] \times [0, 1], \\ T^{-1}([0, x_1] \times (\beta, x_2]) &= [\alpha, \alpha + \beta x_1] \times \left(0, \frac{x_1 - \beta}{\alpha}\right]. \end{aligned}$$

Clearly $\mu([0, \alpha x_1] \times [0, 1]) = F(\alpha x_1, 1)$. Moreover,

$$\begin{aligned} \mu\left([\alpha, \alpha + \beta x_1] \times \left(0, \frac{x_2 - \beta}{\alpha}\right]\right) &= F\left(\alpha + \beta x_1, \frac{x_2 - \beta}{\alpha}\right) \\ &\quad - \mu\left([0, \alpha) \times \left(0, \frac{x_2 - \beta}{\alpha}\right)\right) \\ &= F\left(\alpha + \beta x_1, \frac{x_2 - \beta}{\alpha}\right) - F\left(\alpha, \frac{x_2 - \beta}{\alpha}\right), \end{aligned}$$

completing the proof of (2.11). Since the expanding and contracting directions are the x_1 and x_2 axes, we may separate variable to solve the equation $\hat{\mathcal{T}}F := \hat{F} = F$. In other words, we search for a function $F(x_1,x_2) = F_1(x_1)F_2(x_2)$ such that $\hat{\mathcal{T}}F = F$. Since the Lebesgue measure is invariant for the map S, we may try $F_1(x_1) = x_1$. Substituting this in $\hat{\mathcal{T}}F$ yields $\hat{\mathcal{T}}F(x_1,x_2) = x_1\hat{F}_2(x_2)$ where

$$\mathcal{B}F_2 := \hat{F}_2(x_2) = \begin{cases} \alpha F_2\left(\frac{x_2}{\beta}\right) & 0 \le x_2 < \beta, \\ \alpha + \beta F_2\left(\frac{x_2-\beta}{\alpha}\right) & \beta \le x_2 < 1. \end{cases}$$

Here we are using $F_2(1) = 1$. We are now searching for F_2 such that $\mathcal{B}F_2 = F_2$. It turns out that this equation has a unique solution F_2 that has zero derivative almost everywhere. Hence our invariant measure $\bar{\mu} = \lambda_1 \times \lambda_2$ with λ_1 the Lebesgue measure and λ_2 a singular measure. One can show that the support of the measure λ_2 is of *fractal dimension* $\frac{\alpha\log\alpha+\beta\log\beta}{\alpha\log\beta+\beta\log\alpha} =: \Delta$. To explain this heuristically, let us first propose a definition for the *fractal dimension of a measure*. To motivate our definition, observe that if a measure μ is absolutely continuous with respect to the Lebesgue measure $\lambda = \lambda_1$, with $d\mu = fd\lambda_1$, then by a classical theorem of Lebesgue,

$$\lim_{\delta\to 0}(2\delta)^{-1}\mu(x-\delta, x+\delta) = f(x),$$

for almost all x. This means that if $I_n(x) = \left(x - n^{-1}, x + n^{-1}\right)$, then

$$\lim_{n\to\infty} \frac{\mu\big(I_n(x)\big)}{\lambda_1\big(I_n(x)\big)} = f(x),$$

for almost all x. In the support of μ, we can assert

$$0 < \lim_{n\to\infty} \frac{\mu\big(I_n(x)\big)}{\lambda_1\big(I_n(x)\big)} < \infty,$$

because $f(x) > 0$, μ-almost everywhere. We may say that a measure μ is of fractal dimension D if

$$0 < \limsup_{\delta\to 0} \frac{\mu\big(I_n(x)\big)}{\lambda_1\big(I_n(x)\big)^D} \le \liminf_{\delta\to 0} \frac{\mu\big(I_n(x)\big)}{\lambda_1\big(I_n(x)\big)^D} < \infty, \tag{2.13}$$

μ-almost everywhere. This requires that the fractal measure μ to be rather regular and if true, (2.13) is often hard to establish. To have a less ambitious definition, we may require

$$\lim_{n\to\infty} \frac{\log\mu(I_n(x))}{\log\lambda_1(I_n(x))} = D, \qquad \mu - a.e. \tag{2.14}$$

In fact, we may try to establish (2.14) for a sequence of intervals $I_n(x)$ with $x \in I_n(x)$, $\cap_n I_n(x) = \{x\}$. With this definition in mind, let us write A for the set of points x such that

$$\lim_{n\to\infty} \frac{\log \lambda_2(I_n(x))}{\log \lambda_1(I_n(x))} = \Delta. \tag{2.15}$$

We wish to show that $\lambda_2(A) = 1$. To construct I_n, let us first define a family of intervals $I_{a_1,\dots,a_n}$, with $a_1,\dots,a_n \in \{0,1\}$, so that $I_0 = [0,\beta)$, $I_1 = [\beta,1)$, and if $I_{a_1,\dots,a_n} = [p,q)$, then $I_{a_1,\dots,a_n,0} = [p, p+\beta(q-p))$, and $I_{a_1,\dots,a_n,1} = [p+\beta(q-p), q)$. By induction on n, it is not hard to show

$$\lambda_2(I_{a_1,\dots,a_n}) = \alpha^{L_n}\beta^{R_n}, \qquad \lambda_1(I_{a_1,\dots,a_n}) = \beta^{L_n}\alpha^{R_n}, \tag{2.16}$$

where L_n and R_n denote the number of 0 and 1 in the sequence $a_1,\dots,a_n$, respectively. Given x, we can find a sequence $\omega(x) = (a_1,\dots,a_n,\dots) \in \Omega = \{0,1\}^{\mathbb{N}}$, such that $x \in I_{a_1,\dots,a_n}$ for every n. The transformation $x \mapsto \omega(x)$ pushes forward the measure λ_2 to the product measure λ_2' such that each a_n is 0 with probability α. If $L_n(x)$ and $R_n(x)$ denote the number of 0 and 1 in $a_1,\dots,a_n$ with $\omega(x) = (a_1,\dots,a_n,\dots)$, then by Birkhoff Ergodic Theorem

$$\lambda_2\left(\left\{x : \lim_{n\to\infty} \frac{L_n(x)}{n} = \alpha,\quad \lim_{n\to\infty} \frac{R_n(x)}{n} = \beta\right\}\right) = 1.$$

From this and (2.16) we can readily deduce that $\lambda_2(A) = 1$. Note that the support of $\bar\mu$ is of dimension $1+\Delta$. Evidently $\Delta < 1$ unless $\alpha = \beta = \frac{1}{2}$.

What we have constructed is the *Sinai–Ruelle–Bowen* (SRB) measure $\bar\mu$ of our baker's transformation T. Note that this measure is absolutely continuous with respect to the expanding direction x_1-axis. A remarkable result of Sinai–Ruelle–Bowen asserts

$$\lim_{n\to\infty} \frac{1}{n} \sum_{0}^{n-1} f(T^j(x)) = \int f\, d\bar\mu$$

for almost all x with respect to the Lebesgue measure. This is different from Birkoff's ergodic theorem that only gives the convergence for $\bar\mu$-a.e. and $\bar\mu$ is singular with respect to the Lebesgue measure. We may define the SRB measure as the invariant measure of the *maximum metric entropy*. The metric entropy will be discussed thoroughly in Chapter 4.

We end this section with a discussion regarding the flow-analog of Perron–Frobenious equation. Given a flow ϕ_t associated with the ODE $\frac{dx}{dt} = b(x)$, let us define

$$T_t g = g \circ \phi_t.$$

This defines a group of transformations on the space of real-valued functions g. The dual of T_t acts on measures. More precisely, $T_t^*\mu$ is defined by

$$\int T_t f d\mu = \int f dT_t^* \mu,$$

or equivalently $T_t^*\mu(A) = \mu(\phi_t^{-1}A) = \mu(\phi_{-t}(A))$. The following theorem of *Liouville* gives an infinitesimal description of $T_t^*\mu$ when μ is absolutely continuous with respect to Lebesgue measure.

Theorem 2.3 *Suppose that there exists a differentiable function $\rho(x,t)$ such that $d(T_t^*\mu) = \rho(x,t)dx$. Then ρ satisfies the Liouville's equation*

$$\rho_t + div(\rho b) = 0.$$

Proof Let g be a differentiable function of compact support. We have

$$\begin{aligned} \int g(y)\rho(y,t+h)dy &= \int g(\phi_{t+h}(x))\rho(x,0)dx \\ &= \int g(\phi_h(\phi_t(x)))\rho(x,0)dx \\ &= \int g(\phi_h(y))\rho(y,t)dy \\ &= \int g(y+hb(y)+o(h))\rho(y,t)dy \\ &= \int g(y)\rho(y,t)dy + h\int \nabla\ g(y)\cdot b(y)\rho(y,t)dy \\ &\quad +o(h). \end{aligned}$$

This implies that $\frac{d}{dt}\int g(y)\rho(y,t)dy = \int b(y)\cdot\nabla g(y)\rho(y,t)dy$. After an integration by parts,

$$\frac{d}{dt}\int g(y)\rho(y,t)dy = \int g(\rho_t + \mathrm{div}(\rho b))dy.$$

Since g is arbitrary, we are done. $\square$

In particular a measure ρdx is invariant if

$$\mathrm{div}(\rho b) = 0,$$

or equivalently $\rho\nabla b + \rho$ div $f = 0$. The generalization of this to manifolds is straightforward. If $\mathcal{L}_b$ denotes the *Lie derivative* and f is the velocity of the flow, then $\rho\omega$ is invariant if and only if

$$\mathcal{L}_b\rho + \rho\ \mathrm{div}\ b = 0.$$

Exercises

(i) Prove Proposition 2.1.

(ii) Show that the generalized baker's transformation is reversible in the following sense: If $\Phi(x_1, x_2) = (1 - x_2, 1 - x_1)$ then $\Phi^2 =$ identity and $T^{-1} = \Phi T \Phi$.

(iii) Let T and Φ be as in **(ii)**. Show that if $\mu \in \mathcal{I}_T$, then $\Phi_\sharp \mu \in \mathcal{I}_{T^{-1}}$.

(iv) Let $T : (0, 1] \to (0, 1]$ by $T(x) = \left\{ \frac{1}{x} \right\}$ where $\{\cdot\}$ means the fractional part. Derive the corresponding Perron–Frobenious equation. Show that $\rho(x) = \frac{1}{\log 2} \frac{1}{1+x}$ is a fixed point for the corresponding Perron–Frobenious operator.

(v) Let $T : [0, 1] \to [0, 1]$ by $T(x) = 4x(1 - x)$. Derive the corresponding Perron–Frobenious equation and show that $\rho(x) = \pi^{-1}(x(1 - x))^{-1/2}$ is a fixed point.

(vi) Let $u(x, t) = T_t g(x) = g(\phi_t(x))$. Show that u satisfies $u_t = \mathcal{L} u$ where $\mathcal{L} u = b(x) \cdot \frac{\partial u}{\partial x}$.

(vii) Show that $\mu \in \mathcal{I}_\phi$ iff $\int \mathcal{L} g \, d\mu = 0$ for every $g \in C^1$ of compact support. □

Notes The proof of Theorem 2.2 was taken from [Li]. The example of the generalized baker's transformation was taken from [D].

3 Entropy

Roughly speaking, the entropy measures the exponential rate of increase in dynamical complexity as a system evolves in time. We will discuss two notions of entropy in this section, the *topological entropy* that was defined by Bowen, and (Kolmogorov–Sinai) *metric entropy* that was formulated by Kolmogorov.

We start with some some definition and elementary facts that will prepare us for the definition of the topological entropy.

Definition 3.1(i) Let (X, d) be a compact metric space and $T : X \to X$ be a continuous transformation. Define $B_d(x, r) = \{y : d(x, y) < r\}$, and

$$\begin{aligned} d_n(x,y) = d_n^T(x,y) &= \max\{d(x,y), d(T(x),T(y)), \dots, d(T^{n-1}(x), T^{n-1}(y))\}, \\ B^n(x,r) = B^n_{T,d}(x,r) &= B_{d_n}(x,r) = \{y : d_n(x,y) < r\} \\ &= B_d(x,r) \cap T^{-1}B_d\big(T(x),r\big) \cap \dots \cap T^{1-n}B_d\big(T^{n-1}(x),r\big). \end{aligned}$$

(ii) We define $S^n_{T,d}(r)$ as the smallest number k for which we can find a set A of cardinality k such that $X = \bigcup_{x\in A} B^n_{T,d}(x,r)$.

(iii) We define $\hat{S}^n_{T,d}(r)$ as the smallest number k for which we can find an open cover $\mathcal{O}$ of X cardinality k such that for every $A \in \mathcal{O}$, the diameter of the set A with respect to the metric d_n is at most $2r$.

(iv) We define $N^n_{T,d}(r)$ to be the maximal number of points in X with pairwise d_n-distances at least r. □

As the following Proposition indicates, the numbers $S^n_{T,d}(r)$, $\hat{S}^n_{T,d}(r)$, and $N^n_{T,d}(r)$ are closely related as $r \to \infty$.

Proposition 3.1 *We have*

$$N^n_{T,d}(2r) \le S^n_{T,d}(r) \le N^n_{T,d}(r), \tag{3.1}$$

$$S^n_{T,d}(2r) \le \hat{S}^n_{T,d}(r) \le S^n_{T,d}(r), \tag{3.2}$$

$$\hat{S}^{m+n}_{T,d}(r) \le \hat{S}^m_{T,d}(r)\hat{S}^n_{T,d}(r). \tag{3.3}$$

Proof The first inequality in (3.1) follows from the fact that no d_n-ball of radius r can contain two points that are $2r$-apart. The second inequality in (3.1) follows from the fact that if $N^n(r) = L$ and $\{x_1, \dots, x_L\}$ is a maximal set, then $X = \bigcup_{j=1}^L B_{d_n}(x_j, r)$.

The second inequality in (3.2) is obvious, and the first inequality is true because if $\mathcal{O}$ is an open cover of X such that for every $A \in \mathcal{O}$, the diameter of the set A with respect to the metric d_n is at most $2r$, then we can pick $a_A \in A$ and form a cover of the form

$$X = \cup_{A\in\mathcal{O}} B^n_{T,d}(a_A, 2r).$$

To show (3.3), take two collections of sets $\mathcal{O}$ and $\mathcal{O}'$ with $\mathcal{O}$ (respectively $\mathcal{O}'$) an open cover of X with respect to the metric d_m (respectively d_n) such that

$$A \in \mathcal{O} \implies diam_{d_m}(A) \leq 2r,$$
$$B \in \mathcal{O}' \implies diam_{d_n}(B) \leq 2r.$$

Then $\mathcal{O}''$ consisting of the sets of the form $A \cap T^{-m}(B)$ with $A \in \mathcal{O}$ and $B \in \mathcal{O}'$ is an open cover of X with respect to the metric d_{m+n}. Moreover $diam_{d_{m+n}}\big(A \cap T^{-m}(B)\big) \leq 2r$, follows from

$$d_{m+n}(x,y) = \max\big\{d_m(x,y), d_n\big(T^m(x), T^m(y)\big)\big\}.$$

□

As an immediate consequence of (3.3), we learn that the sequence $a_n = \log \hat{S}^n_{T,d}(r)$ is subadditive. The following standard fact guarantees the convergence of the sequence a_n/n as $n \to \infty$.

Lemma 3.1 *Let a_n be a sequence of numbers such that $a_{n+m} \leq a_n + a_m$. Then $\lim_{n\to\infty} \frac{1}{n} a_n = \inf_n \frac{a_n}{n}$.*

Proof Evidently $\liminf_{n\to\infty} \frac{a_n}{n} \geq \inf_n \frac{a_n}{n}$. On the other hand, if $n = \ell m + r$ with $m, \ell \in \mathbb{N}$, $r \in [0, m)$, then

$$a_n = a_{\ell m + r} \leq a_{\ell m} + a_r \leq \ell a_m + a_r,$$
$$\frac{a_n}{n} \leq \frac{\ell m}{n}\frac{a_m}{m} + \frac{a_r}{n}.$$

After sending $n \to \infty$, we obtain,

$$\limsup_{n\to\infty} \frac{a_n}{n} \leq \frac{a_m}{m}$$

for every $m \in \mathbb{Z}^+$. This completes the proof. □

The topological entropy is define so that "higher entropy" would mean "more orbits". But the number of orbits is usually uncountably infinite. Hence we fix a "resolution" r, so that we do not distinguish points that are of distance less than r. Hence $N^n(r)$ represents the number of distinguishable orbits of length n, and this number grows like $e^{nh_{\mathrm{top}}(T)}$. We are now ready to define our topological entropy.

Definition 3.2 We define

$$h_{\mathrm{top}}(T; d) = h_{\mathrm{top}}(T) = \lim_{r\to 0} \lim_{n\to\infty} \frac{1}{n} \log \hat{S}^n_{T,d}(r) = \sup_{r>0} \inf_n \frac{1}{n} \log \hat{S}^n_{T,d}(r).$$

As an immediate consequence of (3.1) and (3.2), we also have

$$\begin{aligned} h_{\text{top}}(T;d) &= \lim_{r\to 0}\limsup_{n\to\infty}\frac{1}{n}\log S^n_{T,d}(r) = \lim_{r\to 0}\liminf_{n\to\infty}\frac{1}{n}\log S^n_{T,d}(r), \\ &= \lim_{r\to 0}\limsup_{n\to\infty}\frac{1}{n}\log N^n_{T,d}(r) = \lim_{r\to 0}\liminf_{n\to\infty}\frac{1}{n}\log N^n_{T,d}(r). \end{aligned}$$

□

We will see below that $h_{\text{top}}(T;d)$ is independent of the choice of the metric and depends on the topology of the underlying space. Here are some properties of the topological entropy.

Proposition 3.2 **(i)** *If the metrics d and d' induce the same topology, then $h_{\text{top}}(T;d) = h_{\text{top}}(T;d')$.*

(ii) *If $F : X \to Y$ is a homeomorphism, $T : X \to X$, $S : Y \to Y$, and $S \circ F = F \circ T$, then $h_{\text{top}}(T) = h_{\text{top}}(S)$.*

(iii) *$h_{\text{top}}(T^n) = nh_{\text{top}}(T)$. Moreover, if T is a homeomorphism, then $h_{\text{top}}(T) = h_{\text{top}}(T^{-1})$.*

Proof(i) Set $\eta(\epsilon) = \min\{d'(x,y) : d(x,y) \geq \epsilon\}$. Then

$$d'(x,y) < \eta(\epsilon) \Rightarrow d(x,y) < \epsilon.$$

As a result, $\lim_{\epsilon\to 0}\eta(\epsilon) = 0$ and $B^n_{T,d'}(x,\eta(\epsilon)) \subseteq B^n_{T,d}(x,\epsilon)$. Hence $S^n_{T,d'}(\eta(\epsilon)) \geq S^n_{T,d}(\epsilon)$. Thus $h_{\text{top}}(T,d) \leq h_{\text{top}}(T,d')$.

(ii) Given a metric d on X, define a metric d' on Y by $d'(x,y) = d(F^{-1}(x), F^{-1}(y))$. Evidently $h_{\text{top}}(T;d) = h_{\text{top}}(S;d')$.

(iii) Evidently $B^{nk}_{T,d}(x,r) \subseteq B^k_{T^n,d}(x,r)$. Hence

$$S^{nk}_{T,d}(r) \geq S^k_{T^n,d}(r), \quad h_{\text{top}}(T^n) \leq nh_{\text{top}}(T).$$

For the converse, use the continuity of T to find a function $\eta : (0,\infty) \to (0,\infty)$ such that $\eta(r) \leq r$ and $B_d(x,\eta(r)) \subseteq B^n_{T,d}(x,r)$. Then $B^k_{T^n,d}(x,\eta(r)) \subseteq B^{kn}_{T,d}(x,r)$. This implies that $S^k_{T^n,d}(\eta(r)) \geq S^{kn}_{T,d}(r)$, which in turn implies

$$\frac{1}{k}\log S^k_{T^n,d}(\eta(r)) \geq n\frac{k-1}{k}\max_{(k-1)n\leq \ell\leq kn}\frac{1}{\ell}\log S^{\ell}_{T,d}(r).$$

From this, it is not hard to deduce that $h_{\text{top}}(T^n) \geq nh_{\text{top}}(T)$.

For $h_{\text{top}}(T^{-1}) = h_{\text{top}}(T)$, observe that $d^T_n(x,y) = d^{T^{-1}}_n\big(T^{n-1}(x), T^{n-1}(y)\big)$. This means that $T^{n-1}(B^n_{T,d}(x,r)) = B^n_{T^{-1},d}(T^{n-1}(x),r)$. Hence $X = \bigcup_{j=1}^k B^n_{T,d}(x_j,r)$ is equivalent to

$X=\bigcup_{j=1}^{k}B^n_{T^{-1},d}(T^{n-1}(x_j),r)$. From this we deduce $S^n_{T^{-1},d}(r)=S^n_{T,d}(r)$. This implies that $h_{\rm top}(T^{-1})=h_{\rm top}(T)$. □

Example 3.1(i) Let $T:\mathbb{T}^d\to\mathbb{T}^d$ be a translation. Since T is an isometry, $d_n(x,y)=d(x,y)$ for $d(x,y)=|x-y|$. Thus $S^n(r)$ is independent of n and $h_{\rm top}(T)=0$.

(ii) Let $X=\{0,1,\dots,m-1\}^{\mathbb{Z}}$. Given $\omega=(\omega_j:\ j\in\mathbb{Z})\in X$, define $(\tau\omega)_j=\omega_{j+1}$. Consider the metric

$$d(\omega,\omega')=\sum_{j\in\mathbb{Z}}\eta^{-|j|}|\omega_j-\omega'_j|,$$

with $\eta>1$. Fix $\alpha\in X$ and take any $\omega\in X$. Evidently

$$\sum_{|j|>\ell}\eta^{-|j|}|\alpha_j-\omega_j|\le 2(m-1)\sum_{r=\ell+1}^{\infty}\eta^{-r}=\frac{2(m-1)}{\eta^\ell(\eta-1)}.$$

Also, if $\omega_j\ne\alpha_j$ for some $j\in\{-\ell,\dots,\ell\}$, then

$$\sum_{|j|\le\ell}\eta^{-|j|}|\alpha_j-\omega_j|\ge\eta^{-\ell}.$$

Evidently d induces the product topology on X no matter what $\eta\in(1,\infty)$ we pick. Choose η large enough so that $\frac{2(m-1)}{\eta-1}<1$. For such a choice of η,

$$B_d\left(\alpha,\eta^{-\ell}\right)=\left\{\omega:\omega_j=\alpha_j\text{ for }j\in\{-\ell,\dots,\ell\}\right\}.$$

Since

$$\left\{\omega:d(\tau^i(\omega),\tau^i(\alpha))<\eta^{-\ell}\right\}=\left\{\omega:\ \omega_{j+i}=\alpha_{j+i}\text{ for }j\in\{-\ell,\dots,\ell\}\right\},$$

we deduce

$$B_{d_n}\left(\alpha,\eta^{-\ell}\right)=\left\{\omega:\omega_j=\alpha_j\text{ for }j\in\{-\ell,\dots,\ell+n-1\}\right\}.$$

Evidently every two d_n-balls of radius $\eta^{-\ell}$ are either identical or disjoint. As a result, $S^n_{\tau,d}\left(\eta^{-\ell}\right)=m^{2\ell+n}$. Thus

$$h_{\rm top}(T)=\lim_{\ell\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log m^{2\ell+n}=\log m.$$

(iii) Let $T_m:\mathbb{T}\to\mathbb{T}$ be the expansion map as in Example 1.5**(ii)**. From Part **(ii)** and Exercise 3.1**(i)**, we deduce that $h_{top}(T_m)\le\log m$. We will see later in Example 3.4 below that in fact $h_{top}(T_m)=\log m$.

(iv) Let (X,τ) be as in the Part **(ii)** and let $A=[a_{ij}]$ be an $m\times m$ matrix with $a_{ij}\in\{0,1\}$ for all $i,j\in\{0,1,\dots,m-1\}$. Set

$$X_A=\left\{\omega\in X:a_{\omega_i,\omega_{i+1}}=1\text{ for all }i\in\mathbb{Z}\right\}.$$

Evidently X_A is an invariant set and the restriction of τ to X_A gives a dynamical system. Write τ_A for this restriction. To have an irreducible situation, we assume that each row and column of A contains at least one 1 (if for example $a_{0j} = 0$ for all j, we may replace X with $\{1, 2, \dots, m-1\}^{\mathbb{Z}}$). For such A,

$$\begin{aligned}
S^n_{\tau_A,d}\left(\eta^{-\ell}\right) &= \#\text{ of balls of radius } \eta^{-\ell} \text{ with nonempty intersection with } X_A \\
&= \#\text{ of } (\alpha_{-\ell}, \dots, \alpha_{\ell+n-1}) \text{ with } a_{\alpha_i,\alpha_{i+1}} = 1 \text{ for } -\ell \le i < \ell + n - 1 \\
&= \sum_{r,s=0}^{m-1} \#\big\{(\alpha_{-\ell}, \dots, \alpha_{\ell+n-1}) : \ a_{\alpha_i,\alpha_{i+1}} = 1 \text{ for } -\ell \le i < \ell + n - 1 \\
&\qquad\qquad \text{and } \alpha_{-\ell} = r, \ \alpha_{\ell+n-1} = s\big\} \\
&= \sum_{r,s=0}^{m-1} a^{2n+\ell-1}_{r,s} = \|A^{2\ell+n-1}\|
\end{aligned}$$

where $a^k_{r,s}$ is the (r, s) entry of the matrix A^k, and $\|A\|$ denotes the norm of A, i.e., $\|A\| = \sum_{r,s} |a_{r,s}|$. We now claim

$$h_{\text{top}}(\tau_A) = \lim_{\ell\to\infty} \limsup_{n\to\infty} \frac{1}{n} \log \|A^{2\ell+n-1}\| = \log r(A),$$

where $r(A) = \max\big\{|\lambda| : \lambda \text{ an eigenvalue of } A\big\}$. To see this, first observe that if $Av = \lambda v$, then $A^k v = \lambda^k v$. Hence

$$|\lambda|^k \max_j |v_j| \le |\lambda|^k \sum_j |v_j| \le \sum_{i,j} |a^k_{i,j}||v_i| \le \|A^k\| \max_j |v_j|.$$

As a result, $\|A^k\| \ge |\lambda|^k$. This shows that $h_{\text{top}}(\tau_A) \ge \log r(A)$. For the converse, we choose a basis so that the off-diagonal entries in Jordan normal form of A become small. Using this we can show that $|Av| \le (r(A) + \delta)|v|$ which in turn implies that $|A^k v| \le (r(A) + \delta)^k |v|$. From this we deduce that $h_{\text{top}}(\tau_A) \le \log(r(A) + \delta)$. Finally send $\delta \to 0$ to deduce that $h_{\text{top}}(\tau_A) \le \log r(A)$. This completes the proof of $h_{\text{top}}(\tau_A) = \log r(A)$.

(v) Let $T : \mathbb{T}^2 \to \mathbb{T}^2$ is given by $T(x) = Bx \pmod 1$, where B is an integer-valued matrix with eigenvalues λ_1, λ_2 satisfying $|\lambda_2| < 1 < |\lambda_1| = |\lambda_2|^{-1}$. For the sake of definiteness, let us take $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ with eigenvalues $\lambda_1 = \frac{3+\sqrt{5}}{2}$, $\lambda_2 = \frac{3-\sqrt{5}}{2}$ and eigenvectors $v_1 = \begin{bmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ \frac{-\sqrt{5}-1}{2} \end{bmatrix}$. T is a contraction along v_2 and an expansion along v_1. We now draw the eigen lines from the origin and let them intersect several times to separate torus into disjoint rectangles. Let us write R_1 and R_2 for these rectangles and study $T(R_1)$ and $T(R_2)$. We set

$$T(R_1) \cap R_1 = Z_0 \cup Z_1, \quad T(R_1) \cap R_2 = Z_3, \quad R_1 = Z_0 \cup Z_1 \cup Z_2.$$

We then define Z_4 so that $R_2 = Z_3 \cup Z_4$. One can then show that $T(R_2) = Z_2 \cup Z_4$. We now define $X = \{0,1,2,3,4\}^{\mathbb{Z}}$ and $F : X_A \to \mathbb{T}^2$ with

$$A = [a_{ij}] = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix},$$

where $F(\omega) = x$ for $\{x\} = \bigcap_{n\in\mathbb{Z}} T^{-n}(Z_{\omega_n})$. In other words, $F(\omega) = x$ iff $T^n(x) \in Z_{\omega_n}$ for all $n \in \mathbb{Z}$. If τ_A denotes the shift on X_A, then we have $T \circ F = F \circ \tau_A$. Here we are using the fact that if $x \in Z_i$ and $T(x) \in Z_j$, then $a_{ij} = 1$. This also guarantees that $\bigcap_{n\in\mathbb{Z}} T^{-n}(Z_{\omega_n}) \neq \emptyset$. Also, since T is contracting in v_2-direction and T^{-1} is contracting in v_1-direction, then $\bigcap_{n\in\mathbb{Z}} T^{-n}(Z_{\omega_n})$ has at most one point. Clearly the transformation F is onto. However, h is not one-to-one. For example if $\bar{\alpha}$ denotes $\bar{\alpha} = (\omega_n : n \in \mathbb{Z})$ with $\omega_n = \alpha$ for all n, then $\bar{0}, \bar{1}, \bar{4} \in X_A$ (but not $\bar{2}$ and $\bar{3}$). Moreover $\tau_A(\bar{0}) = \bar{0}$, $\tau_A(\bar{1}) = \bar{1}$, $\tau_A(\bar{4}) = \bar{4}$. On the other hand the only x with $T(x) = x$ is $x = 0$. In fact $F(\bar{0}) = F(\bar{1}) = F(\bar{4})$ is equal to the origin. From $T \circ F = F \circ \hat{T}$, Exercise 3.1**(i)** and Example 3.1**(iv)** we conclude that $h_{\text{top}}(T) \leq h_{\text{top}}(\tau_A) = \log r(A)$. A straightforward calculation yields $r(A) = \lambda_1 = \frac{3+\sqrt{5}}{2}$. Later we discuss the metric entropy, and using the metric entropy of T we will show in Example 3.4 below that indeed $h_{\text{top}}(T) = \log \frac{3+\sqrt{5}}{2}$. □

The metric entropy is the measure-theoretic version of the topological entropy. As a preparation, we make a definition.

Definition 3.2 Let $T : X \to X$ be a measurable transformation and take $\mu \in \mathcal{I}_T$.

(i) A countable collection ξ of measurable subsets of X is called a *μ-partition* if $\mu(A \cap B) = 0$ for every two distinct $A, B \in \xi$, and $\mu\left(X \setminus \bigcup_{A\in\xi} A\right) = 0$. We also write $C_\xi(x)$ for the unique $A \in \xi$ such that $x \in A$. Note that $C_\xi(x)$ is well-defined for μ-almost all x.

(ii) If ξ and η are two μ-partition, then their *common refinement* $\xi \vee \eta$ is the partition

$$\xi \vee \eta = \{A \cap B : A \in \xi,\ B \in \eta,\ \mu(A \cap B) > 0\}.$$

Also, if ξ is a μ-partition, then we set

$$T^{-1}\xi = \{T^{-1}(A) : A \in \xi\},$$

which is also a μ-partition because $\mu \in \mathcal{I}_T$.

(iii) For $m < n$, we define

$$\xi^T(m,n) = \xi(m,n) = T^{-m}\xi \vee T^{-m-1}\xi \vee \cdots \vee T^{-n}\xi.$$

$\square$

As we discussed in the introduction, the metric entropy measures the exponential gain in the information. Imagine that we can distinguish two points x and y only if x and y belong to different elements of the partition ξ. Now if the orbits up to time $n-1$ are known, we can use them to distinguish more points. The partition $\xi^T(0,n-1)$ represents the accumulated information gained up to time $n-1$. Except for a set of zero μ-measure, each x belongs to a unique element

$$C_n(x) = C_\xi(x) \cap T^{-1}\big(C_\xi(T(x))\big) \cap \cdots \cap T^{1-n}\big(C_\xi(T^{n-1}(x))\big) \in \xi(0,n-1).$$

Let's have an example.

Example 3.2(i) Let (X,τ) be as in Example 1.1**(iii)**, with $E=\{0,1,\dots,m-1\}$. Choose $\xi = \big\{A_0,\dots,A_{m-1}\big\}$, with $A_i = \big\{\omega:\ \omega_1 = i\big\}$. Given $p=(p_0,\dots,p_{m-1})$, with $p_j \geq 0$, $\sum_j p_j = 1$, recall $\mu_p \in \mathcal{I}_\tau$ is the product measure with $\mu_p(A_i) = p_i$. Systems of the form $(E^{\mathbb{Z}},\tau,\mu_p)$ are known as *Bernoulli Shifts*.

We have

$$C_n(\alpha) = \big\{\omega:\ \omega_i = \alpha_i \ \text{ for } \ i=1,\dots,n\big\}.$$

We certainly have $\mu_p(C_n(\alpha)) = p_{\alpha_1}\dots p_{\alpha_n}$ and

$$\frac{1}{n}\log\mu_p(C_n(\alpha)) = \frac{1}{n}\sum_1^n \log p_{\alpha_j} = \frac{1}{n}\sum_0^{n-1}\log f(\tau^j(\alpha))$$

where $f(\alpha) = p_{\alpha_1}$. By the Ergodic Theorem,

$$\lim_{n\to\infty}\frac{1}{n}\log\mu_p(C_n(\alpha)) = \sum_0^{m-1} p_j\log p_j. \tag{3.4}$$

(ii) Let $(\mathbb{T},T_m)$ be the expansion map as in Example 1.5**(ii)**. Let

$$\xi = \left\{\left[\frac{j}{m},\frac{j+1}{m}\right) : j=0,\dots,m-1\right\}.$$

Then

$$\eta_n = \xi^T(0,n-1) = \big\{\big[\cdot a_1\dots a_n, \cdot a_1\dots a_n + m^{-n}\big):\ a_1\dots a_n \in \{0,1,\dots,m-1\}\big\}.$$

Given x, let $\cdot a_1a_2\dots a_n**\dots$ denote its base m expansion. Note that for points on the boundary of the intervals in η_n, we may have two distinct expansions. Since we have chosen closed-open intervals in ξ, we dismiss expansions which end with infinitely many m. In other words, between $.a_1 \dots a_k(m-1)(m-1)\dots$, with $a_k < m-1$ and $.a_1 \dots a_k' 00 \dots$ for $a_k' = a_k + 1$, we choose the latter. we have

$$C_{\eta_n}(x) = \left[\cdot a_1 \dots a_n, \cdot a_1 \dots a_n + m^{-n}\right).$$

For μ_p as in Example 1.5**(iii)**, we have $\mu_p(C_{\eta_n}(x)) = p_{a_1} \dots p_{a_n}$ and

$$\frac{1}{n} \log \mu_p(C_{\eta_n}(x)) = \frac{1}{n} \sum_1^n \log p_{a_j} = \frac{1}{n} \sum_0^{n-1} \log f(T^j(x))$$

where $f(\cdot a_1 a_2 \dots) = p_{a_1}$. By the Ergodic Theorem,

$$\lim_{n\to\infty} \frac{1}{n} \log \mu_p(C_{\eta_n}(x)) = \sum_0^{m-1} p_j \log p_j.$$

□

In general, since we are interested in the amount of information the partition $\eta_n = \xi(0, n-1)$ carries out, perhaps we should look at $\mu(C_n(x))$ where $C_n(x) = C_{\eta_n}(x)$. This is typically exponentially small in n. Motivated by Example 3.2, we make the following definition.

Definition 3.3(i) Let $\mu \in \mathcal{M}$ and ξ be a μ-partition. The *entropy* of ξ with respect to μ is defined by

$$H_\mu(\xi) = \int I_\xi(x)\ \mu(dx) = -\sum_{C\in\xi} \mu(C) \log \mu(C),$$

where $I_\xi(x) = -\log \mu(C_\xi(x))$.

(ii) Given two μ-partitions η and ξ, the *conditional entropy* of ξ, given η is defined by

$$H_\mu(\xi \mid \eta) = \int I_{\xi|\eta}\ d\mu = -\sum_{A\in\xi, B\in\eta} \mu(A\cap B) \log \frac{\mu(A\cap B)}{\mu(B)},$$

where

$$I_{\xi|\eta}(x) = -\log \mu(C_\xi(x) \mid C_\eta(x)) = -\log \frac{\mu(C_\xi(x) \cap C_\eta(x))}{\mu(C_\eta(x))}.$$

(iii) Given a dynamical system (X, T), an invariant measure $\mu \in \mathcal{I}_T$, and a μ-partition ξ, we define

$$h_\mu(T, \xi) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\xi(0, n-1)),$$

The existence of the limit follows from the subadditivity of $a_n = H_\mu(\xi(0, n-1))$, which is an immediate consequence of Proposition 3.2**(ii)** below.

(iv) The entropy of T with respect to $\mu \in \mathcal{I}_T$ is defined by

$$h_\mu(T) = \sup_\xi h(T, \xi),$$

where the supremum is over all finite μ-partitions.

(v) We write $\alpha \le \beta$ when β is a *refinement* of α. This means that for every $B \in \beta$, there exists a set $A \in \alpha$ such that $\mu(B \setminus A) = 0$. □

Proposition 3.3 *Let ξ, η, and γ be three μ-partitions.*

(i) *We have*

$$I_{\xi\vee\eta} = I_\eta + I_{\xi|\eta}, \qquad H_\mu(\xi \vee \eta) = H_\mu(\eta) + H_\mu(\xi \mid \eta).$$

More generally,

$$I_{(\xi\vee\eta)|\gamma} = I_{\eta|\gamma}(x) + I_{\xi|(\eta\vee\gamma)}, \qquad H_\mu(\xi \vee \eta \mid \gamma) = H_\mu(\eta \mid \gamma) + H_\mu(\xi \mid \eta \vee \gamma)$$

(ii) $H_\mu(\xi \mid \eta) \le H_\mu(\xi)$

(iii) $H_\mu(\xi \vee \eta) \le H_\mu(\xi) + H_\mu(\eta)$.

(iv) *We have $I_{T^{-1}\xi} = I_\xi \circ T$, and $I_{T^{-1}\xi|T^{-1}\eta} = I_{\xi|\eta} \circ T$. Moreover, $H_\mu(T^{-1}\xi) = H_\mu(\xi)$, and $H_\mu\big(T^{-1}\xi \mid T^{-1}\eta\big) = H_\mu(\xi \mid \eta)$.*

(v) *If $\eta \le \gamma$, then*

$$H_\mu(\eta) \le H_\mu(\gamma), \qquad H_\mu(\xi \mid \eta) \ge H_\mu(\xi \mid \gamma).$$

Proof(i) By definition,

$$\begin{aligned}
I_{(\xi\vee\eta)|\gamma}(x) &= -\log \frac{\mu\big(C_{\xi\vee\eta}(x) \cap C_\gamma(x)\big)}{\mu\big(C_\gamma(x)\big)} = -\log \frac{\mu\big(C_\xi(x) \cap C_\eta(x) \cap C_\gamma(x)\big)}{\mu\big(C_\gamma(x)\big)} \\
&= -\log \frac{\mu(C_\xi(x) \cap C_{\eta\vee\gamma}(x)}{\mu(C_\gamma(x))} = -\log \frac{\mu\big(C_\eta(x) \cap C_\gamma(x)\big)}{\mu\big(C_\gamma(x)\big)} - \log \frac{\mu\big(C_\xi(x) \cap C_{\eta\vee\gamma}(x)\big)}{\mu\big(C_{\eta\vee\gamma}(x)\big)} \\
&= I_{\eta|\gamma}(x) + I_{\xi|(\eta\vee\gamma)}(x).
\end{aligned}$$

(ii) Set $\phi(x) = x \log x$ and use the convexity of ϕ to assert

$$\begin{aligned}
\phi(\mu(A)) &= \phi\left(\sum_{B\in\eta} \mu(B) \frac{\mu(A \cap B)}{\mu(B)}\right) \le \sum_{B\in\eta} \mu(B) \phi\left(\frac{\mu(A \cap B)}{\mu(B)}\right) \\
&= \sum_{B\in\eta} \mu(A \cap B) \log \frac{\mu(A \cap B)}{\mu(B)}.
\end{aligned}$$

(iii) This follows from **(i)** and **(ii)**.

(iv) This part is an immediate consequence of the invariance $\mu(T^{-1}(A)) = \mu(A)$ for every $A \in \xi$.

(iv) We only prove the second inequality as the proof of the first inequality is similar. By definition,

$$H_\mu(\xi \mid \alpha) = -\sum_{A\in\alpha, C\in\xi} \mu(A\cap C)\log\frac{\mu(A\cap C)}{\mu(A)} = -\sum_{A\in\alpha, C\in\xi} \mu(A)\varphi\left(\frac{\mu(A\cap C)}{\mu(A)}\right).$$

Fix A and choose a family $J \subseteq \beta$, so that

$$\mu\left(A\Delta\big(\cup\{B : B\in J\}\big)\right) = 0.$$

Hence

$$\varphi\left(\frac{\mu(A\cap C)}{\mu(A)}\right) = \varphi\left(\sum_{B\in J}\frac{\mu(B)}{\mu(A)}\frac{\mu(C\cap B)}{\mu(B)}\right) \leq \sum_{B\in J}\frac{\mu(B)}{\mu(A)}\varphi\left(\frac{\mu(C\cap B)}{\mu(B)}\right).$$

From this we deduce $H_\mu(\xi \mid \alpha) \geq H_\mu(\xi \mid \beta)$. □

We now show that the limits (3.4) in Example 3.2 are always true if μ is ergodic.

Theorem 3.1 *(Shannon–McMillan–Breiman) If $\mu \in \mathcal{I}_T^{er}$, then*

$$\lim_{n\to\infty}\int\left|\frac{1}{n}\log\mu(C_n(x)) + h_\mu(T,\xi)\right| d\mu = 0. \tag{3.5}$$

Proof Recall $\xi(n,m) = T^{-n}\xi \vee T^{-n-1}\xi \vee \cdots \vee T^{-m}\xi$ whenever $n < m$. We have

$$I_{\xi(0,n-1)} = I_{\xi\vee\xi(1,n-1)} = I_{\xi(1,n-1)} + I_{\xi|\xi(1,n-1)} = I_{\xi(0,n-2)}\circ T + I_{\xi|\xi(1,n-1)},$$

because $C_{T^{-1}\eta}(x) = C_\eta(T(x))$. Applying this repeatedly, we obtain

$$\frac{1}{n}I_{\xi(0,n-1)} = \frac{1}{n}\left[I_{\xi|\xi(1,n-1)} + I_{\xi|\xi(1,n-2)}\circ T + \cdots + I_{\xi|\xi(1,2)}\circ T^{n-3} + I_{\xi|T^{-1}\xi}\circ T^{n-2} + I_\xi\circ T^{n-1}\right]$$

If it were not for the dependence of $I_{\xi|\xi(1,n-j)}$ on $n-j$, we could have used the Ergodic Theorem to finish the proof. However, if we can show that $\lim_{m\to\infty} I_{\xi|\xi(1,m)} = \hat{I}$ exists, say in $L^1(\mu)$-sense, then we are almost done because we can replace $I_{\xi|\xi(1,n-j)}$ with $\hat{I}$ with an error that is small in L^1-sense. We then apply the ergodic theorem to assert

$$\lim_{n\to\infty}\frac{1}{n}I_{\xi(0,n-1)} = \int \hat{I}\,d\mu.$$

Note that if we write $\mathcal{F}_\eta$ for the σ–algebra generated by η, then $\mu(C_\xi(x)|C_\eta(x))$ is nothing other than

$$\mu(C_\xi \mid C_\eta)(x) = \sum_{A\in\xi} \mu(A \mid \mathcal{F}_\eta)(x)1\!\!1_A(x),$$

i.e. the conditional expectation of the indicator function of the set C_ξ, given the σ-field $\mathcal{F}_\eta$. Hence, we simply have

$$\hat{I}(x) = -\log\left\{\lim_{n\to\infty}\sum_{A\in\xi}\mu(A \mid \xi(1,n))(x)1\!\!1_A(x)\right\} = -\sum_{A\in\xi}\log\left\{\lim_{n\to\infty}\mu(A \mid \xi(1,n))(x)\right\}1\!\!1_A(x).$$

This suggests studying $\lim_{n\to\infty}\mu(A \mid \xi(1,n))$. The existence and interpretation of the limit involve some probabilistic ideas. We may define $\mathcal{F}_{1,n}$ to be the σ-algebra generated by the partition $\xi(1,n)$. We then have $\mathcal{F}_{1,2} \subseteq \mathcal{F}_{1,3} \subseteq \dots$ and if $\mathcal{F}_{1,\infty}$ is the σ-algebra generated by all $\xi(1,n)$'s, then

$$\lim_{n\to\infty}\mu(A \mid \xi(1,n)) = \mu(A \mid \mathcal{F}_{1,\infty}), \tag{3.6}$$

μ-almost surely and in $L^1(\mu)$–sense. The right-hand side is the conditional measure of A given the σ-algebra $\mathcal{F}_{1,\infty}$. The proof of (3.6) follows the celebrated martingale convergence theorem. We refer the reader to any textbook on martingales for the almost sure convergence. For our purposes, we need something stronger, namely $\log\mu(A \mid \mathcal{F}_{1,n}) \to \log\mu(A \mid \mathcal{F}_{1,\infty})$ in $L^1(\mu)$. The proof of this will be carried out in Lemma 3.2 below. □

Lemma 3.2 *Let $\mathcal{F}_\infty$ be a σ-algebra, and let $\mathcal{F}_n$ be a family of σ-algebras with $\mathcal{F}_n \subseteq \mathcal{F}_\infty$, and $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ for all n. Then for any $A \in \mathcal{F}_\infty$,*

$$\int_A\left(\sup_n\left(-\log\mu(A \mid \mathcal{F}_{1,n})\right)\right)d\mu \le -\mu(A)\log\mu(A) + \mu(A), \tag{3.7}$$

$$\lim_{n\to\infty}\log\mu(A \mid \mathcal{F}_n) = \log\mu(A \mid \mathcal{F}_\infty). \tag{3.8}$$

Proof (3.9) is an immediate consequence of (3.6), (3.7), and the Lebesgue's dominated convergence. As for (3.7), pick $\ell > 0$, and define

$$A_n = \left\{x : \mu\left(A \mid \mathcal{F}_{1,n}\right)(x) < e^{-\ell},\ \ \mu\left(A \mid \mathcal{F}_{1,k}\right)(x) \ge e^{-\ell}\ \text{ for } k = 1,2,\dots,n-1\right\},$$

then $A_n \in \mathcal{F}_{1,n}$ and we can write

$$\begin{aligned}\mu\left\{x \in A : \sup_n\left(-\log\mu(A \mid \mathcal{F}_{1,n})(x)\right) > \ell\right\} &= \mu\left(A \cap \cup_{n=1}^{\infty}A_n\right) = \sum_1^\infty\mu(A\cap A_n)\\ &= \sum_1^\infty\int_{A_n}\mu(A \mid \mathcal{F}_{1,n})d\mu \le \sum_1^\infty\int_{A_n}e^{-\ell}d\mu = e^{-\ell}\sum_1^\infty\mu(A_n) \le e^{-\ell}.\end{aligned}$$

From this we deduce

$$\int_A \left(\sup_n \left(-\log \mu(A \mid \mathcal{F}_{1,n})\right)(x) \right) d\mu = \int_0^\infty \mu \left\{ x \in A : \sup_n \left(-\log \mu(A \mid \mathcal{F}_{1,n})(x)\right) > \ell \right\} \, d\ell$$
$$\leq \int_0^\infty \min\{\mu(A), e^{-\ell}\} d\ell = -\mu(A) \log \mu(A) + \mu(A).$$

This completes the proof of (3.7). □

Remark 3.1 The convergence $n^{-1}\mu\big(C_n(x)\big) \to h_\mu(T, \xi)$ is also true μ-a.e. This can be established with the aid of Corollary 1.2, when T is invertible. To see this, set

$$\eta(m, n) = T^m\xi \wedge T^{m+1}\xi \wedge \cdots \wedge T^n\xi,$$

and observe that since $\xi(m, n) = T^{-n}\eta(m, n)$, we can write

$$I_{\xi(0,n-1)} = I_{\eta(0,n-1)} \circ T^{n-1} = \left[I_{\xi|\eta(1,n-1)} + I_{\xi|\eta(1,n-2)} \circ T^{-1} + \cdots + I_\xi \circ T^{1-n}\right] \circ T^{n-1}$$
$$= I_{\xi|\eta(1,n-1)} \circ T^{n-1} + I_{\xi|\eta(1,n-2)} \circ T^{n-2} + \cdots + I_\xi.$$

On the other hand, since

$$\lim_{n\to\infty} I_{\xi|\eta(1,n-1)} = \tilde{I},$$

exists in $L^1(\mu)$, with

$$\tilde{I} = -\sum_{A\in\xi} \log \big(\mu\big(A \mid \mathcal{F}_\infty^+\big)\big) \, 1\!\!1_A,$$

where $\mathcal{F}_\infty^+$ denotes the s-algebra generated by all $T^i\xi$, $i \in \mathbb{N}$, we can apply Corollary 1.2 to establish the μ-a.e. convergence. □

The proof of Theorem 3.1 suggests an alternative formula for the entropy. In some sense $h_\mu(T, \xi)$ is the entropy of the "present" ξ relative to its "past" $\xi(1, \infty)$. To make this rigorous, first observe that by Proposition 3.2**(i)**,

$$H_\mu\big(\xi(0, n-1)\big) = \sum_{j=1}^{n-1} H_\mu\big(\xi \mid \xi(1, j)\big) \tag{3.9}$$

where $H_\mu\big(\xi \mid \xi(1, 1)\big)$ means $H_\mu(\xi)$. In fact we have

Proposition 3.4 $h_\mu(T, \xi) = \inf_n H_\mu\big(\xi \mid \xi(1, n)\big)$ *and the sequence* $H_\mu\big(\xi \mid \xi(1, n)\big)$ *is nondecreasing.*

Proof The monotonicity of the sequence $a_n = H_\mu\big(\xi \mid \xi(1,n)\big)$ follows from Proposition 3.2**(iv)**. We then use (3.9) to assert

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}H_\mu(\xi(0,n-1)) &= \lim_{n\to\infty}\frac{1}{n}\sum_{1}^{n-1}H_\mu(\xi \mid \xi(1,j)) \\ &= \lim_{n\to\infty}H_\mu(\xi \mid \xi(1,n)) = \inf_n H_\mu(\xi \mid \xi(1,n)).\end{aligned}$$

□

We continue with some basic properties of the entropy.

Proposition 3.5 **(i)** *$h_\mu(T^k) = kh_\mu(T)$ and if T is invertible, then $h_\mu(T) = h_\mu(T^{-1})$.*

(ii) *If $\theta \in [0,1]$, $\mu \perp \nu$ and $\mu, \nu \in \mathcal{I}_T$, then $h_{\theta\mu+(1-\theta)\nu}(T) = \theta h_\mu(T) + (1-\theta)h_\nu(T)$.*

Proof(i) We have

$$\frac{k}{nk}H_\mu\left(\bigvee_{0}^{nk-1}T^{-r}\xi\right) = \frac{1}{n}H_\mu\left(\bigvee_{j=0}^{n-1}(T^k)^{-j}(\xi \vee T^{-1}\xi \vee \cdots \vee T^{-k+1}\xi)\right).$$

Hence $kh_\mu(T,\xi) = h_\mu(T^k,\eta)$ where $\eta = \xi \vee T^{-1}\xi \vee \cdots \vee T^{-k+1}\xi$. Since $\eta \geq \xi$, we deduce that $kh_\mu(T) = h_\mu(T^k)$.

The claim $h_\mu(T^{-1}) = h_\mu(T)$ follows from the invariance of μ and the fact

$$\xi(0,n-1) = \xi \vee \cdots \vee T^{-n+1}\xi = T^{-n+1}(\xi \vee \cdots \vee T^{n-1}\xi).$$

(ii) Let A be such that $\mu(A) = 1$, $\nu(A) = 0$. Set $B = \bigcup_{m=0}^{\infty}\bigcap_{n\geq m}T^{-n}(A)$. We can readily show that $T^{-1}B = B$ and that $\mu(B) = 1$, $\nu(B) = 0$. Set $\beta = \{B, X\setminus B\}$ and given a partition ξ, define $\hat{\xi} = \xi \vee \beta$. If $\gamma = \theta\mu + (1-\theta)\nu$, then

$$H_\gamma(\hat{\xi}_n) = \theta H_\mu(\xi_n) + (1-\theta)H_\nu(\xi_n) - \theta\log\theta - (1-\theta)\log(1-\theta), \tag{3.10}$$

where $\xi_n = \xi(0,n-1)$ and $\hat{\xi}_n = \hat{\xi}(0,n-1)$. To see this, observe that if $C \in \hat{\xi}_n$ and $\phi(z) = z\log z$, then

$$\phi(\gamma(C)) = \begin{cases}\theta\mu(C)\log(\theta\mu(C)) & \text{if } C \subseteq B,\\ (1-\theta)\nu(C)\log((1-\theta)\nu(C)) & \text{if } C \subseteq X\setminus B.\end{cases}$$

This clearly implies (3.10). Hence,

$$h_\gamma(T,\hat{\xi}) = \theta h_\mu(T,\xi) + (1-\theta)h_\nu(T,\xi).$$

From this we deduce

$$h_\gamma(T) \le \theta h_\mu(T) + (1-\theta)h_\nu(T).$$

This and Exercise **(viii)** complete the proof. □

In practice, we would like to know whether $h_\mu(T) = h_\mu(T,\xi)$ for a partition ξ. In the next theorem, we provide a sufficient condition for this.

Theorem 3.2 **(i)** *Let ξ be a finite μ-partition and assume that the smallest σ-algebra consisting of $T^{-n}(C)$, $n \in \mathbb{N}$, $C \in \xi$, equals to the Borel σ-algebra. Then $h_\mu(T) = h_\mu(T,\xi)$.*

(ii) *If T is invertible, then in part* **(i)**, *we only need to assume that the smallest σ-algebra consisting of $T^n(C)$, $n \in \mathbb{Z}$, $C \in \xi$, equals to the Borel σ-algebra.*

As a preparation we prove an inequality.

Lemma 3.3 *For every pair of finite partitions η and ξ we have*

$$h_\mu(T,\xi) \le h_\mu(T,\eta) + H_\mu(\xi \mid \eta).$$

Proof Recall $\xi(m,n) = T^{-m}\xi \vee \cdots \vee T^{-n}\xi$. We certainly have

$$H_\mu(\xi(0,n-1)) \le H_\mu(\eta(0,n-1)) + H_\mu(\xi(0,n-1) \mid \eta(0,n-1)).$$

We are done if we can show that $H_\mu(\xi(0,n-1) \mid \eta(0,n-1)) \le nH_\mu(\xi \mid \eta)$. Indeed using Proposition 3.2**(i)**, we can assert

$$\begin{aligned} H_\mu(\xi(0,n-1) \mid \eta(0,n-1)) &\le H_\mu(\xi \mid \eta(0,n-1)) + H_\mu(\xi(1,n-1) \mid \eta(0,n-1) \vee \xi) \\ &\le H_\mu(\xi \mid \eta) + H_\mu(\xi(1,n-1) \mid \eta(1,n-1)) \\ &\le H_\mu(\xi \mid \eta) + H_\mu(T^{-1}\xi(0,n-2) \mid T^{-1}\eta(0,n-2)) \\ &= H_\mu(\xi \mid \eta) + H_\mu(\xi(0,n-2) \mid \eta(0,n-2)) \\ &\quad \dots \\ &\le nH_\mu(\xi \mid \eta). \end{aligned}$$

□

Proof of Theorem 3.2 We only give a proof for part **(i)**, because **(ii)** can be shown by verbatim argument.

For a given partition η, we apply Lemma 3.1 to assert

$$h_\mu(T,\eta) \le h_\mu(T, \xi \vee \cdots \vee T^{-n+1}\xi) + H_\mu(\eta \mid \xi \vee \cdots \vee T^{-n+1}\xi). \tag{3.11}$$

From the definition, it is not hard to see that indeed $h_\mu(T, \xi \vee \cdots \vee T^{-n+1}\xi) = h_\mu(T, \xi)$. From this and (3.11), it suffices to show that for every partition η,

$$\lim_{n\to\infty} H_\mu(\eta \mid \xi \vee \cdots \vee T^{-n+1}\xi) = 0. \tag{3.12}$$

To believe this, observe that if $\eta \le \alpha$, then $H_\mu(\eta \mid \alpha) = 0$ because

$$I_{\eta|\alpha}(x) = -\log \frac{\mu(C_\eta(x) \cap C_\alpha(x))}{\mu(C_\alpha(x))} = -\log \frac{\mu(C_\alpha(x))}{\mu(C_\alpha(x))} = 0.$$

Now if the σ-algebra generated by all $\xi_n = \xi \vee \cdots \vee T^{-n+1}\xi$, $n \in \mathbb{N}^*$ is the full σ-algebra, then $\eta \le \xi_n$ at least asymptotically. We may prove this by the Martingale Convergence Theorem. In fact if $\mathcal{F}_n$ is the σ-algebra generated by ξ_n, then

$$\begin{aligned} \mu(C_\eta(x) \mid C_{\xi_n}(x)) &= \sum_{A\in\eta} 1\!\!1_A(x)\mu(A \mid \mathcal{F}_n)(x) \\ &\to \sum_{A\in\eta} 1\!\!1_A(x)\mu(A \mid \mathcal{F}_\infty)(x) = \sum_{A\in\eta} 1\!\!1_A(x) 1\!\!1_A(x) = 1, \end{aligned}$$

μ-a.e. By Lemma 3.1, the convergence is also true in $L^1(\mu)$ sense because of the uniform integrability. the This and (3.6) imply that $H_\mu(\eta \mid \xi_n) = -\int \log \mu(C_\eta(x) \mid C_{\xi_n}(x))\mu(dx) \to 0$, which is simply (3.12). □

Example 3.3(i) Consider the dynamical system of Example 3.2. Let ξ be as in Example 3.2. The condition of Theorem 3.2 is satisfied for such ξ and we deduce

$$h_{\mu_p}(T) = -\sum_{0}^{m-1} p_j \log p_j.$$

(ii) Consider a translation $T(x) = x + \alpha \pmod 1$ in dimension 1. If $\alpha \in \mathbb{Q}$, then $T^m =$ identity for some $m \in \mathbb{N}$. This implies that $h_\mu(T) = \frac{1}{m} h_\mu(T^m) = 0$ where μ is the Lebesgue measure. If α is irrational, then set $\xi = \{[0, 1/2), [1/2, 1)\}$. By the denseness of $\{T^{-n}(a) : n \in \mathbb{N}\}$ for $a = 0$ and $1/2$, we deduce that ξ satisfies the condition of Theorem 3.2. As a result, $h_\mu(T) = h_\mu(T, \xi)$. On the other hand $\xi_n := \xi \vee \cdots \vee T^{-n+1}\xi$ consists of $2n$ elements. To see this, observe that if we already know that ξ_n has $2n$ elements, then as we go to ξ_{n+1}, we produce two more elements because $T^n(0)$ and $T^n(1/2)$ bisect exactly two intervals in ξ_n. From this and Exercise **(vi)**, $H_\mu\big(\xi(0, n-1)\big) \le \log(2n)$. As a result, $h_\mu(T, \xi) = 0$, which in turn implies that $h_\mu(T) = 0$. □

In fact we can show that the entropy of a translation is zero using the fact that the topological entropy of a translation zero. More generally we always have the following fundamental formula.

Theorem 3.3 *For any compact metric space X and continuous transformation T, we have*

$$h_{\text{top}}(T) = \sup_{\mu \in \mathcal{I}_T} h_\mu(T) = \sup_{\mu \in \mathcal{I}_T^{ex}} h_\mu(T). \tag{3.13}$$

Note that by the second equality in (3.13) is an immediate consequence of Proposition 3.4**(ii)**.

Motived by the thermodynamics formalism in statistical mechanics, we may formulated a variant of the variational problem (3.12) for which the maximizing measure if exists is a variant of the SRB measure; it may be regarded as the analog of *Gibbs Measures* with respect to the SRB measure. For this we need a variant of Definition 3.1:

Definition 3.4(i) Let (X, d) be a compact metric space and $T : X \to X$ and $f : X \to \mathbb{R}$ be two continuous functions. Define

$$S^n_{T,d}(r; f) = \min \left\{ \sum_{x \in A} e^{n\Phi_n(f)(x)} \ : \ X = \bigcup_{x \in A} B^n_{T,d}(x, r) \right\},$$
$$N^n_{T,d}(r; f) = \max \left\{ \sum_{x \in A} e^{n\Phi_n(f)(x)} \ : \ a, b \in A, a \neq b \implies d_n(a, b) > r \right\}.$$

(ii) Given a continuous *potential function* $f : X \to \mathbb{R}$, its *topological pressure* $P_{top}(f; T)$ is defined by

$$\begin{aligned} P_{\text{top}}(f; T) &= \lim_{r \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N^n_{T,d}(r; f) = \sup_{r > 0} \limsup_{n \to \infty} \frac{1}{n} \log N^n_{T,d}(r; f) \\ &= \lim_{r \to 0} \limsup_{n \to \infty} \frac{1}{n} \log S^n_{T,d}(r; f) = \sup_{r > 0} \limsup_{n \to \infty} \frac{1}{n} \log S^n_{T,d}(r; f). \end{aligned}$$

The third equality is an immediate consequence of Proposition 3.6 below. Evidently, $h_{top}(T) = P_{top}(0; T)$. □

Proposition 3.6 *For any continuous function $f : X \to \mathbb{R}$,*

$$N^n_{T,d}(2r; f) e^{-\omega(r)n} \le S^n_{T,d}(r; f) \le N^n_{T,d}(r; f), \tag{3.14}$$

where $\omega(\cdot)$ denotes the modulus of continuity of f. Moreover, $P_{top}\big(k\Phi_k(f); T^k\big) = kP_{top}(f; T)$ for every $k \in \mathbb{N}$.

Theorem 3.4 *For every continuous dynamical system (X, T), and continuous function f,*

$$P_{\text{top}}(f, T) = \sup_{\mu \in \mathcal{I}_T} \left(\int f \, d\mu + h_\mu(T) \right). \tag{3.15}$$

Proof *(Step 1)* Let $\xi = \{C_1, \dots, C_\ell\}$ be a μ-partition. Pick $\varepsilon > 0$, and choose compact sets $K_1, \dots, K_\ell$ with $K_j \subseteq C_j$ such that $\mu(C_j \backslash K_j) \le \epsilon$ for $j = 1, \dots, \ell$. Let $K_0 = X \backslash K_1 \cup \dots \cup K_\ell$ and put $\eta = \{K_0, K_1, \dots, K_\ell\}$. Evidently η is a partition and

$$\begin{aligned} H_\mu(\xi \mid \eta) &= -\sum_{i=1}^{\ell} \sum_{j=0}^{\ell} \mu(C_i \cap K_j) \log \frac{\mu(C_i \cap K_j)}{\mu(K_j)} = -\sum_{i=1}^{\ell} \mu(C_i \cap K_0) \log \frac{\mu(C_i \cap K_0)}{\mu(K_0)} \\ &= -\mu(K_0) \sum_{i=1}^{\ell} \frac{\mu(C_i \cap K_0)}{\mu(K_0)} \log \frac{\mu(C_i \cap K_0)}{\mu(K_0)} \le \mu(K_0) \log \ell \le \epsilon \ell \log \ell, \end{aligned}$$

by Exercise **(vi)**. From this and Lemma 3.2 we deduce,

$$h_\mu(T, \xi) \le h_\mu(T, \eta) + \varepsilon \ell \log \ell. \tag{3.16}$$

(Step 2) Set $\eta_n = \eta(0, n-1)$, and given $A \in \eta_n$, let

$$M_n(A) = \sup_{x \in A} \Phi_n(f)(x).$$

We certainly have,

$$\begin{aligned} \frac{1}{n} H_\mu(\eta_n) + \int f \, d\mu =& \frac{1}{n} H_\mu(\eta_n) + \int \Phi_n(f) \, d\mu \le \frac{1}{n} \sum_{A \in \eta_n} \big[M_n(A)\mu(A) - \mu(A) \log \mu(A) \big] \\ \le& \frac{1}{n} \log \sum_{A \in \eta_n} e^{M_n(A)}. \end{aligned} \tag{3.17}$$

To bound the right-hand side, set

$$r_0 = \frac{1}{2} \min \big\{ \operatorname{dist}(K_i, K_j) : i \ne j, \ i, j \in \{1, \dots, \ell\} \big\}.$$

and choose $r = r(\varepsilon) \in (0, r_0)$ so that

$$d(x, y) < r \quad \Longrightarrow \quad |f(x) - f(y)| < \varepsilon.$$

This in turn implies

$$d_n(x, y) < r \quad \Longrightarrow \quad \big| \Phi_n(f)(x) - \Phi_n(f)(y) \big| < \varepsilon. \tag{3.18}$$

Pick a set $E_n(r)$ such that

$$\cup_{x\in E_n(r)} B_{d_n}(x,r) = X.$$

Clearly a ball $B_d(x,r)$ intersects at most two elements of η, one K_j with $j \in \{1,\dots,n\}$ and perhaps K_0. We now argue that $B_{d_n}(x,r)$ intersects at most 2^n elements of η_n. To see this, observe

$$B_{d_n}(x,r) = B_d(x,r) \cap T^{-1}\big(B_d(T(x),r)\big) \cap \dots \cap T^{-n+1}\big(B_d(T^{n-1}(x),r)\big).$$

Also, if $A \in \eta_n$, then $A = A_0 \cap T^{-1}(A_1) \cap \dots \cap T^{-n+1}(A_{n-1})$ with $A_j \in \eta$. Now if $B^{d_n}(x,r) \cap A \neq \emptyset$, then $T^{-j}(B_d(T^j(x),r)) \cap T^{-j}(A_j) \neq \emptyset$ for $j = 0,\dots,n-1$. Hence $B_d(T^j(x),r) \cap A_j \neq \emptyset$ for $j = 0,\dots,n-1$. As a result, there are at most 2^n-many choices for A. Recall that we wish to bound $M_n(A)$ with $A \in \eta_n$. Since A is covered by Balls $\big\{B_{d_n}(x,r) :\ x \in E_n(r)|big\}$, we can find $x(A) \in E_n(r)$ such that

$$M_n(A) = M_n\big(B_{d_n}(x(A),r)\big), \qquad A \cap B_{d_n}(x(A),r) \neq \emptyset.$$

By (3.18),

$$M_n(A) = M_n\big(B_{d_n}(x(A),r)\big) \le \Phi_n(f)(x(A)) + \varepsilon, \tag{3.19}$$

and since $A \cap B_{d_n}(x(A),r) \neq \emptyset$,

$$\sharp\big\{A \in \eta_n :\ x(A) = x\big\} \le 2^n,$$

for every $x \in E_n(r)$. From this and (3.19) we deduce

$$\sum_{A\in\eta_n} e^{nM_n(A)} \le 2^n e^{n\varepsilon} \sum_{x\in E_n(r)} e^{n\Phi_n(f)(x)}.$$

From this, and (3.17) we learn

$$\frac{1}{n} H_\mu(\eta_n) + \int f\ d\mu \le \log 2 + \sup_{r>0} \frac{1}{n} \log S^n_{T,d}(r;f).$$

We now send $n \to \infty$ and use (3.16), to deduce

$$h_\mu(T,\xi) + \int f\ d\mu \le P_{top}(f;T) + \varepsilon\ell\log\ell + \varepsilon + \log 2.$$

Sending $\varepsilon \to 0$ yields

$$h_\mu(T,\xi) + \int f\ d\mu \le P_{top}(f;T) + \log 2. \tag{3.20}$$

(Step 3) Taking supremum over partition ξ and invariant measure μ in (3.20) yields

$$\sup_{\mu\in\mathcal{I}_T}\left(h_\mu(T)+\int f\ d\mu\right)\le P_{top}(f;T)+\log 2. \tag{3.21}$$

This is half of (3.15) except for the term $\log 2$ on the right-hand side. To get rid of $\log 2$, we replace T with T^m, and f with $m\Phi_m(f)$ in the equation (3.21):

$$\sup_{\mu\in\mathcal{I}_T}\left(h_\mu(T^m)+\int m\Phi_m(f)\ d\mu\right)\le P_{top}\big(m\Phi_m(f);T^m\big)+\log 2.$$

From this, Proposition 3.6, and Proposition 3.4**(i)** we learn

$$\sup_{\mu\in\mathcal{I}_T}\left(h_\mu(T)+\int f\ d\mu\right)\le P_{top}(f;T)+\frac{\log 2}{m}.$$

After sending m to infinity, we arrive at

$$\sup_{\mu\in\mathcal{I}_T}\left(h_\mu(T)+\int f\ d\mu\right)\le P_{top}(f;T). \tag{3.22}$$

(Step 4) On account of (3.22), it remains to show

$$P_{top}(f;T)\le\sup_{\mu\in\mathcal{I}_T}\left(h_\mu(T)+\int f\ d\mu\right). \tag{3.23}$$

For every $r>0$, we may select a finite set $E_n(r)$ such that

$$x,y\in E_n(r),\ x\neq y\qquad\Longrightarrow\qquad d_n(x,y)\ge r,$$

and

$$Z_n(r):=N^n_{T,d}(r,f)=\sum_{x\in E_n(r)}e^{n\Phi_n(f)(x)}.$$

To prove (3.23), it suffices to show that for every $r>0$, there exists a partition $\xi=\xi(r)$, and an invariant measure $\bar\mu=\bar\mu_{r,f}$, such that

$$\limsup_{n\to\infty}\frac{1}{n}\log Z_n(r)\le h_{\bar\mu}(T,\xi)+\int f\ d\bar\mu=h_{\bar\mu}(T,\xi)+\int\Phi_n(f)\ d\bar\mu. \tag{3.24}$$

To find the measure $\bar\mu$, we first define

$$\mu_n=\frac{1}{Z_n(r)}\sum_{x\in E_n}e^{n\Phi_n(f)(x)}\ \delta_x.$$

Now take a partition ξ, such that $\text{diam}_d(C) < r/2$ for every $C \in \xi$, where $\text{diam}_d(C)$ denotes the diameter of C with respect to the metric d. This implies that $\text{diam}_{d_n}(C) < r/2$ for every $C \in \xi_n = \xi \vee \dots \vee T^{1-n}\xi$. Hence,

$$x \in E_n(r),\ C \in \xi_n \qquad \Longrightarrow \qquad \sharp\big(E_n(r) \cap C\big) \in \{0,1\}.$$

As a result,

$$n^{-1} H_{\mu_n}(\xi_n) + \int \Phi_n(f)\ d\mu_n = n^{-1} \log Z_n(r). \tag{3.25}$$

(Step 5) From a comparison of (3.25) and (3.24), we are tempted to choose $\bar{\mu}$ any limit point of the sequence $\{\mu_n\}_{n \in \mathbb{N}}$. This would not work because such a limit point may not be an invariant measure. Moreover, the function $H_\mu(\eta)$ is not a continuous function of μ with respect to the weak topology. To treat the former issue, we define

$$\hat{\mu}_n = \ \frac{1}{n} \sum_0^{n-1} \mathcal{T}^j \mu_n,$$

where $\mathcal{T}\nu = T_\sharp \nu$ as before. Equivalently

$$\int h\ d\hat{\mu}_n = \frac{1}{n} \sum_0^{n-1} \int h(T^j(x)) \mu_n(dx) = \frac{1}{n Z_n(r)} \sum_0^{n-1} \sum_{x \in E_n} h(T^j(x))\ e^{n\Phi_n(f)(x)},$$

for any continuous function h. Let us choose an increasing subsequence $n_i \to \infty$ such that

$$\limsup_{n \to \infty} \frac{1}{n} \log Z_n(r) = \lim_{i \to \infty} \frac{1}{n_i} \log Z_{n_i}(r), \qquad \text{and} \qquad \lim_{i \to \infty} \hat{\mu}_{n_i} =: \bar{\mu} \ \text{ exists.}$$

It is not hard to show that $\bar{\mu} \in \mathcal{I}_T$ because

$$\mathcal{A}\bar{\mu} - \bar{\mu} = \lim_{n \to \infty} (\mathcal{A}^n \mu_n - \mu_n)/n = 0.$$

Pick a partition ξ such that $\text{diam}(C) < r$ for every $C \in \xi$. We wish to use (3.25) to deduce (3.24). To achieve this, pick k and m such that $0 \le k < m < n = n_i$ and set $a(k) = \left[\frac{n-k}{m}\right]$ so that we can write

$$\{0, 1, \dots, n-1\} = \{k + tm + i : 0 \le t < a(k),\ 0 \le i < m\} \cup R$$

with $R = \{0, 1, \dots, k-1\} \cup \{k + ma(k),\ k + ma(k) + 1, \dots, n-1\} =: R_1 \cup R_2$. Clearly $\# R_1 \le m$, $\# R_2 \le m$. We then write

$$\xi_n = \bigvee_{t=0}^{a(k)-1} T^{-(tm+k)}(\xi \vee \dots \vee T^{-m+1}\xi) \vee \bigvee_{i \in R} T^{-i}\xi.$$

From Proposition 3.3**(ii)** and Exercise **(iv)** below we learn,

$$\begin{aligned}
H_{\mu_n}(\xi_n) &\le \sum_{t=0}^{a(k)-1} H_{\mu_n}(T^{-(tm+k)}\xi_m) + \sum_{i\in R} H_{\mu_n}(T^{-i}\xi)\\
&= \sum_{t=0}^{a(k)-1} H_{\mathcal{T}^{tm+k}\mu_n}(\xi_m) + \sum_{i\in R} H_{\mu_n}(T^{-i}\xi)\\
&\le \sum_{t=0}^{a(k)-1} H_{\mathcal{T}^{tm+k}\mu_n}(\xi_m) + 2m\log(\#\xi).
\end{aligned}$$

This is true for every k. Hence

$$\begin{aligned}
mH_{\mu_n}(\xi_n) &\le \sum_{k=0}^{m-1}\sum_{t=0}^{a(k)-1} H_{\mathcal{T}^{tm+k}\mu_n}(\xi_m) + 2m^2\log(\#\xi)\\
&\le \sum_{j=0}^{n-1} H_{\mathcal{T}^{j}\mu_n}(\xi_m) + 2m^2\log(\#\xi)\\
&\le nH_{\hat\mu_n}(\xi_m) + 2m^2\log(\#\xi),
\end{aligned}$$

where for the last inequality we used Exercise **(viii)** below. As a result,

$$\frac{1}{n}H_{\mu_n}(\xi_n) \le \frac{1}{m}H_{\hat\mu_n}(\xi_m) + 2\frac{m}{n}\log(\#\xi).$$

From this and

$$\int \Phi_n(f)\ d\mu_n = \int f\ d\hat\mu_n,$$

we learn

$$\frac{1}{n}\log Z_n(r) \le \frac{1}{m}H_{\hat\mu_n}(\xi_m) + \int f\ d\hat\mu_n + 2\frac{m}{n}\log(\#\xi).$$

We now send $n = n_i$ to infinity to deduce

$$\limsup_{n\to\infty}\frac{1}{n}\log Z_n(r) = \lim_{j\to\infty}\frac{1}{n_j}\log Z_{n_j}(r) \le \frac{1}{m}H_{\bar\mu}(\xi_m) + \int f\ d\bar\mu,$$

provided that we have

$$\lim_{i\to\infty} H_{\hat\mu_{n_i}}(\xi_m) = H_{\bar\mu}(\xi_m), \tag{3.26}$$

for every m. We now send m to infinity to deduce (3.24). This completes the proof provided that (3.26) holds.

(Step 6) It remains to verify (3.26). For this we need to be more selective about the partition ξ. We first would like to find a partition $\xi = \{C_1 \dots C_\ell\}$ such that $\text{diam}(C_j) \le \delta := r/2$ for $j = 1, \dots, \ell$, and $\bar{\mu}(\partial C_j) = 0$ where ∂C_j denotes the boundary of C_j. The construction of such a partition ξ is straightforward. First, if $B_d(x, a)$ is a ball of radius a, then we consider

$$\bigcup \big\{\partial B_d(x, a') : a - \epsilon \le a' \le a\big\},$$

to observe that there exists $a' \in (a - \epsilon, a)$ such that $\bar{\mu}(\partial B_d(x, a')) = 0$. From this, we learn that we can cover X by finitely many balls B_j, $j = 1, \dots, \ell$ of radius at most $\frac{\delta}{2}$ such that $\bar{\mu}(\partial B_j) = 0$ for $j = 1, \dots, \ell$. We finally define $\xi = \{C_1 \dots C_\ell\}$ by $C_1 = \bar{B}_1$, $C_2 = \bar{B}_2 \setminus \bar{B}_1, \dots, C_n = \bar{B}_n \setminus \bigcup_{j=1}^{n-1} \bar{B}_j$. Since $\partial C_j \subseteq \bigcup_{k=1}^{\ell} \partial B_k$, we are done. We now argue that the partition $\xi_n = \xi \vee \dots \vee T^{-n+1}\xi$ enjoys the same property; $\bar{\mu}(\partial C) = 0$ if $C \in \xi_n$. This is because $\partial C \subseteq \bigcup_{A \in \xi} \bigcup_{k=0}^{n-1} T^{-j}(\partial A)$ and by invariance, $\bar{\mu}(T^{-j}(\partial A)) = \bar{\mu}(\partial A) = 0$. For such a partition we have (3.26) because by Exercise **(ii)** in Chapter 1, $\mu_n(A) \to \bar{\mu}(A)$ for every $A \in \xi_m$. □

Remark 3.2(i) Recall that $\mu \mapsto h_\mu(T)$ is concave (Exercise **(vii)**), and that $f \mapsto P_{top}(f; T)$ is convex. Our Theorem 3.4 establishes a conjugacy between the entropy and the pressure. If $h_\mu(T)$ is also upper semi-continuous, then we also have:

$$h_\mu(T) = \inf_{f \in C(X)} \left(P_{top}(f; T) - \int f \, d\mu \right). \tag{3.27}$$

The upper semi-continuity also guarantees the existence of a maximizer in (3.15). According to a result of Griffith and Ruelle, there exists a unique maximizer in (3.15) in the case of Example 3.1**(i)** (provided that A^k has positive entries for some $k \in \mathbb{N}$).

(ii) If we set

$$I(\mu) = h_{top}(T) - h_\mu(T) = P_{top}(0; T) - h_\mu(T), \quad \hat{P}_{top}(f; T) = P_{top}(f; T) - P_{top}(0; T),$$

then I is convex, and (3.15) and (**??**) can be rewritten as

$$\hat{P}_{top}(f; T) = \sup_{\mu \in \mathcal{I}_T} \left(\int f \, d\mu - I(\mu) \right), \quad I(\mu) = \sup_{f \in C(X)} \left(\int f \, d\mu - \hat{P}_{top}(f; T) \right). \tag{3.28}$$

In the case of the dynamical system $\big(E^{\mathbb{Z}}, \tau\big)$, the functional I serves as the *large deviation rate function* as was demonstrated by Donskar and Varadhan.

Assume $X = E^{\mathbb{Z}}$, with $E = \{0, 1, \dots, m-1\}$, as in Example 3.1**(i)**. We also write ν for the measure of maximum entropy, namely ν is a product measure such that $\nu|big\{x : x_i = j\}\big) = m^{-1}$. Since any continuous function f can be approximated by local functions, and both sides of (3.15) are continuous functionals of f, we may assume that f is local without loss of generality. That is, $f(x) = g(x_0, \dots, x_{k-1})$, for some function $g : E^k \to \mathbb{R}$. Clearly,

$$n\Phi_n(f)(x) = g(x_0, \dots, x_{k-1}) + \dots + g(x_{n-1}, \dots, x_{n+k-2}).$$

Moreover,

$$\int e^{n\Phi_n(f)} \, d\nu = m^{-(n+k-1)} \sum_{(x_0,\dots,x_{n+k-2}) \in E^{n+k-2}} e^{n\Phi_n(f)(x)}.$$

Observe that if the metric d is as in Example 3.1**(i)**, and A is a minimal set with the property

$$X = \cup_{x \in A} B_{d_n}(x, r), \quad \text{with } \ r = \eta^{-\ell},$$

then any distinct pair x and $x' \in A$ must differ on $\{-\ell, \dots, \ell + n - 1\}$. We may assume that all $x \in A$ agree outside the set $\{-\ell, \dots, \ell + n - 1\}$ to avoid a repetition in our covering by d_n-balls. Now if $\ell \geq k$, then for such a set A,

$$\sum_{x \in A} e^{n\Phi_n(f)(x)} = m^{2\ell - k + 1} \sum_{(x_0,\dots,x_{n+k-2}) \in E^{n+k-2}} e^{n\Phi_n(f)(x)} = m^{n+2\ell} \int e^{n\Phi_n(f)} \, d\nu.$$

From this we learn

$$P_{top}(f; \tau) = \log m + \lim_{n \to \infty} \int e^{n\Phi_n(f)} \, d\nu.$$

Hence the first equation in (3.28) means

$$\lim_{n \to \infty} \frac{1}{n} \log \int e^{n\Phi_n(f)} \, d\nu = \sup_{\mu \in \mathcal{I}_\tau} \left(\int f \, d\mu - I(\mu) \right). \tag{3.29}$$

This after some manipulation is an immediate consequence of *Donskar-Varadhan large deviation principle (LDP)*. To explain this, we need some preparations.

Given $x \in X$, we build a n-periodic sequence x^n from it by $x^n_{i+rn} = x_i$, for $i \in \{0, \dots, n-1\}$, and $r \in \mathbb{Z}$. Evidently,

$$\left| \Phi_n(f)(x) - \Phi_n(f)(x^n) \right| \leq k \max |g|.$$

Moreover, if we set

$$\hat{\mu}^x_n := \mu^{x^n}_n,$$

then $\hat{\mu}^x_n \in \mathcal{I}_\tau$, and

$$\Phi_n(f)(x^n) = \int f \, d\hat{\mu}^x_n.$$

Hence, for $n \geq \ell$,

$$\log \sum_{x \in E} e^{n\Phi_n(f)(x)} = \log \sum_{x \in E} e^{n\Phi_n(f)(x^n)} + O(k)$$

Because of this, (3.29) is equivalent to the statement

$$\lim_{n\to\infty}\frac{1}{n}\log\int e^{n\int f\ d\hat{\mu}_n^x}\ \nu(dx)=\sup_{\mu\in\mathcal{I}_\tau}\left(\int f\ d\mu-I(\mu)\right). \tag{3.30}$$

In fact what Donskar-Varadhan LDP entails to a stronger statement, namely for and continuous function $F:\mathcal{I}_\tau\mathbb{R}$,

$$\lim_{n\to\infty}\frac{1}{n}\log\int e^{nF(\hat{\mu}_n^x)}\ \nu(dx)=\sup_{\mu\in\mathcal{I}_\tau}\left(F(\mu)-I(\mu)\right), \tag{3.31}$$

which is the same as (3.30) when F is linear. It turns out that if (3.31) is true for all continuous functions F, then roughly speaking,

$$\nu\left(\{x:\hat{\mu}_n^x\ \text{ is near }\mu\}\right)\approx e^{-nI(\mu)}.$$

As we saw before, if we choose choose $\xi=\{A_0,\dots,A_{m-1}\}$, with $A_i=\{\omega:\omega_0=i\}$, then $h_\mu(\tau)=h_\mu(\tau,\xi)$. Since $\nu(A)=m^{-n}$, for every $A\in\xi_n:=\xi(0,n-1)$, we deduce

$$I(\mu)=\log m-h_\mu(\tau,\xi)=\lim_{n\to\infty}\frac{1}{n}\sum_{A\in\xi_n}\mu(A)\log\frac{\mu(A)}{\bar{\mu}(A)}.$$

Indeed if we write $\mathcal{F}_n$ for the σ-algebra generated by ξ_n, and write h_n for the Radon-Nikodym derivative $\frac{d\mu}{d\nu}$ for the restriction of ν and μ to $\mathcal{F}_n$, then

$$I(\mu)=\lim_{n\to\infty}\frac{1}{n}H_{\mathcal{F}_n}\big(\mu|\bar{\mu}\big).$$

when

$$H_{\mathcal{F}_n}\big(\mu|\nu\big)=\int h_n\log h_n\ d\nu,$$

represents the *relative entropy* of μ with respect to $\bar{\mu}$ in $\mathcal{F}_n$. We refer to [R] for more details.

(iii) If X is a manifold with a volume measure m, then there exists a unique $\bar{\mu}=\mu_{SRB}\in\mathcal{I}_T$ such that $h_{\text{top}}(T)=h_{\bar{\mu}}(T)$, and if $I(\mu)=h_{\text{top}}(T)-h_\mu(T)=h_{\bar{\mu}}(T)-h_\mu(T)$, then, we still have a LDP with rate I as in part **(ii)**. □

Example 3.5 Consider $T:\mathbb{T}^2\to\mathbb{T}^2$, $Tx=Ax \pmod 1$ with A an integer matrix with $\det A=1$. We assume that A is symmetric and its eigenvalues $\lambda_1,\lambda_2=\lambda_1^{-1}$ satisfy $|\lambda_1|>1>|\lambda_2|$. We claim that if μ is the Lebesgue measure, then $h_\mu(T)\ge\log|\lambda_1|$. In case of $T=\begin{bmatrix}2&1\\1&1\end{bmatrix}$, we can use our result $h_{\text{top}}(T)\le\log|\lambda_1|$ from Example 3.6 to conclude that in fact $h_\mu(T)=h_{\text{top}}(T)=\log|\lambda_1|$.

For $h_\mu(T) \geq \log|\lambda_1|$, observe that by the invariance of μ with respect to T, $H_\mu(T^{-n}\xi \vee \cdots \vee T^n\xi) = H_\mu(\xi \vee \cdots \vee T^{-2n}\xi)$. Hence it suffices to study $\lim_{n\to\infty} \frac{1}{2n} H_\mu(T^{-n}\xi \vee \cdots \vee T^n\xi)$. For estimating this, we show that the area of each $C \in \eta_n = T^{-n}\xi \vee \cdots \vee T^n\xi$ is exponentially small. This is achieved by showing that $\operatorname{diam}(C) = O(|\lambda_1|^{-n})$. For example, let us choose $\xi = \{Z_0, \ldots, Z_4\}$ where Z_i's are the rectangles of Example 3.6. It is not hard to see that if the side lengths of Z_i's are all bounded by a constant c, then η_n consists of rectangles with side lengths bounded by $c\lambda_1^{-n}$. Hence $\mu(A) \leq c^2\lambda_1^{2n}$ for every $A \in \eta_n$.

This evidently implies that $\frac{1}{2n} H_\mu(\eta_n) \geq \log|\lambda_1| + o(1)$, and as $n \to \infty$ we deduce that $h_\mu(T) \geq \log|\lambda_1|$. □

We finish this Chapter with a variant of the entropy that was defined by Katok.

Definition 3.5 Given $r, \delta > 0$, we define $S^n_{T,d}(r,\delta)$ to be the smallest k such that there exists a set E with $\#E = k$ and $\mu\left(\bigcup_{x\in E} B^n_{T,d}(x,r)\right) > 1-\delta$. We then define

$$\hat{h}_\mu(T) = \lim_{\delta\to 0} \lim_{r\to 0} \limsup_{n\to\infty} \frac{1}{n} \log S^n_{T,d}(r,\delta).$$

□

Evidently $\hat{h}_\mu(T) \leq h_{\mathrm{top}}(T)$. Moreover,

Theorem 3.5 *(Katok) For every ergodic $\mu \in \mathcal{I}_T$, we have $h_\mu(T) \leq \hat{h}_\mu(T)$.*

Proof Given a partition $\xi = \{C_1, \ldots, C_\ell\}$, build a partition $\eta + \{K_0, K_1, \ldots, K_\ell\}$ as in *Step 1* of the proof of Theorem 3.4, so that (3.13) holds. Recall that by Theorem 3.1,

$$\lim_{n\to\infty} \frac{1}{n} \log \mu(C_n(x)) = -h_\mu(T,\eta)$$

in L^1-sense, when $C_n(x) = C_{\eta_n}(x)$. Pick $\varepsilon' > 0$ and choose a subsequence $\{n_j : j \in \mathbb{N}\}$ such that if

$$X_N = \left\{ x \in X : \frac{1}{n_j} \log \mu(C_{n_j}(x)) \leq -h_\mu(T,\eta) + \varepsilon' \text{ for } n_j > N \right\},$$

then $\mu(X_N) \to 1$ as $N \to \infty$. Pick $\delta > 0$, and find N such that $\mu(X_N) > 1-\delta$. Let

$$r = \frac{1}{2} \min\left\{ \operatorname{dist}(K_i, K_j) : i \neq j,\ i,j \in \{1,\ldots,\ell\} \right\}.$$

As in *Step 2* of the proof of Theorem 3.4, a ball $B_{d_n}(x,r)$ intersects at most 2^n elements of η_n. Now assume that $\mu\left(\bigcup_{x\in E} B_{d_n}(x,r)\right) > 1-\delta$. We would like to bound $\#E$ from below. First observe

$$\begin{aligned} 1-2\delta &\leq \mu\left(\bigcup_{x\in E} B_{d_n}(x,r) \cap X_N\right) \leq \sum_{x\in E} \mu\big(B_{d_n}(x,r) \cap X_N\big) \\ &= \sum_{x\in E} \sum_{A\in\eta_n} \mu\big(B_{d_n}(x,r) \cap X_N \cap A\big). \end{aligned}$$

But if $B_{d_n}(x,r) \cap X_N \cap A \neq \emptyset$ for $n = n_j > N$, then

$$\mu\big(B_{d_n}(x,r) \cap X_N \cap A\big) \leq \mu(A) \leq e^{-n(h_\mu(T,\eta)-\varepsilon')}.$$

As a result,

$$1 - 2\delta \leq 2^n e^{-n(h_\mu(T,\eta)-\varepsilon')}(\#E).$$

Hence

$$h_\mu(T,\eta) \leq \limsup_{n_j \to \infty} \frac{1}{n_j} \log S^{n_j}_{T,d}(r,\delta) + \varepsilon' + \log 2.$$

From this we deduce that $h_\mu(T,\eta) \leq \hat{h}_\mu(T) + \varepsilon' + \log 2$. From this and (3.13) we learn that $h_\mu(T,\xi) \leq \hat{h}_\mu(T) + \varepsilon\ell \log \ell + \varepsilon' + \log 2$. By sending $\varepsilon, \varepsilon' \to 0$ and taking supremum over ξ we deduce

$$h_\mu(T) \leq \hat{h}_\mu(T) + \log 2. \tag{3.32}$$

We wish to modify (3.32) by getting rid of $\log 2$. To achieve this, we would like to replace T with T^m in the equation (3.32). A repetition of the proof of Proposition 3.2(iii) yields $\frac{1}{m}\hat{h}_\mu(T^m) = \hat{h}_\mu(T)$. If $\mu \in \mathcal{I}^{er}_{T^m}$, then we will have

$$h_\mu(T) = \frac{1}{m} h_\mu(T^m) \leq \frac{1}{m}\hat{h}_\mu(T^m) + \frac{\log 2}{m} = \hat{h}_\mu(T) + \frac{\log 2}{m},$$

which is desirable because of the factor m^{-1} in front of $\log 2$; this factor goes to 0 in large m limit. However, it is possible that $\mu \in \mathcal{I}^{er}_T \setminus \mathcal{I}^{er}_{T^m}$. If this is the case, then we may apply Exercise **(viii)** of Chapter 1, to assert that if m a prime number, then all T^m invariant sets come from a finite partition ζ with exactly m elements. This suggests replacing the partition η with $\hat{\eta} = \eta \wedge \zeta$ so that we still have

$$\lim_{n \to \infty} n^{-1} \log \mu\big(\hat{C}_n(x)\big) = -h_{\hat{\eta}}(\mu),$$

where $C_n(x) = C_{\hat{\eta}_n}(x)$, and $\hat{\eta}_n = \eta \wedge T^{-m}\hat{\eta} \wedge \cdots \wedge T^{m(1-n)}\hat{\eta}$. Here we are using the fact that the ergodic theorem is applicable because the limit is constant on members of the partition ζ. Repeating the above proof for T^m, we can only assert that $B_d(x,r)$ can intersect at most $2m$ elements of $\hat{\eta}$, and that $B_{d_n}(x,r)$ can interest at most $(2m)^n$ elements of $\hat{\eta}_n$. This leads to the bound

$$h_\mu\big(T^m,\xi\big) \leq \hat{h}_\mu(T^m) + \varepsilon\ell\log\ell + \varepsilon' + \log(2m),$$

which in turn yields

$$h_\mu(T) \leq \hat{h}_\mu(T) + m^{-1}\log(2m),$$

for every prime number m. We arrive at $h_\mu(T) \leq \hat{h}_\mu(T)$, by sending m to infinity. □

Remark 3.3 Theorem 3.4 provides us with a rather local recipe for calculating the entropy. It turns out that there is another local recipe for calculating the entropy that is related to $\hat{h}_\mu(T)$. A theorem of Brin and Katok[BK] asserts that if $\mu \in \mathcal{I}_T$ is ergodic, then $\frac{1}{n}\log \mu(B_{d_n}(x,r))$ approximates $h_\mu(T)$. More precisely,

$$h_\mu(T) = \lim_{r\to 0} \limsup_{n\to\infty} \left[-\frac{1}{n} \log \mu(B_{d_n}(x,r)) \right]$$

for μ-almost all x. □

Exercises

(i) Let $F : X \to Y$ be a continuous function with $F(X) = Y$. Let $T : X \to X$, $T' : Y \to Y$ be continuous and $F \circ T = T' \circ F$. show that $h_{\text{top}}(T') \leq h_{\text{top}}(T)$.

(ii) Let (X_1, d_1), (X_2, d_2) be two compact metric spaces and let $T_i : X_i \to X_i$, $i = 1, 2$ be two continuous functions. show that $h_{\text{top}}(T_1 \times T_2) = h_{\text{top}}(T_1) + h_{\text{top}}(T_2)$.
Hint: For $T = T_1 \times T_2$ and a suitable choice of a metric d for $X_1 \times X_2$, show that

$$S^n_{T,d}(r) \leq S^n_{T_1,d_1}(r) S^n_{T_2,d_2}(r), \quad N^n_{T,d}(r) \geq N^n_{t,d_1}(r_1) N^n_{T,d_2}(r_2).$$

(iii) Let A be as in Example 3.1**(v)**. show that $r(A) = \frac{3+\sqrt{5}}{2}$.

(iv) Show that if τ_m denotes the shift map of Example 3.1**(ii)** or **(iv)** on m many symbols, then τ_m^k may be regarded as a shift map on a set of m^k many symbols. (Define a homeomorphism $F : X \to \hat{X}$,

$$X = \{0, \dots, m-1\}^{\mathbb{Z}}, \qquad \hat{X} = \left(\{0, \dots, m-1\}^k\right)^{\mathbb{Z}},$$

such that $F \circ \tau_m^k = \hat{\tau} \circ F$, where $\hat{\tau}$ denotes the shift operator on $\hat{X}$.) Use this to show that if $h(m) = h_{top}(\tau_m)$, then $h(m^k) = kh(m)$.

(v) According to the *Perron-Frobenius Theorem*, for any matrix A with non-negative entries we can find an nonnegative eigenvalue with a corresponding nonnegative eigenvector. Use this theorem to show that the matrix A in Example 3.1**(iv)** has a real eigenvalue $\lambda \geq 1$. For such a matrix A, what is necessary and sufficient condition for this eigenvalue to be 1?

(vi) If ξ has m elements, then $0 \leq H_\mu(\xi) \leq \log m$.

(vii) If $\alpha \leq \beta$, then $H_\mu(\alpha) \leq H_\mu(\beta)$ and $h_\mu(T, \alpha) \leq h_\mu(T, \beta)$.

(viii) If $\mu_1, \mu_2 \in \mathcal{I}_T$ and $\theta \in [0, 1]$, then

$$\begin{aligned} H_{\theta\mu_1+(1-\theta)\mu_2}(\xi) &\geq \theta H_{\mu_1}(\xi) + (1-\theta) H_{\mu_2}(\xi), \\ h_{\theta\mu_1+(1-\theta)\mu_2}(T, \xi) &\geq \theta h_{\mu_1}(T, \xi) + (1-\theta) h_{\mu_2}(T, \xi), \\ h_{\theta\mu_1+(1-\theta)\mu_2}(T) &\geq \theta h_{\mu_1}(T) + (1-\theta) h_{\mu_2}(T). \end{aligned}$$

(ix) (Rokhlin Metric) Define $d(\eta, \xi) = H_\mu(\eta \mid \xi) + H_\mu(\xi \mid \eta)$. Show that d is a metric on the space of μ-partitions.

(x) We say that the matrix A in Example 3.1**(iv)** is *irreducible and aperiodic* or *primitive* if A^{n_0} has positive entries for some $n_0 \in \mathbb{N}$. In the case of a primitive A, *Perron-Frobenius Theorem* asserts that the largest eigenvalue $\lambda > 1$ of A is of multiplicity 1 and the corresponding right and left eigenvectors u^r and u^ℓ can be chosen to have positive components. We may assume that $u^r \cdot u^\ell = 1$. Define measure μ on X_A with the following recipe:

$$\mu\left(X_A \cap \left\{\omega : \ \omega_1 = \alpha_1, \dots, \omega_k = \alpha_k\right\}\right) = \pi(\alpha_1) \prod_{i=1}^{k-1} p(\alpha_i, \alpha_{i+1}),$$

where

$$\pi(i) = u_i^\ell u_i^r, \qquad p(i,j) = \frac{a_{ij} u_j^r}{\lambda u_i^r}.$$

Show that the measure μ is well-defined and is invariant for τ. Show

$$h_\mu(T) = -\sum_{i,j=0}^{m-1} \pi(i) p(i,j) \log p(i,j) = \log \lambda.$$

(xi) Work out the measure μ of part **(x)** in the case of Example 3.1**(v)**.

(xii) Show that $P_{top}(f;T)$ is convex in f.

(xiii) Verify Proposition 3.6.

(xiv) Verify

$$\sup_{x \in M} \big(x \cdot a - H(x)\big) = P(a),$$

where M is the set of vectors $x = (x_1, \dots, x_d)$ with $x_i \geq 0$, and $x_1 + \dots + x_k = 1$, and

$$H(x) = \sum_{i=1}^k x_i \log x_i, \qquad P(a) = \log \sum_{i=1}^k e^{a_i}.$$

More generally show

$$\sup_h \left(\int hf \ d\nu - H(h)\right) = P(f),$$

where the supremum is over probability densities $h \geq 0$, $\int h \ d\nu = 1$, and

$$H(h) = \int h \log h \ d\nu, \qquad P(f) = \log \int e^f \ d\nu.$$

□

4 Lyapunov Exponents

For the expanding map of Examples 3.1**(iii)** and 3.3**(i)**,and Arnold's map of Example 3.4 we saw that the entropy was indeed the logarithm of the expansion rate. In this chapter, we learn how to define the exponential rates of expansions and contractions of a transformation with respect to its invariant measures. These rates are called the *Lyapunov Exponents*, and can be used to bound the entropy. In fact for the so-called *hyperbolic* dynamical system, the entropy can be expressed as the sum of positive Lyapunov exponents by Pesin's formula. In general an inequality of Ruelle implies that the entropy is bounded above by the sum of positive Lyapunov exponents.

Consider a transformation $T : X \to X$ where X is a compact C^1 manifold and T is a C^1 transformation. We also assume that M is a Riemannian manifold. This means that for each x there exists an inner product $\langle \cdot, \cdot \rangle_x$ and (an associated norm $| \ |_x$) that varies continuously with x. To study the rate of expansion and contraction of T, we may study $\big(dT^n\big)_x : \mathcal{T}_x X \to \mathcal{T}_{T^n(x)} X$. We certainly have

$$(4.1) \qquad \big(dT^n\big)_x = (dT)_{T^{n-1}(x)} \circ \cdots \circ (dT)_{T(x)} \circ (dT)_x.$$

If we write $A(x) = (dT) : T_x X \to T_{T(x)} X$, then (4.1) can be written as

$$(4.2) \qquad A_n(x) := \big(dT^n\big)_x = A(T^{n-1}(x)) \circ \cdots \circ A(T(x)) \circ A(x).$$

Here we are interested in the long time behavior of the dynamical system associated with $dT : \mathcal{T}X \to \mathcal{T}X$ that is defined by $dT(x, v) = (T(x), (dT)_x(v)) = (T(x), A(x)v)$.

The formula (4.1) suggests an exponential growth rate for $\big(dT^n\big)_x$. Let us examine some examples first.

Example 4.1(i) Consider the dynamical system $T : \mathbb{T}^d \to \mathbb{T}^d$ that is given by $T(x) = Ax \pmod 1$, where A is a $d \times d$ matrix. Identifying $\mathcal{T}\mathbb{T}^d$ with $\mathbb{T}^d \times \mathbb{R}^d$, we learn that $A(x) = A$ is constant and $A_n(x) = A^n$. We may use a Jordan normal form to express A as a diagonal block matrix. More precisely, we can express

$$(4.3) \qquad \mathbb{R}^d = G_1 \oplus \cdots \oplus G_q,$$

where each G_j corresponds to an eigenvalue λ_j of A and $r_j = \dim G_j$ represents the multiplicity of λ_j. If λ_j is complex, we use the same G_j for both λ_j and its complex conjugate, and r_j is twice the multiplicity. For real λ_j, the space G_j is the generalized eigenspace associated with λ_j:

$$G_j = \big\{ v \in \mathbb{R}^d : (A - \lambda_j)^r v = 0 \text{ for some } r \in \mathbb{N} \big\}.$$

In the case of a complex pair of eigenvalues $\alpha_j \pm i\beta_j$, the space G_j is spanned by real and imaginary parts of the generalized eigenvectors. In the case of real λ_j, the restriction of the map $x \mapsto Ax$ to G_j has a diagonal block matrix representation, with each block of the form

$$A' = \begin{bmatrix} \lambda_j & 0 & \dots & 0 & 0 \\ 1 & \lambda_j & \dots & 0 & 0 \\ \vdots & & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \lambda_j \end{bmatrix}$$

In the case the complex eigenvalue $\alpha_j \pm i\beta_j$, in A' we replace λ_j by the 2×2 matrix

$$R_j = \begin{bmatrix} \alpha_j & -\beta_j \\ \beta_j & \alpha_j \end{bmatrix}$$

and the 1's below the diagonal are replaced by the 2×2 identity matrix. For our purposes, we would like to replace the off-diagonal entries with some small number δ. (When A' is $\ell \times \ell$, we make the change of coordinates $(x_1, \dots, x_\ell) \mapsto \big(\delta^{-1}x_1, \dots, \delta^{-\ell}x_\ell\big)$.) From this, it is not hard to show

$$\big(|\lambda_j| - \delta\big)^n |v| \le |A^n v| \le \big(|\lambda_j| + \delta\big)^n |v|.$$

By sending δ to 0 we learn that for $v \in G_j \setminus \{0\}$,

$$\lim_{n\to\infty} \frac{1}{n} \log |A^n v| = \log |\lambda_j|. \tag{4.4}$$

(ii) Consider a dynamical system (X, T), with X a smooth Riemmanian manifold of dimension d. Let us take a fixed point a of T so that $A = A(a) = (dT)_a$ and $A_n(a) = A^n$ map $\mathcal{T}_aX$ to itself. We may identify $\mathcal{T}_aX$ with $\mathbb{R}^d$ and represent A as a $d \times d$ matrix. Using the decomposition of Part **(i)**, and (4.4), we deduce that for $v \in G_j \setminus \{0\}$,

$$\lim_{n\to\infty} \frac{1}{n} \log |A^n(a) v| = \log |\lambda_j|.$$

(iii) Let us now assume that $T : X \to X$ is a diffeomorphism, and the orbit associated with a is periodic of period N. This means that $T^N(a) = a$ and $\big(dT^N\big)_a$ maps $\mathcal{T}_aX$ to itself. If we write $n = mN + r$ with $m \in \mathbb{N}$ and $r \in \{0, \dots, N-1\}$, Then we have $A_n(a) = C \circ B^m$, where

$$B = \big(dT^N\big)_a = A\big(T^{N-1}(a)\big) \circ \dots \circ A(a),$$
$$C = \begin{cases} A\big(T^{r-1}(a)\big) \circ \dots \circ A(a) & \text{for } r > 0, \\ I & \text{for } r = 0. \end{cases}$$

From this, we learn

$$c_0^{-1} |B^m v| \le |A_n(a) v| \le c_0 |B^m v|,$$

for $c_0 = \big(\max(1, \|A\|)\big)^{N-1}$. We use the generalized eigenspaces of B to decompose $\mathcal{T}_a X \equiv \mathbb{R}^d$ as in(4.4). We now have that for $v \in G_j \setminus \{0\}$,

$$\lim_{n\to\infty} \frac{1}{n} \log |A^n(a)v| = \frac{1}{N} \log |\lambda_j|,$$

where λ_j is an eigenvalue of B. □

As a preparation for the definition of *Lyapunov exponents*, let us observe that if we set $S_n(x) = \log \|A_n(x)\|$, then $S_0 = 0$ and

$$S_{n+m}(x) \le S_n(x) + S_m(T^n(x)), \tag{4.5}$$

by (4.2). The following theorem guarantees the existence of the *largest Lyapunov exponent.* This theorem is an immediate consequence of the *Kingman's subadditive ergodic theorem.*

Theorem 4.1 *Let T be a diffeomorphism and assume that $\mu \in \mathcal{I}_T^{ex}$. Then there exists $\ell \in \mathbb{R}$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \log \left\|\left(dT^n\right)_x\right\| = \ell,$$

for μ-almost all x.

Proof 4.1 On the account of Theorem 1.6, we only need to show $\ell \ne -\infty$. Clearly,

$$id = \left(dT^{-n}\right)_{T^n(x)} \left(dT^n\right)_x, \qquad 1 \le \left\|\left(dT^{-n}\right)_{T^n(x)}\right\| \left\|\left(dT^n\right)_x\right\|.$$

On the other hand, if we write

$$\alpha := \sup_x \left\|\left(dT^{-1}\right)_x\right\|,$$

then

$$\left\|\left(dT^{-n}\right)_{T^n(x)}\right\| = \left\|\left(dT^{-1}\right)_x \circ \dots \left(dT^{-1}\right)_{T^{n-1}(x)} \circ \left(dT^{-1}\right)_{T^n(x)}\right\| \le \alpha^n.$$

Hence $\left\|(dT^n)_x\right\| \ge \alpha^{-n}$ which implies that $\ell \ge -\log\alpha$. □

We now state the *Oseledets Theorem* that guarantees the existence of a collection of *Lyapunov exponents.*

Theorem 4.2 *Let $T : X \to X$ be a C^1-diffeomorphism with $\dim X = d$ and let $\mu \in \mathcal{I}_T^{ex}$. Let A be a measurable function such that $A(x) : \mathcal{T}_x X \to \mathcal{T}_{T(x)} X$ is linear for each x and $\log^+ \|A(x)\| \in L^1(\mu)$. Define $A_n(x) = A(T^{n-1}(x)) \circ \dots \circ A(T(x)) \circ A(x)$. Then there exists a set $X' \subseteq X$ with $\mu(X') = 1$, numbers $l_1 < l_2 < \dots < l_k$ and $n_1, \dots, n_k \in \mathbb{N}$ with $n_1 + \dots + n_k = d$, and a linear decomposition $\mathcal{T}_x X = E_x^1 \oplus \dots \oplus E_x^k$ with $x \mapsto (E_x^1, \dots, E_x^k)$ measurable such that*

$$\lim_{n\to\infty} \frac{1}{n} \log |A_n(x)v| = l_j$$

for $x \in X'$ and $v \in F_x^j \setminus F_x^{j-1}$, where $F_x^j := E_x^1 \oplus \dots \oplus E_x^j$.

Example 4.2(i) Let $T : \mathbb{T}^d \to \mathbb{T}^d$ be a translation. Then $dT^n = id$ and the only Lyapunov exponent is zero.

(ii) Let $T : \mathbb{T}^m \to \mathbb{T}^m$ be given by $T(x) = Ax \pmod 1$ with A a matrix of integer entries. Let $\lambda_1, \dots, \lambda_q$ denote the eigenvalues of A. Let $l_1 < l_2 < \cdots < l_k$ be numbers with $\{l_1, \dots, l_k\} = \{\log|\lambda_1|, \dots, \log|\lambda_q|\}$. We also write n_j for the sum of the multiplicities of eigenvalues λ_i with $\log|\lambda_i| = l_j$. The space spanned by the corresponding generalized eigenvectors is denoted by E_j. We certainly have that if $v \in E_j$ then $\lim_{n\to\infty} \frac{1}{n} \log|A^n v| = l_j$.

(iii) If $a \in X$ is a periodic point of period N, then $\mu = N^{-1} \sum_{j=0}^{N-1} \delta_{T^j(a)}$ is an ergodic invariant measure. In this case the Oseledets Theorem follows from our discussion in Example 4.1**(iii)**. Indeed if $\lambda_1, \dots, \lambda_q$ denote the eigenvalues of $B = (dT^N)_a$, then $\ell_1 < \cdots < \ell_k$ are chosen so that $\{\ell_1, \dots, \ell_k\} = \{N^{-1}\log|\lambda_1|, \dots, N^{-1}\log|\lambda_q|\}$ and $E_a^j = \oplus_i \{V_i : N^{-1}\log|\lambda_i| = \ell_j\}$ where $G_i = \{v \in T_a M; (A(a) - \lambda_i)^r v = 0 \text{ for some } r\}$ is the generalized eigenspace associated with λ_i.

(iv) When $d = 1$, Theorem 4.3 (or 4.1) is an immediate consequence of the Ergodic Theorem and the only Lyapunov exponent is $l_1 = \int \log|A(x)| \ \mu(dx)$. □

Remark 4.1(i) The identity $A_n(T(x))A(x)v = A_{n+1}(x)v$ implies

$$A(x)F_x^j \subseteq F_{T(x)}^j,$$

for $j = 1, \dots, k$. By invertiblity, we can also show that

$$A(x)F_x^j \supseteq F_{T(x)}^j.$$

(ii) By Ergodic Theorem,

$$\frac{1}{n} \log|\det A_n(x)| = \frac{1}{n} \sum_0^{n-1} \log|\det A(T^j(x))| \to \int \log|\det(dT)_x| \ d\mu,$$

As we will see later,

$$\int \log|\det(dT)_x| d\mu = \sum_1^k n_j l_j.$$

(iii) Theorem 4.1 allows us to determine the largest Lyapunov exponent, whereas Part **(ii)** offers a way of getting the sum (with multiplicity) of all Lyapunov exponents. A combination of both ideas will be used to obtain all Lyapunov exponents by studying the norm of the exterior powers of A_n, which involves the determinant of submatrices of A_n. □

(iii) It turns out that the most challenging part of Theorem 4.3 is the existence of the limit. Indeed if we define

$$l(x,v) = \limsup_{n\to\infty} \frac{1}{n} \log |A_n(x)v|,$$

then we can show that as in Theorem 4.3 there exists a splitting $\mathcal{T}_x X = E_x^1 \oplus \cdots \oplus E_x^k$ with $l(x,v) = l_j$ for $v \in F_j(x)$. □

Proof of Theorem 4.3 for $\mathbf{d = 2}$ We only prove Theorem 4.3 when $A = dT$. The proof of general case is similar. By Theorem 4.1, there exist numbers l_1 and l_2 such that if

$$X_0 = \left\{ x : \lim_{n\to\infty} \frac{1}{n} \log \left\|(dT^n)_x\right\| = l_2,\ \lim_{n\to\infty} \frac{1}{n} \log \left\|(dT^{-n})_x\right\| = -l_1 \right\}$$

then $\mu(X_0) = 1$. Evidently $|A_n v|^2 = \langle A_n^* A_n v, v\rangle = |B_n v|^2$ where $B_n = (A_n^* A_n)^{1/2}$. Clearly $A_n^* A_n \geq 0$ and B_n is well-defined. Since $B_n \geq 0$, we can find numbers $\mu_2^n(x) \geq \mu_1^n(x) \geq 0$ and vectors $a_1^n(x), a_2^n(x)$ such that $|a_1^n| = |a_2^n| = 1$, $\langle a_1^n, a_2^n\rangle_x = 0$ and $B_n a_j^n = \mu_j^n a_j^n$ for $j = 1, 2$.

Note that since $\|A_n(x)\| = \|B_n(x)\|$,

$$l_2 = \lim_{n\to\infty} \frac{1}{n} \log \mu_2^n. \tag{4.6}$$

To obtain a similar formula for l_1, first observe that

$$\left(dT^n\right)_{T^{-n}(x)} \left(dT^{-n}\right)_x = id \quad \Rightarrow \quad A_{-n}(x) := \left(dT^{-n}\right)_x = \left(A_n\left(T^{-n}(x)\right)\right)^{-1}.$$

If we set $S_{-n}(x) = \log \|A_{-n}(x)\|$ and $R_n(x) = \log \|A_n(x)^{-1}\|$ then both $\{S_{-n}(x) : n \in \mathbb{N}\}$ and $\{R_n(x) : n \in \mathbb{N}\}$ are subadditive;

$$S_{-n-m} \leq S_{-n} \circ T^{-m} + S_{-m}, \qquad R_{n+m} \leq R_n \circ T^m + R_m.$$

Clearly,

$$-l_1 = \lim_{n\to\infty} \frac{1}{n} S_{-n} = \inf_n \frac{1}{n} \int S_{-n}\, d\mu, \qquad \hat{l} = \lim_{n\to\infty} \frac{1}{n} R_n = \inf_n \frac{1}{n} \int R_n\, d\mu.$$

Since $S_{-n} = R_n \circ T^{-n}$, we have $\int R_n\, d\mu = \int S_{-n}\, d\mu$. This in turn implies that $\hat{l} = -l_1$. As a result,

$$-l_1 = \lim_{n\to\infty} \frac{1}{n} \log \|A_n^{-1}\| = \lim_{n\to\infty} \frac{1}{n} \log \|A_n^{*-1}\|.$$

(Recall that $\|A\| = \|A^*\|$.) We then have

$$\begin{aligned} -l_1 &= \lim_{n\to\infty} \frac{1}{n} \log \|(A_n^* A_n)^{-1/2}\| = \lim_{n\to\infty} \frac{1}{n} \log \|B_n^{-1}\| \\ &= -\lim_{n\to\infty} \frac{1}{n} \log(\mu_1^n \wedge \mu_2^n) = -\lim_{n\to\infty} \frac{1}{n} \log \mu_1^n. \end{aligned} \tag{4.7}$$

Naturally we expect E_x^2 to be the limit of the lines $\{ta_2^n : t \in \mathbb{R}\}$ as $n \to \infty$. For this though we need to assume that $l_1 < l_2$. To see this, let us first estimate $|a_2^{n+1}(x) - a_2^n(x)|$. We may assume that $\langle a_2^{n+1}, a_2^n \rangle_x \geq 0$ for all n. Indeed if this is true for all $n < m$ but $\langle a_2^{m+1}, a_2^m \rangle_x < 0$, replace a_{m+1} with $-a_{m+1}$.

We certainly have

$$|a_2^{n+1} - a_2^n|^2 = 2 - 2\langle a_2^{n+1}, a_2^n \rangle, \qquad 1 = |a_2^{n+1}|^2 = \langle a_2^{n+1}, a_1^n \rangle^2 + \langle a_2^{n+1}, a_2^n \rangle^2.$$

From this and the elementary inequality $1 - z^2 \leq \sqrt{1 - z^2}$, we learn

$$\begin{aligned}
|a_2^{n+1} - a_2^n|^2 &= 2 - 2(1 - \langle a_2^{n+1}, a_1^n \rangle^2)^{1/2} \leq 2\langle a_2^{n+1}, a_1^n \rangle^2 \\
&= 2\langle B_{n+1} a_2^{n+1} / \mu_2^{n+1}, a_1^n \rangle^2 = 2(\mu_2^{n+1})^{-2} \langle a_2^{n+1}, B_{n+1} a_1^n \rangle^2 \\
&\leq 2(\mu_2^{n+1})^{-2} |B_{n+1} a_1^n|^2 = 2(\mu_2^{n+1})^{-2} |A_{n+1} a_1^n|^2 \\
&= 2(\mu_2^{n+1})^{-2} |A(T^n(x)) A_n(x) a_1^n(x)|^2 \leq 2(\mu_2^{n+1})^{-2} c_0 |A_n(x) a_1^n(x)|^2 \\
&= 2(\mu_2^{n+1})^{-2} c_0 |B_n a_1^n|^2 = 2c_0 (\mu_2^{n+1} / \mu_1^n)^{-2}
\end{aligned}$$

for $c_0 = \max_x \|A(x)\|$. From this, (4.6) and (4.7) we deduce

$$\limsup_{n \to \infty} \frac{1}{n} \log |a_2^{n+1} - a_2^n| \leq -(l_2 - l_1).$$

Let us now assume that $l_2 - l_1 > \delta > 0$. We then have that for constants c_1, c_2,

$$|a_2^{n+1} - a_2^n| \leq c_1 e^{-\delta n}, \qquad |a_2^{n+r} - a_2^n| \leq c_2 e^{-\delta n}$$

for all positive n and r. As a result, $\lim_{n \to \infty} a_2^n = b_2$ exists for $x \in X$ and

$$|a_2^n - b_2| \leq c_2 e^{-\delta n}$$

for all n. This being true for all $\delta \in (0, l_2 - l_1)$, means

$$\limsup_{n \to \infty} \frac{1}{n} \log |a_2^n - b_2| \leq -(l_2 - l_1). \tag{4.8}$$

We now define $E_2^x = \{tb_2(x) : t \in \mathbb{R}\}$. To show that $\lim_{n \to \infty} \frac{1}{n} \log |A_n(x) b_2(x)| = l_2$, observe

$$\begin{aligned}
|A_n b_2| &\leq |A_n a_2^n| + |A_n(a_2^n - b_2)| \leq |B_n a_2^n| + \|A_n\| |a_2^n - b_2| = \mu_2^n + \|A_n\| |a_2^n - b_2|, \\
|A_n b_2| &\geq |A_n a_2^n| - |A_n(a_2^n - b_2)| \geq |B_n a_2^n| - \|A_n\| |a_2^n - b_2| = \mu_2^n - \|A_n\| |a_2^n - b_2|
\end{aligned}$$

From this and (4.6)-(4.8) we deduce

$$\begin{aligned}
\limsup_{n \to \infty} \frac{1}{n} \log |A_n b_2| &\leq \max\left(\limsup_{n \to \infty} \frac{1}{n} \log \mu_2^n, \ \limsup_{n \to \infty} \frac{1}{n} \log \left(\|A_n\| |a_2^n - b_2| \right) \right) \\
&\leq \max(l_2, l_1) = l_2, \\
l_2 = \lim_{n \to \infty} \frac{1}{n} \log \mu_2^n &\leq \max\left(\liminf_{n \to \infty} \frac{1}{n} \log |A_n b_2|, \ \limsup_{n \to \infty} \frac{1}{n} \log \left(\|A_n\| |a_2^n - b_2| \right) \right) \\
&\leq \max\left(\liminf_{n \to \infty} \frac{1}{n} \log |A_n b_2|, l_1 \right).
\end{aligned}$$

From this we can readily deduce

$$\lim_{n\to\infty} \frac{1}{n} \log |A_n(x)b_2| = l_2,$$

for $x \in X$.

To find E_1^x, replace T with T^{-1} in the above argument. This completes the proof when $l_1 \neq l_2$.

It remains to treat the case $l_1 = l_2$. We certainly have

$$|A_n v|^2 = |B_n v|^2 = \langle v, a_1^n \rangle^2 (\mu_1^n)^2 + \langle v, a_2^n \rangle^2 (\mu_2^n)^2.$$

Hence

$$\mu_1^n |v| \leq |A_n v| \leq \mu_2^n |v|.$$

We are done because $\lim \frac{1}{n} \log \mu_2^n = \lim \frac{1}{n} \log \mu_1^n = l_1 = l_2$. □

From (4.6) and (4.7) we learned that when $d = 2$, we may use the eigenvalues of the matrix

$$\Lambda_n(x) = \big(A_n(x)^* \circ A_n(x)\big)^{\frac{1}{2n}} = B_n(x)^{\frac{1}{n}} : \mathcal{T}_x X \to \mathcal{T}_x X, \tag{4.9}$$

to find the Lyapunov exponents. In fact the same is true in any dimension. To explain this, observe that

$$\lim_{n\to\infty} \frac{1}{n} \log \det B_n(x) = \lim_{n\to\infty} \frac{1}{n} \log \big| \det A_n(x) \big|,$$

should yield a way of getting the sum of all Lyapunov exponents. In dimension 2, and when there are two Lyapunov exponents, we can use the determinant to get our hand on l_1, because by Theorem 4.1 we already have a candidate for l_2. To generalize this idea to higher dimension, we use the notion of the *exterior power* of a linear transformation and a vector space.

Definition 4.1(i) Given a vector space V, its r-fold exterior power $\wedge^r V$ is a vector space consisting of

$$\wedge^r V = \big\{ v_1 \wedge \cdots \wedge v_r : \; v_1, \ldots, v_r \in V \big\}.$$

By convention, $v_1 \wedge \cdots \wedge v_r = 0$ if $v_1, \ldots, v_r$ are not linearly independent. The wedge product is characterized by two properties: it is multilinear and alternative. By the former we mean that for all scalers c and c',

$$(cv_1 + c'v_1') \wedge v_2 \wedge \cdots \wedge v_r = c(v_1 \wedge v_2 \wedge \cdots \wedge v_r) + c'(v_1' \wedge v_2 \wedge \cdots \wedge v_r).$$

By the latter we mean that interchanging two vectors in $a = v_1 \wedge \cdots \wedge v_r$ changes the sign of a. If $\{e_1, \ldots, e_d\}$ is a basis for V, then

$$\{ e_{i_1, i_2, \ldots, i_r} := e_{i_1} \wedge \cdots \wedge e_{i_r} : \; i_r < \cdots < i_r \},$$

is a basis for $\wedge^r V$. In particular $\dim \wedge^r V = \binom{d}{r}$.

(ii) If $\langle\cdot,\cdot\rangle$ is an inner product on the vector space V, then we equip $\wedge^r V$ with the inner product

$$\langle v_1 \wedge \cdots \wedge v_r, v'_1 \wedge \cdots \wedge v'_r\rangle = \det\left[\langle v_i, v'_j\rangle\right]_{i,j=1}^r.$$

The quantity

$$\|v_1 \wedge \cdots \wedge v_r\|^2 = \langle v_1 \wedge \cdots \wedge v_r,\ v_1 \wedge \cdots \wedge v_r\rangle = \det\left[\langle v_i, v_j\rangle\right]_{i,j=1}^r,$$

represents the r-dimensional volume of the parallelepiped generated by vectors $v_1, \ldots, v_r$.

(iii) Let V and V' be two vector spaces and assume that $A : V \to V'$ is a linear transformation. We define

$$\wedge^r A : \wedge^r V \to \wedge^r V',$$

by

$$\wedge^r A(v_1 \wedge \cdots \wedge v_r) = (Av_1) \wedge \cdots \wedge (Av_r).$$

(vi) *(Grassmanian of a vector space)* Given a vector space V of dimension d, we write $Gr(V, r)$ for the set of r-dimensional linear subspaces of V. If V is equipped with an inner product and the corresponding norm is denoted by $|\cdot|$, then we may define a metric d_{Gr} on $Gr(V, r)$ as follows: Given $W, W' \in Gr(V, r)$, write

$$\hat{W} = \left\{w \in W :\ |w| = 1\right\}, \quad \hat{W}' = \left\{w' \in W' :\ |w'| = 1\right\},$$

and set

$$d_{Gr}(W, W') = \max\left(\max_{x\in\hat{W}} \min_{y\in\hat{W}'} |x - y|,\ \max_{y\in\hat{W}'} \min_{x\in\hat{W}} |x - y|\right).$$

(v) Given a smooth d-dimensional Riemannian manifold X, the manifold $\wedge^r X$ is a vector bundle that assign to each point $x \in X$, the vector space $\wedge^r_x X = \wedge^r \mathcal{T}_x X$. The metric on X induces a metric on $\wedge^r X$ by using $\langle\cdot,\cdot\rangle_x$ to produce an the inner on product $\wedge^r_x X$ as Part **(ii)**.

□

What we have in mind is that the r-vector $v_1 \wedge \cdots \wedge v_r$ represents the r-dimensional linear subspace that is spanned by vectors $v_1, \ldots, v_r$. We list a number of straightforward properties of the r-vectors in Proposition 4.1 below. The elementary proof of this proposition is omitted.

Proposition 4.1 **(i)** *Two sets of linearly independent vectors* $\{v_1, \ldots, v_r\}$ *and* $\{v'_1, \ldots, v'_r\}$ *span the same vector space iff* $v_1 \wedge \cdots \wedge v_r = \lambda v'_1 \wedge \cdots \wedge v'_r$ *for some nonzero scalar* λ.

(ii) *If $\langle\cdot,\cdot\rangle$ is an inner product on the vector space V, and $\{a_1,\dots,a_d\}$ is an orthonormal basis for V, then the set $\big\{a_{i_1}\wedge\dots\wedge a_{i_r}:\ 1\le i_1<\dots<i_r\big\}$ is an orthonormal basis for $\wedge^r V$.*

(iii) *If V,V',V'' are three vector spaces and $A:V\to V'$, $B:V'\to V''$ are linear, then $\wedge^r\big(B\circ A\big)=\big(\wedge^r B\circ\wedge^r A\big)$. If A is invertible, then $\wedge^r A^{-1}=\big(\wedge^r A\big)^{-1}$. If V and V' are inner product spaces and $A^*:V'\to V$ is the transpose of A, then $\wedge^r A^*=\big(\wedge^r A\big)^*$.*

(iv) *If $A:\mathbb{R}^d\to\mathbb{R}^d$ is represented by a $d\times d$ matrix, then the transformation $\wedge^r A$ is represented by a $\binom{d}{r}\times\binom{d}{r}$ matrix we obtain by taking the determinants of all $r\times r$ submatrices of A.*

(iv) *Suppose that V is an inner product space of dimension d, and $A:V\to V$ is a symmetric linear transformation. If $\{a_1,\dots,a_d\}$ is an orthonormal basis consisting of eigenvectors, associated with eigenvalues $\lambda_1\le\dots\le\lambda_d$, then the set $\big\{a_{i_1}\wedge\dots\wedge a_{i_r}:\ 1\le i_1<\dots<i_r\big\}$ is an orthonormal basis consisting of eigenvectors of $\wedge^r A$ associated with eigenvalues $\big\{\lambda_{i_1}\dots\lambda_{i_r}:\ 1\le i_1<\dots<i_r\big\}$.*

(v) *For an inner product space V, the space $\big(Gr(V,r),d_{Gr}\big)$ is a compact metric space.*

Proof of Theorem 4.3 for general d *(Step 1)* Recall that if $B_n=(A_n^*A_n)^{1/2}$, then $|A_nv|=|B_nv|$ with $B_n\ge 0$. Let us write $\mu_1^n\le\dots\le\mu_d^n$ for the eigenvalues of B_n and $a_1^n,\dots,a_d^n$ for the corresponding eigenvectors with $|a_i^n|=1,\langle a_i^n,a_j^n\rangle_x=0$ for $i\neq j$. We now claim that

$$\lim_{n\to\infty} n^{-1}\log\mu_i^n(x)=:m_i, \tag{4.10}$$

exists μ-almost surely. Indeed since

$$A_{m+n}(x)=A_n\big(T^m(x)\big)\circ A_m(x),$$

we may apply Proposition 4.1**(ii)** to assert

$$\wedge^r A_{m+n}(x)=\big(\wedge^r A_n\big)\big(T^m(x)\big)\circ\wedge^r A_m(x).$$

This in turn implies

$$\|\wedge^r A_{m+n}(x)\|\le\big\|\big(\wedge^r A_n\big)\big(T^m(x)\big)\big\|\ \ \|\wedge^r A_m(x)\|\,.$$

This allows us to apply Subadditive Ergodic Theorem (Theorem 1.6) to deduce

$$\lim_{n\to\infty}\frac{1}{n}\log\|\wedge^r A_n(x)\|:=l^r, \tag{4.11}$$

exists for μ-almost all x and every $r \in \{1, \dots, d\}$. On the other hand, Proposition 4.1 allows us to write

$$\|\wedge^r A_n(x)\| = \left\| (\wedge^r A_n(x)^* \circ \wedge^r A_n(x))^{1/2} \right\| = \left\| \left[\wedge^r (A_n(x)^* \circ A_n(x)) \right]^{1/2} \right\|$$
$$= \left\| \wedge^r \left[(A_n(x)^* \circ A_n(x))^{1/2} \right] \right\| = \|\wedge^r B_n(x)\| = \mu_n^d \dots \mu_n^{d-r+1}.$$

From this and (4.10) we deduce,

$$\lim_{n\to\infty} \left[\sum_{i=d-r+1}^{d} n^{-1} \log \mu_i^n(x) \right],$$

exists for μ-almost all x and every $r \in \{1, \dots, d\}$. This certainly implies (4.10)

(Step 2) Choose $l_1 < \cdots < l_k$ so that $\{m_1, \dots, m_d\} = \{l_1, \dots, l_k\}$. We also set

$$L_j = \{i : \ m_i \le l_j\}.$$

We define

$$F_x^{j,n} = \left\{ \sum_{i \in L_j} c_i a_i^n(x) : \ c_i \in \mathbb{R} \quad \text{for } i \in L_j \right\} \subseteq \mathcal{T}_x X.$$

We wish to show that the sequence $\{F_x^{j,n}\}$ is convergent with respect to d_{Gr}, so that we can define

$$F_x^j = \lim_{n\to\infty} F_x^{j,n}. \tag{4.12}$$

We only prove this for $j = k-1$; the proof for other j can be carried out in an analogous way.

To establish (4.12), we show first that if $b_n \in F_x^{k-1,n}$, with $|b_n| = 1$, then there exists $u_{n+1} \in F_x^{k-1,n+1}$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \log \left| b_n - \frac{u_{n+1}}{|u_{n+1}|} \right| \le -(l_k - l_{k-1}). \tag{4.13}$$

For this, it suffice to show

$$\limsup_{n\to\infty} \frac{1}{n} \log |b_n - u_{n+1}| \le -(l_k - l_{k-1}), \tag{4.14}$$

because

$$\left| b_n - \frac{u_{n+1}}{|u_{n+1}|} \right| \le |b_n - u_{n+1}| + \left| u_{n+1} - \frac{u_{n+1}}{|u_{n+1}|} \right| = |b_n - u_{n+1}| + |1 - |u_{n+1}|| \le 2\,|b_n - u_{n+1}|.$$

In fact we may simply choose u_{n+1} to be the projection of b_n onto $F_x^{k-1,n+1}$. More precisely, we write $b_n = u_{n+1} + v_{n+1}$ with $u_{n+1} \in F_x^{k-1,n+1}$, $v_{n+1} \perp F_x^{k-1,n+1}$, and show

$$\limsup_{n\to\infty} \frac{1}{n} \log |v_{n+1}| \le -(l_k - l_{k-1}), \tag{4.15}$$

which implies (4.14). For (4.15), observe that we can find scalars c_j^n and c_j^{n+1} such that

$$b_n = \sum_{i\in L_{k-1}} c_i^n a_i^n, \qquad v_{n+1} = \sum_{i\notin L_{k-1}} c_i^{n+1} a_i^{n+1}.$$

As a result,

$$\begin{aligned}\big|A_{n+1}v_{n+1}\big|^2 &= \big|B_{n+1}v_{n+1}\big|^2 = \left|\sum_{i\notin L_{k-1}} c_i^{n+1}\mu_i^{n+1}a_i^{n+1}\right|^2 = \sum_{i\notin L_{k-1}} \big(c_i^{n+1}\big)^2\big(\mu_i^{n+1}\big)^2 \\ &\ge \left(\min_{i\notin L_{k-1}} \mu_i^{n+1}\right)^2 \sum_{i\notin L_{k-1}} \big(c_i^{n+1}\big)^2 = \left(\min_{i\notin L_{k-1}} \mu_i^{n+1}\right)^2 \ |v_{n+1}|^2.\end{aligned}$$

In summary,

$$\left(\min_{i\notin L_{k-1}} \mu_i^{n+1}\right) \ |v_{n+1}| \le \big|A_{n+1}v_{n+1}\big|. \tag{4.16}$$

On the other hand, observe

$$\langle A_{n+1}u_n,\ A_{n+1}v_n\rangle = \langle A_{n+1}^* \circ A_{n+1}u_n,\ v_n\rangle = \langle B_{n+1}^2 u_n,\ v_n\rangle = 0,$$

because the space $F^{k-1,n+1}$ is invariant under the action of B_{n+1}. This and $A_{n+1}b_n = A_{n+1}u_{n+1} + A_{n+1}v_{n+1}$ imply

$$\big|A_{n+1}b_n\big|^2 = \big|A_{n+1}u_n\big|^2 + \big|A_{n+1}v_n\big|^2 \ge \big|A_{n+1}v_n\big|^2. \tag{4.17}$$

Furthermore,

$$\begin{aligned}\big|A_{n+1}b_n\big|^2 &= \big|A\big(T^n(x)\big)\circ A_n(x)b_n\big|^2 \le \big|A\big(T^n(x)\big)\big|^2\ \big|A_n(x)b_n\big|^2 \\ &\le C_0^2\big|A_n(x)b_n\big|^2 = C_0^2\big|B_n(x)b_n\big| = C_0^2\left|\sum_{i\in L_{k-1}} c_i^n\mu_i^n a_i^n\right|^2 \\ &= C_0^2 \sum_{i\in L_{k-1}} \big(c_i^n\big)^2\big(\mu_i^n\big)^2 \le C_0^2\left(\max_{i\in L_{k-1}} \mu_i^n\right)^2 \sum_{i\in L_{k-1}} \big(c_i^n\big)^2 = C_0^2\left(\max_{i\in L_{k-1}} \mu_i^n\right)^2,\end{aligned}$$

where C_0 is the maximum of the function $\|A\|$ over X. From this, (4.16) and (4.17) we deduce

$$|v_{n+1}| \leq C_0 \left(\max_{i \in L_{k-1}} \mu_i^n \right) \left(\min_{i \notin L_{k-1}} \mu_i^{n+1} \right)^{-1}.$$

This implies (4.13).

(Step 3) Repeating the above argument and using the boundedness of $\|A^{-1}\|$ on X, we can show that if $b_{n+1} \in F^{k-1,n+1}$, and u'_n is the projection of b_{n+1} onto $F^{k-1,n}$, then

$$\limsup_{n\to\infty} \frac{1}{n} \log \left| b_{n+1} - \frac{u'_n}{|u'_n|} \right| \leq -(l_k - l_{k-1}).$$

From this and (4.12) we deduce

$$\limsup_{n\to\infty} \frac{1}{n} \log d_{Gr} \left(F^{k-1,n}, F^{k-1,n+1} \right) \leq -(l_k - l_{k-1}). \tag{4.18}$$

This in turn implies that the limit

$$F^{k-1} := \lim_{n\to\infty} F^{k-1,n},$$

exists.

(Step 4) We wish to show that if $v \in F^k \setminus F^{k-1}$, $|v| = 1$, then

$$\lim_{n\to\infty} \frac{1}{n} \log |A_n v| = l_k, \tag{4.19}$$

μ-a.e. We already know,

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} \log |A_n v| &= \limsup_{n\to\infty} \frac{1}{n} \log |B_n v| \leq \limsup_{n\to\infty} \frac{1}{n} \log \left(\|B_n\| |v| \right) \\ &= \limsup_{n\to\infty} \frac{1}{n} \log \|B_n\| = \limsup_{n\to\infty} \frac{1}{n} \log \mu_d^n = l_k. \end{aligned} \tag{4.20}$$

Hence, for (4.19), we only need to show

$$\liminf_{n\to\infty} \frac{1}{n} \log |A_n v| \geq l_k, \tag{4.21}$$

For this we decompose v as

$$v = u_n + v_n, \qquad u_n \in F_x^{k-1,n}, \quad v_n = \sum_{i \notin L_{k-1}} c_i^n a_i^n \perp F_x^{k-1,n}.$$

Since $v \notin F^{k-1}$, we must have

$$\limsup_{n\to\infty} \frac{1}{n} \log |v_n| \geq 0. \tag{4.22}$$

Simply because if (4.22) were not true, then

$$\lim_{n\to\infty} \left| v - \frac{u_n}{|u_n|} \right| = 0,$$

which contradicts $v \notin F^{k-1}$. On the other hand, since $A_n u_n \perp A_n v_n$ as in (4.14), we can assert

$$\begin{aligned} \left|A_n v\right|^2 \geq \left|A_n v_n\right|^2 = \left|B_n v_n\right|^2 = \left| \sum_{i \notin L_{k-1}} c_i^n \mu_i^n a_i^n \right|^2 &= \sum_{i \notin L_{k-1}} \left(c_i^n\right)^2 \left(\mu_i^n\right)^2 \\ \geq \left(\min_{i \notin L_{k-1}} \mu_i^n \right)^2 \sum_{i \notin L_{k-1}} \left(c_i^n\right)^2 &= \left(\max_{i \in L_{k-1}} \mu_i^n \right)^2 |v_n|^2. \end{aligned}$$

This and (4.22) imply (4.21), completing the proof of (4.19).

(Final Step) Inductively, we can construct other F^j. For example, for $j = k-2$, the analog of (4.20) is the statement that if $v \in F^{k-1}$, then

$$\limsup_{n\to\infty} \frac{1}{n} \log |A_n(x) v| \leq l_{k-1}, \tag{4.23}$$

μ-a.e. To prove (4.23), we use the definition of F^{k-1} and (4.18), to find a sequence $\{u_n\}$ such that $u_n \in F^{k-1,n}$ and

$$\limsup_{n\to\infty} \frac{1}{n} \log |v - u_n| \leq -(l_k - l_{k-1}).$$

As a result,

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} \log |A_n v| &\leq \max \left\{ \limsup_{n\to\infty} \frac{1}{n} \log |A_n u_n|,\ \limsup_{n\to\infty} \frac{1}{n} \log \left(\|A_n\|\ |v - u_n| \right) \right\} \\ &\leq \max \left\{ l_{k-1}, \lambda_k + l_{k-1} - l_k \right\} = l_{k-1}, \end{aligned}$$

proving (4.23). □

We now state and prove an inequality of Ruelle.

Theorem 4.3 *Let $T : X \to X$ be C^1 and $\mu \in \mathcal{I}_T^{er}$. Then*

$$h_\mu(T) \le \sum_1^k n_j l_j^+.$$

Proof We only present the proof when $\dim X = d = 2$. First we would like to divide X into "small squares". For this we take a triangulation of X; $X = \cup_i \Delta_i$ where each Δ_i is a diffeomorphic copy of a triangle in $\mathbb{R}^2$ and $\Delta_i \cap \Delta_j$ is either empty, or a common vertex, or a common side. We then divide each triangle into squares of side length ε and possibly triangles of side length at most ε (we need these triangles near the boundary of Δ_i's). The result is a covering of X that is denoted by ξ^ε. Note that we may choose members of ξ^ε such that $\mu(\partial A) = 0$ for $A \in \xi^\varepsilon$. (If this is not the case, move each element of ξ^ε by small amount and use the fact that for some translation of boundary side we get zero measure.) As a result, ξ^ε is a μ-partition. It is not hard to show

$$h_\mu(T) = \lim_{\varepsilon \to 0} h_\mu(T, \xi^\varepsilon). \tag{4.24}$$

Recall that $h_\mu(T, \xi^\varepsilon) = \lim_{k\to\infty} \int I_{\xi^\varepsilon | \xi^{\varepsilon,k}} d\mu$ where $\xi^{\varepsilon,k} = T^{-1}(\xi^\varepsilon) \vee T^{-2}(\xi^\varepsilon) \vee \dots \vee T^{-k}(\xi^\varepsilon)$ and

$$I_{\xi^\varepsilon | \xi^{\varepsilon,k}} = -\sum_{A \in \xi^\varepsilon} \sum_{B \in \xi^{\varepsilon,k}} \frac{\mu(A \cap B)}{\mu(B)} \log \frac{\mu(A \cap B)}{\mu(B)} 1\!\!1_B.$$

Given x, let $B = B_{\varepsilon,k}(x)$ be the unique element of $\xi^{\varepsilon,k}$ such that $x \in B$. Such B is of the form $T^{-1}(C_1) \cap \dots \cap T^{-k}(C_k)$ with $C_1 \dots C_k \in \xi^\varepsilon$, where $C_j = C_{\xi^\varepsilon}(T^j(x))$. Let us write simply write $C_1(x)$ for $C_{\xi^\varepsilon}(T(x))$. We have

$$\begin{aligned} I_{\xi^\varepsilon | \xi^{\varepsilon,k}}(x) &\le \log \#\{A \in \xi^\varepsilon : A \cap B_{\varepsilon,k}(x) \ne \emptyset\} \\ &\le \log \#\{A \in \xi^\varepsilon : A \cap T^{-1}(C_1(x)) \ne \emptyset\}. \end{aligned} \tag{4.25}$$

Each $C_1(x)$ is a regular set; either a diffeomorphic image of a small square or a triangle. Since the volume of C is of order $O(\varepsilon^2)$, we have

$$\mathrm{vol}(T^{-1}(C)) \le c_1 \varepsilon^2 \max_{z \in C} \left| \det(dT)_z^{-1} \right|,$$

for a constant c_1. If $A \cap T^{-1}(C) \ne \emptyset$, then for a constant α_0,

$$A \subseteq \{y : |y - x_0| \le \alpha_0 \varepsilon \text{ for some } x_0 \in T^{-1}(C)\} =: D.$$

We now want to bound $\mathrm{vol}(D)$. The boundary of $T^{-1}(C)$ is a regular curve. Hence its length is comparable to the diameter of $T^{-1}(C)$, and this is bounded above by a multiple of the norm of dT^{-1}. Using this we obtain

$$\mathrm{vol}(D) \le c_2 \max_{z \in C} \left(1 + \left\|(dT)_z^{-1}\right\| + \left|\det(dT)_z^{-1}\right|\right) \varepsilon^2. \tag{4.26}$$

for a constant c_2. (We could have bounded $\mathrm{vol}(A)$ by $\big(\|(dT)^{-1}\|\varepsilon\big)^2$ but (4.26) is a better bound.)

We now use (4.26) to obtain an upper bound for the right-hand side (4.25). Indeed

$$\#\{A \in \xi^\varepsilon :\ A \cap T^{-1}(C) \neq \emptyset\} \leq c_3 \max_{z \in C} \big(1 + \big\|(dT)_z^{-1}\big\| + \big|\det(dT)_z^{-1}\big|\big) \tag{4.27}$$

for a constant c_3. This is because the union of such A's is a subset of D, for two distinct A, B, we have $\mu(A \cap B) = 0$, and for each $A \in \xi^\varepsilon$ we have that $\mathrm{vol}(A) \geq c_4 \varepsilon^2$ for some positive constant c_4. From (4.26) and (4.27) we learn

$$I_{\xi^\varepsilon|\xi^{\varepsilon,k}}(x) \leq c_5 + \log \max_{z \in C} \big(1 + \big\|(dT)_z^{-1}\big\| + \big|\det(dT)_z^{-1}\big|\big)$$

for $C = C_1(x)$. By sending $k \to \infty$ we deduce

$$h_\mu(T, \xi^\varepsilon) \leq c_5 + \int \log \max_{z \in C_{\xi^e}(T(x))} \big(1 + \big\|(dT)_z^{-1}\big\| + \big|\det(dT)_z^{-1}\big|\big)\ d\mu. \tag{4.28}$$

By the invariance of μ,

$$h_\mu(T, \xi^\varepsilon) \leq c_5 + \int \log \max_{z \in C_{\xi^\varepsilon}(x)} \big(1 + \big\|(dT)_z^{-1}\big\| + \big|\det(dT)_z^{-1}\big|\big)\ \mu(dx).$$

Send $\varepsilon \to 0$ to yield

$$h_\mu(T) \leq c_5 + \int \big(1 + \big\|(dT)_x^{-1}\big\| + \big|\det(dT)_x^{-1}\big|\big)\ \mu(dx).$$

The constant c_5 is independent of T. This allows us to replace T with T^{-n} to have

$$n h_\mu(T) \leq c_5 + \int \log \big(1 + \big\|d(T^n)\big| + \big|\det d(T^n)\big|\big)\ d\mu.$$

First assume that there are two Lyapunov exponents. Since

$$\frac{1}{n} \log \big\|d(T^n)\big| \to l_2, \quad \frac{1}{n} \log \big|\det d(T^n)\big| \to l_1 + l_2,$$

μ-a.e., we deduce

$$h_\mu(T) \leq \max(0, l_2, l_1 + l_2) \leq l_1^+ + l_2^+. \tag{4.29}$$

In the same way we treat the case of one Lyapunov exponent. □

The bound (4.29) may appear surprising because $h_\mu(T) > 0$ would rule out the case $l_1, l_2 < 0$. In fact we cannot have $l_1, l_2 < 0$ because we are assuming T is invertible. An invertible transformation cannot be a pure contraction. Moreover if $h_\mu(T) > 0$ we must have a hyperbolic transformation in the following sense:

Corollary 4.1 *If* $\dim X \geq 2$ *and* $h_\mu(T) > 0$*, then there exists a pair of Lyapunov exponents* α, β *such that* $\alpha > 0$*,* $\beta < 0$*. In particular, if* $\dim X = 2$ *and* $h_\mu(T) > 0$*, then* $l_1 < 0 < l_2$*.*

Proof Observe that if $l_1 < \cdots < l_k$ are Lyapunov exponents of T, then $-l_k < \cdots < -l_1$ are the Lyapunov exponents of T^{-1}. Simply because if $A_n(x) = D_x T^n$, then $A_{-n} \circ T^n = A_n^{-1}$. Now by Theorem 4.7,

$$\begin{aligned} h_\mu(T) = h_\mu(T^{-1}) &\leq \sum_i n_i(-l_i)^+ = \sum_i n_i l_i^-, \\ h_\mu(T) &\leq \sum_i n_i l_i^+. \end{aligned}$$

From these we deduce that $-\sum_i l_i^- < 0 < \sum_i l_i^+$ whenever $h_\mu(T) > 0$. □

Pesin's theorem below gives a sufficient condition for having equality in Theorem 4.7. We omit the proof of Pesin's formula.

Theorem 4.4 *Let* X *be a* C^1*-manifold and assume* $T : X \to X$ *is a* C^1 *diffeomorphism. Assume* DT *is Hölder continuous. Let* $\mu \in \mathcal{I}_T$ *be an ergodic measure that is absolutely continuous with respect to the volume measure of* X*. Then*

$$h_\mu(T) = \sum_i n_i l_i^+.$$

In the context of Theorem 4.3, the Lyapunov exponents of T^{-1} are $-l_k < \cdots < -l_1$. Let us write

$$\mathcal{T}_x X = \bigoplus_i \hat{E}_x^i,$$

for the splitting associated with T^{-1}. It is natural to define

$$E_x^s = \bigoplus_{l_i<0} E_x^i, \qquad E_x^u = \bigoplus_{l_i>0} \hat{E}_x^i,$$

If there is no zero Lyapunov exponent, we have $\mathcal{T}_x X = E_x^s \oplus E_x^u$, μ-almost everywhere. If we write $l^\pm = \min_i l_i^\pm$, then we have

$$\lim_{n\to\infty} \frac{1}{n} \log \left|(dT^{-n})_x v\right| \leq -l^+$$

for $v \in E_x^u \setminus \{0\}$, and

$$\lim_{n\to\infty} \frac{1}{n} \log \left|(dT^{n})_x v\right| \leq -l^-$$

for $v \in E^s_x - \{0\}$, μ-almost everywhere. If this happens in a uniform fashion, then we say that μ is an *Anosov measure*. More precisely, we say a that the measure $\mu \in \mathcal{I}^{ex}_T$ is *Anosov* if there exists a decomposition $\mathcal{T}_x X = E^u_x \oplus E^s_x$ and constants $K > 0$ and $\alpha \in (0,1)$ such that

$$\begin{aligned}
&(dT)_x E^u_x = E^u_{T(x)}, \qquad (dT)_x E^s_x = E^s_{T(x)},\\
&\left|(dT)^n_x)v\right| \le K\alpha^n |v| \qquad \text{for } v \in E^s_x,\\
&\left|(dT^{-n})v\right| \le K\alpha^n |v| \qquad \text{for } v \in E^u_x.
\end{aligned}$$

If we define

$$\begin{aligned}
W^s(x) &= \left\{y : \lim_{n\to\infty} d(T^n(x), T^n(y)) = 0\right\}\\
W^u(x) &= \left\{y : \lim_{n\to\infty} d(T^{-n}(x), T^{-n}(y)) = 0\right\}
\end{aligned}$$

with d a metric on X, then we have a nice foliation of X. In fact

$$\begin{aligned}
&W^s(x) \cap W^s(y) \ne \emptyset \quad \Rightarrow \quad W^s(x) = W^s(y),\\
&W^u(x) \cap W^u(y) \ne \emptyset \quad \Rightarrow \quad W^u(x) = W^u(y),\\
&E^u_x = \mathcal{T}_x W^u(x), \qquad E^s_x = \mathcal{T}_x W^s(x).
\end{aligned}$$

We also have a simple formula for the topological entropy:

$$h_{\text{top}}(T) = \int \log\left|\det(dT)_{E^u_x}\right| \ \mu(dx) = \sum_i n_i l_i^+,$$

where $(dT)_{E^u_x}$ denotes the restriction of $(dT)_x$ to E^u_x. An obvious example of an Anosov transformation is the Arnold cat transformation.

In the next section we study the Lyapunov exponents for Hamiltonian systems. As a prelude, we show that the Lyapunov exponents for a Hamiltonian flow come in a pair of numbers of opposite signs.

In the case of a Hamiltonian system, we have a *symplectic transformation* $T : X \to X$. This means that X is equipped with a *symplectic form* ω and if $A(x) = (dT)_x$, then

$$\omega_x(a,b) = \omega_{T(x)}(A(x)a, A(x)b). \tag{4.30}$$

By a symplectic form we mean a closed non-degenerate 2-form. As is well-known, $\dim X = 2d$ is always even, and the volume form associated with ω (namely the d-wedge product of ω) is preserved under T. An example of a symplectic manifold is $X = \mathbb{R}^{2d}$ that is equipped with the standard form $\bar\omega$: $\omega_x(a,b) = \bar\omega(a,b)$ with $\bar\omega(a,b) = Ja \cdot b$, and

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},$$

where I is the $d \times d$ identity matrix. In this case we may represent A as a matrix and the symplactic property means

$$A(x)^t J A(x) = J.$$

As is well-known, this in particular implies that $\det A(x) = 1$. Of course we already know this for Hamiltonian systems by Liouville's theorem, namely the volume is invariant under a Hamiltonian flow.

Theorem 4.5 *Let (X, ω) be a closed symplectic manifold of dimension $2d$. Then the Lyapunov exponents $l_1 < l_2 < \cdots < l_k$ satisfy $l_j + l_{k-j+1} = 0$ and $n_j = n_{k-j+1}$ for $j = 1, 2, \ldots, [k/2]$. Moreover the space $F_x^j := \bigoplus_{i=1}^{j1} E_x^j$ is ω–orthogonal complement of $\hat{E}_x^{k-j+1}$.*

Proof. Write $l(x, v) = \lim_{n\to\infty} \frac{1}{n} \log |A_n(x)v|$ where $A_n(x) = (dT^n)_x$ and $v \in \mathcal{T}_x X$. Note that since X is compact, we can find a constant c_0 such that

$$|\omega_x(a, b)| \leq c_0 |a||b|$$

for all $a, b \in T_x M$ and all $x \in M$. As a result,

$$|\omega_x(a, b)| = |\omega_{T^n(x)}(A_n(x)a, A_n(x)b)| \leq c_0 |A_n(x)a||A_n(x)b|,$$

and if $\omega_x(a, b) \neq 0$, then

$$l(x, a) + l(x, b) \geq 0. \tag{4.31}$$

By Theorem 4.4, we can find numbers $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_{2d}$ and spaces

$$\{0\} = V_0 \subseteq V_1(x) \subseteq \cdots \subseteq V_{2d-1}(x) \subseteq V_{2d}(x) = \mathcal{T}_x X$$

such that $\dim V_j(x) = j$ and if $v \in V_{j+1}(x) \setminus V_j(x)$, then $l(x, v) = \beta_j$. Of course $l_1 < \cdots < l_k$ are related to $\beta_1 \leq \cdots \leq \beta_{2d}$ by $\{l_1, \ldots, l_k\} = \{\beta_1, \ldots, \beta_{2d}\}$ and $n_j = \#\{s : \beta_s = l_j\}$. Note that if W is a linear subspace of $T_x M$ and

$$W^{\mathrm{II}} = \{b \in \mathcal{T}_x X : \omega(a, b) = 0 \text{ for all } a \in W\},$$

then one can readily show that $\dim W + \dim W^{\mathrm{II}} = 2d$. As a result, we can use $\dim V_j + \dim V_{2d-j+1} = 2d + 1$ to deduce that there exist $a \in V_j$ and $b \in V_{2d-j+1}$ such that $\omega(a, b) \neq 0$. Indeed the set

$$\Lambda = \big\{(a, b) \in (\mathcal{T}_x X)^2 : \ a \in V_j, \ b \in V_{2d-j+1}, \ \omega_x(a, b) \neq 0\big\}$$

is a nonempty open subset of $V_j \times V_{2d-j+1}$. Hence

$$\tilde{\Lambda} = \big\{(a, b) \in (\mathcal{T}_x X)^2 : \ a \in V_j \setminus V_{j-1}, \ b \in V_{2d-j+1} \setminus V_{2d-j}, \ \omega_x(a, b) \neq 0\big\}$$

is also nonempty. As a result, we can use (4.31) to assert

$$\beta_j + \beta_{2d-j+1} \geq 0, \tag{4.32}$$

for $j \in \{1, 2, \ldots, d\}$. On the other hand

$$\sum_{j=1}^{d} (\beta_j + \beta_{2d-j+1}) = \sum_i n_i l_i = 0,$$

by Remark 4.1**(ii)** because the volume is preserved. From this and (4.32) we deduce that

$$\beta_j + \beta_{2d-j+1} = 0.$$

From this we can readily deduce that $l_j + l_{k-j+1} = 0$ and $n_j = n_{k-j+1}$.

For the last claim, observe that since $l_j + l_{k-j+1} = 0$, we have $l_j + l_i < 0$ whenever $i + j \leq k$. From this and (4.31) we learn that if $i + j \leq k$ and $(a, b) \in E_x^i \times E_x^j$, then $\omega_x(a, b) = 0$. Hence $\hat{E}_x^{j-1} \subseteq (\hat{E}_x^{k-j+1})^{\perp}$. Since

$$n_1 + \cdots + n_{k-j+1} + n_1 + \cdots + n_{j-1} = n_1 + \cdots + n_{k-j+1} + n_k + \cdots + n_{k-j+2} = 2d,$$

we deduce that

$$\dim \hat{E}_x^{j-1} = \dim(\hat{E}_x^{k-j+1})^{\mathrm{II}}.$$

This in turn implies that $\hat{E}_x^{j-1} = (\hat{E}_x^{k-j+1})^{\mathrm{II}}$. □

We continue with a description of an approach that would allow us to approximate E^u and E^s. Recall that $\big(dT^n\big)_x E_x^u = E_{T^n(x)}^u$. For simplicity, let us assume that $d = 2$, so that if we have two Lyapunov exponents $l_1 < 0 < l_2$, then both E_x^u and E_x^s are straight lines. Now imagine that we can find a sector C_x such that

$$E_x^u \subset C_x, \qquad E_x^s \cap C_x = \{0\}, \qquad (dT)_x C_x \subsetneq C_{T(x)}.$$

(Recall that the E_x^s component of each vector $v \in C_x$ shrinks under the transformation $A(x) = (dT)_x$.) In other words, $A(x)C_x$ is a slimmer sector than $C_{T(x)}$. As we repeat this, we get a very slim sector $\big(dT^n\big)_x C_x$ at the point $T^n(x)$ that is approximating $E_{T^n(x)}^u$. To approximate E_x^u, we may try

$$C_x^n := \big(dT^n\big)_{T^{-n}(x)} C_{T^{-n}(x)}.$$

Observe

$$\begin{aligned} C_x^{n+1} &= \big(dT^{n+1}\big)_{T^{-n-1}(x)} C_{T^{-n-1}(x)} = \big(dT^n\big)_{T^{-n}(x)} \, (dT)_{T^{-n-1}(x)} C_{T^{-n-1}(x)} \\ &\subsetneq \big(dT^n\big)_{T^{-n}(x)} C_{T^{-n}(x)} = C_x^n. \end{aligned}$$

From this we expect

$$E^u_x = \cap_{n=0}^{\infty} C^n_x.$$

Similarly, if we can find a sector C'_x such that

$$E^s_x \subset C'_x, \qquad E^u_x \cap C'_x = \{0\}, \qquad \left(dT^{-1}\right)_x C'_x \subsetneq C_{T^{-1}(x)},$$

then we expect to have

$$E^s_x == \cap_{n=0}^{\infty} C^{-n}_{,}$$

where

$$C^{-n}_x := \left(dT^{-n}\right)_{T^n(x)} C_{T^n(x)}.$$

The existence of such sectors also guarantee that we have nonzero Lyapunov exponents. To see how this works in principle, let us examine an example.

Example 4.3 Consider a matrix-valued function $A(x)$, $x \in \mathbb{T}^2$ such that for almost all x, A has positive entries and $\det A(x) = 1$. Let $T : \mathbb{T}^2 \to \mathbb{T}^2$ be invariant with respect to the Lebesgue measure μ and define $l(x,v) = \lim_{n\to\infty} \frac{1}{n} \log |A_n(x)v|$, where

$$A_n(x) = A(T^{n-1}(x))A(T^{n-2}(x)) \cdots A(T(x))A(x).$$

Define the sector $C(x) \equiv C = \left\{ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} : v_1 v_2 > 0 \right\}$. Note that if $\begin{bmatrix} v'_1 \\ v'_2 \end{bmatrix} = A(x) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, and $A(x) = \begin{bmatrix} a(x) & b(x) \\ c(x) & d(x) \end{bmatrix}$, then

$$\begin{aligned} Q(v'_1, v'_2) = v'_1 v'_2 = (av_1 + bv_2)(cv_1 + dv_2) &= (1+2bc)v_1 v_2 + acv_1^2 + bdv_2^2 \\ &\geq (1+2bc)v_1 v_2 + 2\sqrt{acbd}\; v_1 v_2 =: \lambda Q(v_1, v_2). \end{aligned}$$

We can also show that A maps C onto a sector which lies strictly inside C. If

$$A_n(x) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1^{(n)} \\ v_2^{(n)} \end{bmatrix},$$

then

$$|A_n(x)v|^2 \geq 2v_1^{(n)} v_2^{(n)} \geq 2v_1 v_2 \prod_{i=0}^{n-1} \left[\lambda(T^i(x))\right],$$

$$\liminf_{n\to\infty} \frac{1}{n} \log |A_n(x)v| \geq \frac{1}{2} \int \log \lambda \; d\mu =: \bar{l} > 0,$$

whenever $v_1 v_2 > 0$. This implies that $l_2 > 0$. Since $\det A_n \equiv 1$, we know that $l_1 + l_2 = 0$. So $l_1 < 0 < l_2$. As for E_x^u, let us write $\theta_n(x)$ for the slope of the lower boundary of C_x^n. Since

$$C_x^{n+1} = A\big(T^{-1}(x)\big)C^n_{T^{-1}(x)},$$

we learn

$$\theta_{n+1}(x) = F\left(T^{-1}(x), \theta_n\big(T^{-1}(x)\big)\right), \tag{4.33}$$

where

$$F(x,\theta) = \frac{c(x) + d(x)\theta}{a(x) + b(x)\theta} = \frac{c(x)}{a(x)} + \frac{1}{a(x)}\frac{1}{b(x) + a(x)\theta^{-1}}. \tag{4.34}$$

This follows from the fact that if $\theta = v_2/v_1$ and $\theta' = v_2'/v_1'$, then $\theta' = F(x,\theta)$. The slope of E_x^u is given by

$$\theta^s := \lim_{n\to\infty} \theta_n(x),$$

with $\theta_0(x) = 0 < \theta_1(x) = c(x)/a(x)$. The last expression of (4.34) may be used to give an expression for θ^s as a continued fraction. Observe that the sequence θ_n is strictly increasing with $\theta_n < (d/b) \circ T^{-1}$ for all n. The latter follows fron the fact that F is increasing in θ: If we already know that $\theta_{n-1} < \theta_n$, then

$$\theta_{n+1}(x) = F\left(T^{-1}(x), \theta_n\big(T^{-1}(x)\big)\right) < F\left(T^{-1}(x), \theta_{n-1}\big(T^{-1}(x)\big)\right) = \theta_n(x).$$

The bound $\theta_n < (d/b) \circ T^{-1}$ is an immediate consequnce of the monoticity $F(x,\theta) < F(x,\infty) = d/b$. We may use the whole past history $\left\{T^{-n}(x) : n \in \mathbb{N}\right\}$ to express θ^s as a continued fraction

$$\theta^s = A_1 + \cfrac{1}{B_1 + \cfrac{C_1}{A_2 + \cfrac{1}{B_2 + \cfrac{C_2}{A_3 + \cfrac{1}{\ddots}}}}}, \tag{4.35}$$

where

$$A_i = \frac{c}{a} \circ T^{-i}, \qquad B_i = (ab) \circ T^{-i}, \qquad C_i = a^2 \circ T^{-i}.$$

□

In the continuous case the Lyapunov exponents are defined likewise. Consider a group of C^1-transformations $\{\phi_t : t \in \mathbb{R}\}$. Here each ϕ_t is from an m-dimensional manifold M onto itself. We then pick an ergodic measure $\mu \in \mathcal{I}_\phi$ and find a splitting $T_xM = E_x^1 \oplus \cdots \oplus E_x^k$ such that for $v \in E_x^1 \oplus \cdots \oplus E_x^j \setminus E_x^1 \oplus \cdots \oplus E_x^{j-1}$,

$$\lim_{t\to\infty} \frac{1}{t} \log |(d\phi_t)_x v| = l_j.$$

It turns out that we always have a zero Lyapunov exponent associated with the flow direction. More precisely, if $\frac{d}{dt}\phi_t(x)|_{t=0} = \xi(x)$, then

$$\lim_{t\to\infty} \frac{1}{t} \log |(d\phi_t)_x \xi| = 0.$$

Intuitively this is obvious because two phase points that lie close to each other on the same trajectory do not separate exponentially.

With the aid of the subadditive ergodic theorem, we managed to define Lyapunov exponents for C^1 diffeomorphism. Needless to say that the Lyapunov exponents and the corresponding decomposition of the tangent fibers provide us with valuable information about the underlying dynamical system. We now take a closer look at the type of linear transformations $A_n(x)$ that we encountered in Theorem 4.3. We write $\mathbb{G}_d = GL_d(\mathbb{R})$ for the set of invertible $d \times d$ invertible matrix.

Theorem 4.6 *Let (X, T) be a dynamical system, and take $\mu \in \mathcal{I}_T$. Let $A : X \to \mathbb{G}_d$ be a Borel function with $\log \|A^{\pm}\| \in L^1(\mu)$, and set*

$$A_n = \big(A \circ T^{n-1}\big) \dots \big(A \circ T\big) A.$$

Then

$$\Lambda(x) := \lim_{n\to\infty} \big(A_n(x)^t A_n(x)\big)^{\frac{1}{2n}}, \tag{4.36}$$

exists μ-a.e. Moreover,

$$\lim_{n\to\infty} \frac{1}{n} \log \big\|A_n(x)\Lambda(x)^{-1}\big\| = \lim_{n\to\infty} \frac{1}{n} \log \Big\|\big(A_n(x)\Lambda(x)^{-1}\big)^{-1}\Big\| = 0. \tag{4.37}$$

Observe that for any vector $v \in \mathbb{R}^d$,

$$\lim_{n\to\infty} \frac{1}{n} \log \big|\Lambda(x)^n v\big| = \lim_{n\to\infty} \frac{1}{n} \log \big|A_n(x) v\big|, \tag{4.38}$$

which exists by Theorem 4.3. However, (4.39) guarantees the existence of the limit

$$\log \Lambda(x) = \lim_{n\to\infty} \frac{1}{2n} \log\left(A_n(x)^t A_n(x)\right).$$

We may interpret $A_n(x)$ as a non-commutative ergodic average (or in a probabilist language, a non-commutative random walk) and (4.36) as a non-commutative analog of our Ergodic Theorem. To achieve (4.36), we first define a suitable metric on the space of positive definite matrices.

Definition 4.2(i) We write $\mathbb{M}_d$ for the space of $d \times d$ matrices, and $\mathbb{S}_d$ for the set of $d \times d$ symmetric matrices. Note that $\mathbb{S}_d$ is a subspace of $\mathbb{M}_d$ of dimension $d(d+1)/2$. The set of positive definite matrices is denotes by $\mathbb{P}_d$. On $\mathbb{M}_d$, we define the inner product and norm

$$\langle A, B\rangle = \mathrm{tr}(A^t B), \qquad \|A\| = \mathrm{tr}\left(A^t A\right)^{1/2}.$$

(ii) We regard $\mathbb{P}_d$ as a Riemannian manifold with tangent fiber $\mathcal{T}_P\mathbb{P}_d = \left\{P^{-1}A :\ A \in \mathbb{S}_d\right\}$, and the metric

$$\langle A, B\rangle_P = \langle P^{-1}A, P^{-1}B\rangle, \qquad \|A\|_P = \langle A, A\rangle_P^{1/2}.$$

Note

$$\|A\|_P^2 = \mathrm{tr}\left(P^{-1}AP^{-1}A\right) = \mathrm{tr}\left(P^{-1/2}AP^{-1/2}P^{-1/2}AP^{-1/2}\right) = \|P^{-1/2}AP^{-1/2}\|^2.$$

The Riemannian metric in turn induces a Riemannian distance on $\mathbb{P}_d$:

$$D(P_1, P_2) = \inf\left\{\int_0^1 \|\dot\gamma(t)\|_{\gamma(t)}\, dt :\ \gamma(0) = P_1,\ \gamma(1) = P_2,\ \gamma \text{ is } C^1\right\}.$$

(iii) For every $A \in \mathbb{G}_d$, we define the action $\varphi_A : \mathbb{S}_d \to \mathbb{S}_d$, by $\varphi_A(B) := A \odot B := A^t BA$. Evidently, $\varphi_A\left(\mathbb{P}_d\right) = \mathbb{P}_d$. □

Proposition 4.2 **(i)** *For every $P_1, P_2 \in \mathbb{P}_d$, and $A \in \mathbb{G}_d$, we have $D(P_1, P_2) = D\left(\varphi_A(P_1), \varphi_A(P_2)\right)$.*

(ii) *For every $P_1, P_2 \in \mathbb{P}_d$,*

$$D(P_1, P_2) = \left\|\log\left(P_1^{-1}P_2\right)\right\| = \left\|\log\left(P_2^{1/2}P_1^{-1}P_2^{1/2}\right)\right\| = \left(\sum_{i=1}^n (\log\lambda_i)^2\right)^{1/2}, \tag{4.39}$$

where $\lambda_1, \dots, \lambda_d$ are the eigenvalues of the matrix $P_1^{-1}P_2$. (Note that $\lambda_1, \dots, \lambda_d > 0$, because the matrices $P_1^{-1}P_2$ and $P_2^{1/2}P_1^{-1}P_2^{1/2}$ are similar, and $P_2^{1/2}P_1^{-1}P_2^{1/2} \in \mathbb{P}_d$.)

Proof(i) This is an immediate consequence of

$$\begin{aligned}\left\|\varphi_A(B)\right\|^2_{\varphi_A(C)} &= \left\|A^{-1}C^{-1}BA\right\|^2 = \operatorname{tr}\left(\left(A^{-1}C^{-1}BA\right)^t\left(A^{-1}C^{-1}BA\right)\right)\\ &= \operatorname{tr}\left(\left(C^{-1}B\right)^t\left(C^{-1}B\right)\right) = \|B\|^2_C.\end{aligned}$$

(ii) *(Step 1)* Let us write $\hat{D}(P_1, P_2)$ for the right-hand side of (4.39). Evidently, $\hat{D}(P_1, P_2) = \hat{D}\big(\varphi_A(P_1), \varphi_A(P_2)\big)$, because

$$\varphi_A(P_1)^{-1}\varphi_A(P_2) = A^{-1}P_1^{-1}P_2A,$$

and $P_1^{-1}P_2$ are similar. Hence, we only need to verify (4.39) for $P_1 = I$ because

$$\begin{aligned}D(P_1, P_2) &= D\big(\varphi_{P_1^{-1/2}}(P_1), \varphi_{P_1^{-1/2}}(P_2)\big) = D\big(I, \varphi_{P_1^{-1/2}}(P_1)\big),\\ \hat{D}(P_1, P_2) &= \hat{D}\big(\varphi_{P_1^{-1/2}}(P_1), \varphi_{P_1^{-1/2}}(P_2)\big) = \hat{D}\big(I, \varphi_{P_1^{-1/2}}(P_1)\big).\end{aligned}$$

We may also assume that $P_2 = e^A$ for some $A \in \mathbb{S}_d$. In summary, we only need to show: $D(I, e^A) = \|A\|$, for $A \in \mathbb{S}_d$. Observe that if $\gamma(t) = e^{tA}$, then

$$\int_0^1 \|\dot{\gamma}(t)\|_{\gamma(t)}\ dt = \|A\|.$$

Hence $D(I, e^A) \le \|A\|$. It remains to show

$$D(I, e^A) \ge \|A\|. \tag{4.40}$$

(Step 2) For (4.40), it suffices to show that if L_B is the derivative of the exponential map:

$$L_B(C) = \lim_{\delta \to 0} \delta^{-1}\big(e^{B+\delta C} - e^B\big),$$

then

$$\left\|e^{-B}L_B(C)\right\| \ge \|C\|. \tag{4.41}$$

Indeed, if (4.43) holds, then we can argue that for any C^1 path $\gamma(t) = e^{A(t)}$ with $A(0) = 0,\ A(1) = A$,

$$\int_0^1 \|\dot{\gamma}(t)\|_{\gamma(t)}\ dt = \int_0^1 \left\|e^{-A(t)}L_{A(t)}(\dot{A}(t))\right\|\ dt \ge \int_0^1 \|\dot{A}(t)\|\ dt \ge \left\|\int_0^1 \dot{A}(t)\ dt\right\| = \|A\|.$$

It remains to verify (4.43). Note that if $B = U^tDU, C = U^tEU$, for a unitary matrix U, then

$$\left\|e^{-B}L_B(C)\right\| = \left\|U^te^{-D}L_D(E)U\right\| = \left\|e^{-D}L_D(E)\right\|.$$

Hence, it suffices to verify

$$\left\|e^{-D}L_D(E)\right\| \geq \|E\| = \|C\|, \tag{4.42}$$

for any diagonal matrix $D = diag(\lambda_1, \dots, \lambda_d)$, and symmetric matrix E. Let us write E_{ij} for the matrix that has 1 for its (i,j)-th entry, and 0 for any other entries. We certainly have $e^{-D}L_D(E_{ii}) = E_{ii}$, which implies (4.42) when $E = E_{ii}$. On the other hand, when $i \neq j$,

$$\begin{aligned} e^{D+\delta E_{ij}} &= \sum_{k=0}^{\infty} \frac{(D+\delta E_{ij})^k}{k!} = e^D + \delta E_{ij} + \delta \sum_{k=2}^{\infty} (k!)^{-1}\big(\lambda_i^{k-1} + \lambda_i^{k-2}\lambda_j + \cdots + \lambda_j^{k-1}\big)E_{ij} \\ &= e^D + \delta E_{ij} + \delta \sum_{k=2}^{\infty} \frac{\lambda_i^k - \lambda_j^k}{k!(\lambda_i - \lambda_j)} E_{ij} = e^D + \delta \frac{e^{\lambda_i} - e^{\lambda_j}}{\lambda_i - \lambda_j} E_{ij}, \end{aligned}$$

because

$$DE_{ij} = \lambda_i E_{ij}, \quad E_{ij}D = \lambda_j E_{ij}, \quad E_{ij}^2 = 0.$$

As a result,

$$e^{-D}L_D E_{ij} = \frac{e^{\lambda_i} - e^{\lambda_j}}{\lambda_i - \lambda_j} e^{-D} E_{ij} = e^{-\lambda_i} \frac{e^{\lambda_i} - e^{\lambda_j}}{\lambda_i - \lambda_j} e^{-D} E_{ij}.$$

This in turn implies

$$e^{-D}L_D(E_{ij} + E_{ji}) = \frac{e^{\lambda_i} - e^{\lambda_j}}{\lambda_i - \lambda_j}\big(e^{-\lambda_i}E_{ij} + e^{-\lambda_j}E_{ji}\big).$$

Hence

$$\begin{aligned} \left\|e^{-D}L_D(E_{ij} + E_{ji})\right\|^2 &= \left(\frac{e^{\lambda_i} - e^{\lambda_j}}{\lambda_i - \lambda_j}\right)^2 \left\|e^{-\lambda_i}E_{ij} + e^{-\lambda_j}E_{ji}\right\|^2 \\ &= \left(\frac{e^{\lambda_i} - e^{\lambda_j}}{\lambda_i - \lambda_j}\right)^2 \big(e^{-2\lambda_i} + e^{-2\lambda_j}\big). \end{aligned}$$

To verify (4.42) for $E = E_{ij} + E_{ji}$, we need to check

$$\big(e^{\lambda_i} - e^{\lambda_j}\big)\big(e^{-2\lambda_i} + e^{-2\lambda_j}\big) \geq 2(\lambda_i - \lambda_j)^2.$$

Equivalently,

$$2 + e^{2z} + e^{-2z} - 2e^z - 2e^{-z} \geq 2z^2,$$

for $z = \lambda_i - \lambda_j$, which is straightforward to check. Finally, since L_D is linear, the collection

$$\big\{E_{ij} + E_{ji} : \ i,j = 1, \dots, n\big\},$$

is a orthogonal basis for $\mathbb{S}_d$, and (4.40) is true for members of this basis, we are done. $\square$

We now formulate a general setting for a geometric treatment of a non-commutative ergodic theorem of Karlsson and Margulis.

(i) As before, we have a dynamical system (X, T), and another metric space $(\mathbb{P}, D)$ that is *proper* (every bounded closed set is compact).

(ii) We have a topological group $\mathbb{G}$ that acts on $\mathbb{P}$. More precisely, for each $A \in \mathbb{G}$, there exists an invertible map $\varphi_A : X \to X$ that is an isometry:

$$D\big(\varphi_A(P_1), \varphi_A(P_2)\big) = D(P_1, P_2).$$

To ease the notation, we also use the notation $A \odot P = \varphi_A(P)$.

(iii) A measurable map $\mathcal{A} : X \to \mathbb{G}$ is given, and we define

$$\mathcal{A}_n(x) = \mathcal{A}(x)\mathcal{A}(T(x)) \dots \mathcal{A}(T^{n-1}(x)).$$

For our non-commutative Ergodic Theorem, we fix $I \in \mathbb{P}$, and examine

$$\bar{A} := \lim_{n\to\infty} \frac{1}{n} D\big(I, \mathcal{A}_n(x) \odot I\big) =: \lim_{n\to\infty} \frac{1}{n} S_n(x). \tag{4.43}$$

Note that since

$$\begin{aligned} S_{m+n}(x) &\le D\big(I, \mathcal{A}_m(x) \odot I\big) + D\big(\mathcal{A}_m(x) \odot I, \big(\mathcal{A}_m(x)\mathcal{A}_n(T^m(x))\big) \odot I\big) \\ &= S_m(x) + S_n\big(T^m(x)\big), \end{aligned}$$

we may apply the subadditive ergodic theorem to assert that the limit in (4.42) exists.

Definition 4.3(i) For each $P \in \mathcal{P}$ we set

$$h_P(Q) = D(Q, P) - D(I, P).$$

If we write $\mathcal{L}$ for the set of Lipschitz functions h of Lipschitz constant 1, such that $h(I) = 0$, then $\Psi : \mathbb{P} \to \mathcal{L}$, defined by $\Psi(P) = h_P$, is a continuous injective map, provided that we equip $\mathcal{L}$ with the topology of local uniform convergence (which is metrizable). We regard $\mathcal{H} := \overline{\Psi(\mathbb{P})}$ as a compactification of $\mathbb{P}$ (note that $\mathcal{L}$ is a compact metric space).

(ii) The group $\mathbb{G}$ is also acting on $\mathcal{H}$; Given $A \in \mathbb{G}$, we define

$$(\varphi'_A h)(Q) := \big(A \odot' h\big)(Q) := h\big(A^{-1} \odot Q\big) - h\big(A^{-1} \odot I\big).$$

for $h \in \mathcal{H}$. Note

$$\begin{aligned} h_{A\odot P}(Q) &= D\big(Q, A \odot P\big) - D\big(I, A \odot P\big) \\ &= D\big(A^{-1} \odot Q, P\big) - D\big(A^{-1} \odot I, P\big) = \big(A \odot' h_P\big)(Q), \end{aligned}$$

which explain the reason behind our definition of $\hat{\varphi}_A$.

(iii) We now define a dynamical system on $\hat{X} := X \times \mathcal{H}$: A map $\hat{T} : \hat{X} \to \hat{X}$ is defined by

$$\hat{T}(x,h) = \big(T(x), \mathcal{A}(x)^{-1} \odot' h\big) = \big(T(x), F(x,h)\big),$$

where $F(x,h)(P) = h\big(\mathcal{A}(x) \odot P\big) - h\big(\mathcal{A}(x) \odot I\big)$. Observe

$$\hat{T}^n(x,h) = \big(T^n(x), \mathcal{A}_n(x)^{-1} \odot' h\big) =: \big(T^n(x), F_n(x,h)\big),$$

with $F_n(x,h)(P) = h\big(\mathcal{A}_n(x) \odot P\big) - h\big(\mathcal{A}_n(x) \odot I\big)$. □

The main idea of Karlsson and Margulis is that this limit can be rewritten as an ergodic average. To see this, observe that if $f(x,h) = h\big(\mathcal{A}(x) \odot I\big)$, then

$$\begin{aligned} f(x,h) + f\big(\hat{T}(x,h)\big) &= h\big(\mathcal{A}(x) \odot I\big) + F(x,h)\big(\mathcal{A}(T(x)) \odot I\big) \\ &= h\big(\mathcal{A}(x) \odot I\big) + h\big(\mathcal{A}(x) \odot \big(\mathcal{A}(T(x)) \odot I\big)\big) - h\big(\mathcal{A}(x) \odot I\big) \\ &= h\big(\mathcal{A}_2(x) \odot I\big). \end{aligned}$$

More generally,

$$h\big(\mathcal{A}_n(x) \odot I\big) = \sum_{i=0}^{n-1} f\big(\hat{T}^i(x,h)\big). \tag{4.44}$$

Let us observe

$$D(Q) := D(I,Q) = -\inf_{h \in \mathcal{H}} h(Q), \tag{4.45}$$

because

$$-h_P(Q) = D(I,P) - D(Q,P) \le D(I,Q), \quad h_Q(Q) = -D(I,Q).$$

From this, (4.42) and (4.44) we learn

$$\bar{A} = \lim_{n\to\infty} \frac{1}{n} D\big(I, \mathcal{A}_n(x) \odot I\big) = -\lim_{n\to\infty} \inf_{h \in \mathcal{H}} \frac{1}{n} \sum_{i=0}^{n-1} f\big(\hat{T}^i(x,h)\big). \tag{4.46}$$

Exercise (i) Verify the following properties of $l(x,v)$ of Remark 4.1**(iii)** directly (without using Theorem 4.3):

(i) $l(x, \alpha v_1) = l(x, v_1)$, $l(x, v_1 + v_2) \le \max(l(x,v_1), l(x,v_2))$ for every x, v_1, and v_2 and scalar $\alpha \neq 0$.

(ii) $l(T(x), A(x)v) = l(x, v)$

(iii) We have $l(x, v) \in [-\infty, +\infty)$.

(iv) The space $\{v : l(x, v) \leq t\} = V_x(t)$ is linear and that $V_x(s) \subseteq V_x(t)$ for $s \leq t$, $A(x)V_x(t) \subseteq V_{T(x)}(t)$.

(v) There exists $k(x) \in \mathbb{N}$, numbers $l_1(x) < l_2(x) < \cdots < l_{k(x)}(x)$ and splitting $T_xM = E_x^1 \oplus \cdots \oplus E_x^{k(x)}$ such that if $v \in E_x^1 \oplus \cdots \oplus E_x^j \setminus E_x^1 \oplus \cdots \oplus E_x^{j-1}$ then $l(x, v) = l_j$. Indeed $E_x^1 \oplus \cdots \oplus E_x^j = V_x(l_j)$.

(ii) Prove Parts **(i)** and **(v)** of Proposition 4.1. □

5 Lorentz Gases and Billiards

So far we have discussed various statistical notions such as ergodicity, entropy and Lyapunov exponents for dynamical systems. We have examined these notions for a rather limited number of examples, namely toral automorphisms, translations (or free motions) and one-dimensional expansions. In this section we study examples coming from classical mechanics. A *Lorentz gas* is an example of a gas in which heavy molecules are assumed to be immobile and light particles are moving under the influence of forces coming from heavy particles. The dynamics of a light particle with position $q(t)$ is governed by the Newton's law

$$\frac{d^2q}{dt^2} = -\nabla V(q),$$

where $V(q) = \sum_j W(|q - q_j|)$ with q_j denoting the center of immobile particles and $W(|z|)$ represents a central potential function. For simplicity we set the mass of the light particle to be 1. We may rewrite (5.1) as

$$\frac{dq}{dt} = p, \qquad \frac{dp}{dt} = -\nabla V(q). \tag{5.1}$$

Recall that the total energy $H(q,p) = \frac{1}{2}|p|^2 + V(q)$ is conserved. Because of this, we may wish to study the ergodicity of our system restricted to an energy shell

$$\{(q,p) : H(q,p) = E\}.$$

When W is of compact support, we may simplify the model by taking

$$W(|z|) = \begin{cases} 0 & \text{if } |z| > \varepsilon, \\ \infty & \text{if } |z| \le \varepsilon. \end{cases} \tag{5.2}$$

To interpret (5.1) for W given by (5.2), let us first assume that the support of $W(|q - q_i|)$, $i \in \mathbb{Z}$ are non overlapping. Assume a particle is about to enter the support of $W(|q - q_i|)$. For such a scenario, we may forget about other heavy particles and assume that the potential energy is simply given by $W(|q - q_i|)$. For such a potential we have two conservation laws:

$$\text{conservation of energy: } \frac{d}{dt}\left(\frac{1}{2}|p|^2 + V(|q - q_i|)\right) = 0$$

$$\text{conservation of angular momentum: } \frac{d}{dt}\big(p \times (q - q_i)\big) = 0.$$

Let us assume that a particle enters the support at a position q with velocity p and exits the support at a position q' with velocity p'. For a support choose a ball of center q_i and diameter ε. If $n = \frac{q - q_i}{|q - q_i|}$ and $n' = \frac{q' - q_i}{|q' - q_i|}$, then we can use the above conservation laws to conclude that $|p'| = |p|$ and the angle between (p, n) is the negation of the angle between (p', n').

The same conservation laws hold for the case (5.2). We are now ready for interpretation of dynamics when W is given by (5.2). Draw a ball of diameter ε and center q_i for each i. Then the phase space is

$$X = \left(\mathbb{R}^d \setminus \bigcup_i B_{\varepsilon/2}(q_i)\right) \times \mathbb{R}^d = \big\{(q, p) : |q - q_i| \geq \varepsilon \text{ for all } i, \text{ and } p \in \mathbb{R}^d\big\}.$$

For $q \notin \partial X$ we simply have $\frac{dq}{dt} = p$. When $|q - q_i| = \varepsilon$ then the dynamics experiences a jump discontinuity in p-component. More precisely

$$|q(t) - q_i| = \varepsilon \quad \text{implies} \quad p(t+) = p(t-) - 2p(t-) \cdot n_i(t) n_i(t), \tag{5.3}$$

where $n_i(t) = \frac{q(t) - q_i}{|q(t) - q_i|}$. As our state, we may consider

$$\begin{aligned} X &= \{q : |q - q_i| \geq \varepsilon \text{ for all } i\} \times \{p : |p| = 1\} \\ &=: Y_\varepsilon \times \mathbb{S}^{d-1}. \end{aligned}$$

Classically two possibilities for the configurations of q_i's are considered. As the first possibility, imagine that the q_i's are distributed periodically with period 1. Two cases may occur: Either $\varepsilon < 1$ which corresponds to an *infinite horizon* because a light particle can go off to infinity; or $\varepsilon \geq 1$ which corresponds to a finite horizon.

As our second possibility we distribute q_i's randomly according to a Poissonian probability distribution.

In this section we will study Lorentz gases on tori. In the periodic case of an infinite horizon, we simply have a dynamical system with phase space

$$M = \big(\mathbb{T}^d \setminus B_\varepsilon\big) \times \mathbb{S}^{d-1} =: Y_\varepsilon \times \mathbb{S}^{d-1},$$

where $\mathbb{T}^d \setminus B_\varepsilon$ represents a torus from which a ball of radius $\varepsilon/2$ is removed. In the case of finite horizon our $M = Y_\varepsilon \times \mathbb{S}^{d-1}$ but now Y_ε is a region confined by 4 concave arcs. In the random case we may still restrict the dynamics to a torus. For example, we select N points $q_1, \dots, q_j$ randomly and uniformly from the set

$$X_\varepsilon = \{(q_1, \dots, q_N) : |q_i - q_j| > \varepsilon \text{ for } i \neq j\},$$

and then we set

$$Y_\varepsilon = \{q : |q - q_i| \geq \varepsilon \text{ for } i = 1, \dots, N\}.$$

A Lorentz gas can be regarded as an example of a billiard. For the sake of definiteness let us focus on the case of finite horizon that can be recast as a billiard in a bounded domain with piecewise smooth boundary. More generally, let us take a bounded region Y^o in $\mathbb{R}^d$ with piecewise smooth boundary and set $X = Y \times \mathbb{S}^{d-1}$, where Y is the topological closure of Y^o. We set set

$$\partial^\pm X = \big\{(q,p): \ q \in \partial Y, \ \pm(p \cdot \nu(q)) \geq 0\big\},$$

where $\nu(q)$ represents the unit inward normal to ∂Y at q. Points in $\partial^- X$ and $\partial^+ X$ represent the pre and post collisional states respectively in our billiard. We now define two closely related dynamical systems.

(i) A continuous dynamical system $\phi_t(q,p) = (q(t), p(t))$, that is defined in the following way: so long as $q(t) \in Y^\circ = Y \setminus \partial Y$, we have

$$\frac{dq}{dt}(t) = p(t), \qquad \frac{dp}{dt}(t) = 0.$$

However, when $q(t)$ reaches a boundary point, $p(t)$ experiences a jump discontinuity. More precisely, whenever $q(t) \in \partial Y$ with $p(t) \cdot \nu\big(q(t)\big) \leq 0$, then

$$p'(t) := p(t+) = R_{q(t)} p(t-) := p(t-) - 2\left(p(t-) \cdot \nu\big(q(t)\big)\right) \nu\big(q(t)\big).$$

(ii) A discrete dynamical system on $\partial^+ X$ associated with a map $T : \partial^+ X \to \partial^+ X$. The map $T(q,p) = (Q,P)$ is defined by the following recipe:

$$Q = q + \tau(q,p)p, \qquad P = p - 2\left(p \cdot \nu\big(Q\big)\right) \nu\big(Q\big),$$

where $\tau(q,p)$ is the smallest $\tau > 0$ such that $\phi_\tau(q,p)$ reaches the boundary.

Next we find an invariant measure for the dynamical system $(q(t), p(t))$ and the map T. Regarding the flow ϕ_t as a Hamiltonian flow, we expect that the normalized Lebesgue measure $m(dx) = Z^{-1} dq\ dp$ where Z is a normalizing constant, dq is the Lebesgue measure on Y, and dp is the standard volume measure on $\mathbb{S}^{d-1}$ (compare with Example 1.7**(ii)**). To prove the invariance of m, let us take a smooth test function $\zeta : X \to \mathbb{R}$ such that $\zeta\big(q, R_q p\big) = \zeta(q,p)$ whenever $(q,p) \in \partial^- X$. Such a test function produces

$$(T_t \zeta)(x) = u(x,t) = \zeta(\phi_t(x)),$$

that is continuous in (x,t). In fact u satisfies a Liouville-type equation with boundary conditions:

$$\begin{cases} u_t = p \cdot u_q, & x \in X \setminus \partial X; \\ u\big(q, R_q p, t\big) = u(q,p,t), & t \geq 0, \ (q,p) \in \partial^- X. \end{cases} \tag{5.4}$$

We expect (5.4) to be true weakly; if K is a smooth function, then the expression

$$-\frac{d}{dt} \int_X u(x,t) K(x)\ dq dp,$$

equals

$$\int_X u(x,t)p\cdot K_q(x)\ dqdp + \int_{\partial X} u(x,t)K(x)\big(p\cdot\nu(q)\big)\ \sigma(dq)dp, \tag{5.5}$$

where $\sigma(dq)$ represents the surface integration on ∂Y. To verify this, let us write

$$\tau_0(x) = 0 < \tau_1(x) < \tau_2(x) < \dots$$

for a sequence of functions, such that $\phi_t(x)\in X\setminus\partial X$ for $t\in(\tau_j(x),\tau_{j+1}(x))$, $\phi_{\tau_j(x)}(x)\in\partial X$ if $j>0$, and each finite interval $[0,T]$ can have only finitely many τ_i's. Let us explain this further.

Note that $u(x,t) = J(\phi_t(x))$ is as smooth as J in (x,t) provided that $\phi_t(x)\notin\partial X$. This means that u is as smooth as J with $u_t = p\cdot u_q$, provided $(x,t)\in X\times(0,\infty)\setminus\bigcup_j S_j$, where

$$S_j = \{(x,t):\tau_j(x)=t\}.$$

Note that when t is restricted to a finite interval $[0,T]$, then finitely many S_j's are relevant, each S_j is of codimension 1 in $X\times(0,T)$, and different S_j's are well-separated. It is a general fact that if u is continuous and $u_t = p\cdot u_q$ off $\bigcup_j S_j$, then $u_t = p\cdot u_q$ weakly in X. To see this, take a test function $R(x,t)$ with support in an open set U such that exactly one of the S_j's bisect U into U^+ and U^-. We then have $\int u(R_t - p\cdot R_q)dx\ dt = \int_{U^+} + \int_{U^-}$ and that if we integrate by parts on each $U^\pm$ we get two contributions. One contribution comes from carrying out the differentiation on u, i.e., $\int_{U^\pm}(-u_t + p\cdot u_q)R\ dx\ dt$, which is 0 because $u_t = p\cdot u_q$ in $U^\pm$. The other contribution comes from the boundary of $U^\pm$, and they cancel each other out by the continuity of u.

As a consequence of (5.4) we have that the Lebesgue measure m is invariant. In fact if initially x is selected according to a probability measure $d\mu = f^0(x)dx$, then at later times $x(t)$ is distributed according to $d\mu_t = f(x,t)dx$ where $f(x,t) = f^0(\phi_{-t}(x))$. To see this observe that if we choose $K\equiv 1$ in (5.5), we have

$$\frac{d}{dt}\int \zeta(\phi_t(x))dx = -\int_{\partial X} u(q,p)\big(p\cdot\nu(q)\big)\ \sigma(dq)dp. \tag{5.6}$$

If we integrate over p first and make a change of variable $p\mapsto p' = p - 2p\cdot n\ n$, for $n=\nu(q)$, then u does not change and $p\cdot n$ becomes $p'\cdot n = -p\cdot n$. Also the Jacobian of such a transformation is 1. As a result, the right-hand side of (5.7) is equal to its negation. This implies

$$\int J(\phi_t(x))dx = \int J(x)dx, \tag{5.7}$$

for every t and every J continuous with $J(q,p') = J(q,p)$ on ∂X. If K and f^0 have the same property and we choose

$$J(x) = f^0(\phi_{-t}(x))K(x),$$

then we deduce

$$\int K(x)f^0(\phi_{-t}(x))dx = \int K(\phi_t(x))f^0(x)dx.$$

From this we conclude

$$f(x,t) = f^0(\phi_{-t}(x)), \tag{5.8}$$

as was claimed before.

Remark 5.1 If ϕ_t is the flow of the ODE (5.1), and $f(x,t) == f^0(\phi_{-t}(x))$, then the function f satisfies the Liouville's equation

$$f_t + p \cdot f_q - \nabla V(q) \cdot f_p = 0.$$

The partial derivatives $\alpha = f_q$ and $\beta = f_p$ satisfy

$$\begin{cases} \alpha_t + \alpha_q p - \alpha_p \nabla V(q) = D^2 V(q)\beta, \\ \beta_t + \beta_q p - \beta_p \nabla V(q) = -\alpha. \end{cases}$$

Note that if $Q(x,t) = f_q(x,t) \cdot f_p(x,t) = \alpha(x,t) \cdot \beta(x,t)$, then

$$Q_t + p \cdot Q_q - \nabla V(q) \cdot Q_p = D^2 V(q)\beta \cdot \beta - |\alpha|^2$$

or equivalently

$$\bar{Q}_t = D^2 V(q(t))\bar{\beta} \cdot \bar{\beta} - |\bar{\alpha}|^2$$

for $\bar{Q}(x,t) = Q(\phi_t(x),t)$. In the case of a billiard, the function f satisfies

$$\begin{cases} f_t + p \cdot f_q = 0 & \text{inside } Y \times \mathbb{R}^d, \\ f(q,p,t-) = f(q,p',t+) & \text{on } \partial Y \times \mathbb{R}^d, \end{cases}$$

where $q \in \partial Y$ and t is a collision time. Setting $a(q,p,t) = f_q(q,p,t)$, $b(q,p,t) = f_p(q,p,t)$, we then have

$$\begin{cases} a_t + pD_q a = 0, \\ b_t + pD_p a = -a. \end{cases}$$

Later in this chapter we will learn how to relate $\alpha(q,p,t-)$ to $\beta(q,p',t+)$ on the boundary ∂X as we study the evolution of $d\phi_t$. □

Identity (5.7) suggests that the invariant measure m of ϕ_t induces an invariant measure

$$d\mu = \big(p \cdot \nu(q)\big)\ \sigma(dq)dp,$$

on $\partial^+ X$. To explain this, let us is let us define

$$\hat{X} = \{(x,t) = (q,p,t) : \ (q,p) \in \partial^+ X, \ 0 \le t < \tau(q,p)\},$$

and $F : \hat{X} \to X$ by $F(x,t) = \phi_t(x)$. It is not hard to see that F is invertible. In fact F is an automorphism between the measure spaces (X, dm) and $(\hat{X}, \mu(dx)dt)$. This simply follows from the fact that the Jacobian of the transformation $(q,t) \mapsto q + pt$, equals $p \cdot \nu(q)$. The transformation F provides us with a useful representation of points in X. Using this representation we can also represent our dynamical system in a special form that is known as *special flow representation.* Let us study $F^{-1} \circ \phi_\theta \circ F$:

$$\hat{\phi}_\theta := F^{-1} \circ \phi_\theta \circ F(x,t) = \begin{cases} (x, \theta + t) & \theta + t < \tau(x) \\ (T(x), \theta + t - \tau_1(y)) & \theta + t - \tau(x) < \tau(T(x)) \\ \vdots \end{cases} \tag{5.9}$$

The measure $\mu(dx)dt$ is an invariant measure for the flow $\hat{\phi}_\theta$. We choose A sufficiently small in diameter so that we can find θ_1, θ_2 and θ_3 with the following property:

$$t \in [\theta_1, \theta_2] \Rightarrow \tau(x) < \theta_3 + t < \tau(T(x))$$

for every $x \in A$. This means

$$\hat{\phi}_\theta(A \times [\theta_1, \theta_2]) = \{(T(x), \theta_3 + t - \tau_1(x) : y \in A, \ t \in [\theta_1, \theta_2]\}.$$

Since $\hat{\phi}_\theta$ has $d\mu dt$ for an invariant measure,

$$(\theta_2 - \theta_1)\mu(A) = (\theta_2 - \theta_1)\mu(T(A)).$$

Since T is invariant, we deduce that μ is invariant.

We say a billiard table Y is *dispersive* if there exists $\delta > 0$ such that

$$(d\nu)_q \ge \delta I, \tag{5.10}$$

for every $q \in \partial Y$ at which ν is differentiable. The function ν is the celebrated *Gauss map*, and the operator $d\nu$ is known as the *shape operator* of ∂Y.

As we will see below all *dispersive billiards* have some positive Lyapunov exponents. This was shown by Sinai when $d = 2$, and by Chernov and Sinai when $d \ge 3$.

Theorem 5.1 *Let Y be a dispersive billiard table satisfying* (5.10).

(i) *There exists a T-invariant function $g \ge 0$ with $\int g \, d\mu > 0$, such that for μ-almost every $(q,p) \in \partial^+ X$ and any $(\hat{q}, \hat{p}) \in \mathcal{T}_{(q,p)} \partial^+ X$, with $\hat{q} \cdot \hat{p} > 0$, we have*

$$\liminf_{n \to \infty} n^{-1} \log \left| (dT^n)_{(q,p)}(\hat{q}, \hat{p}) \right| \ge g(q,p). \tag{5.11}$$

(ii) *There exists a ϕ-invariant function $h \geq 0$ with $\int h \, dm > 0$, such that for m-almost every $(q,p) \in X$ and any $(\hat{q},\hat{p}) \in \mathbb{R}^{2d}$, with $p \cdot \hat{p} = p \cdot \hat{q} = 0$, and $\hat{q} \cdot \hat{p} > 0$, we have*

$$\liminf_{t \to \infty} t^{-1} \log \left| (d\phi_t)_{(q,p)}(\hat{q},\hat{p}) \right| \geq h(q,p). \tag{5.12}$$

Note that since ϕ_t is discontinuous, it is not clear that $d\phi_t$ and dT are well-defined. As we will see in Proposition 5.1 both $d\phi_t$ and dT can be defined almost everywhere with respect to the invariant measure m and μ respectively. In fact we have a very precise meaning for the flow $d\phi_t$ that will be described shortly.

Recall that if ϕ_t is the flow associated with an ODE of the form $\frac{dx}{dt} = b(x)$, then the matrix-valued function $(d\phi_t)_x$ solves

$$\frac{dA}{dt} = (db)_{\phi_t(x)} \circ A.$$

Hence, $\hat{x}(t) = A(x,t)\hat{x}$ solves the equation

$$\frac{d\hat{x}}{dt}(t) = B(x,t)\hat{x}(t),$$

where $B(x,t) = (db)_{\phi_t(x)}$. In the case of a Hamiltonian flow of the form (5.2), we have $b(q,p) = (p, -\nabla V(q))$ and $\hat{x}(t) = (\hat{q}(t),\hat{p}(t))$ solves

$$\frac{d\hat{q}}{dt}(t) = \hat{p}(t), \quad \frac{d\hat{p}}{dt}(t) = -D^2V(q(t)) \; \hat{q}(t).$$

For our billiard model, some care is needed because $\phi_t(x)$ is not even continuous. We wish to derive an evolution equation for

$$\hat{x}(t) = (d\phi_t)_x\big(\hat{x}(0)\big).$$

We think of $(x,\hat{x}) \in \mathcal{T}X$ as the initial data for the path

$$(\phi_t(x), (d\phi_t)_x(\hat{x}))\,.$$

For our purposes, we take a path $\big(x^*(\theta) : \ \theta \in (-\delta_0,\delta_0)\big)$ with $x^*(0) = x, \dot{x}^*(0) = \hat{x}$, and keep track of

$$\hat{x}(t) = \hat{\phi}_t(\hat{x}) := x^*_\theta(t,0). \tag{5.13}$$

where $x^*(t,\theta) = \phi_t\big(x^*(\theta)\big)$. We use (5.13) to define $\hat{x}(t)$. In the same fashion, we define dT: Take a path $x^*(\theta) = \big(q^*(\theta),p^*(\theta)\big)$, that lies on $\partial^+ X$, such that $q^*(0) = q, p^*(0) = p$, and

$\dot{q}(0) = \hat{q} \in \partial Y, \dot{p}(0) = \hat{p} \in p^\perp$, and if we write $T(q,p) = (Q,P)$, then $Q = q + \tau(q,p)p$, and $P = R_Q p$. We then set $\big(Q(\theta), P(\theta)\big) = T\big(x^*(\theta)\big)$, and define

$$(dT)_{(q,p)}(\hat{q},\hat{p}) := \big(\dot{Q}(0), \dot{P}(0)\big). \tag{5.14}$$

In analogy with the Riemannian geometry, we may regard $\hat{x}(t)$ as the Jacobi field associated with $x(t)$, and we wish to derive the corresponding Jacobi's equation. This will be achieved in the following Proposition.

Proposition 5.1 **(i)** *Let $\hat{x}(t) = \big(\hat{q}(t), \hat{p}(t)\big)$ be as in* (5.13). *Then in between collisions, we simply have*

$$\frac{d\hat{q}}{dt} = \hat{p}, \qquad \frac{d\hat{p}}{dt} = 0. \tag{5.15}$$

Moreover, at a collision, the precollisional coordinates $(q,p,\hat{q},\hat{p})$, with $x = (q,p) \in \partial^- X$, become $(q,p',\hat{q}',\hat{p}')$ right after collision, with $\hat{q}' = R_q\hat{q}$, and

$$\hat{p}' = R_q\hat{p} + 2\big(p\cdot\nu(q)\big)^-\big(R_q V_x^t (d\nu)_q V_x\big)\hat{q}, \tag{5.16}$$

where

$$R_q = I - 2\nu(q)\otimes\nu(q), \qquad V_{q,p} = I - \frac{p\otimes\nu(q)}{p\cdot n}.$$

(ii) *For every $(q,p,\hat{q},\hat{p}) \in \mathcal{T}\partial^+ X$, we have $T(q,p) = (Q,P)$, and $(dT)_{(q,p)}(\hat{q},\hat{p}) = (\hat{Q},\hat{P})$, with*

$$\hat{Q} = V_{Q,p}\big(\hat{q}+\tau(q,p)\hat{p}\big), \qquad \hat{P} = R_Q\hat{p} + 2\big(p\cdot\nu(Q)\big)^-\big(R_Q V_{Q,p}^t (d\nu)_Q V_{Q,p}\big)\big(\hat{q}+\tau(q,p)\hat{p}\big). \tag{5.17}$$

Proof **(i)** If we take a path $\big(x^*(\theta) : \ \theta \in (-\delta_0,\delta_0)\big)$ with $x^*(0) = x, \dot{x}^*(0) = \hat{x}$, and write $\big(q^*(t,\theta), p^*(t,\theta)\big)$, for $\phi_t\big(x^*(\theta)\big)$, then in between collisions, $p^*(t,\theta) = p^*(\theta)$ does not change, and $q^*(t,\theta) = q^*(\theta) + tp^*(\theta)$. This implies (5.15).

For the dynamics at a collision, let us write $\tau(\theta)$, for the first time $q^*(t,\theta)$ reaches the boundary of Y. Without loss of generality, we may assume that $\tau(0) = 0$ and that $\tau(\theta) > 0$ for $\theta > 0$. We also write

$$Q(\theta) := q^*\big(\tau(\theta),\theta\big) = q^*(\theta) + \tau(\theta)p^*(\theta), \qquad n(\theta) := \nu\big(Q(\theta)\big),$$

for the hitting location and the normal vector at time τ. Differentiating these equations and evaluating the derivatives at $\theta = 0$ yield

$$\hat{Q} = \hat{q} + \hat{\tau}p, \qquad \hat{n} = (d\nu)_q\hat{Q}, \tag{5.18}$$

where $(\hat{\tau}, \hat{Q}, \hat{n}) = (\dot{\tau}(0), \dot{Q}(0), \dot{n}(0))$. Since the path $\big(Q(\theta) : \ \theta \in (-\delta_0, \delta_0)\big)$ lies on ∂Y, we have $n \cdot \hat{Q} = 0$, where $n = n(0)$. From this and (5.18) we learn

$$\hat{\tau} = -\frac{\hat{q} \cdot n}{p \cdot n}, \qquad \hat{Q} = V_{q,p}\hat{q} = V\hat{q} = \left(I - \frac{p \otimes n}{p \cdot n}\right)\hat{q}, \qquad \hat{n} = (d\nu)_q V\hat{q}. \tag{5.19}$$

We note that the operator V is the p-projection onto $n^\perp$. That is $(I - V)\hat{q}$ is parallel to p and $V\hat{q} \cdot n = 0$. We are now ready to determine $\hat{p}'$ and $\hat{q}'$.

Observe that for $\theta > 0$ and $t > \tau(\theta)$,

$$q^*(t, \theta) = Q(\theta) + \big(t - \tau(\theta)\big)p^{*\prime}(\theta) = Q(\theta) - \tau(\theta)p^{*\prime}(\theta) + tp^{*\prime}(\theta) := b(\theta) + tp^{*\prime}(\theta).$$

As we differentiate with respect to θ, and set $\theta = 0$, we deduce,

$$\hat{q}(t) = \dot{b}(0) + t\hat{p}'.$$

We learn from this

$$\hat{q}' = \dot{b}(0) = \hat{Q} - \hat{\tau}p' = \hat{q} + \hat{\tau}p - \hat{\tau}\big(p - 2(p \cdot n)n\big) = \hat{q} - 2(\hat{q} \cdot n)n.$$

Hence $\hat{q}' = R_q\hat{q}$.

We now turn our attention to $\hat{p}'$. From differentiating

$$p^{*\prime}(\theta) = Rp^*(\theta) := \big(I - 2n(\theta) \otimes n(\theta)\big)p^*(\theta),$$

and evaluating the derivative at $\theta = 0$, we arrive at

$$\hat{p}' = R\hat{p} - 2\big(\hat{n} \otimes n + n \otimes \hat{n}\big)p.$$

On the other hand,

$$\big(\hat{n} \otimes n + n \otimes \hat{n}\big)p = (p \cdot n)\hat{n} + (n \otimes p)\hat{n} = (p \cdot n)\left(I + \frac{n \otimes p}{p \cdot n}\right)\hat{n} := (p \cdot n)\hat{V}\hat{n}.$$

As a result,

$$\hat{p}' = R\hat{p} - 2(p \cdot n)\hat{V}(d\nu)_q V\hat{q} =: R\hat{p} - 2A\hat{q}. \tag{5.20}$$

Note that $|\nu| = 1$ implies that $(d\nu)_q$ map $n^\perp$ onto $n^\perp$. Also the range of V is $n^\perp$ and $V : p^\perp \to n^\perp$ is an isomorphism. Moreover, $\hat{V}$ restricted to $n^\perp$ equals $I - \frac{n \otimes p'}{p' \cdot n}$, and that $\hat{V} : n^\perp \to p'^\perp$ is an isomorphism, which simply n-projects onto $p'^\perp$. Indeed since $Rn = -n$ and $R = I$ on $n^\perp$,

$$R\hat{V} = R + R\,\frac{n \otimes p}{n \cdot p} = R - \frac{n \otimes p}{n \cdot p}, \qquad R\hat{V} \restriction_{n^\perp} = I - \frac{n \otimes p}{n \cdot p},$$

and $R\hat{V} = V^t$ is the transpose of V because

$$w \cdot (R\hat{V})w' = w \cdot \left(I - \frac{n \otimes p}{n \cdot p}\right) w' = w \cdot w' - \frac{(p \cdot w')(n \cdot w)}{n \cdot p} = (Vw) \cdot w',$$

for every $w, w' \in n^\perp$. As a result,

$$A = (p \cdot n) R V^t (d\nu)_q V \hat{q}.$$

This and (5.20) imply (5.16).

(ii) If we take a path $x^\dagger(\theta) = \big(q^\dagger(\theta), p^\dagger(\theta)\big)$, that lies on $\partial^+ X$, such that $q^\dagger(0) = q, p^\dagger(0) = p$, and $\dot{q}^\dagger(0) = \hat{q} \in \mathcal{T}_q(\partial Y), \dot{p}^\dagger(0) = \hat{p} \in p^\perp$, and if we write $T(q,p) = (Q,P)$, then $Q = q + \tau^0 p$, and $P = R_Q p$, where $\tau^0 = \tau(q,p)$. Let us write $\tau(\theta)$ for $\tau\big(x^\dagger(\theta)\big)$. Without loss of generality, we may assume that $\tau(\theta) > 0$ for $\theta > 0$ and small. Note that if we set

$$x^*(\theta) = \phi_{\tau^0}\big(x^\dagger(\theta)\big),$$

then we x^* is as part **(i)** except that $\hat{q}$ is replaced with $\hat{q} + \tau^0 \hat{p}$. In other words, $\dot{x}^*(0) = \big(\hat{q} + \tau^0 \hat{p}, \hat{p}\big)$. From this, the middle equation in (5.19), and (5.20), we deduce (5.18). □

Proof of Theorem 5.1(i) Fix $x = (q,p) \in \partial^+ X$, set $(\hat{Q}, \hat{P}) = T(q,p)$, and $\tau = \tau(q,p)$. To explore the dispersive behavior of a dispersive billiard, we study the evolution of the quadratic form

$$\mathcal{Q}(\hat{q}, \hat{p}) = \mathcal{Q}_x(\hat{q}, \hat{p}) = \hat{q} \cdot \hat{p},$$

along a T orbit. Here $\mathcal{Q}_x$ is defined for $\hat{q} \in \mathcal{T}_q Y$, and $\hat{p} \in p^\perp$. By (5.17),

$$\begin{aligned}
\mathcal{Q}\left((dT)_{(q,p)}(\hat{q},\hat{p})\right) &= [V_{Q,p}(\hat{q} + \tau\hat{p})] \cdot \Big[R_Q \hat{p} + 2\big(p \cdot \nu(Q)\big)^- \big(R_Q V_{Q,p}^t (d\nu)_Q V_{Q,p}\big)\big(\hat{q} + \tau\hat{p}\big)\Big] \\
&= (\hat{q} + \tau\hat{p}) \cdot \hat{p} + 2\big(p \cdot \nu(Q)\big)^- [V_{Q,p}(\hat{q} + \tau\hat{p})] \cdot \big[\big(V_{Q,p}^t (d\nu)_Q V_{Q,p}\big)\big(\hat{q} + \tau\hat{p}\big)\big] \\
&= \mathcal{Q}(\hat{q},\hat{p}) + \tau|\hat{p}|^2 + 2\big(p \cdot \nu(Q)\big)^- [V_{Q,p}(\hat{q} + \tau\hat{p})] \cdot \big[(d\nu)_Q V_{Q,p}\big(\hat{q} + \tau\hat{p}\big)\big] \\
&\geq \mathcal{Q}(\hat{q},\hat{p}) + \tau|\hat{p}|^2 + 2\delta\big(p \cdot \nu(Q)\big)^- |V_{Q,p}(\hat{q} + \tau\hat{p})|^2.
\end{aligned}$$

Here for the second equality we used $R_Q V_{Q,p} = V_{Q,p}$ (which is true because the restriction of R_Q to $\mathcal{T}_Q Y$ is identity), and that $V_{Q,p} z \cdot \hat{p} = z \cdot \hat{p}$ (which is true because $V_{Q,p} z - z$ is parallel to p, and $p \cdot \hat{p} = 0$), and for the third equality we used $V_{Q,p}^2 = V_{Q,p}$. Observe that if $\hat{q} = a + b$ with $a \in p^\perp$, and b parallel to p, then $V_{Q,p}(\hat{q} + \tau\hat{p}) = V_{Q,p}(a + \tau\hat{p})$, and

$$|V_{Q,p}(\hat{q} + \tau\hat{p})|^2 = |V_{Q,p}(a + \tau\hat{p})|^2 \geq |a + \tau\hat{p}|^2 \geq 4\tau a \cdot \hat{p} = 4\tau \mathcal{Q}(\hat{q},\hat{p}).$$

Hence,

$$\mathcal{Q}\left((dT)_{(q,p)}(\hat{q},\hat{p})\right) \geq f(q,p)\mathcal{Q}(\hat{q},\hat{p}), \tag{5.21}$$

where

$$f(q,p) = 1 + 8\delta\big(p\cdot\nu(Q)\big)^{-}\tau(q,p),$$

for $Q = q + \tau(q,p)p$. Inductively (5.21) yields

$$\mathcal{Q}\left((dT^n)_{(q,p)}(\hat{q},\hat{p})\right) \geq \prod_{i=0}^{n-1} f\big(T^i(q,p)\big)\mathcal{Q}(\hat{q},\hat{p}). \tag{5.22}$$

We now assume that $\mathcal{Q}(\hat{q},\hat{p}) > 0$. From (5.22) and the Ergodic Theorem we deduce

$$\liminf_{n\to\infty} n^{-1}\log\mathcal{Q}\left((dT^n)_{(q,p)}(\hat{q},\hat{p})\right) \geq \lim_{n\to\infty} n^{-1}\sum_{i=0}^{n-1}\log f\big(T^i(q,p)\big) =: g_0(q,p), \tag{5.23}$$

where $g_0 = P_\mu \log f$, with

$$\int g_0 \; d\mu = \int \log f \; d\mu > 0.$$

Finally, given any $(\hat{q},\hat{p})$ with $\mathcal{Q}(\hat{q},\hat{p}) > 0$, use (5.23) and

$$\left|(dT^n)_{(q,p)}(\hat{q},\hat{p})\right| \geq \sqrt{2}\mathcal{Q}\left((dT^n)_{(q,p)}(\hat{q},\hat{p})\right)^{\frac{1}{2}},$$

to deduce (5.11) for $g = 2^{-1}g_0$.

(ii) By Proposition 5.1, the quantity $\lambda(t) := p(t)\cdot\hat{q}(t)$ is independent of time because in between collisions $\dot{\lambda}(t) = p(t)\cdot\hat{p}(t) = 0$, and at a collision time t,

$$\lambda(t+) = R_{q(t)}p(t-)\cdot R_{q(t)}\hat{q}(t-) = p(t-)\cdot\hat{q}(t-) = \lambda(t-).$$

As a result, if initially $p\cdot\hat{q} = 0$, then $p(t)\cdot\hat{q}(t) = 0$, for all $t \geq 0$. If we write $\mathcal{Q}(t) = \mathcal{Q}(\hat{q}(t),\hat{p}(t))$, then in between collisions, $\frac{d\mathcal{Q}}{dt} = |\hat{p}|^2$ and at a collision,

$$\begin{aligned}
\mathcal{Q}(t+) = \hat{q}'\cdot\hat{p}' &= R_q\hat{q}\cdot\Big[R_q\hat{p} + 2\big(p\cdot\nu(q)\big)^{-}\big(R_qV_{q,p}^t(d\nu)_qV_{q,p}\big)\hat{q}\Big] \\
&= \mathcal{Q}(t-) + 2\big(p\cdot\nu(q)\big)^{-}\big(V_{q,p}\hat{q}\big)\cdot\big((d\nu)_qV_{q,p}\hat{q}\big) \\
&\geq \mathcal{Q}(t-) + 2\delta\big(p\cdot\nu(q)\big)^{-}\big|V_{q,p}\hat{q}\big|^2,
\end{aligned}$$

where $(q,p) = (q(t),p(t))$. We note that since $V_{q,p}\hat{q} - \hat{q}$ is parallel to p, and $p\perp\hat{q}$, we learn

$$\big|V_{q,p}\hat{q}\big|^2 \geq |\hat{q}|^2.$$

As a result,

$$\mathcal{Q}(t+) \geq \mathcal{Q}(t-) + 2\delta\big(p(t-)\cdot\nu(q(t))\big)^{-}\;\big|\hat{q}(t-)\big|^2 \tag{5.24}$$

Let us write τ_i, for the time at which $q(t)$ has reached the boundary for the i-th times, and set $\bar\tau_i = \tau_i - \tau_{i-1}$. By (5.24),

$$\begin{aligned}
\mathcal{Q}\big(\tau_{i+1}+\big) &\geq \mathcal{Q}\big(\tau_i+\big) + \int_{\tau_i}^{\tau_{i+1}} \big|\hat p(t)\big|^2\,dt + 2\delta\big(p_i\cdot n_{i+1}\big)^-\,\big|\hat q(\tau_{i+1}-)\big|^2\\
&\geq \mathcal{Q}\big(\tau_i+\big) + 2\delta\big(p_i\cdot n_{i+1}\big)^-\,\big|\hat q_i + \bar\tau_i\hat p_i\big|^2\\
&\geq \mathcal{Q}\big(\tau_i+\big) + 8\delta\bar\tau_i\big(p_i\cdot n_{i+1}\big)^-\,\big(\hat q_i\cdot\hat p_i\big)\\
&= \Big(1+8\delta\bar\tau_i\big(p_i\cdot n_{i+1}\big)^-\Big)\,\mathcal{Q}\big(\tau_i+\big),
\end{aligned}$$

where $q_{i+1} = q(\tau_{i+1}), p_i = p(\tau_i+), \hat q_i = \hat q(\tau_i+), \hat p_i = \hat p(\tau_i+)$ and $n_{i+1} = \nu(q_{i+1})$. From this, we can readily deduce that for a function h_0 with $\int h_0\,dm > 0$,

$$\liminf_{n\to\infty} n^{-1}\log\mathcal{Q}\big(\tau_n+\big) \geq g_0(\tau_1). \tag{5.25}$$

It is not hard to show

$$\lim_{n\to\infty} n^{-1}\tau_n = k,$$

exists and is positive m-almost everywhere. This and (5.25) imply

$$\liminf_{t\to\infty} t^{-1}\log\mathcal{Q}(t) \geq h_1,$$

for a nonnegative and nonzero function h_1. This in turn implies (5.12) as in Part **(i)**. □

Remark 5.2(i) Write V for $V_{q,p}$ with $q\in\partial Y$. The operator $W = V^tV$ on $p^\perp$ has a simple geometric interpretation. For $\hat q\in p^\perp$,

$$\begin{aligned}
W\hat q &= \left(I - \frac{n\otimes p}{p\cdot n}\right)\left(I - \frac{p\otimes n}{p\cdot n}\right)\hat q = \left(I - \frac{n\otimes p}{p\cdot n}\right)\left(\hat q - \frac{n\cdot\hat q}{n\cdot p}n\right)\\
&= \hat q - \frac{n\cdot\hat q}{n\cdot p}\left(p - \frac{|p|^2}{n\cdot p}n\right) = \hat q - \frac{n\cdot\hat q}{n\cdot p}\,V^tp\\
&= \left(I - \frac{(V^tp)\otimes n}{p\cdot n}\right)\hat q,
\end{aligned}$$

where V^tp is the n-projection of p onto $p^\perp$. Moreover,

$$|V\hat q|^2 = W\hat q\cdot\hat q = |\hat q|^2 + \left(\frac{n\cdot\hat q}{n\cdot p}\right)^2|p|^2.$$

(ii) For the flow ϕ_t, we have a zero Lyapunov exponent in the direction of the flow. To avoid this direction, we assume that $\hat q\cdot p = 0$ in Theorem 5.1. This suggests that we restrict $(\hat q,\hat p)$

to $W(x)=\{(\hat q,\hat p):\hat q\cdot p=\hat p\cdot p=0\}=p^{\perp}$ for $x=(q,p)$. Note that if $(\hat q,\hat p)\in W(x)$ initially, then $(\hat q(t),\hat p(t))\in W(\phi_t(x))$ at later times. Let us define a sector

$$C_x=\{(\hat q,\hat p)\in W(x):\hat q\cdot\hat p>0\}.$$

What we have learned so far is that

$$(d\phi_t)_x(C_x)\subsetneqq C_{\phi_t(x)}. \tag{5.26}$$

Note that $\hat q$ is gaining in size in between collisions. However the gain in the size of $\hat p$ is occurring only at collisions.

(iii) As we mention in Remark 5.1, the function $f(q,p,t)=f^0(\phi_{-t}(q,p))$, satisfies the equation $f_t+p\cdot f_q=0$ strictly inside X. To derive an equation for the evolution of $\alpha=f_q$ and $\beta=f_p$, observe that if $\Lambda_0(c)=\big\{x:\ f^0(x)=c\big\}$, then

$$\Lambda_t(c)=\big\{(q,p):f(q,p,t)=c\big\}=\phi_t\big(\Lambda_0(c)\big).$$

This means that if $x\in\Lambda_t(c)$, then $x=\phi_t(y)$, for some $y\in\Lambda_0(c)$, and the normal $z(0):=\nabla f^0(y)$ is transported to $z(t)=\big(\alpha(x,t),\beta(x,t)\big)=\nabla f(x,t)$ after t units of time. This suggests studying the evolution of a normal vector to a surface of codimention one that evolves with ϕ_t. More generally, take a surface Λ of codimension one in X. The manifold $\mathcal{T}\Lambda\subseteq\mathcal{T}X$ evolves to $\mathcal{T}\phi_t(\Lambda)$ and we study its evolution by keeping track of its corresponding unit normal. If $z(t)=(a(t),b(t))\in\mathcal{T}X$ is normal to $\mathcal{T}(\phi_t(\Lambda))$ at all times, then we would like to derive an evolution equation for it. The vector (a,b) is chosen so that for every t,

$$a(t)\cdot\hat q(t)+b(t)\cdot\hat p(t)=0,$$

where $(\hat q(t),\hat p(t))\in\mathcal{T}_{x(t)}\Lambda_t$ with $\Lambda_t=\phi_t(\Lambda)$. In between collisions, $\hat x(t)=(\hat q+t\hat p,\hat p)$ and $a(t)\cdot(\hat q+t\hat p)+b(t)\cdot\hat p=0$, or $a(t)\cdot\hat q+(ta(t)+b(t))\cdot\hat p=0$. Hence if initially $(a(0),b(0))=(a,b)$, then $a(t)=a$ and $b(t)=b-ta$. So in between collisions we simply have $\frac{da}{dt}=0$, $\frac{db}{dt}=-a$. At a collision (a,b) experiences a jump discontinuity. If after a collision the normal vector is given by (a',b'), then

$$\begin{aligned}&a'\cdot(R\hat q)+b'\cdot(R\hat p-2A\hat q)=0,\\ &(Ra')\cdot\hat q+(Rb')\cdot\hat p-2(A^tb')\cdot\hat q=0.\end{aligned}$$

This suggests

$$b'=Rb,\qquad a'=Ra+2RA^tRb=:Ra+2Bb. \tag{5.27}$$

Note that if $Q(t)=a(t)\cdot b(t)$, then in between collisions,

$$\frac{dQ}{dt}=-|a|^2,$$

and at a collision

$$\begin{aligned}\mathcal{Q}(t+) &= a'\cdot b' = (Ra + 2RA^tRb)\cdot Rb\\ &= \mathcal{Q}(t-) + 2A^tRb\cdot b\\ &= \mathcal{Q}(t-) + 2b\cdot RAb\\ &= \mathcal{Q}(t-) + 2(p\cdot n)(d\nu)_Q(Vb)\cdot(Vb),\end{aligned}$$

and in the case of a dispersive billiard,

$$\mathcal{Q}(t+) - \mathcal{Q}(t-) \le 2\delta(p\cdot n)|Vb|^2 < 0.$$

Hence $\mathcal{Q}(t)$ is decreasing. □

As we mentioned in Chapter 4, we may use sectors to find the stable and unstable directions for a dynamical systems. We wish to use Proposition 5.1 to determine how certain sectors get slimmer along the flow. As a first step, let us observe that since the flow ϕ_t or the map are symplectic, then both the stable and unstable subspaces are *Lagrangian.* Recall that if $\hat{x} = (\hat{q},\hat{p})$ and $\hat{x}' = (\hat{q}',\hat{p}')$, then

$$\omega\big(\hat{x},\hat{x}'\big) = \hat{p}\cdot\hat{q}' - \hat{q}\cdot\hat{p}'.$$

Also recall for $x = (q,p)$,

$$W_x = \big\{(\hat{q},\hat{p}):\ \hat{q}\cdot p = \hat{p}\cdot p = 0\big\} = p^\perp\times p^\perp.$$

We think of $W = \big(W_x:\ x\in X\big)$ as a vector bundle of dimension $2d-2$. A sub-bundle $L = \big(L_x:\ x\in X\big)$ is called *Lagrangian* if $\omega(a,b) = 0$ for every $a,b\in L_x$. Here we think of ω as a symplectic form on W.

Proposition 5.2 **(i)** *The billiard flow is symplectic.*

(ii) *Both stable and unstable bundles $E^{s(u)}$ are Lagrangian.*

(iii) *If there exist symmetric linear maps $S^s_x, S^u_x : p^\perp\to p^\perp$ such that*

$$E^{u(s)}_x = \big\{(\hat{q},\hat{p}):\ \hat{p} = S^{s(u)}_x\hat{q},\ \hat{q}\in p^\perp\big\},$$

then in between collisions,

$$\frac{d}{dt}S^{s(u)}_{\phi_t(x)} + \big(S^{s(u)}_{\phi_t(x)}\big)^2 = 0, \tag{5.28}$$

or equivalently,

$$S^{s(u)}_{\phi_t(x)} = S^{s(u)}_x\big(I + tS^{s(u)}_x\big)^{-1} = \Big(tI + \big(S^{s(u)}_x\big)^{-1}\Big)^{-1}, \tag{5.29}$$

where the last equality holds whenever $S_x^{s(u)}$ is invertible. At a collision, $S^{s(u)}$ changes to $S^{s(u)\prime}$, where

$$S_x^{s(u)\prime} = R_q S_x^{s(u)} R_q + R_q V_{q,p}^t (d\nu)_q V_{q,p} R_q =: R_q S_x^{s(u)} R_q + \Gamma_{q,p}. \tag{5.30}$$

(iv) *Assume that there exist positive symmetric linear maps $S_x^s, S_x^u : \mathcal{T}_q \partial^+ Y \to p^\perp$ such that*

$$E_x^{u(s)} = \big\{ (\hat{q}, \hat{p}) : \ \hat{p} = S_x^{u(s)} \hat{q}, \ \hat{q} \in \mathcal{T}_q \partial^+ Y \big\}. \tag{5.31}$$

Then

$$S_{T(x)}^{s(u)} = \Gamma_{Q,p} + R_Q \left(\tau(x) I + \big(S_x^{s(u)}\big)^{-1} \right)^{-1} R_Q, \tag{5.32}$$

where $T(q,p) = (Q,P)$ and $x = (q,p)$.

Proof(i) For symplectic property observe that if $\hat{x} = (\hat{q}, \hat{p}), \hat{x}_* = (\hat{q}_*, \hat{p}_*) \in W_x$ change to $\hat{x}' = (\hat{q}', \hat{p}'), \hat{x}_*' = (\hat{q}_*', \hat{p}_*') \in W_x$ at a collision, then

$$\begin{aligned}
\omega\big(\hat{x}', \hat{x}_*'\big) &= \hat{p}' \cdot \hat{q}_*' - \hat{q}' \cdot \hat{p}_*' \\
&= \big(R_q \hat{p} + 2(p\cdot n)^- R_q V_{q,p}^t (d\nu)_q V_{q,p} \hat{q}\big) \cdot \big(R_q \hat{q}_*\big) \\
&\quad - \big(R_q \hat{p}_* + 2(p\cdot n)^- R_q V_{q,p}^t (d\nu)_q V_{q,p} \hat{q}_*\big) \cdot \big(R_q \hat{q}\big) \\
&= \big(\hat{p} + 2(p\cdot n)^- V_{q,p}^t (d\nu)_q V_{q,p} \hat{q}\big) \cdot \hat{q}_* - \big(\hat{p}_* + 2(p\cdot n)^- V_{q,p}^t (d\nu)_q V_{q,p} \hat{q}_*\big) \cdot \hat{q} \\
&= \omega\big(\hat{x}, \hat{x}_*\big),
\end{aligned}$$

where for the last equality, we used the symmetry of $V_{q,p}^t (d\nu)_q V_{q,p}$.

(ii) $a, b \in E_x^s$, then by symplectic property

$$\omega_x(a,b) = \lim_{t\to\infty} \omega_{\phi_t(x)}\big((d\phi_t)_x a, (d\phi_t)_x b\big) = 0,$$

because both $\big|(d\phi_t)_x a\big|$ and $\big|(d\phi_t)_x b\big|$ decay exponentially fast.

(iii) Note that since $\hat{p}(t)$ stays constant in between collisions, we

$$0 = \frac{d}{dt} S(t)\hat{q}(t) = \dot{S}(t)\hat{q}(t) + S(t)\frac{d\hat{q}}{dt}(t) = \dot{S}(t)\hat{q}(t) + S(t)\hat{p}(t) = \dot{S}(t)\hat{q}(t) + S(t)^2\hat{q}(t),$$

where $S(t)$ denotes $S_{\phi_t(x)}^{s(u)}$. This implies the first equation (5.29) because $\hat{q}(t)$ can take any vector. The second equation of (5.29) also implies the first by differentiation.

Since $\big(d\phi_t\big)_x E_x^{s(u)} = E_{\phi_t(x)}^{s(u)}$, and in between collisions $\big(d\phi_t\big)_x (\hat{q}, \hat{p}) = (\hat{q} + t\hat{p}, \hat{p})$, we must have

$$S_{\phi_t(x)}^{s(u)} \big(\hat{q} + t S_x^{s(u)} \hat{q}\big) = S_x^{s(u)} \hat{q}.$$

This is exactly the second equation in (5.29). As for (5.30), we use (5.16) to write

$$S_x^{s(u)\prime}\big(R_q\hat q\big) = R_q S_x^{s(u)}\hat q + R_q V_{q,p}^t (d\nu)_q V_{q,p}\hat q,$$

which yields (5.30).

(iv) The formula (5.32) is an immediate consequence of (5.30) and (5.29). □

Remark 5.3(i) A particularly nice example of Λ as in Remark 5.2(ii) is a normal bundle of a q-surface. More precisely, suppose Θ is a surface of codimension one in Y and set

$$\Lambda = \{(q,p) : q \in \Theta,\ p \text{ is the normal vector at } q\}.$$

Here we are assuming that Λ is orientable and a normal vector p at each $q \in \Theta$ is specified. In this case $(q,p,\hat q,\hat p) \in \mathcal{T}\Lambda$ means that $\hat q \in \mathcal{T}_q\Theta$ and that $\hat p = C_q \hat q$ for a suitable matrix $C(q)$ which is known as the *curvature matrix.* (If $p = \nu(q)$ is the normal vector, then $C_q = (d\nu)_q$.) The evolution of C along an orbit is goverened by (5.29) and (5.30).

(ii) If the billiard map is hyperbolic, the we would have $\dim E^u = \dim E^s = d-1$. This is the highest dimension a Lagrangian bundle can have because $\dim W = 2d-2$. In fact the assumptions (5.28) and (5.31) are based on the fact that if we assume that a Lagrangian subspace L is non-degenerate, then it must be a graph of a symmetric matrix. To guess what the stable fiber at a point $x = (q,p)$ is, observe that if we replace x with $T^{-1}(x)$ in (**??**), we obtain

$$S_x^s = \Gamma_{q,p_1} + R_q\left(\tau_1(x)I + \big(S^s_{T^{-1}(x)}\big)^{-1}\right)^{-1} R_q = \Gamma_{q,R_qp} + R_q\frac{I}{\tau_1 I + \frac{I}{S^s_{x_1}}}R_q,$$

where $x_1 = T^{-1}(x) = T(q,-p) = (q_1,p_1)$ and $\tau_1 = \tau\big(T^{-1}(x)\big)$. From this we guess

(5.33)
$$S_x^s = \hat\Gamma_1 + \hat R_0 \frac{I}{\tau_1 I + \frac{I}{\hat\Gamma_2 + \hat R_1 \frac{I}{\tau_2 I + \frac{I}{\ddots}}\hat R_1}}\hat R_0.$$

Here we write $x_i = (q_i,p_i)$ for $T^{-i}(x)$, and $\tau_i = \tau(x_i)$, $\hat R_i = R_{q_i}$, and $\hat\Gamma_i = \Gamma_{p_i,q_{i-1}}$. To verify (5.33), we need to make sure that the continued fraction is convergent. This is more or less equivalent to the convergence

(5.34)
$$S_x^{s,n} = \big(dT^n\big)_{T^{-n}(x)} S^s_{T^{-n}(x)}.$$

(iii) It is instructive to compare the billiard flow with our Example 4.3. In Example 4.3, we had the sector $C_x = \big\{(\hat q,\hat p) :\ \hat q\hat p \ge 0\big\}$, that got slimmer under dT^n. Its lower boundary

$L = \{(\hat{q}, 0) : \ \hat{q} \in \mathbb{R}\}$ yielded a sequence of lines $L_x^n = (dT^n)_{T^{-n}(x)} L$ with increasing slopes $\theta_n(x) \to \theta^s$. The limit θ^s gave the slope of the stable line. We have a similar scenario for our billiard: the sector

$$C_x = C_{q,p} = \big\{(\hat{q}, \hat{p}) : \ \hat{q}, \hat{p} \in p^\perp\big\},$$

yields a family

$$C_x^t := (d\phi_t)_{\phi_{-t}(x)} C_{\phi_{-t}(x)},$$

of nested sectors as $t \to \infty$. Its lower boundary

$$L_x^t := (d\phi_t)_{\phi_{-t}(x)} L_x, \qquad \text{with} \qquad L_x = \big\{(\hat{q}, 0) : \ \hat{q} \in p^\perp\big\},$$

is a d-dimensional Lagrangian subspace that is expected to converge to the stable fiber E_x^s. Similarly, the sector

$$C_x = C_{q,p} = \big\{(\hat{q}, \hat{p}) : \ \hat{q} \in \mathcal{T}_q \partial Y, \ \hat{p} \in p^\perp\big\},$$

yields a family

$$C_x^n := (dT^n)_{T_{-n}(x)} C_{T_{-n}(x)},$$

of nested sectors as $n \to \infty$. Its boundaries are given by

$$\begin{aligned} L_x^{-,n} &= (dT^n)_{T_{-n}(x)} L_x^-, \qquad \text{with} \qquad L_x^- = \big\{(\hat{q}, 0) : \ \hat{q} \in \mathcal{T}_q \partial Y\big\}, \\ L_x^{+,n} &= (dT^n)_{T_{-n}(x)} L_x^+, \qquad \text{with} \qquad L_x^+ = \big\{(0, \hat{p}) : \ \hat{p} \in p^\perp\big\}. \end{aligned}$$

These are d-dimensional Lagrangian subspaces that are expected to converge to the stable fiber E_x^s for the billiard map T. □

To establish the convergence of C_x^n as $n \to \infty$, we may use the strict nested property of these sequences. For this, we need to define a nice metric for the set of Lagrangian subspaces so that the strict nested property guarantees the convergence. Indeed if we write

$$(dT^n)_{T_{-n}(x)} = \begin{bmatrix} A_n(x) & B_n(x) \\ C_n(x) & D_n(x), \end{bmatrix}$$

then the monotonicity

$$(dT)_x C_x \subsetneq C_{T(x)}, \qquad C_x^{n+1} \subsetneq C_x^n,$$

means $C_n^* B_n > 0$, and if λ^n denotes its smallest eigenvalue, then $\lambda^n \to \infty$ as $n \to \infty$. We then define a metric $\mathcal{D}$ on the set of Lagrangian subspaces such that

$$\mathcal{D}\big(L_x^{-,n}, L_x^{+,n}\big) \le c_0 \big(\lambda^n\big)^{-1},$$

for a constant c_1

Theorem 5.2 *For μ-almost all x, we have that $\cap_n C_x^n =: E_x^s$ is a Lagrangian subspace.*

Proof Since

$$C_x^{n+1} = (dT)_{T^{-1}(x)} C^n_{T^{-1}(x)},$$

we learn

$$L_x^{\pm,n+1} = (dT)_{T^{-1}(x)} L^{\pm,n}_{T^{-1}(x)}.$$

If

$$L_x^{\pm,n} = \left\{ \left(\hat q, S_x^{\pm,n}\hat q\right) : \ \hat q \in \mathcal{T}_q\partial Y \right\},$$

then from (5.31) we learn

$$S^{\pm,n+1}_{Q,P} = \Gamma_{Q,p} + R_Q \left(\tau(q,p)I + \left(S^{\pm,n}_{(q,p)}\right)^{-1}\right)^{-1} R_Q =: \mathcal{G}\left((q,p), S^{\pm,n}_{(q,p)}\right), \tag{5.35}$$

where $T^{-1}(Q,P) = (q,p)$ so that $Q = q + \tau(q,p)p$, and $P = R_Q p$. (Compare this to (4.33).) Note that the function $\mathcal{G}(x,A)$ is monotonically increasing in A. Since $S^{-,0} = 0$, and $S^{-,1} > 0$, we deduce that the sequence $\left\{S^{-,n}\right\}_n$ is increasing.

Note that since

$$\begin{aligned} L^{+,1}_{(Q,P)} &= \left\{ \left(\tau(q,p)V_{Q,p}\hat p, R_Q\hat p + 2\big(p\cdot\nu(Q)\big)^- \tau(q,p)\big(V^{-1}_{Q,p}(d\nu)_Q V_{Q,p}\big)\hat p\right) : \ \hat p \in p^\perp \right\} \\ &= \left\{ \left(\hat q, \tau(q,p)^{-1}V^{-1}_{Q,p}\hat q + 2\big(p\cdot\nu(Q)\big)^- \big(V^{-1}_{Q,p}(d\nu)_Q\big)\hat q\right) : \ \hat q \in \mathcal{T}_q\partial Y \right\}, \end{aligned}$$

we have

$$S^{+,1}_{(Q,P)} = \tau(q,p)^{-1}V^{-1}_{Q,p} + 2\big(p\cdot\nu(Q)\big)^- V^{-1}_{Q,p}(d\nu)_Q.$$

In some sense $S^{+,0} = \infty > S^{+,1}$, which in turn implies that the sequence $\left\{S^{+,n}\right\}_n$ is decreasing.

We now define a metric on the set of Lagrangian subspaces with respect to the sequence $\left\{S^{\pm,n}\right\}_n$ converges. Since we are dealing with Lagrangian subspaces associated with positive definite matrices, we define a metric on the set of positive matrices. This metric yields a metric on the set of corresponding Lagrangian. We write

$$\mathcal{L}(A) = \left\{(a, Aa) : \ a \in \mathbb{R}^d\right\},$$

for the Lagrangian space associated with A. We define

$$\mathcal{D}\big(\mathcal{L}(A), \mathcal{L}(B)\big) = \mathcal{D}'(A,B) = \sup_{a\neq 0}\left|\log\frac{Aa\cdot a}{Ba\cdot a}\right|.$$

We next study the effect of an invertible symplectic matrix on such Lagrangian subspaces. If

$$G = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

is symplectic, then the the Lagrangian subspace associated with S is mapped to a subspace associated with

$$S' = (C + DS)(A + BS)^{-1}.$$

Lemma 5.1 *There exist invertible matrices F_1 and F_2 and a diagonal matrix K such that*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} F_1^{-1} & 0 \\ 0 & F_1^* \end{bmatrix} \begin{bmatrix} I & I \\ K & I+K \end{bmatrix} \begin{bmatrix} F_2^{-1} & 0 \\ 0 & F_2^* \end{bmatrix} =: G_{F_1}\Lambda(K)G_{F_2}.$$

Moreover the matrix K has the same eigenvalues as C^*B.

Note that $G_F L_x^{\pm} = L_x^{\pm}$. From this, Lemma 5.1, and the elementary facts,

$$G_F\mathcal{L}(A) = \mathcal{L}\big(F^*AF\big), \qquad \mathcal{D}'\big(F^*AF, F^*BF\big) = \mathcal{D}'(A,B),$$

we can assert

$$\begin{aligned} \mathcal{D}\big(GL_x^-, GL_x^+\big) &= \mathcal{D}\big(G_{F_1}\Lambda(K)L_x^-, G_{F_1}\Lambda(K)L_x^+\big) \\ &= \mathcal{D}\big(\Lambda(K)L_x^-, \Lambda(K)L^+x\big) = \mathcal{D}(K, I+K) \\ &= \mathcal{D}\big(I+K^{-1}, I\big) = \log\big(1+\lambda_1(G)^{-1}\big), \end{aligned}$$

where $\lambda_1(G)$ is the smallest eigenvalue of C^*B. On the other hand, since $\mathcal{Q}\big(G_F\hat{x}\big) = \mathcal{Q}(\hat{x})$,

$$\begin{aligned} \inf_{\hat{x}\in C_x} \frac{\mathcal{Q}\big(G\hat{x}\big)}{\mathcal{Q}\big(\hat{x}\big)} &= \inf_{\hat{x}\in C_x} \frac{\mathcal{Q}\big(\Lambda(K)\hat{x}\big)}{\mathcal{Q}\big(\hat{x}\big)} = \inf_{\hat{x}\in C_x} \frac{K(\hat{q}+\hat{p})\cdot(\hat{q}+\hat{p}) + \hat{p}\cdot(\hat{q}+\hat{p})}{\mathcal{Q}\big(\hat{x}\big)} \\ &\le \inf_{a,b>0} \frac{\lambda_1(a+b)(a+b) + b(a+b)}{ab} = \inf_{b>0}\Big[\lambda_1\left(b+b^{-1}\right)^2 + b\left(b+b^{-1}\right)\Big] \\ &= \inf_{b>0}\big[(1+\lambda_1)b^2 + \lambda_1 b^{-2} + 2\lambda_1 + 1\big] = 2\sqrt{\lambda_1(\lambda_1+1)} + 2\lambda_1 + 1 \end{aligned}$$

Now if

$$G = G_n = (dT^n)_{T^{-n}(x)},$$

we are done if we can show

$$\lim_{n\to\infty} \lambda_1(G_n) = \infty.$$

For this it suffices to show

$$\lim_{n\to\infty} \sigma(G_n) := \lim_{n\to\infty} \inf_{\hat{x}\in C_x} \frac{\mathcal{Q}\big(G_n\hat{x}\big)}{\mathcal{Q}\big(\hat{x}\big)} = \infty. \tag{5.36}$$

Since the set

$$\big\{\hat{x}\in C_x :\ \mathcal{Q}(\hat{x}) = 1\big\},$$

is compact, (5.36) follows if we can show that for each $\hat{x}\in C_x$,

$$\lim_{n\to\infty} \frac{\mathcal{Q}\big(G_n\hat{x}\big)}{\mathcal{Q}\big(\hat{x}\big)} = \infty.$$

This is a consequence of Theorem 5.1. $\square$

Exercises

(i) Consider a billiard inside a planar disc D. Write down a explicit formula for the billiard map $T : \partial^+ X \to \partial^+ X$, where $X = D \times \mathbb{S}$. Write (q, θ) for a point on $\partial^+ X$, with $q \in \partial D$, and $\theta \in (-\pi, \pi)$ for the angle p makes with the tangent vector at q. Show that the set of $\{(q, \theta) : \; q \in \partial D\}$ is invariant for T.

(ii) Consider a billiard inside a planar domain D, and assume that the set D is strictly convex. Write (q, θ) for a point on $\partial^+ X$, with $q \in \partial D$, and $\theta \in (-\pi, \pi)$ for the angle p makes with the tangent vector at q. Given $q, Q \in \partial D$, write $S(q, Q) = |q - Q|$. Show that if $T(q, \theta) = (Q, \Theta)$, then $S_q = -\cos\theta$ and $S_Q = \cos\Theta$. $\square$

6 Ergodicity of Hyperbolic Systems

Lyapunov exponents can be used to measure the hyperbolicity of dynamical systems. Anosov measures (systems) are examples of uniformly or strongly hyperbolic systems which exhibit chaotic and stochastic behavior. In reality, dynamical systems are rarely strongly hyperbolic and those coming from Hamiltonian systems are only weakly (or even partially) hyperbolic.

An argument of Hopf shows that hyperbolicity implies ergodicity. We examine this argument for two models in this sections; Examples 6.1 and 6.2. To explain Hopf's argument, let us choose the simplest hyperbolic model with expansion and contraction, namely Arnold cat transformation, and use this argument to prove its ergodicity. In fact in Example 1.6 we showed the mixing of Arnold's cat transformation which in particular implies the ergodicity. But our goal is presenting a second proof of ergodicity which is the key idea in proving ergodicity for examples coming from Hamiltonian systems.

Example 6.1 Let $A = \begin{bmatrix} 1 & \alpha \\ \alpha & 1+\alpha^2 \end{bmatrix}$ with $\alpha \in \mathbb{Z}$. Let $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ be the projection $\pi(a) = a(\operatorname{mod} 1)$ and define $T : \mathbb{T}^2 \to \mathbb{T}^2$ by $T \circ \pi = \pi \circ \hat{T}$ where $\hat{T}(a) = Aa$. Since $\alpha \in \mathbb{Z}$ and $\det A = 1$, we know that T is continuous and that the normalized Lebesgue measure μ on $\mathbb{T}^2$ is invariant for T. The eigenvalues of A are

$$\lambda_1 = \lambda(\alpha) = \frac{1}{2}[2 + \alpha^2 - \alpha\sqrt{4+\alpha^2}] < 1 < \lambda_2 = (\lambda(\alpha))^{-1},$$

provided that $\alpha > 0$. The corresponding eigenvectors are denoted by v_1 and v_2. Define

$$\hat{W}^s(a) = \{a + tv_1 : t \in \mathbb{R}\},\ \hat{W}^u(a) = \{a + tv_2 : t \in \mathbb{R}\}.$$

We then have that $W^s(x)$ and $W^u(x)$ defined by

$$W^s(\pi(a)) = \pi(\hat{W}^s(a)),\ W^u(\pi(a)) = \pi(\hat{W}^a(a))$$

are the stable and unstable manifolds. Take a continuous periodic $\hat{f} : \mathbb{R}^2 \to \mathbb{R}$. This induces a continuous $f : \mathbb{T}^2 \to \mathbb{R}$ such that $f \circ \pi = \hat{f}$. We have that $f \circ T^n \circ \pi = \hat{f} \circ \hat{T}^n$. Define $\hat{X}^{\pm}$ to be the set of points a such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_0^{n-1} \hat{f}(\hat{T}^{\pm j}(a)) =: \hat{f}^{\pm}(a)$$

exists. Then $\pi(\hat{X}^{\pm}) = X^{\pm}$ with $X^{\pm}$ consisting of points x such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_0^{n-1} f(T^{\pm j}(x)) =: f^{\pm}(x)$$

exists with $f^\pm = \hat f^\pm \circ \pi$. Evidently $f^\pm \circ T = f^\pm$ on $X^\pm$ and $\hat f^\pm \circ \hat T = \hat f^\pm$ on $\hat X^\pm$. From definition, we see that if $b \in \hat W^s(a)$ (resp. $b \in \hat W^u(a)$), then

$$|\hat T^n(b) - \hat T^n(a)| = \lambda^n |a-b|,$$

$$(\text{resp. } |\hat T^{-n}(b) - \hat T^{-n}(a)| = \lambda^n |a-b|).$$

for $n \in \mathbb{N}$. Hence $a \in \hat X^+$ (resp. $\hat X^-$) implies that $\hat W^s(a) \subseteq \hat X^+$ (resp. $\hat W^u(a) \subseteq \hat X^-$). Let $d(\cdot,\cdot)$ be the standard distance on the torus. More precisely,

$$d(x,y) = \min\{|a-b| : \pi(a) = x,\ \pi(b) = y\}.$$

Again if $y \in W^s(x)$ (resp. $y \in W^u(x)$), then

$$d(T^n(x), T^n(y)) = \lambda^n d(x,y),$$

$$(\text{resp. } d(T^{-n}(x), T^{-n}(y)) = \lambda^n d(x,y))$$

for $n \in \mathbb{N}$. Similarly $x \in X^+$ (resp. X^-) implies that $W^s(x) \subseteq X^+$ (resp. $W^u(x) \subseteq X^-$). Let Y (respectively $\hat Y$ denote the set of points $x \in X^- \cap X^+$ (respectively $x \in \hat X^- \cap \hat X^+$ such that $f^+(x) = f^-(x)$ (respectively $\hat f^+(x) = \hat f^-(x)$). By Remark 1.3**(iii)**, $\mu(Y) = 1$ and the Lebesgue measure of the complement of $\hat Y$ is zero. Choose a point x_0 such that $\hat W^s(x_0) \setminus \hat Y$ is a set of 0 length. The function $\hat f^+$ is constant on $\hat W^s(x_0)$. The function $\hat f^-$ is constant on $\hat W^u(y)$ for every $y \in \hat W^s(x_0) \cap \hat Y$ and this constant coincides with the value $\hat f^+$ at y. Hence $\hat f^+ = \hat f^-$ is a constant on the set

$$\bigcup_{y \in \hat W^s(x_0) \cap \hat Y} \hat W^u(y).$$

But this set is of full measure. So $\hat f^+ = \hat f^-$ is constant a.e. and this implies that $f^+ = f^-$ is constant a.e. □

Let us call a discrete dynamical system *hyperbolic* if its Lyapunov exponents are nonzero. According to a result of *Pesin*, a hyperbolic diffeomorphism with a smooth invariant measure has at most countably many ergodic components. Pesin's theory also proves the existence of stable and unstable manifolds for hyperbolic systems.

Sinai studied the issue of ergodicity and hyperbolicity for a system of colliding balls in the late 60's. These systems can be regarded as hyperbolic systems with discontinuities. To get a feel for Sinai's method, we follow a work of Liverani and Wojtkowski [LiW] by studying a toral transformation as in Example 6.1 but now we assume that $\alpha \notin \mathbb{Z}$ so that the induced transformation is no longer continuous. As we will see below, the discontinuity of the transformation destroys the uniform hyperbolicity of Example 6.1 and, in some sense our system is only weakly hyperbolic.

Example 6.2 As in Example 6.1, let us write $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ for the $\pmod 1$ projection onto the torus and set $\hat{T}(a) = Aa$, for

$$A = \begin{bmatrix} 1 & \alpha \\ \alpha & 1+\alpha^2 \end{bmatrix}.$$

This induces $T : \mathbb{T}^2 \to \mathbb{T}^2$, with $T \circ \pi = \pi \circ \hat{T}$. If $0 < \alpha < 1$, then T is discontinuous. However the Lebesgue measure μ is still invariant for T. To understand T, let us express $\hat{T} = \hat{T}_1 \circ \hat{T}_2$, $T = T_1 \circ T_2$, $\hat{T}_i(a) = A_i a$ for $i = 1, 2$, where

$$A_1 = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}.$$

If we regard $\mathbb{T}$ as $[0, 1]$ with $0 = 1$, then

$$T_1\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ \alpha x_1 + x_2 \pmod 1 \end{bmatrix}, \quad T_2\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + \alpha x_2 \pmod 1 \\ x_2 \end{bmatrix}$$

with $x_1, x_2 \in [0, 1]$. Note that T_i is discontinuous on the circle $x_i \in \{0, 1\}$. As a result, T is discontinuous on the circle $x_2 \in \{0, 1\}$ and on the curve $x_1 + \alpha x_2 \in \mathbb{Z}$. One way to portray this is by introducing the sets

$$\begin{aligned} \Gamma^+ &= \{(x_1, x_2) : 0 \le x_1 + \alpha x_2 \le 1,\ 0 \le x_2 \le 1\} \\ \Gamma^- &= \{(x_1, x_2) : 0 \le x_1 \le 1,\ 0 \le -\alpha x_1 + x_2 \le 1\} \end{aligned}$$

and observing that $\hat{T}$ maps Γ^+ onto Γ^- but T is discontinuous along $S^+ = \partial\Gamma^+$. Moreover $\hat{T}^{-1} = \hat{T}_2^{-1} \circ \hat{T}_1^{-1}$ with $\hat{T}_i^{-1}(a) = A_i^{-1} a$ for $i = 1, 2$, where

$$A_1^{-1} = \begin{bmatrix} 1 & 0 \\ -\alpha & 1 \end{bmatrix}, \ A_2^{-1} = \begin{bmatrix} 1 & -\alpha \\ 0 & 1 \end{bmatrix}.$$

Since T_1^{-1} is discontinuous on the circle $x_2 \in \{0, 1\}$ and T_2^{-1} is discontinuous on the circle $x_1 \in \{0, 1\}$, we deduce that T^{-1} is discontinuous on $S^- = \partial\Gamma^-$. Note that the line $x_2 = 0$ is mapped onto the line $x_2 = ax_1$ and the line $x_2 = 1$ is mapped onto the line $x_2 = ax_1 + 1$. Also note that distinct points on S^+ which correspond to a single point on $\mathbb{T}^2$ are mapped to distinct points on $\mathbb{T}^2$.

We now examine the stable and unstable manifolds. For the unstable manifold, we need to have that if $y \in W^u(x)$, then $d(T^{-n}(x), T^{-n}(y)) \to 0$ as $n \to +\infty$. We may try

$$W_0^n(x) = \{\pi(a + v_2 t) : t \in \mathbb{R}\}$$

where a is chosen so that $\pi(a) = x$ and v_2 is the expanding direction. This would not do the job because of the discontinuity. Indeed the discontinuity set S^- cut the set $W_0^u(x)$ into pieces. Let us write $W_1^u(x)$ for the connected component of $W_0^u(x)$ inside Γ^-. Since crossing S^- causes a jump discontinuity for T^{-1}, we have that $d(T^{-n}(x), T^{-n}(y)) \not\to 0$ if $y \in W_0^u(x) \setminus W_1^u(x)$. However note that if $y \in W_1^u(x)$, then $d(T^{-1}(x), T^{-1}(y)) = \lambda d(x, y)$. As a result, $d(T^{-1}(x), T^{-1}(y))$ gets smaller than $d(x, y)$ by a factor of size λ. To have $d(T^{-n}(x), T^{-n}(y)) = \lambda^n d(x, y)$, we need to make sure that the segment joining $T^{-n}(x)$ to $T^{-n}(y)$ is not cut into pieces by S^-. That is, the segment xy does not intersect $T^n(S^-)$. Motivated by this, let us pick $x \in \mathbb{T}^2 \setminus \cup_{i=0}^{\infty} T^i(S^-)$ and define $W_j^u(x)$ to be the component of $W_0^u(x)$ which avoids $\cup_{i=0}^{j} T^i(S^-)$. We now claim that for μ-almost all points, $W^u(x) = \cap_{j=0}^{\infty} W_j^u(x)$ is still a nontrivial segment. (This would be our unstable manifold.) More precisely, we show that for μ-almost all x, there exists a finite $N(x)$ such that

$$W^u(x) = \bigcap_{j=0}^{\infty} W_j^u(x) = \bigcap_{j=0}^{N(x)} W_j^u(x).$$

To see this, let us observe that for example

$$W_2^u(x) = T(T^{-1} W_1^u(x) \cap W_1^u(T^{-1}(x))).$$

In other words, we take $W_1^u(x)$ which is a line segment with endpoints in S^-. We apply T^{-1} on it to get a line segment $T^{-1} W_1^u(x)$ with $T^{-1}(x)$ on it. This line segment is shorter than $W_1^u(x)$; its length is λ times the length of $W_1^u(x)$. If this line segment is not cut by S^-, we set $W_2^u(x) = W_1^u(x)$; otherwise we take the connected component of $T^{-1} W_1^u(x)$ which lies inside S^- and has $T^{-1}(x)$ on it. This connected component lies on $W_1^u(T^{-1}(x))$. We then map this back by T. Note that $W_2^u(x) \neq W_1^u(x)$ only if $d(T^{-1}(x), S^-) =$ distance of $T^{-1}(x)$ from S^- is less than

$$\text{length}(T^{-1} W_1^u(x)) = \lambda \text{ length}(W_1^u(x)).$$

More generally,

$$W_{i+1}^u(x) = T^i(T^{-i} W_i^u(x) \cap W_1^u(T^{-i}(x)),$$

and $W_{i+1}^u(x) \neq W_i^u(x)$ only if

$$d(T^{-i}(x), S^-) < \lambda^i \text{ length } (W_i^u(x)).$$

Since length $(W_i^u(x)) \leq$ length $(W_1^u(x)) =: c_0$, we learn that if $W^u(x) = \{x\}$, then

$$d(T^{-i}(x), S^-) < c_0 \lambda^i,$$

for infinitely many i. Set $S_\delta^- = \{x \in \Gamma^- : d(x, S^-) < \delta\}$. We can write

$$\begin{aligned}
\{x : W^u(x) = \{x\}\} &\subseteq \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} T^i(S^-_{c_0\lambda^i}), \\
\mu\left(\{x : W^u(x) = \{x\}\}\right) &\leq \lim_{n\to\infty} \sum_{i=n}^{\infty} \mu(T^i(S^-_{c_0\lambda^i})) \\
&= \lim_{n\to\infty} \sum_{i=n}^{\infty} \mu(S^-_{c_0\lambda^i}) \\
&\leq \lim_{n\to\infty} \sum_{i=n}^{\infty} c_1 c_0 \lambda^i = 0
\end{aligned}$$

for some constant c_1. From this we deduce that for μ-almost all points x, the set $W^u(x)$ is an interval of positive length with endpoints in $\bigcup_{i=0}^{\infty} T^i(S^-)$. Moreover, if $y \in W^u(x)$, then

$$d(T^{-n}(y), T^{-n}(x)) = \lambda^n d(x, y) \to 0$$

as $n \to \infty$. In the same fashion, we construct $W^s(x)$.

We now apply the Hopf's argument. To this end, let us take a dense subset $\mathcal{T}$ of $C(\mathbb{T}^2)$ and for $f \in C(\mathbb{T}^2)$ define $f^{\pm}$ as in Example 6.1. Set $Y = \cap\{Y_f : f \in \mathcal{A}\}$, where

$$Y_f = \{x \in \mathbb{T}^2 : f^{\pm}(x), W^s(x), W^u(x) \text{ are well-defined and } f^+(x) = f^-(x)\}$$

So far we know that $\mu(Y) = 1$. Regarding $\mathbb{T}^2$ as $[0,1]^2$ with $0 = 1$ and slicing $\mathbb{T}^2$ into line segments parallel to v_i for $i = 0, 1$, we learn that each stable or unstable leaf intersects Y on a set of full length, except for a family of leaves of total μ-measure 0. Let us pick a leaf $W^s(x_0)$ which is not one of the exceptional leaf and define

$$Z(x_0) = \bigcup \{W^u(y) : y \in W^s(x_0) \text{ and } y \in Y\}.$$

Since $W^u(y)$ is of positive length, for each $y \in W^s(x)$, we deduce that $\mu(Z(x_0)) > 0$. On the other hand f^+ is constant on $W^s(x_0)$ and f^- is constant on each $W^u(y)$, $y \in W^s(x_0) \cap Y$. Since $f^+ = f^-$ on $W^s(x_0)$, we deduce that $f^+ = f^-$ is constant on $Z(x_0)$ for every $f \in \mathcal{T}$.

With the aid of Hopf's argument, we managed to show that $f^{\pm}$ is constant on a set of positive μ-measure. But for ergodicity of μ, we really need to show this on a set of μ-full measure. This is where Hopf's argument breaks down, however it does show that μ has at most countably many ergodic components. Indeed if we define

$$Z'(x_0) = \{x : f^{\pm}(x) \text{ exist and } f^{\pm}(x) = f^{\pm}(x_0)\},$$

then $\mu\big(Z'(x_0)\big) > 0$ because $Z'(x_0) \supseteq Z(x_0)$. Since this is true for μ-almost all x_0, we deduce that μ can only have countably many ergodic components.

We now explain how Sinai's method can be used to prove the ergodicity of μ. To this end, let us take a box B with boundary lines parallel to v_1 and v_2 and define

$$W^u(B) = \big\{y \in B \cap Y' : W^u(y) \cap Y \text{ is of full length and } W^u(y) \\ \text{reaches the boundary of } B \text{ on both ends}\big\}$$

where

$$Y' = Y'(f) = \big\{y : f^+(y) \text{ and } f^-(y) \text{ are defined and } f^+(y) = f^-(y)\big\}.$$

In the same fashion we define $W^s(B)$. We now claim that f^+ is constant on $W^s(B)$, f^- is constant on $W^u(B)$, and these constants coincide. To see this, we fix $W^u(y) \subseteq W^u(B)$ and take all $z \in W^u(y) \cap Y'$. We have that f^- is constant on $W^u(y)$ and that $f^-(z) = f^+(z)$ for such $z \in W^u(y) \cap Y'$. Since f^+ is constant on each $W^s(z)$, we deduce that f^+ is constant on $\bigcup_{z \in W^u(y) \cap Y'} (W^s(z) \cap Y')$ and this constant coincides with $f^-(y)$. By varying $y \in W^u(B)$, we obtain the desired result. (Here we are using the fact that if $W^u(y) \subseteq W^u(B)$ and $W^s(z) \subseteq W^s(B)$, then $W^u(y)$ and $W^s(z)$ intersect.)

Let us regard the vertical and horizontal axis as the stable and unstable directions. We now take two boxes which overlap. For example, imagine that $B_1 = I_1 \times J_1$, $B_2 = I_2 \times J_2$ in the (v_2, v_1) coordinates, where either $J_1 = J_2$ and $I_1 \cap I_2 \neq \emptyset$, or $I_1 = I_2$ and $J_1 \cap J_2 \neq \emptyset$. We wish to show that the constant $f^\pm$ of $W^{u(s)}(B_1)$ equal the constant $f^\pm$ of $W^{u(s)}(B_2)$. We know that f^+ is constant on $W^s(B_1) \cup W^s(B_2)$ and that f^- is constant on $W^u(B_1) \cup W^u(B_2)$. We also know that $f^+ = f^-$ in Y'. Clearly if $J_1 = J_2$, $I_1 \cap I_2 \neq \emptyset$ and $W^s(B_1) \cap W^s(B_2) \neq \emptyset$ (respect. $I_1 = I_2$, $J_1 \cap J_2 \neq \emptyset$ and $W^u(B_1) \cap W^u(B_2) \neq \emptyset$), then the constant f^+ (respect. f^-) for $W^s(B_1)$ (respect. $W^u(B_1)$) coincides with the constant f^+ (respect. f^-) for $W^s(B_2)$ (respect. $W^u(B_2)$). Let us identify a scenario for which $\mu\big(W^s(B_1) \cap W^s(B_2)\big) > 0$. Given $\beta \in (0,1)$, let us call a box B *β-uconnected* if the set

$$B^u = \big\{x \in B : W^u(x) \text{ is defined and reaches the boundary of } B \text{ on both ends}\big\}$$

satisfies $\mu(B^u) > \beta\mu(B)$. The set B^s is defined in a similar way and we say that B is β-sconnected if $\mu(B^s) > \beta\mu(B)$. Note that if $\mu(B^{u(s)}) > \beta\mu(B)$, then $\mu(W^{u(s)}(B)) > \beta\mu(B)$ because Y' is of full-measure. (Here we are using Fubini's theorem to write the measures of Y' as an integral of the lengths of v_1 or v_2 slices of Y'.) Now assume that B_1 and B_2 satisfy the following conditions:

- $B_1 = I_1 \times J_1$, $B_2 = I_2 \times J_2$, with $J_1 = J_2$,
- β-sconnected (respect. uconnected),
- , B_2 is to the right of B_1 (respect. B_2 is on the top of B_1),

- $\mu(B_1 \cap B_2) \geq (1-\beta)\max(\mu(B_1), \mu(B_2))$,
- $\mu\big(W^u(B_1)\big), \mu(\big(W^u(B_2)\big) > 0$ (respect. $\mu\big(W^s(B_1)\big), \mu(\big(W^s(B_2)\big) > 0)$,

then for sure $\mu\big(W^s(B_1)\cap W^s(B_2)\big) > 0$ (respect. $\mu\big(W^u(B_1)\cap W^u(B_2)\big) > 0$). Simply because $W^s(B_1) \cap B_2 \subseteq W^s(B_2)$, and $\mu\big(W^s(B_1) \cap B_2\big) > 0$.

Based on this observation, let us take a box $\bar{B}$ and cover it by overlapping small boxes. Pick $\beta \in (0, 1/2)$ and take a grid

$$\left\{\frac{\beta}{n} i \in \bar{B} : i \in \mathbb{Z}^2\right\}$$

and use the points of this grid as the center of squares of side length $\frac{1}{n}$. Each such square has area $\frac{1}{n^2}$, and two adjacent squares overlap on a set of area $(1-\beta)\frac{1}{n^2}$. Let us write $\mathcal{B}_n^\beta(\bar{B})$ for the collection of such overlapping squares. We now state a key result of Sinai regarding the α-u(s)connected boxes.

Theorem 6.3 *There exists $\alpha_0 < 1$ such that for every $\beta \in (0, \alpha_0)$,*

$$\lim_{n\to\infty} n\mu\left(\cup\{B \in \mathcal{B}_n^\beta(\bar{B}) : B \textit{ is not either } \beta\textit{-uconnected or } \beta\textit{-sconnected } \}\right) = 0.$$

We now demonstrate how Theorem 6.3 can be used to show that f^+ and f^- are constant almost everywhere in $\bar{B}$. We choose $\beta < \alpha < \alpha_0$ and would like to show that if $y, z \in Y_f' \cap \bar{B}$, then $f^-(y) = f^+(z)$.

To prove this, we first claim that there exists a full column of boxes in $\mathcal{B}_n^\beta(\bar{B})$ such that each box B in this column is α-uconnected and $W^u(y)$ reaches two boundary sides of a box in the column provided that n is sufficiently large. Here y is fixed and since $W^u(y)$ is a nontrivial interval, it crosses $c_1 n$ many columns of total area. If each such column has a box which is not α-uconnected, then

$$\mu(\cup\{B \in \mathcal{B}_n^\beta(\bar{B}) : B \text{ is not } \alpha\text{-uconnected}\}) \geq c_3 n \cdot \frac{1}{n^2}$$

for some $c_3 > 0$ (note that a point x belongs to at most $\left(\frac{1}{2\beta} + 1\right)^2$ many boxes). This contradicts Theorem 2.2 for large n. Hence such a column exists. Similarly, we show that there exists a full row of boxes in $B_n^\beta(\bar{B})$ such that each box is α-sconnected and at least one box in this row is fully crossed by $W^s(z)$. Since $\beta < \alpha$, we now that f^- is constant (with the same constant) on $\cup W^s(B)$ with the union over the boxes B on that row, and that f^+ is constant on $\cup W^u(B)$ with union over the boxes B on that column. Since the row and the column intersect on a box, we deduce that $f^+(y) = f^-(z)$. This completes the proof of $f^+ = f^- =$ constant a.e. in $\bar{B}$. We now turn to the proof of Theorem 6.3.

Proof of Theorem 6.3**.** First we define a sector

$$\mathcal{C} = \{(a,b) \in \mathbb{R}^2 : |a| \le \gamma|b|\}$$

which is symmetric about the unstable line v_2 and contains the two directions of sides of Γ^-. We use the explicit value of the slope of v_2 to see that in fact γ can be chosen in $(0,1)$. We now argue that all the line segments in $\bigcup_0^\infty T^i(S^-)$ have directions in the sector $\mathcal{C}$. This is because $\mathcal{C}$ already has the directions of S^-. On the other hand, since the sides of S^- are not parallel to v_1, T^i pushes these lines toward v_2.

Now let us measure the set of points not in $W^u(B)$ for a box in $\mathcal{B}_n^\beta(B)$. Note that if a point $x \in B$ is not in $W^u(B)$, it means that $W^u(x)$ is cut by one of $T^i(S^-)$, $i \in \mathbb{N}^*$ inside B. Let us first consider the case when B is intersected by precisely one line segment of $\bigcup_i T^i(S^-)$. Since this line segment is in sector $\mathcal{C}$, we learn that $\mu(B - W^u(B)) \le \frac{\gamma}{n^2}$. This means

$$\mu(W^u(B)) \ge (1-\gamma)\mu(B).$$

Let us choose $\alpha_0 = \frac{1}{2}(1-\gamma)$ so that if $\beta < \alpha_0$ and B is not β-uconnected, then B must intersect at least two segments in $\bigcup_i T^i(S^-)$. (This would be true even when $\beta < 1-\gamma$ but we need a smaller β later in the proof.) We now look at $R_L = \bigcup_{i=0}^{L-1} T^i(S^-)$ and study those boxes which intersect at least two line segments in R_L. Note that each box B is of length $1/n$ and the line segments in R_L are distinct. So, a box $B \in \mathcal{B}_n^\beta$ intersects at least two lines in R_L only if it is sufficiently close to an intersection point of two lines in R_L. More precisely, we can find a constant $c_1(L)$ such that such a box is in a $\frac{c_1(L)}{n}$ neighborhood of an intersection point. (In fact $c_1(L)$ can be chosen to be a constant multiple of $L^2 e^{c_0 L}$ because there are at most $4L(4L-1)$ intersection points and the smallest possible angle between two line segment in R_L is bounded below by $e^{-c_0 L}$ for some constant c_0.) Hence the total area of such boxes is $c_1(L)n^{-2}$. Now we turn to those boxes which intersect at most one line in R_L and at least one line in $R'_L = \bigcup_{i=L}^\infty T^i(S^-)$. Let us write $\mathcal{D}_L$ for the set of such boxes. Let us write $B - W^u(B) = B'_L \cup B''_L$, where

$$\begin{aligned} B'_L &= \{x \in B : W^u(x) \cap B \cap R_L \ne \emptyset\} \\ B''_L &= \{x \in B : W^u(x) \cap B \cap R'_L \ne \emptyset\}. \end{aligned}$$

If $B \in \mathcal{D}_L$, then B can intersect at most one line segment in R_L. Hence $\mu(B'_L) \le \gamma\mu(B) \le (1-2\beta)\mu(B)$. If $B \in \mathcal{D}_L$ is not β-uconneted, then

$$(1-\beta)\mu(B) \le \mu(B - W^u(B)) \le (1-2\beta)\mu(B) + \mu(B''_L).$$

From this we deduce

$$\begin{aligned} \mu\left(\cup\left\{B \in \mathcal{D}_L : B \text{ is not } \beta\text{-uconnected}\right\}\right) &\le \sum \left\{\mu(B) \in \mathcal{D}_L : B \text{ is not } \beta\text{-uconnected}\right\} \\ &\le \beta^{-1} \sum \left\{\mu(B''_L) \in \mathcal{D}_L : B \text{ is not } \beta\text{-uconnected}\right\} \\ &\le \beta^{-1} c c(\beta) \mu\left(\cup\left\{B''_L \in \mathcal{D}_L : B \text{ is not } \beta\text{-uconnected}\right\}\right), \end{aligned}$$

where for the last inequity we have used the fact that each point belongs to at most $c(\beta) = (1/(2\beta)+1)^2$ many boxes in $\mathcal{B}_n^\beta$. Let $x \in B_L''$ for some $B \in \mathcal{D}_L$. This means that $W^u(x) \cap B$ intersects $T^i(S^-)$ for some $i \geq L$. Hence $T^{-i}(W^u(x) \cap B) \cap S^- \neq \emptyset$. Note that $T^{-i}(W^u(x) \cap B)$ is a line segment of length at most $\lambda^{-i} n^{-1}$. As a result, $T^{-i}(x)$ must be within $\lambda^i n^{-1}$-distance of S^-. That is, $x \in T^i(S^-_{\lambda^i n^{-1}})$. So,

$$\mu\left(\cup\left\{B_L'' : B \in \mathcal{D}_L\right\}\right) \leq \mu\left(\bigcup_{i=L}^{\infty} T^i\left(S^-{}_{\lambda^i n^{-1}}\right)\right) \leq \sum_{i=L}^{\infty} \mu\left(T^i\left(S^-{}_{\lambda^i n^{-1}}\right)\right)$$
$$= \sum_{i=L}^{\infty} \mu\left(S^-{}_{\lambda^i n^{-1}}\right) \leq c_2 \sum_{i=L}^{\infty} n^{-1}\lambda^i \leq c_3 n^{-1}\lambda^L.$$

This yields

$$\mu\left(\cup\left\{B \in \mathcal{B}_n^\beta(\bar{B}) : B \text{ is not } \alpha\text{-usconnected}\right\}\right) \leq c_1(L) n^{-2} + c_4(\beta) n^{-1}\lambda^L$$

for every n and L. By choosing $L = \eta \log n$ for $\eta = (c_0 - \log\lambda)^{-1}$, we get

$$c_5 n^{-1+\eta\log\lambda}(\log n)^2,$$

for the right-hand side. This completes the proof of Theorem 6.3. □

7 Classification of Dynamical Systems

Newtonian ODEs of the celestial mechanics are examples of Hamiltonian system that exhibit both deterministic and stochastic behaviors. A prime example of a deterministic dynamical system is a rotation (Example **1.1(i)** or its infinite dimensional on $\mathbb{T}^{\mathbb{N}}$). The simplest example of a stochastic dynamical system is a shift (Example **1.1(iii)**). A dynamical system of positive entropy has always a stochastic subsystem as the following result confirms.

Theorem 7.1 (Sinai) *Let (X, T, μ) be an ergodic dynamical system, and let $(E^{\mathbb{Z}}, \tau, \mu_p)$ be a Bernoulli shift as in Example* **3.2(i)**. *If $h_\mu(T) \geq h_{\mu_p}(\tau)$, then there exists a measurable factor map $F : X \to E^{\mathbb{Z}}$.*

With the aid of the entropy we can completely classify Bernoulli shifts:

Theorem 7.2 (Ornstein) *Two Bernoulli shifts of equal entropy are isomorphic.*

Kolmogorov formulated a class of K-automorphisms that includes Bernoulli shifts. A K-automorphism yields a dynamical system is purely stochastic and has no deterministic factor. Pinsker had conjectured that any dynamical system can be *split* into a K-automorphism and a system of zero entropy. In 1973 Ornstein showed that Pinsker's conjecture is not true. However the following weaker version of Pinsker's conjecture was established in 2017:

Theorem 7.3 (Austin) *Given $\varepsilon > 0$, an ergodic system (X, T, μ), we can find an isomorphism between $(Y \times Z, S \times \tau, \nu \times \hat{\mu})$ such that $h_S(\nu) \le \varepsilon$, and $(Z, \tau, \hat{\mu})$ is a Bernoulli shift.*

Definition 7.1(i) Consider a measure space $(X, \mathcal{F}, \mu)$ and a measurable automorphism $T : X \to X$ such that $T^{-1}\mathcal{F} = T\mathcal{F} = \mathcal{F}$, $T^\sharp \mu = \mu$. The *Pinsker class* $\mathcal{P}(T)$ is defined to be the set of $A \in \mathcal{F}$ such that $h_\mu(T, \xi(A)) = 0$, where $\xi(A) = \{A, X \setminus A\}$.

(ii) We say T has *completely positive entropy* if the σ-algebra $\mathcal{P}(T)$ is trivial. Equivalently, $h(T, \xi) > 0$ for every non-trivial partition ξ.

(iii) Given a partition ξ, we write $\mathcal{F}_\xi^\pm$ for the σ-algebra generated by all $\{T^{\pm n}\xi : n \ge 1\}$. We also write $\mathcal{F}_\xi^n$ for the σ-algebra generated by all $\{T^k \xi : k \le n\}$, and

$$\mathcal{F}_\xi^{-\infty} = \cap_n \mathcal{F}_\xi^n, \qquad \mathcal{F}_\xi^{\infty} = \cup_n \mathcal{F}_\xi^n.$$

(iv) Given a partition ξ, we write $\hat{\xi}(m, n)$ for $T^m \xi \vee \cdots \vee T^n \xi$.

(v) We say $(X, T, \mathcal{B}, \mu)$ is a (Kolmogorov) K-automorphism if there exists a (countable) partition such that $\mathcal{F}_\xi^{-\infty} = \{\emptyset, X\}$, $\mathcal{F}_\xi^\infty = \mathcal{B}$. Equivalently, if we set $\mathcal{K} = \mathcal{F}_\xi^-$, then

$$T^{-1}\mathcal{K} \subseteq \mathcal{K}, \quad \cap_{n=1}^\infty T^{-n}\mathcal{K} = \{\emptyset, X\}, \quad \cup_{n=1}^\infty T^n \mathcal{K} = \mathcal{B}.$$

□

Theorem 7.4 **(i)** (Kolmogorov) *Let $\mathcal{K}$ be a sub-σ algebra such that*

$$T^{-1}\mathcal{K} \subseteq \mathcal{K}, \qquad \cup_{n=1}^\infty T^n \mathcal{K} = \mathcal{B}.$$

Then

$$\mathcal{P}(T) \subseteq \cap_{n=1}^\infty T^{-n}\mathcal{A}. \tag{7.1}$$

In particular any K-automorphism is completely positive entropy.

(ii) (Rokhlin-Sinai) *There exists a sub-σ algebra $\mathcal{A}$ with the following properties:*

$$T^{-1}\mathcal{A} \subseteq \mathcal{A}, \qquad \cup_{n=1}^\infty T^n \mathcal{A} = \mathcal{B}, \qquad \cap_{n=1}^\infty T^{-n}\mathcal{A} = \mathcal{P}(T). \tag{7.2}$$

In particular, any transformation of completely positive entropy is a K-automorphism.

Proof(i) Let ξ be a finite partition with $\xi \subset \mathcal{P}(T)$. Then $\xi \subset \hat{\xi}(-\infty,-1)$, which means that $\hat{\xi}(-\infty,-1) = \hat{\xi}(-\infty,0)$. Inductively, $\hat{\xi}(-\infty,-n) = \hat{\xi}(-\infty,0)$ for every $n \in \mathbb{N}$. Choose $m \in \mathbb{N}$ such that $\xi \subset T^m\mathcal{K}$. Hence $T^{-k-\ell}\xi \subset T^{-\ell}\mathcal{K}$ for every $k \geq m$. Hence

$$\xi \subset \hat{\xi}(-\infty,-m-\ell) \subset T^{-\ell}\mathcal{K}.$$

Since ℓ is arbitrary, we are done. □

Intuitively, $\mathcal{P}(T)$ represents the deterministic part of the dynamics. After all if $h_\mu(T,\xi) = 0$, then the information contained in the past $\mathcal{F}_\xi^-$ determines the present ξ, μ-almost surely. We will see later that there exists ξ that generates the full σ-algebra $\mathcal{B}$, and the set of its the remote past events $\mathcal{F}_\xi^{-\infty}$ coincides with $\mathcal{P}(T)$. The following will prepare us for the proof of Theorems 7.4 and 7.5.

Proposition 7.1 **(i)** *The class $\mathcal{P}(T)$ is a T-invariant σ-algebra.*

(ii) *For every $n \in \mathbb{Z}$, $\mathcal{P}(T) = \mathcal{P}(T^n)$.*

(iii) $h_\mu(T,\xi) = n^{-1}H_\mu\big(\xi \vee \cdots \vee T^{n-1}\xi \big| \mathcal{F}_\xi^-\big)$.

(iii) $h_\mu(T,\xi \vee \eta) = h_\mu(T,\xi) + H_\mu\big(\eta \big| \mathcal{F}_\xi^\infty \vee \mathcal{F}_\eta^-\big)$.

(iv) *We have*

$$\mathcal{P}(T) = \bigvee \left\{\mathcal{F}_\xi^{-\infty} : \ H_\mu(\xi) < \infty\right\}. \tag{7.3}$$

(v) *Let $\mathcal{A}$ be sub-σ algebra such that*

$$T^{-1}\mathcal{A} \subseteq \mathcal{A}, \qquad \cup_{n=1}^\infty T^n\mathcal{A} = \mathcal{B}$$

Then,

$$\mathcal{P}(T) \cap_{n=1}^\infty T^{-n}\mathcal{A} =: \mathcal{A}_{-\infty}. \tag{7.4}$$

Proof(i) Evidently if $A \in \mathcal{P}(T)$, then $X \setminus A \in \mathcal{P}(T)$. Next if $A, B \in \mathcal{P}(T)$, then $\xi(A \cup B) \leq \xi(A) \vee \xi(B)$, and

$$h_\mu(T,\xi(A \cup B)) \leq h_\mu(T,\xi(A) \vee \xi(B)) \leq h_\mu(T,\xi(A)) + h_\mu(T,\xi(B)) = 0.$$

Hence $A \cup B \in \mathcal{P}(T)$. Finally, if $(A_n : \ n \in \mathbb{N})$ is an increasing sequence of sets in $\mathcal{P}(T)$ with $A = \cup_n A_n$, then

$$h_\mu(T,\xi(A)) \leq h_\mu(T,\xi(A_n)) + H_\mu(A|A_n) = H_\mu(A|A_n),$$

which goes to 0 in large n limit.

Finally, if $h_\mu(T,\xi)=0$, then using

$$H_\mu\big((T^{-1}\xi)(0,n)\big)=H_\mu\big(T^{-1}\big(\xi(0,n)\big)\big)=H_\mu\big(\xi(0,n)\big),$$

we deduce that $h_\mu(T,T^{-1}\xi)=0$.

(ii) Assume that $\xi\subset\mathcal{P}(T)$, and define $\eta=\xi\vee T^{-1}\xi\vee\dots\vee T^{-n+1}\xi\subset\mathcal{P}(T)$. Hence as in the proof of Proposition 3.5**(i)**,

$$h_\mu(T^n,\eta)=nh_\mu(T,\xi)=0.$$

This implies $\mathcal{P}(T)\subseteq\mathcal{P}(T^n)$. Conversely,

(iii) Given $m\ge n$, we write $\hat{\xi}(m,n)$ for $T^m\xi\vee\dots T^n\xi$. We have

$$\begin{aligned}
H\big(\xi(0,n-1)\big|\mathcal{F}_\xi^-\big)&=\lim_{m\to\infty}H_\mu\big(\xi(0,n-1)\big|\xi(-m,-1)\big)\\
&=\lim_{m\to\infty}\big[H_\mu\big(\xi\big|\xi(-m,-1)\big)+H_\mu\big(\xi(1,n-1)\big|\xi(-m,0)\big)\big]\\
&=\lim_{m\to\infty}\big[H_\mu\big(\xi\big|\xi(-m,-1)\big)+H_\mu\big(\xi(0,n-2)\big|\xi(-m-1,-1)\big)\big]\\
&=h_\mu(T,\xi)+h_\mu(T,\xi(0,n-2))=\dots=nh_\mu(T,\xi).
\end{aligned}$$

(iii) We have

$$\begin{aligned}
h_\mu(T,\xi\vee\eta)&=(n+1)^{-1}H_\mu\big(\xi(0,n)\vee\eta(0,n)\big|\mathcal{F}_\xi^-\vee\mathcal{F}_\eta^-\big)\\
&=(n+1)^{-1}\big[H_\mu\big(\xi(0,n)\big|\mathcal{F}_\xi^-\vee\mathcal{F}_\eta^-\big)+H_\mu\big(\eta(0,n)\big|\mathcal{F}_\xi^-\vee\mathcal{F}_\eta^-\vee\xi(0,n)\big)\big]\\
&=(n+1)^{-1}\big[H_\mu\big(\xi(0,n)\big|\mathcal{F}_\xi^-\vee\mathcal{F}_\eta^-\big)+H_\mu\big(\eta(0,n)\big|\mathcal{F}_\eta^-\vee\xi(-\infty,n)\big)\big].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
H_\mu\big(\eta(0,n)\big|\mathcal{F}_\eta^-\vee\xi(-\infty,n)\big)&=H_\mu\big(\eta\big|\mathcal{F}_\eta^-\vee\xi(-\infty,n)\big)+H_\mu\big(\eta(1,n)\big|\mathcal{F}_\eta^-\vee\eta\vee\xi(-\infty,n)\big)\\
&=H_\mu\big(\eta\big|\mathcal{F}_\eta^-\vee\xi(-\infty,n)\big)+H_\mu\big(\eta(0,n-1)\big|\mathcal{F}_\eta^-\vee\xi(-\infty,n-1)\big)\\
&=\dots=\sum_{k=0}^{n}H_\mu\big(\eta\big|\mathcal{F}_\eta^-\vee\xi(-\infty,k)\big).
\end{aligned}$$

We are done if we can show

$$h_\mu(T,\xi)=\lim_{n\to\infty}(n+1)^{-1}H_\mu\big(\xi(0,n)\big|\mathcal{F}_\xi^-\vee\mathcal{F}_\eta^-\big),\tag{7.5}$$

$$H_\mu\big(\eta\big|\mathcal{F}_\eta^-\vee\mathcal{F}_\xi^\infty\big)=\lim_{k\to\infty}H_\mu\big(\eta\big|\mathcal{F}_\eta^-\vee\xi(-\infty,k)\big).\tag{7.6}$$

The claim (**??**) is an immediate consequence of Lemma 3.2. As for (7.6), let us write $\hat{h}$ for the right-hand side. By part **(ii)**, we certainly have $h_\mu(T,\xi) \ge \hat{h}$. For the converse, let us write $\gamma = \xi \vee \eta$, and claim

$$\lim_{n\to\infty}(n+1)^{-1}H_\mu\big(\gamma(0,n)\big|\mathcal{F}_\gamma^-\big) = \lim_{n\to\infty}(n+1)^{-1}H_\mu\big(\gamma(0,n)\big|\mathcal{F}_\xi^-\big). \tag{7.7}$$

To see this, observe

$$\begin{aligned}
H_\mu\big(\gamma(0,n)\big|\mathcal{F}_\xi^-\big) &= H_\mu\big(\gamma\big|\mathcal{F}_\xi^-\big) + H_\mu\big(T\gamma\big|\mathcal{F}_\xi^- \vee \gamma\big) + \cdots + H_\mu\big(T^n\gamma\big|\mathcal{F}_\xi^- \vee \gamma(0,n-1)\big) \\
&= \sum_{j=0}^n H_\mu\left(\gamma\big|T^{-j}\big(\mathcal{F}_\xi^- \vee \gamma(0,j-1)\big)\right) = \sum_{j=0}^n H_\mu\left(\gamma\big|\xi(-\infty,-j-1)\vee\gamma(-j,-1)\right)
\end{aligned}$$

This implies (7.7) because

$$\begin{aligned}
&\gamma(-n,-1) \le \xi(-\infty,-n-1)\vee\gamma(-n,-1) \le \gamma(-\infty,-1),\\
&\lim_{n\to\infty}\big[\xi(-\infty,-n-1)\vee\gamma(-n,-1)\big] = \gamma(-\infty,-1).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
H_\mu\big(\gamma(0,n)\big|\mathcal{F}_\gamma^-\big) &= H_\mu\big(\gamma(0,n)\vee\xi(0,n)\big|\mathcal{F}_\gamma^-\big) = H_\mu\big(\xi(0,n)\big|\mathcal{F}_\gamma^-\big) + H_\mu\big(\gamma(0,n)\big|\mathcal{F}_\gamma^-\vee\xi(0,n)\big)\\
H_\mu\big(\gamma(0,n)\big|\mathcal{F}_\xi^-\big) &= H_\mu\big(\gamma(0,n)\vee\xi(0,n)\big|\mathcal{F}_\xi^-\big) = H_\mu\big(\xi(0,n)\big|\mathcal{F}_\xi^-\big) + H_\mu\big(\gamma(0,n)\big|\mathcal{F}_\xi^-\vee\xi(0,n)\big).
\end{aligned}$$

From this, (7.7), and part **(ii)** we deduce

$$\begin{aligned}
\hat{h} &= \lim_{n\to\infty}(n+1)^{-1}H_\mu\big(\xi(0,n)\big|\mathcal{F}_\gamma^-\big)\\
&= \lim_{n\to\infty}(n+1)^{-1}\left[H_\mu\big(\gamma(0,n)\big|\mathcal{F}_\gamma^-\big) - H_\mu\big(\gamma(0,n)\big|\mathcal{F}_\gamma^-\vee\xi(0,n)\big)\right]\\
&= \lim_{n\to\infty}(n+1)^{-1}\left[H_\mu\big(\gamma(0,n)\big|\mathcal{F}_\xi^-\big) - H_\mu\big(\gamma(0,n)\big|\mathcal{F}_\gamma^-\vee\xi(0,n)\big)\right]\\
&\ge \lim_{n\to\infty}(n+1)^{-1}\left[H_\mu\big(\gamma(0,n)\big|\mathcal{F}_\xi^-\big) - H_\mu\big(\gamma(0,n)\big|\mathcal{F}_\xi^-\vee\xi(0,n)\big)\right]\\
&= \lim_{n\to\infty}(n+1)^{-1}H_\mu\big(\xi(0,n)\big|\mathcal{F}_\xi^-\big) = h_\mu(T,\xi),
\end{aligned}$$

as desired.

(iv) Given a partition ξ with $H_\mu(\xi) < \infty$, observe that for any partition η,

$$\eta \subset \mathcal{F}_\xi^{-\infty} = \cap_n T^{-n}\mathcal{F}_\xi^- \quad\implies\quad \eta \le \xi,\;\; \mathcal{F}_\eta^\infty \subseteq \mathcal{F}_\xi^-.$$

As a result

$$h_\mu(T,\xi) = h_\mu(T,\xi\vee\eta) = h_\mu(T,\eta) + H_\mu\big(\xi\big|\mathcal{F}_\xi^-\vee\mathcal{F}_\eta^\infty\big) = h_\mu(T,\eta) + H_\mu\big(\xi\big|\mathcal{F}_\xi^-\big),$$

which means that $h_\mu(T,\eta)=0$, or $\eta\subset\mathcal{P}(T)$. As a result

$$\bigvee\left\{\mathcal{F}_\xi^{-\infty}:\ H_\mu(\xi)<\infty\right\}\subseteq\mathcal{P}(T).$$

For the converse, let us take any $A\in\mathcal{P}(T)$. Then for $\eta=\xi(A)$, we have $\eta<\eta(-\infty,-1)$. Equivalently, $\eta(-\infty,0)=\eta(-\infty,-1)$. Inductively, $\eta(-\infty,0)=\eta(-\infty,-n)$, for every $n\in\mathbb{N}$. As a result $\mathcal{F}_\eta^-=\mathcal{F}_\eta^{-\infty}$. This implies

$$A\in\eta\subset\mathcal{F}_\eta^-=\mathcal{F}_\eta^{-\infty}\subseteq\bigvee\left\{\mathcal{F}_\xi^{-\infty}:\ H_\mu(\xi)<\infty\right\}.$$

This completes the proof.

(v) For (7.5), we need to show that for every partition ξ,

$$\xi\subset\mathcal{P}(T)\vee\mathcal{A}_{-\infty}\quad\implies\quad\xi\subset\mathcal{A}_{-\infty}.$$

Equivalently,

$$H_\mu\big(\xi\big|\mathcal{P}(T)\vee\mathcal{A}_{-\infty}\big)=H_\mu\big(\xi\big|\mathcal{A}_{-\infty}\big). \tag{7.8}$$

□

Definition 7.2(i) Given two spaces $(X_i,\mathcal{B}_i)$, $i=1,2$, by a *kernel* we mean a measurable map $\theta:X_1\to\mathcal{M}(X_2)$. Given a kernel θ and a measure $\pi\in\mathcal{M}(X_1)$, by a *hookup* of θ and π, we mean a measure on $X_1\times X_2$ of the form

$$\int f\ d(\pi\ltimes\theta)=\int\int f(x_1,x_2)\ \theta(x_1,dx_2)\ \pi(dx_1).$$

(ii) Let $(X_i,\mathcal{B}_i,T_i,\mu_i)$, $i=1,\dots,k$, a collection of dynamical systems. We say $(X,\mathcal{B},T,\mu)$ is a *joining* of this collection, if

$$X=\prod_{i=1}^d X_i,\qquad\mathcal{B}=\otimes_{i=1}^d\mathcal{B}_i,\qquad T=\prod_{i=1}^d T_i,$$

and μ is an invariant measure for T with marginals $\mu_1,\dots,\mu_n$.

(iii) When $k=2$, a kernel θ is called a *stationary channel* iff $\mu_1\ltimes\theta$ is a joining of μ_1 and μ_2. □

Example 7.1(i) In the setting of Definition 7.2(i), $\mu=\prod_{i=1}^d\mu_i$ is a joining.

(ii) Assume that $k = 2$ in the setting of Definition 7.2(i), and given a measurable map $h : X_1 \to X_2$, consider the measure μ on $X_1 \times X_2$ by

$$\int f\ d\mu = \int f(x_1, h(x_1))\ \mu_1(dx_1).$$

The marginals of this map are $(\mu_1, h_\sharp \mu_1) =: (\mu_1, \mu_2)$. Observe that for $T = (T_1, T_2)$,

$$\int f\ d\mu = \int f(x_1, h(x_1))\ \mu_1(dx_1) = \int f\big(T_1(x_1), (h \circ T_1)(x_1)\big)\ \mu_1(dx_1),$$
$$\int f \circ T\ d\mu = \int f\big(T_1(x_1), (T_2 \circ h)(x_1)\big)\ \mu_1(dx_1).$$

Hence μ is a joining iff h is a factor, i.e., $h \circ T_1 = T_2 \circ h$, μ_1-almost surely. □

Proposition 7.2 (Kakutani-Rokhlin) *Let $(X, \mathcal{B}, T, \mu)$ be an ergodic dynamic system. Assume that μ is atomless. Then for every $n \in \mathbb{N}$, and $\varepsilon > 0$, there exists a measurable set A such that $A, T(A), \dots, T^{n-1}(A)$ are disjoint, and*

$$\mu\big(A \cup T(A) \cup \cdots \cup T^{n-1}(A)\big) \geq 1 - \varepsilon. \tag{7.9}$$

Proof *(Step 1)* Note that the desired set A necessarily satisfies $\mu(A) \leq n^{-1}$.

To start, we pick $m \in \mathbb{N}$, and a set $B \in \mathcal{B}$ such that $0 < \mu(B) =: c_0 < m^{-1}$. Since by the ergodic theorem

$$\lim_{\ell \to \infty} \ell^{-1} \sum_{i=0}^{\ell-1} 1\!\!1_B\big(T^i(x)\big) = \mu(B),$$

μ-a.e., we deduce that the set $\cup_{i \geq 0} T^{-i}(B)$ is of full measure. Put

$$C = B \setminus \big(T(B) \cup \cdots \cup T^m(B)\big).$$

Observe that since $T^j(C) \subseteq T^j(B)$, and

$$T^j(C) = T^j(B) \setminus \big(T^{j+1}(B) \cup \cdots \cup T^{j+m}(B)\big),$$

we learn that $T^i(C) \cap T^j(C) = \emptyset$ whenever $1 \leq i < j \leq m$. Hence the collection of sets $T^j(C)$, $j = 1, \dots m$ are disjoint. We claim that $\mu(C) > 0$. To see this, suppose to the contrary

$$B \subseteq T(B) \cup \cdots \cup T^m(B),$$
$$T^{-1}(B) \subseteq B \cup T(B) \cup \cdots \cup T^{m-1}(B) \subseteq T(B) \cup \cdots \cup T^m(B),$$

modulo a μ-null set. Inductively,

$$T^{-j}(B) \subseteq T(B) \cup \cdots \cup T^m(B).$$

for all $j \in \mathbb{N}$. Since the set $\cup_{i \geq 0} T^{-i}(B)$ is of full measure, the set $T(B) \cup \cdots \cup T^m(B)$ is also of full measure. But this is absurd because

$$\mu\big(T(B) \cup \cdots \cup T^m(B)\big) \leq mc_0 < 1.$$

In summary $\mu(C) > 0$, and the sets $C, T(C), \ldots, T^m(C)$ are disjoint.

(Step 2) Choose $m \geq \max\{n, \varepsilon^{-1}\}$, and consider the set C that was constructed in Step 1. We then define

$$\theta(x) = \min\{k \geq 0 : \ T^k(x) \in C\}.$$

Since $\mu(C) > 0$, the function $\theta < \infty$, μ-a.e. We next define

$$A = \big\{x : \ \theta(x) = kn \quad \text{for some} \ \ k \in \mathbb{N}\big\}.$$

Note that since $\theta(x) \geq n$ for $x \in A$, we have $\theta(T^j(x)) = \theta(x) - j$ for $j = 0, 1, \ldots, n-1$. (Here we are using the fact that the sets $C, T(C), \ldots, T^m(C)$ are disjoint.) In other words $y \in T^j(A)$ iff $\theta(y) = j \pmod n$, $\theta(y) \geq 1$ for every $j = 0, 1, \ldots, n-1$. This means that the sets $A, T(A), \ldots, T^{n-1}(A)$ are disjoint, and

$$D := A \cup T(A) \cup \cdots \cup T^{n-1}(A) = \{x : \ \theta(x) \geq 1\} = X \setminus C.$$

On the other hand,

$$\mu(D) = 1 - \mu(C) \geq 1 - \mu(B) \geq 1 - m^{-1} \geq 1 - \varepsilon,$$

as desired. □

Definition 7.3(i) Given a Polish metric space (X, d), define a distance $\bar{d}$ on $\mathcal{M}(X)$ by

$$\bar{d}(\mu, \nu) = \inf \left\{ \int_{X \times X} d(x, y) \alpha(dx, dy) : \ \alpha \in \mathcal{M}(X \times X), \ \mu \text{ and } \nu \text{ are the marginals of } \alpha \right\}.$$

(ii) Given a Polish metric space (E, d), we define the metric d_n on E^n by

$$d_n(\omega, \omega') = d_n\big((\omega_1, \ldots, \omega_n), (\omega'_1, \ldots, \omega'_n)\big) := \ n^{-1} \sum_{i=1}^n d(\omega_i, \omega'_i).$$

The corresponding measure on $\mathcal{M}(A^n)$ is denoted by $\bar{d}_n$.

(iii) For (E,d) as above, consider the dynamical system $\Omega=(E^{\mathbb{Z}},\tau)$. The *Ornstein metric* d^O is defined on $\mathcal{I}_\tau$ by

$$\bar{d}^O(\mu,\nu)=\inf\left\{\int d(\omega_0,\omega_0')\ \lambda(d\omega,d\omega'):\ \lambda \text{ is a joining of } \mu \text{ and } \nu\right\}.$$

□

Note that the product $(E^{\mathbb{Z}},\tau)\times(E^{\mathbb{Z}},\tau)$ is isomorphic to $((E\times E)^{\mathbb{Z}},\tau)$. Hence a joining of two τ-invariant measures is simply a τ-invariant measure of $(E\times E)^{\mathbb{Z}}$. From this, it is not hard to see that when d is a bounded metric, then

$$\bar{d}^O(\mu,\nu)=\lim_{n\to\infty}\bar{d}_n(\mu^n,\nu^n), \tag{7.10}$$

where μ^n and ν^n are the law of $\omega^n=(\omega_0,\dots,\omega_{n-1})$ with respect to μ and ν respectively.

Lemma 7.1 **(i)** (Fano) *Let E be a finite set, and $\alpha\in\mathcal{M}(E^2)$ with marginals $\mu,\nu\in\mathcal{M}(E)$. Then*

$$H(\mu|\nu)\le -p\log p-(1-p)\log(1-p)+p\log(|E|-1), \tag{7.11}$$

where

$$p=\int 1\!\!1(x\ne y)\alpha(dx,dy).$$

Proof It is more convenient to think of α as the law of E^2-valued random variable (X,Y). If we write $Z=1\!\!1(X\ne Y)$, then

$$H(\mu|\nu)=H(X|Y)=H\big((X,Z)\big|Y\big)=H(Z|Y)+H\big(X\big|(Y,Z)\big)\le H(Z)+H\big(X\big|(Y,Z)\big).$$

Evidently $H(Z)=-p\log p-(1-p)\log(1-p)$. On the other hand,

$$H\big(X\big|(Y,Z)\big)=(1-p)H\big(X\big|Y,1\!\!1(Z=1)\big)+pH\big(X\big|Y,1\!\!1(Z=0)\big)=pH\big(X\big|Y,1\!\!1(Z=0)\big)\le p\log(|E|-1)$$

References

[AB] A. Avila and J. Bochi, On the subadditive ergodic theorem. preprint, 2009.

[BC] A. Bellow, A. Calderon, A weak type inequality for convolution products, Harmonic analysis and partial differential equations,(Chicago, IL, 1996), 41-48, *Chicago Lectures in Math.*, Univ. Chicago Press, Chicago, IL, (1999).

[BK] M. Brin and A. Katok, On local entropy. Geometric dynamics (Rio de Janeiro, 1981), 30–38, Lecture Notes in Math., 1007, Springer, Berlin, 1983.

[D] J. R. Dorfman, "An Introduction to Chaos in Nonequilibrium Statistical Mechanics", Cambridge Lecture Notes in Physics.

[Li] C. Liverani, Decay of Correlations, Annals of Math., Vol. 142, 239–301 (1995).

[LaYo] A. Lasota and J. A. Yorke, On the existence of invariant measures for piecewise monotonic transformation, TAMS, Vol. 186, 481–488 (1973).

[LiW] C. Liverani, and M. P. Wojtkowski, Ergodicity in Hamiltonian systems. Dynamics reported, 130–202, Dynam. Report. Expositions Dynam. Systems (N.S.), 4, Springer, Berlin, 1995.

[P] K. R. Parthasarathy, "Probabilty measures on Metric Spaces", AMS Chelsea Publishing, 1967.

[Po1] H.J. Poincare. Sur le probleme des trois corps et les equations de la dynamique. Acta Mathematica, 13, 1-270, 1890.

[Po2] H.J. Poincare. Les methodes nouvelles de la mecanique celeste, Vols 1-3. Gauthiers-Villars, Paris, 1892, 1893, 1899.

[R] F. Rezakhanlou, "Lectures on Large Deviation Principle".