Stochastic Growth and KPZ Equation

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1 Introduction

Various phenomena in physics and biology, such as the formation of crystals and the spread of infections are modeled by stochastic growth models. Many of such growth models are macroscopically described by Hamilton-Jacobi partial differential equations. In these models, a random interface separates regions associated with different phases and the interface can be locally approximated by the graph of a solution to a HamiltonJacobi equation. Such a solution gives us a macroscopic description of the interface. Microscopically though, the interface is rough and fluctuates about the macroscopic solution. A central limit theorem should provide us with a better description of the interface.

Perhaps the simplest example of a stochastic growth model is the Eden-Richardson model that was studied in a biological context. In this model each lattice site $i \in \mathbb{Z}^d$ represents the center of a cubical cell and the set $A(t)$ denotes the union of the infected cells, where a healthy cube outside $A(t)$ becomes infected with a rate proportional to the number of adjacent infected cells. Richardson shows that the set $A(t)$ grows linearly in$t$, and as ε goes to zero,

$$\varepsilon A(t/\varepsilon) \approx \big\{x \in \mathbb{R}^d : \ N(x) \leq t\big\}, \tag{1.1}$$

for a suitable norm $N(\cdot)$ associated with the model. The proof of Richardson's theorem is by now rather standard and follows from the celebrated *Subadditive Ergodic Theorem.* It is conjectured that the error in the approximation (1.1) is of order $O\big(\varepsilon^{2/3}\big)$. Put it differently the fluctuation of set $A(t)$ about the set $\big\{x \in \mathbb{R}^d : \ N(x) \leq t\big\}$ is of order $O(t^{1/3})$. This conjecture is expected to be true for a wide class of planar stochastic growth models and is still wide open for Eden-Richardson model. There are few *exactly solvable* models for which not only the conjecture has been established, a lot is known for the $O\big(\varepsilon^{2/3}\big)$ correction term in (1.1). In all these models the growth can only occur in one direction; this feature simplify the geometry of the interface drastically. We now describe several models that include all the exactly solvable examples. These examples will be examined thoroughly in subsequent chapters. In all these examples the boundary of the growing set A is given by a graph of a function $h(x, t)$.

- As our first set models, we consider a family of growth models known as exclusion processes with $h : \mathbb{Z}^d \times [0, \infty) \to \mathbb{Z}$.
- As our second model, we discuss the celebrated *Hammersley-Aldous-Diaconis (HAD) process*, where $h : \mathbb{R}^d \times [0, \infty) \to \mathbb{Z}$. $x \in \mathbb{R}^d$, $t \in [0, \infty)$, and $h \in \mathbb{Z}$.
- In the third model, our height function $h : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$ satisfies a classical PDE called Hamilton-Jacobi Equation (HJE) for which the Hamiltonian function is random.

1.1 Exclusion Processes

One of the simplest and most studied growth model is *Simple Exclusion Process (SEP)*. The configuration space Γ consists of the height functions $h : \mathbb{Z} \to \mathbb{Z}$ such that

$$0 \leq h(i+1) - h(i) \leq 1,$$

for all $i \in \mathbb{Z}$. With rate $\lambda \in [0,1]$ (respectively $1-\lambda$), each $h(i)$ increases (respectively decreases) by one unit provided that the resulting configuration does not leave the configuration space; otherwise the growth is suppressed. More precisely, the process $h(i,t)$ is a Markov process with the infinitesimal generator

$$(\mathcal{A}F)(k) = \sum_{i \in \mathbb{Z}} \left[(1-\lambda) 1\!\!1(k^i \in \Gamma)(F(k^i) - F(k)) + \lambda 1\!\!1(k_i \in \Gamma)(F(k_i) - F(k)) \right]$$

where $F : \mathbb{Z}^{\mathbb{Z}} \to \mathbb{R}$ is any cylindrical function ($F(k)$ depends on finitely many $k(i)$'s) and k^i, k_i are defined by

$$k_i(j) = \begin{cases} k(i) - 1 & if \;\; j = i, \\ k(j) & if \;\; j \neq i \;, \end{cases} \qquad k^i(j) = \begin{cases} k(i) + 1 & if \;\; j = i, \\ k(j) & if \;\; j \neq i \;. \end{cases}$$

When $\lambda \in \{0, 1\}$, we refer to the process h as the *Totally Asymmetric Simple Exclusion Process (TASEP)*.

Note that if we set $v(i) = i^+ \in \Gamma$, then $k \in \Gamma$ iff $k(j) - k(i) \leq v(j-i)$. More generally, we may start from any nonnegative additive function $v : \mathbb{Z}^d \to \mathbb{Z}$, and define the state space

$$\Gamma(v) = \left\{ h : \mathbb{Z}^d \to \mathbb{Z} : \;\; h(j) - h(i) \leq v(j-i) \quad \text{for all} \quad i, j \in \mathbb{Z}^d \right\}.$$

By a v-exclusion process, we mean a Markov process with a generator of the form

$$(\mathcal{A}F)(k) = \sum_{i \in \mathbb{Z}^d} \left[(1-\lambda) 1\!\!1(k^i \in \Gamma(v))(F(k^i) - F(k)) + \lambda 1\!\!1(k_i \in \Gamma(v))(F(k_i) - F(k)) \right].$$

As in (1.1), we may define

$$h^\varepsilon(x,t) = \varepsilon h\big([x/\varepsilon], t/\varepsilon\big),$$

for $(x,t) \in \mathbb{R} \times [0, \infty)$. We are hoping to establish

$$h^e(x,t) = u(x,t) + \varepsilon^{2/3} Z(x,t) + o\big(\varepsilon^{2/3}\big), \tag{1.2}$$

where u is the macroscopic profile that satisfies a suitable Hamilton-Jacobi PDE, and Z is a suitable stochastic process that should be universal.

When necessary, we write $h(i,t) = h(i,t;k)$ for the process with the initial configuration k, i.e., $h(i,0;k) = k(i)$. Some of the important features of the process $h(\cdot,t;k)$ are as follows:

(i) Any v-exclusion process is *monotone* in the following sense: If $k \le k'$, then $h(\cdot,t;k) \le h(\cdot,t;k')$ for all $t \ge 0$.

(ii) When $\lambda = 1$, any v-exclusion process is *strongly monotone* in the following sense:

$$h\big(\cdot,t;k \vee k'\big) = h\big(\cdot,t;k\big) \vee h\big(\cdot,t;k'\big), \tag{1.3}$$

Likewise, when $\lambda = 0$,

$$h\big(\cdot,t;k \wedge k'\big) = h\big(\cdot,t;k\big) \wedge h\big(\cdot,t;k'\big), \tag{1.4}$$

(iii) Assume that $v(i) = i^+$, and $d = 1$ (the case of SEP), we have an explicit candidate for the *invariant measures.* Given $\rho \in [0,1]$, we may consider the random initial height function k^ρ that is specified with properties that $k^\rho(0) = 0$, and $\big(k^\rho(i+1) - k^\rho(i) : \ i \in \mathbb{Z}\big)$ are independent with $\mathbb{P}\big(k^\rho(i+1) - k^\rho(i) = 1\big) = \rho$. Then this property persists at later times. In other words, $\big(h(i+1,t;k^\rho) - h(i,t;k^\rho) : \ i \in \mathbb{Z}\big)$ are independent with $\mathbb{P}\big(h(i+1,t;k^\rho) - h(i,t;k^\rho) = 1\big) = \rho$. We note that by Donsker Invariance Principle,

$$k^{\rho,\varepsilon}(x) := \varepsilon k^\rho\big([x/\varepsilon]\big) = \rho x + \sqrt{\varepsilon} B_\rho(x) + o\big(\sqrt{\varepsilon}\big),$$

where $B_\rho(x)$ is a Brownian motion with variance

$$\mathbb{E} B_\rho(x)^2 = \rho(1-\rho)x.$$

(iv) When $\lambda = 0$, and $\hat{v}(i) = -v(-i)$, then $\hat{v} \in \Gamma(v)$, and $h(i,t;\hat{v}) = \hat{v}(i)$. That is, the height function $\hat{v}(\cdot)$ does not change with time. If $\lambda \in [0,1/2)$, we can construct an invariant measure that coincides with the delta measure at $\hat{v}(\cdot)$ when $\lambda = 1$. This invariant measure is not translation invariant; in fact it is concentrated on the set of heights $h \in \Gamma(v)$ such that $h \le \hat{v}$, and for any such h,

$$\mu_\lambda(h) = z^{-1} \left(\frac{1-\lambda}{\lambda}\right)^{\sum_i (h(i) + v(-i))}, \tag{1.5}$$

where z is the normalizing constant. Note that $\mu_0 = \delta_{\hat{v}}$ if we interpret $0^0 = 1$.

(v) When $d = 1$, there is a simple description of the dynamics of the height differences

$$\eta(i,t) = h(i+1,t) - h(i,t).$$

In particular, when $v(i) = i^+$, we may interpret $\eta(i,t) = 1$ as the presence of a particle at site i. Similarly, the site i is vacant at time t when $\eta(i,t) = 0$. Hence $\eta \in \{0,1\}^{\mathbb{Z}}$ at all times. We then have a Markovian particle system such that a particle jumps to the right/left with rates λ and $1-\lambda$ respectively. Though this jump is suppressed when the outcome is not in our state space $\{0,1\}^{\mathbb{Z}}$. It is also common to regard SEP as a Markov process with state space $\{-1,1\}^{\mathbb{Z}}$. The advantage of this interpretation is that some figures associated with SEP would like nicer. After all if we linear interpret $h:\mathbb{Z}\to\mathbb{Z}$ with the convention that when $h(i+1)-h(i)=\pm 1$ the graph of h has a piece of slope $\pm$, then with rate λ (respectively $1-\lambda$) a $\wedge$ (respectively $\vee$) corner changes to a $\vee$ (respectively $\wedge$) corner.

(vi) When $\lambda=0$ and the initial height function is v, then there is a variational formula for $h(\cdot,t;v)$ which is very useful. We only describe this variational formula in the case of TASEP. For our purposes it is more convenient to consider the set

$$A(t)=\big\{(i,j):\ j\le h(i,t;v)\big\},$$

and examine its growth as t increases. For $a\notin A(0)$, set

$$T(a)=\inf\big\{t:\ a\in A(t)\big\}.$$

Take a collection of independent unit mean exponential random variables

$$\big\{\theta_{i,j}:\ v(i)<j\big\}.$$

Then $T(a)$ is given by

$$\sup\left\{\sum_{i=1}^{\ell}\theta_{z_i}:\ z_1=(0,1),z_\ell=a,z_{i+1}-z_i\in\{(-1,0),(1,1)\}\text{ for }i=1,\dots\ell-1\right\}. \tag{1.6}$$

(vii) Assume that $\lambda=0$. Let us write $v(\cdot;j)$ for a translation of v: $v(i)=v(i-j)$. We also set $w(i,t;j):=h(i,t;v(\cdot;j)$. Evidently for any $k\in\Gamma(v)$, we have

$$k(\cdot)=\inf_j\big\{h(j)+v(\cdot;j)\big\}.$$

When then use the strong monotonicity to write

$$h(i,t;k)=\inf_j\big\{k(j)+w(i,t;j)\big\}. \tag{1.7}$$

1.2 Hammersley-Aldous-Diaconis (HAD) Process

HAD process is the analog of TASEP with the randomness now is coming from a Poisson point process of intensity one in $\mathbb{R}\times(0,\infty)$. The state space Γ consists of functions $h:\mathbb{R}\to\mathbb{Z}$ such that for a discrete set $\{x_i:\ i\in I\}\subset\mathbb{R}$, we have

$$h'(x)=\sum_{i\in I}\delta_{x_i}.$$

In words, h is nondecreasing that increases for one unit at points in a discrete subset of $\mathbb{R}$. If for example $I = \mathbb{Z}$, and $x_{i-1} < x_i$ for all $i \in \mathbb{Z}$, then x_i jumps to a point y that is selected uniformly from the interval (x_{i-1}, x_i). Let us write Γ' for the set of sequences $\mathbf{x} = (x_i :\ i \in \mathbb{Z})$, such that

$$\lim_{i \to \pm\infty} x_i = \pm\infty.$$

Then we regard HAD as a Markov process that is defined on Γ or Γ'. In the latter case the generator is given by

$$(\mathcal{A}F)(\mathbf{x}) = \sum_{i \in \mathbb{Z}} \int_{x_{i-1}}^{x_i} \big(F(\mathbf{x}^{i,y})) - F(\mathbf{x})\big)\ dy,$$

where $\mathbf{x}^{i,y}$ is the configuration we get from $\mathbf{x}$ by moving the i-th particle x_i to y. Similarly for the process $h(\cdot, t) \in \Gamma$, we have a generator (with a slight abuse of notation, we use the same notation)

$$(\mathcal{A}F)(k) = \sum_{i \in \mathbb{Z}} \int_{x_{i-1}(k)}^{x_i(k)} \big(F(k^{i,y})) - F(k)\big)\ dy,$$

where $x_i(k)$ is the u-th point of increase of k, and $k^{i,y}$ is the height function we get from k by increasing k by 1 over the interval $(y, x_i(k)]$. Let us write $h(\cdot, t; k)$ for the height function at time t that initially is given by k. Similarly, we write $\mathbf{x}(t; \mathbf{y})$ for the particle system associated with $\mathcal{A}$ that is given by $\mathbf{y}$ initially. We now address several properties of the HAD process:

(i) HAD process is strongly monotone:

$$h(x, t; k \vee k') = h(\cdot, t; k) \vee h(\cdot, t; k').$$

(ii) Given $\rho > 0$, write $\mathbf{y}^\rho$ for a Poisson point process of density/intensity ρ. Then for every $t > 0$, the particle configuration $\mathbf{x}\big(t; \mathbf{y}^\rho\big)$ is again a Poisson point process with intensity ρ.

(iii) Let us write k^ρ for a height function such that $k^\rho(0) = 0$, and its increases occur exactly at the points of $\mathbf{y}^\rho$. Given any $a \in \mathbb{R}$, the increase points of process $t \mapsto h(a, t; k^\rho)$ form a Poisson point process of intensity ρ^{-1}. Intuitively, this has to do with the fact that product of the intensity measure $\rho\ dx$ and $\rho^{-1}\ dt$ is the intensity measure $dx\ dt$ of the Poisson point process we started from.

(iv) Note that by the classical Law of Large Number and Donsker Invariance Principle,

$$\varepsilon k^\rho(x/\varepsilon) = \rho x + \sqrt{\varepsilon} B_\rho(x) + o(\sqrt{\varepsilon}), \qquad \varepsilon h(0, t/\varepsilon; k^\rho) = \rho^{-1} x + \sqrt{\varepsilon} B'_\rho(t) + o(\sqrt{\varepsilon}),$$

where B_ρ and B'_ρ are Brownian motions with

$$\mathbb{E} B_\rho(x)^2 = \rho |x|, \quad \mathbb{E} B'_\rho(t)^2 = \rho^{-1} t.$$

From this for sure we have

$$\varepsilon h(x/\varepsilon, t/\varepsilon; k^\rho) = \rho x + \rho^{-1} t + o(1).$$

We can say more. Since the noise coming from the dynamics causes fluctuations of order $O(\varepsilon^{2/3})$, such fluctuations cannot be felt for corrections of order $O(\sqrt{\varepsilon})$. In fact the initial fluctuations is transport with linear speed with time as the following formula indicates:

$$\varepsilon h(x/\varepsilon, t/\varepsilon; k^\rho) = \rho x + \rho^{-1} t + \sqrt{\varepsilon} B_\rho\big(x - \rho^{-2} t\big) + \varepsilon^{2/3} Z_\rho(x,t) + o(\varepsilon^{2/3}), \tag{1.8}$$

for a suitable stochastic process Z_ρ that will be analyzed later. For now, let us write $H(\rho) = \rho^{-1}$ and observe that the Brownian noise will be transported with speed that is nothing other than $H'(\rho)$. Observe that according to (1.8),

$$B'_\rho(t) = B_\rho\big(-\rho^{-2} t\big),$$

and this is consistent with the following calculation:

$$\mathbb{E} B_\rho\big(-\rho^{-2} t\big)^2 = \rho\rho^{-2} t = \rho^{-1} t.$$

It is worth mentioning that we may get rid of $O(\sqrt{\varepsilon})$ in (1.8) by choosing $x = \rho^{-2} t$.

(v) In general, if initially

$$\varepsilon k(x/\varepsilon) = g(x) + o(\varepsilon^{2/3}),$$

then we expect

$$\varepsilon h(x/\varepsilon, t/\varepsilon; k) = u(x,t) + \varepsilon^{2/3} Z(x,t) + o(\varepsilon^{2/3}), \tag{1.9}$$

where u solves a Hamilton-Jacobi PDE of the form

$$u_t = H(u_x), \qquad u(x,0) = g(x), \tag{1.10}$$

with $H(\rho) = \rho^{-1}$. (Equivalently $u_t u_x = 1$). One of the main goal of these notes is the derivation of the correction term involving Z. As we will see later, we need to rescale the correction term in order to produce an interesting stochastic process that is often referred to as an *Airy Process*. Roughly speaking, the rescaled height function $h^\varepsilon(x,t) = \varepsilon h(x/\varepsilon, t/\varepsilon)$ satisfies

$$h^\varepsilon_t = H(h^\varepsilon_x) + O(\varepsilon^{2/3}). \tag{1.11}$$

If we accept this, then any correction of the form

$$h^\varepsilon(x,t) = u(x,t) + \sqrt{\varepsilon} B(x,t) + o(\sqrt{\varepsilon}),$$

leads to an equation of the form

$$B_t = H'(u_x)B_x.$$

This equation has a solution of the form

$$B(x,t) = B_\rho\big(x + H'(\rho)t\big) = B_\rho\big(x - \rho^{-2}t\big),$$

provided that $u(x,t) = \rho x + \rho^{-1}t$, and initially $B(x,0) = B_\rho(x)$.

(vi) There is a candidate for the analog of v function of the exclusion process that we now describe. Though our v function is defined for $x > 0$ only. Imagine that $v(x) = \infty 1\!\!1(x = \infty)$ with the interpretation that initially there is no particle in $(0,\infty)$, and infinitely many particles are lined up at ∞. Instantaneously these particles are rushed to finite points in $(0,\infty)$. The description of the particle trajectories $x_1(t) > x_2(t) > \dots$ (with $x_i(0) = \infty$ for all i) is in order. For the first particle, find an up-left path $X_1 = \big((x_1(t),t) :\ t > 0\big)$ such that at each L corner there is a Poisson point, and there is no Poisson point between the set $\big\{(x,t) :\ xt = 0, x,t \geq 0\big\}$ and X_1. Inductively we define an up-left path $X_{n+1} = \big((x_{n+1}(t),t) :\ t > 0\big)$ such that at each L corner there is a Poisson point, and there is no Poisson point between the set X_n and X_{n+1}. Once the family $X_1, X_2, \dots$ are identified, we then set a height function $w : (0,\infty)^2 \to \mathbb{Z}$ by

$$w(x,t) = \sum_{n=1}^{\infty} n 1\!\!1\big(x \in [x_n(t), x_{n+1}(t)\big).$$

There is an obvious variational description for w. Indeed

$$w(x,t) = \max\big\{\ell :\ \text{there exist Poisson points } (x_1,t_1) < \dots < (x_\ell,t_\ell) \text{ in } (0,x)\times(0,t)\big\}.$$

(v) Let us write $w(x,t;y)$ for the analog of $w(x,t)$ where 0 is replaced with $y \in \mathbb{R}$; the height function $w(x,t;y)$ is defined for $x > y$ and uses the Poisson points to the right of y. Using the identity

$$k(x) = \sup_{y \leq x} \big\{k(y) + v(x-y)\big\},$$

and the strong monotonicity we deduce

$$h(x,t;k) = \sup_{y \leq x} \big\{k(y) + w(x,t;y)\big\}. \tag{1.12}$$

This is the analog of (1.2) for HAD model.

1.3 Stochastic Hamilton-Jacobi Equation

As our microscopic model we may use a Hamilton-Jacobi equation. More precisely, we assume that the microscopic height function $h : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$ satisfies

$$h_t + H(x,t,h_x) = 0, \qquad h(x,0) = k,$$

for a Hamiltonian function $H : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ that is selected according to a probability measure that is stationary and ergodic with respect to the spatial and temporal translations:

$$\tau_{a,s}H(x,t,p) = H(x+a,t+s,p).$$

Example 1.1 As a classical example, consider

$$H(x,t,p) = \frac{1}{2}|p|^2 + V(x,t),$$

where V is selected according to probability measure that is τ-invariant and ergodic. as an example of V, pick a smooth function $W : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ of compact support and set

$$V(x,t) = \sum_i W(x - x_i, t - t_i),$$

where $\{(x_i,t_i) : \ i \in I\}$ is a Poisson point process of intensity one. Another classical example is

$$V(x,t) = \sum_i V_i(x)\dot{B}_i(t),$$

where V_i's are periodic functions, and B_i are independent Brownian motions. □

Again, we are interested in $h^\varepsilon(x,t) = \varepsilon h(x/\varepsilon, t/\varepsilon)$, that now satisfies

$$h_t^\varepsilon + H\big(x/\varepsilon, t/\varepsilon, h_x^\varepsilon\big) = 0.$$

Let us write $h(x,t;k)$ for the solution with initial data k. Here are some of the features the growth model h:

(i) The process h is monotone: If $k \le k'$, then $h(x,t;k) \le h(x,t;k')$.

(ii) When $H(x,t,p)$ is convex in p, then the process h is strongly monotone: $h(x,t;k \wedge k') = h(x,t;k) \wedge h(x,t;k')$. Similarly, when $H(x,t,p)$ is concave in p, then process h is strongly monotone: $h(x,t;k \vee k') = h(x,t;k) \vee h(x,t;k')$.

(iii) When $H(x,t,p)$ is convex in p, then there is a variational description for solutions:

$$h(x,t;k) = \inf\left\{ g(y(0)) + \int_0^t L(y(s),s,\dot{y}(s))\, ds : \ y \in C^1\big([0,1],\mathbb{R}^d\big), y(t) = x \right\},$$

where L is the Legendre Transform of H:

$$L(x,t,v) = \sup_p \big(p \cdot v - H(x,t,p)\big).$$

Similarly, when $H(x,t,p)$ is concave in p, then

$$h(x,t;k) = \sup\left\{g(y(0)) + \int_0^t \hat{L}(y(s),s,\dot{y}(s))\ ds:\ y \in C^1\big([0,1],\mathbb{R}^d\big), y(t)=x\right\},$$

where

$$\hat{L}(x,t,v) = \inf_p \big(p \cdot v - H(x,t,p)\big).$$

□

1.4 KPZ Equation

KPZ equation is a stochastic PDE that is used to study the fluctuations of a stochastic interface. The KPZ equation can be used as a model of a stochastic height function: We assume that the height function $h : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ satisfies the KPZ equation

$$h_t = h_{xx} + \frac{1}{2}h_x^2 + \xi.$$

We write $\xi(x,t)$ for the space time white noise. In other words, ξ is Gaussian distribution with

$$\mathbb{E}\xi(x,t)\xi(y,s) = \delta_0(x-y)\delta_0(t-s), \qquad \mathbb{E}\xi(x)\xi(y) = \delta_0(x-y).$$

To explain this further, let us write $\mathcal{D}_k = \mathcal{D}(\mathbb{R}^k)$ for the space of smooth functions of compact support in $\mathbb{R}^k$, then $\xi : \mathcal{D}_2 \to \mathbb{R}$ is a bounded linear map such that for each $J \in \mathcal{D}_2$, the random variable $\xi(J)$ is a Gaussian random variable with $\mathbb{E}\xi(J) = 0$, and

$$\mathbb{E}\xi(J)^2 = \int J(x,t)^2\ dxdt.$$

Note that since delta function δ_0 satisfies the scaling relation

$$(\lambda\lambda')^{-1}\delta_0(x/\lambda, t/\lambda') = \delta_0(x,t),$$

we deduce that the white noise ξ enjoys the following scaling:

$$(\lambda\lambda')^{-1/2}\xi(x/\lambda, t/\lambda') =^D \xi(x,t).$$

We use this to examine the scaling behavior of KPZ equation: If we set

$$h^\varepsilon(x,t) = \varepsilon^\alpha h\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\beta},\right).$$

with $\alpha, \beta > 0$, then l^ε satisfies

$$h_t^\varepsilon = \frac{1}{2}\varepsilon^{2-\alpha-\beta}\big(h_x^\varepsilon\big)^2 + \varepsilon^{2-\beta}h_{xx}^\varepsilon + \varepsilon^{\alpha-\frac{\beta}{2}+\frac{1}{2}}\xi.$$

Observe that if $\alpha + \beta = 2$, then we have

$$h_t^\varepsilon = \frac{1}{2}\big(h_x^\varepsilon\big)^2 + \varepsilon^{\alpha}h_{xx}^\varepsilon + \varepsilon^{\frac{3\alpha}{2}-\frac{1}{2}}\xi. \tag{1.13}$$

If $\alpha \in (1/3, 2)$ and $\beta = 2 - \alpha$, then the coefficients of the second order term and the white noise both go to 0 in small ε limit and we expect that $h^\varepsilon \to \bar{h}$ with $\bar{h}$ satisfying

$$\bar{h}_t = \frac{1}{2}\bar{h}_x^2. \tag{1.14}$$

As it turns our (1.14) does not posses a classical solutions even when the initial data is smooth, and the question is what we mean by a solution. As we will see, the appropriate notion of solution we need very much depends on the value of α. Let us examine two special cases:

- When $\alpha = 1$, we can ignore the white noise contribution and we will have a viscosity solution with a solution given by Hopf-Lax-Oleinik formula

$$\bar{h}(x,t) = \sup_y \left(h(y,0) - \frac{(x-y)^2}{2t}\right). \tag{1.15}$$

- When $\alpha = 1/2$, then $\bar{h}$ is not a viscosity solution and since the coefficient of white noise is now $\varepsilon^{1/4}$ which is the square root of the coefficient of the second order term, the white noise contribution cannot be ignored; in some sense a ghost of the white noise, called *white ghost* in these notes will survive. In other words, most likely the limiting $\bar{h}$ would satisfy

$$\bar{h}_t = \frac{1}{2}\bar{h}_x^2 + \hat{\xi}, \tag{1.16}$$

  where $\hat{\xi}$ is our white ghost. We do not know about the exact nature of $\hat{\xi}$. In fact we have a candidate for the analog of (1.15) (see [CQR]), namely

$$\bar{h}(x,t) = \sup_y \left(h(y,0) - \frac{(x-y)^2}{2t} + A\,(x,y;t)\right), \tag{1.17}$$

  where $A(x,y;t)$ is an Airy sheet for each t. Indeed $(x,y) \mapsto A(x,y;t)$ has the same law as

$$t^{1/3}A\left(\frac{x}{t^{2/3}}, \frac{y}{t^{2/3}}\right),$$

  where $A(x,y) := A(x,y;1)$. We refer to the (1.17) as *KPZ Fixed Point.*

We still need to explain why KPZ equation is relevant for our growth models. So far we know that may stochastic growth models can be described by a Hamilton-Jacobi PDE as the first approximation. An error of order $O\big(\varepsilon^{2/3}\big)$ must be taken into account as we examine the fluctuations of the microscopic height function about the macroscopic height function given by (1.10). Roughly the rescaled microscopic height function $h^\varepsilon(x,t) = \varepsilon h\big(x/\varepsilon, t/\varepsilon\big)$ is expected to satisfies

$$h_t^\varepsilon = H\big(h_x^\varepsilon\big) + \varepsilon a\big(h_x^\varepsilon\big)\gamma\big(h_x^\varepsilon\big)_x + \varepsilon\xi, \tag{1.18}$$

with ξ a white noise, and $a, \gamma : \mathbb{R}, \mathbb{R}$ possibly nonlinear functions with $a, \gamma' > 0$. Let us explain this in the case of SEP: If we take a smooth function J of compact support and look at

$$\int J(x)h^\varepsilon(x,t)\ dx \approx \varepsilon^2 \sum_i J(\varepsilon i)h(i, t/e) =: \Omega(t),$$

then

$$d\Omega = A\ dt + dM,$$

where

$$\begin{aligned}
A &= \varepsilon \sum_i \big[(1-\lambda)\eta(i)(1-\eta(i-1)) - \lambda\eta(i-1)(1-\eta(i))\big] J(\varepsilon i) \\
&= (1-2\lambda)\varepsilon \sum_i \eta(i)(1-\eta(i-1))J(\varepsilon i) + \varepsilon\lambda \sum_i \big[\eta(i) - \eta(i-1)\big] J(\varepsilon i) \\
&= \varepsilon \sum_i \eta(i)(1-\eta(i-1))J(\varepsilon i) - \varepsilon\lambda \sum_i \big[J(\varepsilon i) - J(\varepsilon(i+1))\big]\eta(i) \\
&\approx \int \big[H\big(h^\varepsilon(x,t)\big) + \lambda h_{xx}^\varepsilon\big]\ J(x)\ dx,
\end{aligned}$$

where $H(p) = (1-2\lambda)p(1-p)$. On the other hand for the Martingale term we have (see Exercise **(v)** of Chapter 2),

$$\begin{aligned}
\mathbb{E}M(t)^2 &= \mathbb{E}\varepsilon^{-1} \int_0^t \sum_i \big[(1-\lambda)\eta(i,t)(1-\eta(i-1,s)) + \lambda\eta(i-1,s)(1-\eta(i,s))\big]\big(\varepsilon^2 J(\varepsilon i)\big)^2\ ds \\
&\approx \varepsilon^2 \int H\big(h^\varepsilon(x,s)\big)J^2(x)\ ds.
\end{aligned}$$

These formal approximation confirms the formula (1.18) provided that we allow a possibly inhomogeneous white noise ξ, and in the case of exclusion process, we simply have $a\gamma' = \lambda$ a constant function. (The function a and λ have physical interpretations that we do not discuss here and in the case of exclusion process, $a(p) = \lambda p(1-p)$, and $\lambda(p) = p\log p + (1-p)\log(1-p)$ is the entropy.)

In models like SEP and HAD, we know that the equilibrium states correspond to solutions of the form

$$h^\varepsilon(x,t) = xp + tH'(p) + \varepsilon^{1/2}B(x,t) + o(\varepsilon^{1/2}), \tag{1.19}$$

with $x \mapsto B(x,t)$ a Brownian motion for each t. This explain why the choice of $\alpha = 1/2$ in (1.13) should play a special role. More generally, if we take any solution of the form

$$h^\varepsilon(x,t) = xp + tH'(p) + \varepsilon^{1/2}w^\varepsilon(x,t) + o(\varepsilon^{1/2}),$$

then w^ε solves

$$w^\varepsilon_t = H'(p)w^\varepsilon_x + \frac{1}{2}\varepsilon^{1/2}H''(p)\big(w^\varepsilon_x\big)^2 + \varepsilon d(p)w^\varepsilon_{xx} + \varepsilon^{1/2}\xi + O(\varepsilon),$$

where $d(p) = a(p)\lambda'(p)$. We can get ride of the term $H'(p)w^\varepsilon_x$ by a linear translation: The function

$$\hat{w}^\varepsilon(x,t) = w^\varepsilon\big(x - H'(p)t, t\big),$$

satisfies

$$\hat{w}^\varepsilon_t = \frac{1}{2}\varepsilon^{1/2}H''(p)\big(\hat{w}^\varepsilon_x\big)^2 + \varepsilon d(p)\hat{w}^\varepsilon_{xx} + \varepsilon^{1/2}\xi' + O(\varepsilon),$$

where $\xi'(x,t) = \xi(x - ct, t)$, with $c = H'(p)$ is again a white noise. To have a non-trivial limit, we consider

$$u^\varepsilon(x,t) = \hat{w}^\varepsilon\big(x, \varepsilon^{-1/2}t\big).$$

The function u^ε satisfies

$$u^\varepsilon_t == \frac{1}{2}H''(p)\big(\hat{w}^\varepsilon_x\big)^2 + \varepsilon^{1/2}d(p)\hat{w}^\varepsilon_{xx} + \varepsilon^{1/4}\xi + O\big(\varepsilon^{1/2}\big).$$

We may wonder whether or not the small ε limit of u^ε exists. If this limit exists and is denoted by $\bar{h}$ it is expected to satisfy the KPZ fixed point:

$$\bar{h}_t = \frac{1}{2}H''(p)\bar{h}^2_x.$$

In fact (1.21) implies that if $x \mapsto \bar{h}(x,0)$ is a Brownian motion, then $x \mapsto \bar{h}(x,t)$ is also a Brownian motion of the same diffusion coefficient. That is, the Wiener measure is invariant. This peculiar feature of fixed point KPZ confirms our earlier assertion that $\bar{h}$ cannot be a viscosity solution: According to a classical result of Groenboom, an initial Brownian motion for the viscosity solution of (1.14) becomes a piecewise quadratics function instantaneously!

From the above discussion we formulate a scaling law that is one of the main air of these notes: Suppose that we have a stochastic growth model in dimension one associated with macroscopic PDE $u_t = H(u_x)$. Assume that initially

$$\lim_{\varepsilon \to 0} \varepsilon^{-1/2}\left(\varepsilon h\left(\left[\frac{x}{\varepsilon}\right], 0\right) - xp\right) = \bar{h}(x,0).$$

Then at later times

$$\lim_{\varepsilon \to 0} \varepsilon^{-1/2} \left(\varepsilon h \left(\left[\frac{x}{\varepsilon} - \frac{tH'(p)}{\varepsilon^{3/2}} \right], \frac{t}{\varepsilon^{3/2}} \right) - \left(xp - t\varepsilon^{-1/2} \left(pH'(p) - H(p) \right) \right) \right) = \bar{h}(x,t), \tag{1.20}$$

with $\bar{h}$ satisfying the KPZ fixed point.

1.5 Stochastic PDE

As we mentioned earlier the fluctuation of the interface in two dimensional growth models are governed by a stochastic PDE known as KPZ equation. This is subsection we give a quick review of stochastic PDEs in general.

We write $\xi(x,t)$ for the space time white noise and ξ for the space white noise. In other words, both ξ and ξ are Gaussian distributions with

$$\mathbb{E}\xi(x,t)\xi(y,s) = \delta_0(x-y)\delta_0(t-s), \qquad \mathbb{E}\xi(x)\xi(y) = \delta_0(x-y).$$

To explain this further, let us write $\mathcal{D}_k = \mathcal{D}(\mathbb{R}^k)$ for the space of smooth functions of compact support in $\mathbb{R}^k$, then $\xi : \mathcal{D}_{d+1} \to \mathbb{R}$ is a bounded linear map such that for each $J \in \mathcal{D}$, the random variable $\xi(J)$ is a Gaussian random variable with $\mathbb{E}\xi(J) = 0$, and

$$\mathbb{E}\xi(J)^2 = \int J(x,t)^2 \; dxdt.$$

The definition of ξ is similar.

Definition 1.1 Given a distribution $u \in \mathcal{D}'_{d+1}$, we write $[u] = \gamma$ if

$$\varepsilon^\gamma u \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) =^D u(x,t).$$

□

Example 1.2 For the space-time white noise we have $[\xi] = -(d+2)/2$, and for the space white noise we have $[\xi] = -d/2$. To see the latter, observe that if $\hat{\delta}_0(x,t) = \delta_0(t)\delta_0(x)$, then

$$\varepsilon^{-d-2}\delta_0 \left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) = \delta_0(x,t).$$

From this we can readily deduce that $[\xi] = -(d+2)/2$. □

Our goal is to make sense of various SPDEs that appear in statistical mechanics. To make sense of these equation, we need to study the regularity of their solutions. To figure out what is the best regularity we can hope for, we first search for *stochastically self similar solutions (SSS)*. By this we mean a solution u for which

$$u^\varepsilon(x,t) = \varepsilon^\gamma u\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right),$$

is also solution. We call γ the scaling exponent of the SPDE. For our regularity question, we use spaces $\mathcal{C}^\alpha$ with the following properties:

- When $\alpha \in (0,1)$, the space $\mathcal{C}^\alpha$ is set of local α-H'older continuous functions.
- When $\alpha \in (k, k+1)$ for $k \in \mathbb{N}$, the set $\mathcal{C}^\alpha$ consists of C^k function with all k-th derivative in $\mathcal{C}^{\alpha-k}$.
- The space $\mathcal{C}^1$ is the Caldron-Zygmund space and is slightly larger and the space of locally Lipschitz functions.

Definition 1.2(i) We set $|(x,t)| = |x| + \sqrt{t}$. Note that $|(x/\varepsilon, t/\varepsilon^2)| = |(x,t)|/\varepsilon$.
(ii) We write $\mathcal{B} = \mathcal{B}_r$ for the space of C^r functions φ such that the support of φ is contained in $B(0,1)$, and $|D^a\varphi| \le 1$ for a with $|a| \le k$. Given $\varphi \in \mathcal{B}$, we set

$$\varphi_z^\lambda(z') = \lambda^{-d-2}\varphi\left(\frac{x'-x}{\lambda}, \frac{t'-t}{\lambda^2}\right).$$

(iii) For $\alpha > 0$, we write $\mathcal{C}^\alpha$ for the space of functions u such that

$$\int u(\varphi_z^\lambda)dx \le c\lambda^\alpha,$$

for every $\varphi \in \mathcal{B}$ with

$$\int \varphi(P)\ dx = 0,$$

for every polynomial P of degree $[\alpha]$. The constant c is independent of φ and z so long as z is in a bounded set.
(iv) For $\alpha < 0$, we write $\mathcal{C}^\alpha$ for the space of distribution u such that

$$u(\varphi_z^\lambda) \le c\lambda^\alpha,$$

for every $\varphi \in \mathcal{B}$. The constant c is independent of φ and z so long as z is in a bounded set. □

If we have a random process $u(x)$, a standard way to show that $u \in \mathcal{C}^\alpha$ for some $\alpha \in \mathbb{R}$ is the following generalization of Kolmogorov Theorem.

Theorem 1.1 *Let $u \in \mathcal{D}'(\mathbb{R}^k)$ be a random distribution with*

$$[\mathbb{E}\left|u(\varphi_z^\lambda)\right|^p]^{\frac{1}{p}} \le c\lambda^\alpha, \tag{1.21}$$

for a constant c that is independent $\varphi \in \mathcal{B}$ and z so long as z is in a bounded domain. Then there is a version of $u \in \mathcal{C}^\beta$ for $\beta = \alpha - p/d$. In particular, if (1.21) is true for every $p > 1$, then $u \in \mathcal{C}^{\alpha^-}$.

Corollary 1.1 *Suppose that $u \in L^p(\mathbb{P})$ for every $p \ge 1$. If $[u] = \alpha$, then $u \in \mathcal{C}^{\alpha^-}$.*

Proof For every $p \ge 1$,

$$[\mathbb{E}\left|u(\varphi_0^\lambda)\right|^p]^{\frac{1}{p}} = \lambda^\alpha\, [\mathbb{E}\left|u(\varphi)\right|^p]^{\frac{1}{p}} = c\lambda^\alpha.$$

□

We now describe four examples of SPDEs:

(1) *(Stochastic Heat Equation)* The SHE in $\mathbb{R}^d$ is the PDE

$$u_t = \Delta u + \xi, \tag{1.22}$$

where $u : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$ and ξ is the space-time white noise. We can readily find the scaling exponent of solutions. Suppose that SSS for SHE is α. We note that if we have a solution with $[u] = \alpha$, then $[u_t] = \alpha - 2$, and $[\Delta u] = \alpha - 2$. This exponent must match $[\xi] = -(d+2)/3$. Hence SSS for SHE is $\alpha = 1 - d/2$. From this we expect that a solution to SHE to be in $C^{\alpha-}$. Based on this heuristic reasoning, we expect u to be a function only when $d = 1$.

(2) Given $k \in \mathbb{N}$, consider

$$u_t = \Delta u + u^k + \xi. \tag{1.23}$$

The main challenge for making sense of (1.22) is the term u^k when $d \ge 2$ because we expect u to be a distribution. To understand better the role of the nonlinearity, let us for now pretend that we already know how to make sense of u^k. Now if $[u] = \alpha$ as in **(1)**, then we expect $[u^k] = k\alpha$. as we will learn later, it is much easier to treat (1.22), when u^k has a better regularity that the white noise. That is

$$k\left(1 - \frac{d}{2}\right) > -\frac{d}{2} - 1.$$

Equivalently, $d < 2(k+1)/(k-1)$. If this is the case, then we say that (1.23) is *subcritical.* For example, when $k = 3$, then $d < 4$ is subcritical.

(3) *(Anderson Model)* Consider

$$u_t = \Delta u + u\xi. \tag{1.24}$$

Here the main challenge comes from the product term $u\xi$. The point is that if we assume $u \in \mathcal{C}^{\alpha^-}$, then since $\xi \in C^{-(\delta/2)^-}$, we can make sense of the product only if $\alpha > \delta/2$. If we instead look at $u_t = \Delta u + \xi$, we get $\alpha = 2 - \frac{d}{2}$. Even for this α, we end up with the restriction $d < 2$. We will see later that $d < 4$ is subcritical.

(4) *(One dimensional SHE on a manifold)* In local coordinates

$$u_t^i = u_{xx}^i + \Gamma_{jk}^i(u)u_x^j u_x^k + \sigma_\ell^i \xi_\ell. \tag{1.25}$$

is subcritical. Since the dimension is one, we expect $u \in \mathcal{C}^{1/2^-}$. This makes the term $\Gamma_{jk}^i(u)u_x^j u_x^k$ highly singular. $\square$

2 Markov Processes

Our models v-exclusion and HAD processes are examples of continuous time Markov processes. In this Chapter we give an overview of such processes. We always consider *Feller Processes* $\big(x(t) : \ t \in [0, \infty)\big)$ with state space E that is locally compact and separable complete metric space. The process $x(t)$ is always right continuous with left limit; the existence of such realization is standard and can be guaranteed under some general conditions on the transition probability or the infinitesimal generator of x.

Given a Borel set $A \subset E$, $x \in E$, and $t \geq 0$, the transition function $p_t(x, A)$ denotes the probability that $x(t) \in A$, conditioned that the starting point is $x(0) = x$. We write $C_b(E)$ for the space of bounded continuous functions $f : E \to \mathbb{R}$. We also write $C_0(E)$ for $f \in C_b(E)$ that vanishes at infinity. We define linear operators $T_t : t \geq 0$ by

$$T_t f(x) = \int p_t(x, dy) f(y) = \mathbb{E}^{x(0)=x} f(x(t)),$$

for every $f \in C_b(E)$. By Feller property we mean that $T_t f \in C_0(E)$ and

$$\lim_{t \to 0} T_t f = f,$$

for every $f \in C_0(E)$. The Markov property means that $T_t \circ T_s = T_{t+s}$ for all $s, t \geq 0$. The *infinitesimal generator* of the process $x(\cdot)$ is an operator $\mathcal{L} : Dom \to C_0(E)$, that is defined as

$$\mathcal{L} f = \lim_{h \to 0} \big(T_t f - f\big). \tag{2.1}$$

Here denotes the set of $f \in C_0(E)$ for which the limit in (2.1) exists, and is dense subset of $C_0(E)$. From the semigroup property, we can readily show that if $f \in Dom$, then $T_t f \in Dom$ for all $t \geq 0$. On the other hand, from the identity

$$h^{-1}\big(T_{t+h} f - T_t f\big) = T_t \left(h^{-1}\big(T_h f - f\big)\right) = h^{-1}\big(T_h(T_t f) - T_t f\big).$$

As a result

$$\frac{d}{dt} T_t f = L T_t f = T_t L f. \tag{2.2}$$

When state space is$E$ is finite, the (2.2) is an ODE, and we may regard T_t and $\mathcal{L}$ as matrices and we simply have $T_t = e^{t\mathcal{L}}$. In fact writing $u(x, t) = T_t f(x)$, then u is the unique solution to the initial value problem

$$u_t = Lu, \quad u(x, 0) = f(x).$$

The equation (2.2) is known as Kolmogorov Backward Equation. The dual of (2.2) is a evolution equation for the distribution of the process $x(t)$. More precisely, assume that $x(0)$

is selected randomly according to a probability measure μ^0. Then at later time the law of $x(t)$ is a probability measure $\mu^t = T_t^*\mu^0$ that satisfies the following identity for every $f \in C_b(E)$:

$$\mathbb{E}^{\mu^0} f(x(t)) := \int \left[\mathbb{E}^{x(0)=x} f(x(t))\right] \ \mu^0(dx) = \int T_t f(x) \ \mu^0(dx) = \int f \ d\mu^t.$$

When $f \in \mathit{Dom}$, we may differentiate both sides with respect to t to assert

$$\int f \ d\left(\frac{d}{dt}\mu^t\right) = \int T_t(\mathcal{L}f)(x) \ \mu^0(dx) = \int \mathcal{L}f \ d\mu^t =: \int f \ d\big(\mathcal{L}^*\mu^t\big).$$

In short,

$$\frac{d}{dt}\mu^t = \mathcal{L}^*\mu^t. \tag{2.3}$$

We say a probability measure ν is *invariant* if $T_t^*\nu = 0$. In view of (2.3), a measure ν is invariant if $\mathcal{L}^*\nu = 0$, or equivalently

$$\int \mathcal{L}f \ d\nu = 0,$$

for every $f \in \mathit{Dom}$. It is often more convenient to choose a *core* $\mathit{Dom}' \subseteq \mathit{Dom}$ for the generator $\mathcal{L}$. This means that for every $f \in \mathit{Dom}$, we can find a sequence $f_n \in \mathit{Dom}'$ such that

$$\lim_{n\to\infty} f_n = f, \qquad \lim_{n\to\infty} \mathcal{L}f_n = \mathcal{L}f.$$

We say an invariant measure ν is *reversible*, if

$$\int f\mathcal{L}g \ d\nu = \int g\mathcal{L}f \ d\nu,$$

for all $f, g \in \mathit{Dom}'$.

Example 2.1(i) When E is countable, the generator takes the form

$$\mathcal{L}f(x) = \sum_{y\in E} c(x,y)\big(f(y) - f(x)\big),$$

for a suitable $c : E \times E \to [0,\infty)$. We think of $c(x,y)$ as the rate of a jump from state x to state y. The time it takes to jump from x to y is an exponential random variable and $c(x,y)$ is the exponential parameter of this variable. With a slight abuse of notation, we write $\mu(x)$ for $\mu\{x\}\big)$. Clearly

$$\int \mathcal{L}f \ d\mu = \sum_{x,y\in E} m(x,y)\big(f(y) - f(x)\big),$$

where $m(x,y)=c(x,y)\mu(x)$. As an immediate consequence

$$\mathcal{L}^*\mu(x)=\sum_{y\in E}\big[c(y,x)\mu(y)-c(x,y)\mu(x)\big]. \tag{2.4}$$

From this or the previous display it is clear that if $m(x,y)$ is symmetric i.e.,

$$\mu(x)c(x,y)=\mu(y)c(y,x), \tag{2.5}$$

then μ is invariant. In fact such an invariant measure is reversible:

$$\mathcal{E}(f,g):=-\int f\mathcal{L}g\ d\mu=-\int g\mathcal{L}f\ d\nu=\frac{1}{2}\sum_{x,y\in E}m(x,y)\big(f(y)-f(x)\big)\big(g(y)-g(x)\big). \tag{2.6}$$

The way to think about (2.5) is that we have a *detailed balance*: μ is invariant even when the dynamics is restricted to jumps from x to y and back.

(ii) In the case of a diffusion, $\mathcal{L}$ is a second order elliptic operator. For a d-dimensional Brownian motion $\mathcal{L}=\frac{1}{2}\Delta$. In the case of a Compound Poisson process, the generator is

$$\mathcal{L}f(x)=\int\big(f(x+y)-f(x)\big)\ \ell(dy).$$

More generally, if $x(\cdot)$ is a one dimensional Lévy process with Lévy measure ℓ, then $\mathcal{L}$ is a Pseudo Differential operator of the form

$$\mathcal{L}f(x)=\int\big(f(x+y)-f(x)-\mathbb{1}(|y|\le 1)yf'(x)\big)\ \ell(dy).$$

□

Given $f\in Dom$, we may use integrated form of (2.2) to write

$$\mathbb{E}^{x(s)=a}\left[f(x(t))-f(x(s))-\int_s^t\mathcal{L}f(x(\theta))\ d\theta\right]=0,$$

for every $a\in E$ and (s,t) with $0\le s\le t$. As a result, if we take any bounded continuous $F:E^k\to\mathbb{R}$ and any $s_1<\dots<s_k\le s$, then by Markov property

$$\begin{aligned}\mathbb{E}F\big(x(s_1),\dots,x(s_k)\big)&\left[f(x(t))-f(x(s))-\int_s^t\mathcal{L}f(x(\theta))\ d\theta\right]\\&=\mathbb{E}F\big(x(s_1),\dots,x(s_k)\big)\mathbb{E}^{x(s)=a}\left[f(x(t))-f(x(s))-\int_s^t\mathcal{L}f(x(\theta))\ d\theta\right]=0.\end{aligned}$$

As a result, the process

$$M_f(t) = f(x(t)) - f(x(0)) - \int_0^t \mathcal{L}f(x(\theta))\ d\theta,$$

is a martingale with respect to the σ-algebras $\big(\mathcal{F}_t :\ t \geq 0\big)$, where $\mathcal{F}_t$ is the σ-algebra generated from $\{x(\theta) :\ \theta \leq t\}$. In the same fashion we can show that for any $f : E \times [0,T] \to \mathbb{R}$ that is C^1 in t and $f(\cdot,t) \in Dom$, the process

$$M_f(t) = f(x(t),t) - f(x(0),0) - \int_0^t (\partial_\theta + \mathcal{L}) f(x(\theta),\theta)\ d\theta, \tag{2.7}$$

is a Martingale. We end this section with an application of such martingales:

Example 2.2(i) Assume that E is countable, let U be a subset of E and set

$$\tau = \inf \big\{t > 0 :\ x(t) \notin U\big\}.$$

We wish to evaluate $\mathbb{P}^{x(0)=y}\big(\tau > t, x(t) = a\big)$. For this, we find a function $u(x,s) = u(x,s;a,t); (x,s) \in E \times [0,t]$ that solves the following equation

$$\begin{cases} u_s(x,s) + \mathcal{L}u(x,s) = 0, & s < t, \\ u(x,t) = 1\!\!1(x = a), & \\ u(x,s) = 0, & s \leq t,\ x \notin U. \end{cases}$$

Then $M(s) := u\big(x(s \wedge \tau), s \wedge \tau\big)$ is a Martingale. As a result,

$$\begin{aligned} u(y,0;a,t) &= \mathbb{E}^{x(0)=y} M(0) = \mathbb{E}^{x(0)=y} M(t \wedge \tau) = \mathbb{E}^{x(0)=y} u\big(x(t \wedge \tau), t \wedge \tau\big) \\ &= \mathbb{E}^{x(0)=y} u\big(x(t),t\big) 1\!\!1(\tau > t) = \mathbb{P}^{x(0)=y}\big(\tau > t, x(t) = a\big). \end{aligned}$$

We can easily express u in terms of the *fundamental solution* of the equation (2.2) in the domain U. More precisely if $w(x,t) = w(x,t;a)$ is the unique solution of

$$\begin{cases} w_t(x,t) = \mathcal{L}w(x,t), & t > 0, \\ w(x,0) = 1\!\!1(x = a), & \\ w(x,t) = 0, & t \geq 0,\ x \notin U, \end{cases}$$

then $u(x,s;a,t) = w(x,t-s;a)$, and $u(y,0;a,t) = w(y,t;a)$.

In the case of a diffusion in $\mathbb{R}^d$, we may take an open subset $U \subset \mathbb{R}^d$, and examine the law $x(t)$ provided that it never leave U up to time t. Then

$$\mathbb{P}^{x(0)=y}\big(x(t) \in dx, t < \tau\big) = w(x,t;y)\ dx,$$

where $w(x,t) = w(x,t;y)$ solves

$$\begin{cases} w_t(x,t) = \mathcal{L}w(x,t), & t > 0, \\ w(x,0) = \delta_y(dx), & \\ w(x,t) = 0, & t \geq 0,\ x \in \partial U. \end{cases} \tag{2.8}$$

(ii) Let U be any bounded domain in E and let $g : E \to \mathbb{R}$ be a continuous function. Set

$$\tau = \tau_U = \inf\big\{t > 0 :\ x(t) \notin U\big\}.$$

We wish to solve the equation

$$\begin{cases} \mathcal{L}v(x) = 0, & x \in U, \\ v(x) = g(x), & x \in \partial U, \end{cases}$$

If v is a solution, then $M(t) := v\big(x(t \wedge \tau)\big)$ is a martingale. As a result,

$$v(x) = \mathbb{E}^{x(0)=x} M(0) = \mathbb{E}^{x(0)=x} M(\tau) = \mathbb{E}^{x(0)=x} g\big(x(\tau)\big),$$

which expresses the desired solution in terms of the boundary data g. As an example, take $A \subset U^c$ and choose $g(x) = \mathbb{1}(x \in A)$. Then the corresponding v has the following interpretation:

$$v(x) = \mathbb{P}^{x(0)=x}\big(x(\tau) \in A\big).$$

□

Given $T \in (0,\infty]$ and $a \in E$, we write $\mathcal{D}_T^a = D_a([0,T],E)$ for the set of $x(\cdot)$ in the Skorohod space $D([0,T],E)$, such that $x(0) = a$. Recall that $\mathcal{F}_t$ denotes the σ-algebra generated by $\big(x(s) :\ s \in [0,t]\big)$. The Markov process $x(\cdot)$ may be regarded as a probability measure $\mathbb{P}^a$ on the set $\mathcal{D}_T^a$. We wish to study other Markov processes with laws absolutely continuous with respect to $\mathbb{P}^a$. Note that if $\mathbb{Q}^a \ll \mathbb{P}^a$, and

$$M(t) = \frac{d\mathbb{P}_T^a}{d\mathbb{Q}_T^a}\Big|_{\mathcal{F}_t},$$

then the process $M(t)$ is a non-negative martingale. We can readily construct positive martingales that can be used to construct a family of Markov process associated with a given Markov process. The key to this construction is the F*Feynman-Kac Formula.*

Theorem 2.1 *Given a Markov process $x(\cdot)$ with generator $\mathcal{L}$, and a bounded continuous function $V : E \to \mathbb{R}$, define*

$$T_t^V f(x) = \mathbb{E}^{x(0)=x}\ f(x(t))\ e^{\int_0^t V(x(s))\ ds}.$$

Then for any $f \in \mathcal{D}$,

$$\frac{d}{dt} T_t^V f = T_t^V \mathcal{L}_V f, \tag{2.9}$$

where $\mathcal{L}_V f = \mathcal{L} f + V f$.

Proof Given $h > 0$,

$$\begin{aligned}
T_{t+h}^V f(x) &= \mathbb{E}^{x(0)=x} \; f(x(t+h)) \; e^{\int_0^{t+h} V(x(s)) \; ds} \\
&= \mathbb{E}^{x(0)=x} \; f(x(t+h)) \; e^{\int_0^t V(x(s)) \; ds + hV(x(t)) + o(h)} \\
&= \mathbb{E}^{x(0)=x} \; f(x(t+h)) \; e^{\int_0^t V(x(s)) \; ds} \big(1 + hV(x(t)) + o(h)\big) \\
&= \mathbb{E}^{x(0)=x} \; e^{\int_0^t V(x(s)) \; ds} \big(1 + hV(x(t)) + o(h)\big) \; \mathbb{E}^{x(t)} f(x(t+h)) \\
&= \mathbb{E}^{x(0)=x} \; e^{\int_0^t V(x(s)) \; ds} \big(1 + hV(x(t)) + o(h)\big) \; \big(f(x(t)) + h\mathcal{L} f(x(t)) + o(h)\big) \\
&= T_t^V f(x) + hT_t^V \mathcal{L}_V f + o(h).
\end{aligned}$$

□

Corollary 2.1 *For every positive $f \in \mathcal{D}$, the process*

$$N_f(t) = f(x(t)) \; e^{-\int_0^t \frac{\mathcal{L} f}{f}(x(s)) \; ds},$$

is a martingale.

Proof Observe that if we choose $V = -\mathcal{L} f / f$, then $\mathcal{L}_V f = 0$. For this choice of V we have that for every t,

$$T_t^V f(x) = f(x),$$

by (2.9). Using this we can assert

$$\begin{aligned}
\mathbb{E}\big(N(t) \mid \mathcal{F}_{t'}\big) &= \mathbb{E}\left(f(x(t)) \; e^{\int_0^t V(x(s)) \; ds} \middle| \mathcal{F}_{t'}\right) \\
&= \; e^{\int_0^{t'} V(x(s)) \; ds} \; \mathbb{E}^{x(t')} \left(f(x(t)) \; e^{\int_{t'}^t V(x(s)) \; ds}\right) \\
&= e^{\int_0^{t'} V(x(s)) \; ds} \; f(x(t')) = N(t'),
\end{aligned}$$

whenever $t' < t$. □

Remark 2.1 More generally, if $f : E \times \mathbb{R} \to \mathbb{R}$ is C^1 in t, then

$$\frac{d}{dt} T_t^V f(\cdot, t) = T_t^V (\partial_t + \mathcal{L}_V) f(\cdot, t),$$

and the process

$$N_f(t) = f(x(t),t)e^{-\int_0^t \frac{(\partial_s+\mathcal{L})f}{f}(x(s),s)\ ds}, \tag{2.10}$$

is a martingale. □

As we mentioned earlier, we may wonder what does the measure

$$d\mathbb{Q}_T^a = f(a)^{-1}N_f(T)\ d\mathbb{P}_T^a, \tag{2.11}$$

represent.

Theorem 2.2 *The measure* $\mathbb{Q}_T^a$ *defined in* (2.11) *is the law of a Markov process with a generator of the form*

$$\mathcal{L}^f g = \frac{\mathcal{L}(fg)}{f} - g\frac{\mathcal{L}f}{f}.$$

We sketch the proof in Exercise **(v)** and **(vi)**. We may also allow the function f to depend on time and use the martingale N_f of (2.10) in the definition of $\mathbb{Q}$ in (2.11):

$$d\mathbb{Q}_T^a = f(a,0)^{-1}N_f(T)\ d\mathbb{P}_T^a, \tag{2.12}$$

The resulting process is again Markov but possibly inhomogeneous with a time dependent generator $\mathcal{L}_t := \mathcal{L}^{f(\cdot,t)}$.

Remark 2.2 Note that for the definition of $\mathbb{Q}$ and the martingale, we need to assume that $f > 0$. This requirement can be relaxed: When $f \in Dom$ is not positive, then we may set

$$U = \{x:\ f(x) > 0\}, \qquad \tau = \inf\{t > 0:\ x(t) \notin U\},$$

then we can still talk about the martingale

$$N_f(t\wedge\tau) = f(x(t\wedge\tau))\ e^{-\int_0^{t\wedge\tau} \frac{\mathcal{L}f}{f}(x(s))\ ds}.$$

□

A particular important special case is when $f \in Dom$ satisfies $\mathcal{L}f = 0$. The corresponding $\mathbb{Q}$ measure is known as *Doob h-transform.* The measure $\mathbb{Q}$ now takes the form

$$d\mathbb{Q}_T^a = f(a)^{-1}f(x(T))\ d\mathbb{P}_T^a. \tag{2.13}$$

More generally, if we take a time dependent f that satisfies

$$(\partial_t + \mathcal{L})f = 0,$$

then the corresponding process takes the form

$$d\mathbb{Q}_T^a = f(a,0)^{-1} f(x(T),T)\ d\mathbb{P}_T^a. \tag{2.14}$$

The corresponding generator is

$$\mathcal{L}^f(g) = \frac{\mathcal{L}(fg)}{g} - \frac{g\mathcal{L}f}{f} = \frac{\mathcal{L}(fg)}{g} + \frac{gf_t}{f}.$$

Here are some important examples.

Example 2.3(i) Let U be any bounded domain in E with E countable. Set

$$\tau = \tau_U = \inf\big\{t > 0 :\ x(t) \notin U\big\}.$$

Let $x(\cdot)$ be a Markov process in E with generator $\mathcal{L}$. Pick $T > 0$ and write $\mathbb{P}_T$ for law of the process $x(\cdot)$ in the interval $[0,T]$. We wish to study the measure

$$\mathbb{Q}_T(A) = \mathbb{P}_T\big(A \bigm| T < \tau\big) = \mathbb{P}_T^a(A;\ T < \tau)/\mathbb{P}_T^a(T < \tau), \qquad A \in \mathcal{F}_T.$$

Consider the martingale $M(t) = h\big(x(t \wedge \tau), T - t \wedge \tau\big)$ with h satisfying

$$\begin{cases} h_t = \mathcal{L}h, & t > 0,\ x \in U \\ h(x,0) = 1 & x \in U \\ h(x,t) = 0 & t \ge 0,\ x \in \partial U. \end{cases}$$

We claim

$$\begin{aligned} d\mathbb{Q}_T &= h(x(0),T)^{-1}\ h\big(x(T\wedge\tau), T - T\wedge\tau\big)\ d\mathbb{P}_T \\ &= h(x(0),T)^{-1}\ h(x(T),0)\ 1\!\!1(T<\tau) d\mathbb{P}_T \\ &= h(x(0),T)^{-1}\ 1\!\!1(T<\tau)\ d\mathbb{P}_T. \end{aligned}$$

To verify this, it suffices to show

$$h(a,T) = \mathbb{P}_T^{x(0)=a}(T < \tau).$$

Indeed using the $\mathbb{P}$-martingale M,

$$\mathbb{P}_T^{x(0)=a}(T<\tau) = \mathbb{E}^{x(0)=a}\ M\big(T\wedge\tau\big) = \mathbb{E}^{x(0)=a}\ M(0) = h(a,T).$$

The relationship between h and w of Example 2.2**(i)** is that in the discrete setting,

$$h(x,t) = \sum_{a\in U} w(x,t;a).$$

Likewise, when $E = \mathbb{R}^d$, and $x(\cdot)$ is a diffusion,

$$h(x,t) = \int_U w(x,t;a)\ da.$$

(ii) Let us assume that the generator $\mathcal{L}$ is symmetric with respect to a measure $m(dx)$. In this case we may solve the equation $u_t = \mathcal{L}u$ by finding the eigenvalues and eigenfunctions of $\mathcal{L}$. When U is bounded, then the point spectrum of $\mathcal{L}$ consists of eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \dots.$$

We choose an orthonormal basis for $L^2(m)$ consisting of the corresponding eigenfunctions $w_1, w_2, \dots,$ with

$$\begin{cases} \mathcal{L}w_i + \lambda_i w_i = 0, & \text{in } U \\ w_i = 0 & \text{on } \partial U, \end{cases}$$

for each i. For the sake of definiteness, let us assume that $x(\cdot)$ is a diffusion and that m is the Lebesgue measure. Since for any $f \in L^2(m)$ we can write

$$f(x) = \sum_{i=1}^{\infty} w_i(x) \int w_i(y) f(y)\ dy,$$

we learn

$$\delta_y(dx) = \sum_i w_i(y) w_i(x).$$

This in turn implies that the solution w of (2.8) is of the form

$$w(x,t;y) = \sum_{i=1}^{\infty} e^{-\lambda_i t} w_i(y) w_i(x).$$

This means that the function h takes the form

$$h(x,t) = \sum_{i=1}^{\infty} c_i e^{-\lambda_i t} w_i(x), \qquad \text{with} \quad c_i = \int_U w_i(y)\ dy.$$

We wish to calculate

$$\mathbb{Q}_\infty = \lim_{T \to \infty} \mathbb{Q}_T,$$

for the measure $\mathbb{Q}_T$ of part **(i)**. If this limit exists, we interpret $\mathbb{Q}_\infty$ as the probability law of the process $x(\cdot)$ conditioned to stay in U forever. Indeed for any pair (t,T) with $0 < t < T$,

$$\mathbb{Q}_T(A) = \int_A \frac{h(x(t), T-t)}{h(x(0), T)}\ d\mathbb{P},$$

for any $A \in \mathcal{F}_t$. Hence, for determining $\mathbb{Q}_\infty$, we need to calculate

$$\lim_{T\to\infty} \frac{h(x, T-t)}{h(y, T)}.$$

For this, let us assume that the smallest eigenvalue λ_1 is of multiplicity one. Assuming this, we calculate

$$\lim_{T\to\infty} \frac{h(x, T-t)}{h(y, T)} = \lim_{T\to\infty} \frac{e^{-\lambda_1(T-t)} w_1(x)}{e^{-\lambda_1 T} w_1(y)} = e^{\lambda_1 t}\ \frac{w_1(x)}{w_1(y)}.$$

From this we learn

$$d\mathbb{Q}_\infty\Big|_{\mathcal{F}_t} = e^{\lambda_1 t}\ \frac{w_1(x(t))}{w_1(x(0))}\ \ d\mathbb{P}\Big|_{\mathcal{F}_t}.$$

This means that we have a Doob transform associated with the function $k(x,t) = w_1(x) + \lambda_1 t$. The corresponding generator is

$$\mathcal{L}_\infty f = w_1^{-1}(\mathcal{L} + \lambda_1)(w_1 f).$$

When U is unbounded, we need to understand the behavior of $x(\cdot)$ at ∞. This is closely related to the *Martin boundary* of U. Martin boundary points can be studied by looking at the boundary point of the convex set

$$\big\{h : \bar{U} \to \mathbb{R} :\ \ h = 0 \ \text{ on } \ \partial U, \quad h > 0 \ \text{ in } \ U\big\}.$$

$\square$

2.1 Duality

Assume that $x(\cdot)$ and $y(\cdot)$ are two Markov processes with state spaces E and E' and generators $\mathcal{L}_x$ and $\mathcal{L}_y$ respectively. Given a function $H : E \times E' \to \mathbb{R}$, we say that the processes $x(\cdot)$ and $y(\cdot)$ are dual with respect to H if the following conditions are true:

(i) $H(\cdot, y)$ (respectively $H(x, \cdot)$) is in the domain of the definition of $\mathcal{L}_x$ (respectively $\mathcal{L}_y$) for every $y \in E'$ (respectively $x \in E$).

(ii) For every $(x, y) \in E \times E'$,

$$\mathbb{E}^{x(0)=x}\ H(x(t), y) = \mathbb{E}^{y(0)=y}\ H(x, y(t)). \tag{2.15}$$

The following criterion gives us a practical way of verifying (2.15).

Theorem 2.3 *Assume that* **(i)** *is true. Then* **(ii)** *is true iff*

$$\big(\mathcal{L}_x H(\cdot, y)\big)(x) = \big(\mathcal{L}_y H(x, \cdot)\big)(y). \tag{2.16}$$

Proof Write $T_t = e^{t\mathcal{L}_x}$ and $\hat{T}_t = e^{t\mathcal{L}_y}$ for semigroup associated with the processes $x(\cdot)$ and $y(\cdot)$. Assume that (2.16) holds. Write $u(x,y,t)$ and $\hat{u}(x,y,t)$ for the left and right hand side of (2.15) respectively. Evidently

$$u_t = \mathcal{L}_x u, \quad \hat{u} = \mathcal{L}_y \hat{u}, \quad u(x,y,0) = \hat{u}(x,y,t) = H(x,y).$$

By uniqueness of the solutions for the same initial data, we are done if we can show

$$\hat{u} = \mathcal{L}_x \hat{u}. \tag{2.17}$$

Here is the proof of (2.17):

$$\begin{aligned}
\hat{u}_t(x,y,t) &= \mathbb{E}^{y(0)=y}\ \mathcal{L}_y H(x,y(t)) = \mathbb{E}^{y(0)=y}\ \mathcal{L}_x H(x,y(t)) \\
&= \int \mathcal{L}_x H(x,y')\ \hat{p}_t(y,dy') = \mathcal{L}_x \int H(x,y')\ \hat{p}_t(y,dy') \\
&= \mathcal{L}_x \hat{u}(x,y,t),
\end{aligned}$$

where $\hat{p}_t(y,dy') = \mathbb{P}\big(y(t) \in dy' \mid y(0) = y\big)$. □

Exercises

(i) Let $v : \mathbb{Z}^d \to \mathbb{Z}$ be a nonnegative subadditive function with $v(0) = 0$ and $v(i) + v(-i) > 0$ for all $i \neq 0$. Recall $\hat{v}(i) = -v(-i)$. Show that $v, \hat{v} \in \Gamma(v)$ and that $v^i \in \Gamma(v)$ iff $i = 0$. Moreover $v_i, \hat{v}^i \notin \Gamma(v)$ for all $i \in \mathbb{Z}^d$.

(ii) Write $\mathcal{A}_\lambda$ for the generator of the height differences $\eta \in E = \{0,1\}^{\mathbb{Z}}$ in the case of SEP, and write ν^ρ for the equilibrium measure associated with density ρ. Show

$$\int f\mathcal{A}_\lambda g\ d\nu^\rho = \int g\mathcal{A}_{1-\lambda} f\ d\nu^\rho,$$

for any pair of functions $f, g : E \to \mathbb{R}$ that depend on finitely many $(\eta(i) :\ i \in \mathbb{Z})$.

(iii) Write $\mathcal{A}_\lambda$ for the generator of height function h of the v-exclusion process. Recall the measure μ_λ that was defined in (1.5). Show

$$\int f\mathcal{A}_\lambda g\ d\mu_\lambda = \int g\mathcal{A}_\lambda f\ d\mu_\lambda,$$

for any pair of functions $f, g : \Gamma(v) \to \mathbb{R}$ that depend on finitely many $(h(i) :\ i \in \mathbb{Z})$.

(iv) Let us write $\mathcal{A}$ for the generator of process associated with the gap between particle in HAD process: If $z_i = x_i - x_{i+1}$, then the dynamics of $\mathbf{z} = \big(z_i :\ i \in \mathbb{Z}\big) \in (0,\infty)^{\mathbb{Z}}$ is Markovian with generator

$$\mathcal{A}F(\mathbf{z}) = \sum_{i\in\mathbb{Z}} \int_0^{z_i} \big(F(\mathbf{z}_i^a) - F(\mathbf{z})\big)\ da,$$

where $\mathbf{z}_i^a$ is the configuration we obtain from $\mathbf{z}$ by changing z_i and z_{i+1} to $z_i - a$ and $z_{i+1} + a$, and leaving other z_j unchanged. Show that the product of exponential measures $\lambda e^{-\lambda z_i}\, dz_i$: $i \in \mathbb{Z}$ is invariant for $\mathcal{A}$ and find the adjoint of $\mathcal{A}$ with respect to this measure.

(v) Suppose that we have a probability measure $\mathbb{P}$ on $\mathcal{D}_T$ such that for every $f \in \mathcal{D}$, the process $N_f(t)$ is a martingale. From this deduce that the processes

$$M_f(t) = f(x(t)) - f(x(0)) - \int_0^t \mathcal{L}f(x(\theta))\, d\theta,$$
$$M_f'(t) = M_f(t)^2 - \int_0^t (\mathcal{L}f^2 - 2f\mathcal{L}f)(x(\theta))\, d\theta,$$

are martingales. (*Hint:* Use martingales $(N_{e^{\lambda f}} :\ \lambda \in \mathbb{R})$.)

(vi) Let $\mathbb{Q}_T^a$ be as in (2.10). Show that for very $g \in \mathcal{D}$, the process

$$N_{f,g}(t) = g(x(t)) e^{-\int_0^t \frac{\mathcal{L}^f g}{g}(x(s))\ ds},$$

is a martingale with respect to $\mathbb{Q}_T^a$.

(vii) Let $\mathcal{L}$ be the generator of Markov process on a coutable state E with jump rate $c(x, y)$ as in Example 2.1**(i)**. Given any bounded positive $f : E \to \mathbb{R}$, the process associated with $\mathcal{L}^f$ of Theorem 2.2 is a Markov process with jump rate $c^f(x, y)$. Determine c^f.

(viii) Let $x(\cdot)$ be a standard Brownian motion in $\mathbb{R}$ with generator $\mathcal{L}f = 2^{-1}f''$. For any C^2 function $h(x, t)$, determine $\mathcal{L}^h$. Given $y \in \mathbb{R}$ and $T > 0$, define $h : \mathbb{R} \times [0, T] \to \mathbb{R}$ by

$$h(x, t) = (2\pi(T - t))^{-1/2}\ e^{-\frac{(x-y)^2}{2(T-t)}}.$$

Determine the generator $\mathcal{L}^h$ in this case and interpret the corresponding Markov process.

$\square$

3 Determinantal Processes

In Chapter 2 we discussed some basic properties of Markov processes. In this chapter we explore dterminantal processes that will play central role in our study of TASEP in Chapters 5 and 6, and also for other exactly solvable models. We will show in Chapter 5 below that in fact TASEP is an example of a determinantal process. As a preparation we give an overview of determinantal processes in this chapter.

Let X be a discrete (countable) set and write $\mathcal{X} = 2^X$ for the set of subsets of X. By a *Point process* in X, we mean a probability measure $\mathbb{P}$ on $\mathcal{X}$. In other words, a set $\mathbf{x} \in \mathcal{X}$ is selected according to the law $\mathbb{P}$. The set X is equipped with the σ-field generated by sets of the form $\big\{\mathbf{x} \in \mathcal{X} : \ A \subset \mathbf{x}\big\}$ with $A \in \mathcal{X}$. Any point processes in X is uniquely determined by its correlation functions $\rho_k : X^k \to \mathbb{R}, k \in \mathbb{N}$; each ρ_k is a symmetric function that is defined by

$$\rho_k(x_1, \dots, x_k) = \mathbb{P}\left(\big\{x_1, \dots, x_k\big\} \subseteq \mathbf{x}\right),$$

for any distinct $x_1, \dots, x_k \in X$. Note that if $f : X^k \to \mathbb{R}$ is any bounded function, then

$$\mathbb{E} \sum_{x_1 \neq \dots \neq x_k} f(x_1, \dots, x_k) 1\!\!1\left(\big\{x_1, \dots, x_k\big\} \subseteq \mathbf{x}\right) = \sum_{x_1 \neq \dots \neq x_k} f(x_1, \dots, x_k) \rho_k(x_1, \dots, x_k),$$

where $x_1 \neq \dots \neq x_k$ means that $x_1, \dots, x_k$ are distinct. Another useful probability is the *Janossy density* that is defined by

$$J_{k,A}(x_1, \dots, x_k) = \mathbb{P}\left(A \cap \mathbf{x} = \big\{x_1, \dots, x_k\big\}\right),$$

for every $A \in \mathcal{X}$. Given a finite set A with $\sharp A = \ell$, we have

$$\rho_k(x_1, \dots, x_k) = \sum_{r=1}^{\ell-k} \frac{1}{r!} \sum_{y_1, \dots, y_r \in A} J_{k+r,A}(x_1, \dots, x_k, y_1, \dots, y_r), \tag{3.1}$$

for every distinct $x_1, \dots, x_k \in A$. From this, it is not hard to deduce

$$J_{k,A}(x_1, \dots, x_k) = \sum_{r=0}^{\ell-k} \frac{(-1)^r}{r!} \sum_{y_1, \dots, y_r \in A} \rho_{k+r}(x_1, \dots, x_k, y_1, \dots, y_r), \tag{3.2}$$

for every distinct $x_1, \dots, x_k \in A$.

We also use the compact notation

$$\rho(\mathbf{a}) = \mathbb{P}\big(\mathbf{a} \subseteq \mathbf{x}\big), \qquad J_A(\mathbf{a}) = \mathbb{P}\big(\mathbf{x} \cap A = \mathbf{a}\big).$$

We can then write

$$\rho(\mathbf{a}) = \sum_{\mathbf{a} \subseteq \mathbf{b} \subseteq A} J_A(\mathbf{b}), \qquad J_A(\mathbf{a}) = \sum_{\mathbf{a} \subseteq \mathbf{b} \subseteq A} (-1)^{|b|-|a|} \rho(\mathbf{b}), \tag{3.3}$$

for every $a \subseteq A$. Here we are writing $|\mathbf{a}|$ for the cardinality of the set $\mathbf{a}$.

Example 3.1(i) As a simple example, imagine that the variables $\big(1\!\!1(x \in \mathbf{x}) :\ x \in X\big)$ are independent with $\mathbb{P}(x \in \mathbf{x}) = \rho_1(x)$. Then

$$\rho(\mathbf{a}) = \prod_{a \in \mathbf{a}} \rho_1(a), \qquad J_A(\mathbf{a}) = \prod_{a \in \mathbf{a}} \rho_1(a) \prod_{b \in A \setminus \mathbf{a}} (1 - \rho_1(b)).$$

(ii) As a simple example, choose $f = 1\!\!1_A$ to obtain

$$\mathbb{E}\ \sharp\big(A \cap \mathbf{x}\big) = \sum_{x \in A} \rho_1(x).$$

$\square$

Given a symmetric function $K : X \times X \to \mathbb{R}$, we say a Point process $\mathbb{P}$ is *determinantal* with *correlation kernel* K, if

$$\rho_k(x_1, \dots, x_k) = \det\big[K(x_i, x_j)\big]_{i,j=1}^{k},$$

for any distinct $x_1, \dots, x_k \in X$. Note that the right-hand side is symmetric by the definition of a determinant. It is useful to think of K as a matrix for which the rows and columns are labeled by the elements of X:

$$K = K_X = \big[K(x, y)\big]_{x,y \in X}.$$

Then we may write $\rho(\mathbf{a}) = \mathbb{P}\big(\mathbf{a} \subseteq \mathbf{x}\big) = \det K_{\mathbf{a}}$, where $K_{\mathbf{a}}$ denotes the restriction of K to $\mathbf{a} \times \mathbf{a}$. If we regard K as a matrix, then it matters how we labels points in $\mathbf{a}$. Though relabeling columns and rows would not affect $\det K_{\mathbf{a}}$. Regarding K as an operator acting on $\ell^2(X)$, then K_A corresponds to acting the operator on functions with support in $\mathbf{a}$. With the latter interpretation, it wouldn't matter how points in $\mathbf{a}$ are labeled.

Remark 3.1(i) Note that if $K(x, y) = \rho_1(x)1\!\!1(x = y)$, we recover the independent point process of Example 3.1**(i)**. However in general for a distinct pair $x, y \in X$,

$$\mathbb{P}\big(\{x, y\} \subseteq \mathbf{x}\big) = K(x, x)K(y, y) - K(x, y)^2 < K(x, x)K(y, y) = \mathbb{P}\big(\{x\} \subseteq \mathbf{x}\big)\mathbb{P}\big(\{y\} \subseteq \mathbf{x}\big),$$

whenever $K(x, y) \neq 0$. This means that if the matrix K is not diagonal, then the pair correlation is negative i.e., the interaction between particles is repulsive.

(ii) If K is the correlation function and $\lambda : X \to \mathbb{R}$ is a function, then the kernel K^λ defined by

$$K^\lambda(x, y) = \lambda(x)^{-1}K(x, y)\lambda(y),$$

is also a correlation kernel for the same determinantal process because for any $\mathbf{a} \subset X$,

$$\det\left[K^\lambda(x,y)\right]_{x,y\in\mathbf{a}} = \det\left[K(x,y)\right]_{x,y\in\mathbf{a}} \prod_{x\in\mathbf{a}} \lambda(x)^{-1} \prod_{y\in\mathbf{a}} \lambda(y) = \det\left[K(x,y)\right]_{x,y\in\mathbf{a}}.$$

Note that if

$$\mathcal{K}_\lambda f(x) = \sum_{y\in X} K^\lambda(x,y)f(y), \qquad \mathcal{M}_\lambda f(x) = \lambda(x)f(x),$$

then $\mathcal{K}_\lambda = \mathcal{M}_\lambda^{-1}\mathcal{K}\mathcal{M}_\lambda$. In other words, we obtain the operator $\mathcal{K}_\lambda$ from $\mathcal{K}$ by *conjugating* $\mathcal{K}$ with the multiplication operator $\mathcal{M}_\lambda$. $\square$

The following elementary calculation will be used in several occasions below.

Lemma 3.1 **(i)** *Let $A = A_I = \left[a_{ij}\right]_{i,j\in I}$ and $B = B_I = \left[b_{ij}\right]_{i,j\in I}$ be two matices. Then for any $K \subset I$*

$$\det(A+B) = \sum_{I'\subseteq I} \det\left(\chi_{I\setminus I'}A + \chi_{I'}B\right), \tag{3.4}$$

where the (i,j)-th entry of $\chi_{I'}B$ is $1\!\!1(i\in I')b_{ij}$.

(ii) *Let B be as in part*(**i**)*, and write $1\!\!1_I$ for the identity matrix $\left[\delta_{ij}\right]_{i,j\in I}$. Then for any $K \subset I$*

$$\det\left(1\!\!1_{K^c} + B\right) = \sum_{K\subseteq I'\subseteq I} \det B_{I'}. \tag{3.5}$$

Proof(i) Write S_I for the set of permutations of the set I. We have

$$\begin{aligned}\det(A+B) &= \sum_{\sigma\in S_I} \varepsilon(\sigma) \prod_{i\in I}\left(a_{i\sigma(i)} + b_{i\sigma(i)}\right) = \sum_{\sigma\in S_I}\varepsilon(\sigma) \sum_{I'\subseteq I}\ \prod_{i\in I\setminus I'} a_{i\sigma(i)} \prod_{i\in I'} b_{i\sigma(i)} \\ &= \sum_{I'\subseteq I}\ \sum_{\sigma\in S_I} \varepsilon(\sigma) \prod_{i\in I}\left(1\!\!1(i\notin I')a_{i\sigma(i)} + 1\!\!1(i\in I')b_{i\sigma(i)}\right) \\ &= \sum_{I'\subseteq I} \det\left(\chi_{I\setminus I'}A + \chi_{I'}B\right).\end{aligned}$$

(ii) We have

$$\begin{aligned}
\det\big(\mathbb{1}_{K^c}+M\big) &= \sum_{\sigma\in S_I}\varepsilon(\sigma)\prod_{i\in I}\big(\delta_{i\sigma(i)}\mathbb{1}(i\notin K)+m_{i\sigma(i)}\big)\\
&= \sum_{\sigma\in S_I}\varepsilon(\sigma)\sum_{I'\subseteq I}\prod_{i\in I\setminus I'}\big(\delta_{i\sigma(i)}\mathbb{1}(i\notin K)\big)\prod_{i\in I'}m_{i\sigma(i)}\\
&= \sum_{\sigma\in S_I}\varepsilon(\sigma)\sum_{K\subseteq I'\subseteq I}\prod_{i\in I\setminus I'}\delta_{i\sigma(i)}\prod_{i\in I'}m_{i\sigma(i)}\\
&= \sum_{\sigma\in S_I}\varepsilon(\sigma)\sum_{K\subseteq I'\subseteq I}\mathbb{1}\big(\sigma|_{I\setminus I'}=id\big)\prod_{i\in I'}m_{i\sigma(i)}\\
&= \sum_{K\subseteq I'\subseteq I}\sum_{\tau\in S_{I'}}\varepsilon(\tau)\prod_{i\in I'}m_{i\tau(i)}=\sum_{K\subseteq I'\subseteq I}\det M_{I'}.
\end{aligned}$$

□

When X is not a finite set, and assuming

$$\|K\|_2^2=\sum_{x,y\in X}K(x,y)^2<\infty, \tag{3.6}$$

sometime it is useful to think of K as an operator $\mathcal{K}:\ell^2(X)\to\ell^2(X)$, that is defined by

$$\mathcal{K}f(x)=\sum_{y\in X}K(x,y)f(y).$$

We may write this an integral operator that is an example of a Hilbert-Schmidt operator. In fact

Proposition 3.1 **(i)** *Let $\mathbb{P}$ be determinantal with a kernel K. For any function $u:X\to\mathbb{R}$ of finite support,*

$$\mathbb{E}\prod_{x\in\mathbf{x}}(1-u(x))=\det\big(\mathbb{1}-uK\big), \tag{3.7}$$

where $(uK)(x,y)=u(x)K(x,y)$. Moreover when $u\geq 0$, we may define

$$\big(u^{1/2}Ku^{1/2}\big)(x,y)=u^{1/2}(x)K(x,y)u^{1/2}(y).$$

In terms of $u^{1/2}Ku^{1/2}$,

$$\det\big(\mathbb{1}-uK\big)=\det\big(\mathbb{1}-\big(u^{1/2}Ku^{1/2}\big)\big).$$

(ii) *Let A be a finite subset of X. We have $\mathbb{P}\big(\mathbf{x}\cap A=\emptyset\big)=\det\big(\mathbb{1}_A-K_A\big)$. More generally, if $A_1,\dots,A_k$ are disjoints finite subsets of X, then the probability*

$$\mathbb{P}\big(|\mathbf{x}\cap A_1|=r_1,\dots,|\mathbf{x}\cap A_k|=r_k\big), \tag{3.8}$$

is given by

$$\frac{(-1)^{\sum_{i=1}^k r_i}}{\prod_{i=1}^k r_i!}\ \frac{\partial^{\sum_{i=1}^k r_i}}{\partial_{z_1}^{r_1}\dots\partial_{z_k}^{r_k}}\det\big(\mathbb{1}_A-z_1K_{A_1}-\dots-z_kK_{A_k}\big)\Big|_{z_1=\dots=z_k=1}, \tag{3.9}$$

for $A=A_1\cup\dots\cup A_k$.

(iii) *For every $\mathbf{a}\subseteq A$,*

$$J_A(\mathbf{a})=(-1)^{|\mathbf{a}|}\det\big(\mathbb{1}_{A\backslash\mathbf{a}}-K_A\big)=\det\big(\chi_{A\backslash\mathbf{a}}(\mathbb{1}-K)_A+\chi_{\mathbf{a}}K_A\big), \tag{3.10}$$

where $\chi_{\mathbf{a}}(x)=\mathbb{1}(x\in\mathbf{a})$.

(iv) *Assume that $\mathbf{a}\subseteq A$. If K_A is invertible, then*

$$J_A(\mathbf{a})=\det K_A\ \ \det\big(L^{-1}\big)_{A\backslash\mathbf{a}},$$

and if $\mathbb{1}_A-K_A$ is invertible, then

$$J_A(\mathbf{a})=\det\big(\mathbb{1}_A-K_A\big)\ \ \det L_{\mathbf{a}},$$

where $L=L_A=(\mathbb{1}_A-K_A)^{-1}K_A=(\mathbb{1}_A-K_A)^{-1}-\mathbb{1}_A$.

(v) *If K is of trace class, then in part* **(ii)** *we may choose a set A that is not finite.*

Proof(i) Using (3.12) and (3.11),

$$\begin{aligned}\mathbb{E}\prod_{x\in\mathbf{x}}(1-u(x))&=1+\sum_{n\ge1}(-1)^n\ \mathbb{E}\sum_{\{x_1,\dots,x_n\}\subseteq\mathbf{x}}\ \prod_{i=1}^n u(x_i)\\&=\sum_{\mathbf{a}\subseteq X}(-1)^{|\mathbf{a}|}\left(\prod_{a\in\mathbf{a}}u(a)\right)\det K_{\mathbf{a}}\\&=\sum_{\mathbf{a}\subseteq X}(-1)^{|\mathbf{a}|}\det(uK)_{\mathbf{a}}=\det\big(\mathbb{1}-uK\big).\end{aligned}$$

(ii) By choosing $u=\mathbb{1}_A$ in (3.13) we learn

$$\mathbb{P}\big(\mathbf{x}\cap A=\emptyset\big)=\det\big(\mathbb{1}_X-(\mathbb{1}_AK\mathbb{1}_A)\big)=\det\big(\mathbb{1}_A-K_A\big).$$

More generally, if we choose

$$u = \sum_{i=1}^{k} z_i \mathbb{1}_{A_i},$$

in (3.6), we obtain

$$\mathbb{E} \prod_{x \in \mathbf{x}} \left(\sum_{i=1}^{k} (1 - z_i) \mathbb{1}_{A_i}(x) + \mathbb{1}_{X \setminus A}(x) \right) = \det \big(\mathbb{1} - z_1 K_{A_1} - \cdots - z_k K_{A_k} \big).$$

We then apply the differential operator that appears in (3.9) and evaluate both sides at $z_1 = \cdots = z_k = 1$. For example, when $k = 1$, we differentiate both sides with respect to $z = z_1$; the left-hand side yields

$$- \sum_{a \in X} \mathbb{E} \prod_{x \in \mathbf{x}, x \neq a} \big((1 - z) \mathbb{1}_A(x) + \mathbb{1}_{X \setminus A}(x) \big) \mathbb{1}\big(a \in \mathbf{x} \cap A\big),$$

which is $-\mathbb{P}\big(|\mathbf{x} \cap A| = 1\big)$ at $z = 1$.

(iii) From (3.5) and (3.4),

$$\begin{aligned} J_A(\mathbf{a}) &= \sum_{\mathbf{a} \subseteq \mathbf{b} \subseteq A} (-1)^{|\mathbf{b}| - |\mathbf{a}|} \det K_{\mathbf{b}} = (-1)^{-|\mathbf{a}|} \sum_{\mathbf{a} \subseteq \mathbf{b} \subseteq A} \det(-K)_{\mathbf{b}} \\ &= (-1)^{|\mathbf{a}|} \det \big(\mathbb{1}_{A \setminus \mathbf{a}} - K_A \big), \end{aligned}$$

for every $\mathbf{a} \subseteq A$. Alternatively, let us set

$$\hat{K}(x, y) = K(x, y) \mathbb{1}\big(x \in A \setminus \mathbf{a}\big) - K(x, y) \mathbb{1}(x \in \mathbf{a}), \qquad \text{or in short} \qquad \hat{K} = \chi_{A \setminus \mathbf{a}} K - \chi_{\mathbf{a}} K.$$

Then for $\mathbf{b}$ with $\mathbf{a} \subseteq \mathbf{b} \subseteq A$,

$$(-1)^{|\mathbf{b}| - |\mathbf{a}|} \det K_{\mathbf{b}} = (-1)^{|\mathbf{b}|} \det \hat{K}_{\mathbf{b}} = \det \big(- \hat{K} \big)_{\mathbf{b}}.$$

As a result,

$$\begin{aligned} J_A(\mathbf{a}) &= \sum_{\mathbf{a} \subseteq \mathbf{b} \subseteq A} (-1)^{|\mathbf{b}| - |\mathbf{a}|} K_{\mathbf{b}} = \sum_{\mathbf{a} \subseteq \mathbf{b} \subseteq A} \det(-\hat{K})_{\mathbf{b}} = \det \big(\mathbb{1}_{A \setminus \mathbf{a}} - \hat{K}_A \big) \\ &= \det \big(\chi_{A \setminus \mathbf{a}} (\mathbb{1} - K)_A + \chi_{\mathbf{a}} K_A \big). \end{aligned}$$

(iv) By (3.10),

$$\begin{aligned} J_A(\mathbf{a}) &= \det \big(\chi_{A \setminus \mathbf{a}} (\mathbb{1} - K)_A + \chi_{\mathbf{a}} K_A \big) = \det K_A \ \det \big(\chi_{A \setminus \mathbf{a}} (\mathbb{1} - K)_A K_A^{-1} + \chi_{\mathbf{a}} \big), \\ &= \det K_A \ \det \big(\chi_{A \setminus \mathbf{a}} L^{-1} + \chi_{\mathbf{a}} \big) = \det(\mathbb{1} - K)_A \ \det \big(L^{-1} \big)_{A \setminus \mathbf{a}} \\ J_A(\mathbf{a}) &= \det \big(\chi_{A \setminus \mathbf{a}} (\mathbb{1} - K)_A + \chi_{\mathbf{a}} K_A \big) = \det(\mathbb{1} - K)_A \ \det \big(\chi_{A \setminus \mathbf{a}} + \chi_{\mathbf{a}} (\mathbb{1}_A - K_A)^{-1} K_A \big) \\ &= \det(\mathbb{1} - K)_A \ \det \big(\chi_{A \setminus \mathbf{a}} + \chi_{\mathbf{a}} L \big) = \det(\mathbb{1} - K)_A \ \det L_{\mathbf{a}}. \end{aligned}$$

$\square$

We now construct some examples of determinantal processes. In the first two examples the probability of a configuration is given by a determinant and a configuration (restricted to a finite set when X is infinite) can have a varying cardinality. In the remaining examples, the cardinality of all configurations is fixed though the probability of a configuration is the determinant of the product of two matrices of a special form.

Example 3.2(i) Assume that X is finite and let $L : X \times X \to \mathbb{R}$ be a symmetric function such that, regarding L as a matrix, we have $\det L_{\mathbf{a}} > 0$ for every nonempty $\mathbf{a} \subseteq X$. (This condition is certainly true if L is positive definite.) We define a point process on X by

$$\mathbb{P}(\mathbf{x} = \mathbf{a}) = \det(\mathbb{1} + L)^{-1} \ \det L_{\mathbf{a}}. \tag{3.11}$$

for every $\mathbf{a} \in \mathcal{X}$. By Lemma 3.1, (3.18) is a probability measure. We now claim that the probability measure $\mathbb{P}$ given by (3.11) is determinantal with correlation kernel

$$K = L(\mathbb{1} + L)^{-1} = \mathbb{1} - (\mathbb{1} + L)^{-1}. \tag{3.12}$$

Indeed, by (3.5)

$$\begin{aligned}
\rho(\mathbf{a}) &= \det(\mathbb{1} + L)^{-1} \sum_{\mathbf{a} \subseteq \mathbf{b} \subseteq X} \det L_{\mathbf{b}} = \det(\mathbb{1} + L)^{-1} \ \det \big(\mathbb{1}_{\mathbf{a}^c} + L\big) \\
&= \det \big((\mathbb{1} + L)^{-1} \ \big(\mathbb{1}_{\mathbf{a}^c} + L\big)\big) = \det\left((\mathbb{1} - K) \ \mathbb{1}_{\mathbf{a}^c} + K\right) = \det\left(\mathbb{1}_{\mathbf{a}^c} + K \mathbb{1}_{\mathbf{a}}\right) \\
&= \det \begin{bmatrix} K_{\mathbf{a}} & 0 \\ K_{\mathbf{a}^c,\mathbf{a}} & \mathbb{1}_{\mathbf{a}^c} \end{bmatrix} = \det K_{\mathbf{a}},
\end{aligned}$$

as desired. Here we have used the decomposition $X = \mathbf{a} \cup \mathbf{a}^c$ to express the matrix K in a block form

$$K = \begin{bmatrix} K_{\mathbf{a}} & K_{\mathbf{a},\mathbf{a}^c} \\ K_{\mathbf{a}^c,\mathbf{a}} & K_{\mathbf{a}^c} \end{bmatrix}.$$

Alternatively we may use a generalization of Cramer's formula (see Example A1**(ii)** below) to verify $\rho(\mathbf{a}) = \det K_{\mathbf{a}}$: For K as in (3.11),

$$\begin{aligned}
\det K_{\mathbf{a}} &= \sum_{\mathbf{b} \subseteq \mathbf{a}} (-1)^{|\mathbf{b}|} \ \det \big((\mathbb{1} + L)^{-1}\big)_{\mathbf{b}} = \det(\mathbb{1} + L)^{-1} \sum_{\mathbf{b} \subseteq \mathbf{a}} (-1)^{|\mathbf{b}|} \ \det \big(\mathbb{1} + L\big)_{\mathbf{b}^c} \\
&= \det(\mathbb{1} + L)^{-1} \sum_{\mathbf{b} \subseteq \mathbf{a}} (-1)^{|\mathbf{b}|} \sum_{\mathbf{c} \subseteq \mathbf{b}^c} \det L_{\mathbf{c}} \\
&= \det(\mathbb{1} + L)^{-1} \sum_{\mathbf{c} \subseteq X} \det L_{\mathbf{c}} \sum_{\mathbf{b}} (-1)^{|\mathbf{b}|} \ \mathbb{1}\big(\mathbf{b} \subseteq \mathbf{a} \setminus \mathbf{c}\big) \\
&= \det(\mathbb{1} + L)^{-1} \sum_{\mathbf{c} \subseteq X} \det L_{\mathbf{c}} \ \ \mathbb{1}\big(\mathbf{a} \setminus \mathbf{c} = \emptyset\big) = \rho(\mathbf{a}).
\end{aligned}$$

Here for the first and third equality we used (3.5); for the second equality we used (A.7) below in the form

$$\det\left(A^{-1}\right)_{\mathbf{a}} = (\det A)^{-1}\ \det A_{\mathbf{a}^c};$$

for the fifth equality we used the fact that if $\mathbf{a}' = \mathbf{a} \setminus \mathbf{c} \neq \emptyset$, then

$$\sum_{\mathbf{b}} (-1)^{|\mathbf{b}|}\ \mathbb{1}\big(\mathbf{b} \subseteq \mathbf{a}'\big) = \sum_{k=0}^{|\mathbf{a}'|} (-1)^k \binom{|\mathbf{a}'|}{k} = (1-1)^{|\mathbf{a}'|} = 0.$$

(ii) More generally, we choose any set $X_0 \subseteq X$ and define a point process $\mathbf{y}$ in the set X_0 by the formula

$$\mathbb{P}_{X_0}(\mathbf{y} = \mathbf{a}) = \mathbb{P}\big(\mathbf{x} = \mathbf{a} \cup X_0^c \big| X_0^c \subseteq \mathbf{x}\big) = \frac{\det L_{\mathbf{a}\cup X_0^c}}{\det\big(\mathbb{1}_{X_0} + L\big)}. \tag{3.13}$$

Note that $\mathbb{P}_{X_0} = \mathbb{P}$ for $X_0 = X$. We now claim that $\mathbb{P}_{X_0}$ is a determinantal process in X_0 with the correlation kernel $\hat{K} : X_0 \times X_0 \to \mathbb{R}$, that is given by

$$\widehat{K} = \mathbb{1}_{X_0} - \left(\left(\mathbb{1}_{X_0} + L\right)^{-1}\right)_{X_0}. \tag{3.14}$$

As in part **(i)**, we may use (3.12) and (A.7) to assert that for any $\mathbf{a} \subseteq X_0$

$$\begin{aligned}
\det \widehat{K}_{\mathbf{a}} &= \sum_{\mathbf{b}\subseteq\mathbf{a}} (-1)^{|\mathbf{b}|}\ \det\left(\left(\mathbb{1}_{X_0} + L\right)^{-1}\right)_{\mathbf{b}} \\
&= \det\left(\mathbb{1}_{X_0} + L\right)^{-1} \sum_{\mathbf{b}\subseteq\mathbf{a}} (-1)^{|\mathbf{b}|}\ \det\left(\mathbb{1}_{X_0} + L\right)_{\mathbf{b}^c} \\
&= \det\left(\mathbb{1}_{X_0} + L\right)^{-1} \sum_{\mathbf{b}\subseteq\mathbf{a}} (-1)^{|\mathbf{b}|} \sum_{X_0^c\subseteq\mathbf{c}\subseteq\mathbf{b}^c} \det L_{\mathbf{c}} \\
&= \det\left(\mathbb{1}_{X_0} + L\right)^{-1} \sum_{X_0^c\subseteq\mathbf{c}\subseteq X} \det L_{\mathbf{c}} \sum_{\mathbf{b}} (-1)^{|\mathbf{b}|}\ \mathbb{1}\big(\mathbf{b} \subseteq \mathbf{a}\setminus\mathbf{c}\big) \\
&= \det\left(\mathbb{1}_{X_0} + L\right)^{-1} \sum_{X_0^c\subseteq\mathbf{c}\subseteq X} \det L_{\mathbf{c}}\ \ \mathbb{1}\big(\mathbf{a}\setminus\mathbf{c} = \emptyset\big) \\
&= \det\left(\mathbb{1}_{X_0} + L\right)^{-1} \sum_{\mathbf{a}\cup X_0^c\subseteq\mathbf{c}\subseteq X} \det L_{\mathbf{c}} \\
&= \det\left(\mathbb{1}_{X_0} + L\right)^{-1} \sum_{\mathbf{a}\subseteq\mathbf{b}\subseteq X_0} \det L_{X_0^c\cup\mathbf{b}} = \mathbb{P}_{X_0}\big(\mathbf{a} \subseteq \mathbf{y}\big),
\end{aligned}$$

as desired. Note that if we write

$$L = \begin{bmatrix} L_{X_0} & L_{X_0,X_0^c} \\ L_{X_0^c,X_0} & L_{X_0^c} \end{bmatrix},$$

and assume that $L, \mathbb{1}_{X_0} + L_{X_0}, L_{X_0^c}$ are invertible, then by (A.9) of Appendix B,

$$\begin{aligned}\hat{L} := \left(\mathbb{1}_{X_0} - \hat{K}\right)^{-1} - \mathbb{1}_{X_0} &= \left(\left(\left(\mathbb{1}_{X_0} + L\right)^{-1}\right)_{X_0}\right)^{-1} - \mathbb{1}_{X_0} \\ &= L_{X_0} - L_{X_0, X_0^c}\, L_{X_0^c}^{-1}\, L_{X_0^c, X_0}.\end{aligned}$$

(By approximation, we can show that this formula is valid whenever $L_{X_0^c}$ is invertible.) As a result, the process $\mathbf{y}$ is also of the type we defined in **(i)**, with the kernel $\hat{L}$ playing the role of L:

$$\mathbb{P}_{X_0}(\mathbf{y} = \mathbf{a}) = \frac{\det \hat{L}_{\mathbf{a}}}{\det\left(\mathbb{1}_{X_0} + \hat{L}\right)}.$$

There is also a variational description for $\hat{L}$: For every $v \in \ell^2(X_0)$,

$$\hat{L}v \cdot v = \inf_{w \in \ell^2(X_0^c)} L(v + w) \cdot (v + w).$$

This is a straightforward consequence of the identity

$$L(v + w) \cdot (v + w) = L_{X_0}\; v \cdot v + L_{X_0^c}\; w \cdot w + L_{X_0, X_0^c}\; v \cdot w + L_{X_0^c, X_0}\; w \cdot v.$$

We may regard $\mathbb{P}$ as a Gibbs measure with possibly long correlation. Its potential functions are simply given by

$$U\big(\mathbf{a} \mid X_0^c\big) = -\log\det \hat{L}_{\mathbf{a}},$$

where $\hat{L} = L\big(\,\cdot \;\mid X_0^c\big)$ is defined as above.

(iii) For our next example, construct a determinantal process that is defined on

$$\mathcal{X}_n = \big\{\mathbf{a} \in \mathcal{X} : \; |\mathbf{a}| = n\big\}.$$

First we construct a point process with the following recipe: we take a measure $\mu : X \to [0, \infty)$ and two families of functions $\phi_i, \psi_i : X \to \mathbb{R}, i \in \mathbb{N}$ and define a probability measure

$$\begin{aligned}\mathbb{P}\big(\mathbf{x} = \{x_1, \dots, x_n\}\big) &= Z^{-1} \det\left[\phi_i(x_j)\right]_{i,j=1}^{k} \left[\psi_i(x_j)\right]_{i,j=1}^{k} \prod_{i=1}^{n} \mu(x_i) \\ &= Z^{-1} \det\left[K(x_i, x_j)\right]_{i,j=1}^{k} \prod_{i=1}^{n} \mu(x_i). \qquad (3.15)\end{aligned}$$

where Z is the normalizing constant, and

$$K(x, y) = \sum_{i=1}^{n} \phi_i(x)\psi_i(y). \qquad (3.16)$$

Here we are using $(\det A)(\det B) = \det(A^* B)$ for the second equality in (3.15). The normalizing constant is easily calculated:

$$
\begin{aligned}
Z &= (n!)^{-1} \sum_{x_1,\ldots,x_n \in X} \det\big[\phi_i(x_j)\big]_{i,j=1}^k \big[\psi_i(x_j)\big]_{i,j=1}^k \prod_{i=1}^n \mu(x_i) \\
&= (n!)^{-1} \sum_{\sigma,\tau \in S_n} \varepsilon(\sigma)\varepsilon(\tau) \prod_{i=1}^n \sum_{x_i \in X} \phi_{\sigma(i)}(x_i)\psi_{\tau(i)}(x_i)\ \mu(x_i) \\
&= (n!)^{-1} \sum_{\sigma,\tau \in S_n} \varepsilon(\sigma)\varepsilon(\tau) \prod_{i=1}^n \sum_{x \in X} \phi_{\sigma(i)}(x)\psi_{\tau(i)}(x)\ \mu(x) \\
&= (n!)^{-1} \sum_{\sigma,\tau \in S_n} \varepsilon(\tau\sigma^{-1}) \prod_{i=1}^n \sum_{x \in X} \phi_i(x)\psi_{\tau\sigma^{-1}(i)}(x)\ \mu(x) \\
&= \sum_{\gamma \in S_n} \varepsilon(\gamma) \prod_{i=1}^n \sum_{x \in X} \phi_i(x)\psi_{\gamma(i)}(x)\ \mu(x).
\end{aligned}
$$

In summary

$$
Z = \det\left[\int \phi_i \psi_j\ d\mu\right]_{i,j=1}^n. \tag{3.17}
$$

Note that $\mathbb{P}(\mathbf{x} = \mathbf{a})$ is still given by a determinant as in **(i)**, except that the support of $\mathbb{P}$ is now X_n. A natural question is whether or not $\mathbb{P}$ is a determinantal process. Note that the expression

$$
\mathbb{P}\big(\{x_1, \ldots, x_k\} \subset \mathbf{x}\big), \tag{3.18}
$$

equals

$$
\begin{aligned}
&Z^{-1}\big[(n-k)!\big]^{-1} \sum_{x_{k+1},\ldots,x_n \in X} \det\big[\phi_i(x_j)\big]_{i,j=1}^k \big[\psi_i(x_j)\big]_{i,j=1}^k \prod_{i=k+1}^n \mu(x_i) \\
&\quad = Z^{-1}\big[(n-k)!\big]^{-1} \sum_{\sigma,\tau \in S_n} \varepsilon(\sigma)\varepsilon(\tau) \left[\prod_{j=1}^k \phi_{\sigma(j)}(x_j)\psi_{\tau(j)}(x_j)\right] \left[\prod_{i=k+1}^n \sum_{x_i \in X} \phi_{\sigma(i)}(x_i)\psi_{\tau(i)}(x_i)\ \mu(x_i)\right].
\end{aligned}
$$

To have a more tractable expression, we assume the families $\big\{\phi_i :\ i \in \mathbb{N}\big\}$ and $\big\{\psi_i :\ i \in \mathbb{N}\big\}$ are *biorthogonal* with respect to μ:

$$
\sum_{x \in X} \phi_i(x)\psi_j(x)\ \mu(x) = \delta_{ij}. \tag{3.19}
$$

Under this assumption, $Z = 1$ and (3.18) equals

$$\big[(n-k)!\big]^{-1} \sum_{\sigma,\tau \in S_n} \varepsilon(\sigma)\varepsilon(\tau) \left[\prod_{j=1}^{k} \phi_{\sigma(j)}(x_j)\psi_{\tau(j)}(x_j)\right] \mathbb{1}\big(\sigma = \tau \quad \text{on} \quad \{k+1,\dots,n\}\big).$$

For any pair $(\sigma,\tau) \in S_n$ with $\sigma(i) = \tau(i)$, for $i > k$, we have

$$\big\{\sigma(1),\dots,\sigma(k)\big\} = \big\{\tau(1),\dots,\tau(k)\big\} = \big\{\alpha_1,\dots,\alpha_k\big\},$$

with $1 \le \alpha_1 < \dots < \alpha_k \le n$. As a result

$$\begin{aligned} \mathbb{P}\big(\{x_1,\dots,x_k\} \subset \mathbf{x}\big) &= \sum_{1 \le \alpha_1 < \dots < \alpha_k \le n} \ \sum_{\sigma',\tau' \in S_k} \varepsilon(\sigma')\varepsilon(\tau') \left[\prod_{j=1}^{k} \phi_{\alpha_{\sigma'(j)}}(x_j)\psi_{\alpha_{\tau'(j)}}(x_j)\right] \\ &= \sum_{1 \le \alpha_1 < \dots < \alpha_k \le n} \det\big[\phi_{\alpha_l}(x_j)\big]_{j,l=1}^{k} \det\big[\psi_{\alpha_l}(x_j)\big]_{j,l=1}^{k}. \end{aligned} \tag{3.20}$$

If we write

$$A = \big[\phi_i(x_j)\big]_{i,j=1}^{n}, \qquad B = \big[\psi_i(x_j)\big]_{i,j=1}^{n},$$

then by Cauchy-Binet (see (A.5)), we know that

$$\Lambda^r C := \Lambda^r\big(A^* B\big) = \big(\Lambda^r A\big)^*\big(\Lambda^r B\big).$$

In particular

$$\det C_{\mathbf{a}} = \det C_{\mathbf{aa}} = \sum_{\mathbf{b} \in \hat{I}_k} \det A_{\mathbf{ba}} \ \det B_{\mathbf{ba}}.$$

This applied to (3.20) yields

$$\mathbb{P}\big(\{x_1,\dots,x_k\} \subset \mathbf{x}\big) = \det\big[K(x_i,x_j)\big]_{i,j=1}^{k}.$$

(iv) Let us assume that $\mathbb{P}$ is still given by (3.15) with no biorthogonality assumption. We may hope to find $\hat\phi_1,\dots,\hat\phi_n$ and $\hat\psi_1,\dots,\hat\psi_n$ such that

$$\hat\phi_i \in span\ \big\{\phi_1,\dots,\phi_n\big\}, \qquad \hat\psi_i \in span\ \big\{\psi_1,\dots,\hat\psi_n\big\},$$

for every i, and $\langle \hat\phi_i, \hat\psi_j\rangle = \delta_{ij}$, where $\langle\cdot,\cdot\rangle$ denotes the inner product of $L^2(\mu)$. If this is the case, then there are coefficients $A = \big[a_{ij}\big]_{i,j=1}^{n}$ and $B = \big[b_{ij}\big]_{i,j=1}^{n}$ such that

$$\hat\phi_i = \sum_{j=1}^{n} a_{ij}\phi_j, \qquad \hat\psi_i = \sum_{j=1}^{n} b_{ij}\psi_j.$$

Writing $A^{-1} = \big[a'_{ij}\big]_{i,j=1}^n$ and $B^{-1} = \big[b'_{ij}\big]_{i,j=1}^n$, and observing

$$\phi_i(x_j) = \sum_{i'} a'_{ii'} \hat{\phi}_{i'}(x_j), \qquad \psi_i(x_j) = \sum_{i'} b'_{ii'} \hat{\psi}_{i'}(x_j),$$

we may write

$$\begin{aligned}\mathbb{P}\big(\mathbf{x} = \{x_1, \dots, x_n\}\big) &= \big(Z \det A \det B\big)^{-1} \det \big[\hat{\phi}_i(x_j)\big]_{i,j=1}^k \big[\hat{\psi}_i(x_j)\big]_{i,j=1}^k \prod_{i=1}^n \mu(x_i) \\ &= \hat{Z}^{-1} \det \big[\hat{K}(x_i, x_j)\big]_{i,j=1}^k \prod_{i=1}^n \mu(dx_i),\end{aligned}$$

where

$$\hat{K}(x,y) = \sum_{i=1}^n \hat{\phi}_i(x) \hat{\psi}_i(y).$$

Note that since the pair $\big(\hat{\phi}, \hat{\psi}\big)$ is biorthogonal, we learn that $\hat{Z} = 1$, or $Z \det A \det B = 1$. Moreover

$$c_{ij} = \langle \phi_i, \psi_j \rangle = \sum_{k,\ell} a'_{ik} b'_{j\ell} \langle \hat{\phi}_k, \hat{\psi}_\ell \rangle = \sum_k a'_{ik} b'_{jk}.$$

This means that the matrix $C = [c_{ij}]$ satisfies

$$[d_{jk}]_{j,k=1}^d := \big(C^{-1}\big)^* = A^* B, \qquad \text{or} \qquad d_{jk} = \sum_{i=1}^n a_{ij} b_{ik}.$$

This allows us to express $\hat{K}$ in terms of ϕ and ψ:

$$\hat{K}(x,y) = \sum_{ijk=1}^n a_{ij} b_{ik} \phi_j(x) \psi_k(y) = \sum_{j,k=1}^d d_{jk} \phi_j(x) \psi_k(y). \tag{3.21}$$

This expression however is hard to use in practice when n is large because it involves the inverse of the matrix C.

(v) In the previous example, we could have assumed that $\mu \equiv 1$ for the price of replacing the pair (ϕ_i, ψ_i) with $(\sqrt{\mu}\ \phi_i, \sqrt{\mu}\ \psi_i)$. Let us assume that the family $\{\phi_i\}_{i=1}^n$, $\{\psi_i\}_{i=1}^n$ are as in part **(iv)** and that $\mu \equiv 1$. We now claim that the example of part **(iv)** can be recast as a conditional determinantal example as in **(ii)**. To explain this, augment the set X to

$$\hat{X} = \{1, \dots, n\} \sqcup X =: [n] \sqcup X.$$

Define $L : \hat{X} \times \hat{X} \to \mathbb{R}$ as follows

$$L(a,b) = \begin{cases} \phi_a(b) & \text{if } \ a \in [n], \ b \in X, \\ \psi_b(a) & \text{if } \ b \in [n], \ a \in X, \\ 0 & \text{otherwise.} \end{cases}$$

Expressing L as a matrix, set

$$\Phi = \big[\phi_i(x)\big]_{(i,x)\in[n]\times X}, \qquad \Psi = \big[\psi_i(x)\big]_{(i,x)\in[n]\times X},$$

then

$$L = \begin{bmatrix} 0 & \Phi \\ \Psi^* & 0 \end{bmatrix}.$$

Note that since Φ and Ψ are not square matrices, it is natural to augment X to $\hat{X}$ and consider the matrix L as above that is a square matrix. Given $\mathbf{a} \subseteq X$,

$$L_{\mathbf{a}\cup[n]} = \begin{bmatrix} 0 & \Phi_{[n],\mathbf{a}} \\ \big(\Psi_{[n],\mathbf{a}}\big)^* & 0 \end{bmatrix}.$$

Or as a function,

$$L_{\mathbf{a}\cup[n]} : \big(\mathbf{a} \cup [n]\big) \times \big(\mathbf{a} \cup [n]\big) \to \mathbb{R},$$

given by

$$L_{\mathbf{a}\cup[n]}(a,b) = \begin{cases} \phi_a(b) & \text{if } \ a \in [n], \ b \in \mathbf{a}, \\ \psi_b(a) & \text{if } \ b \in [n], \ a \in \mathbf{a}, \\ 0 & \text{otherwise.} \end{cases}$$

Let us consider a point process $\mathbf{x}$ in X with

$$\mathbb{P}(\mathbf{x} = \mathbf{a}) = \det\big(\mathbb{1}_X + L\big)^{-1} \ \det L_{\mathbf{a}\cup[n]},$$

as in part **(ii)**. We claim that this is the same point process we studied in part **(iv)** with $\mu(x) = 1$ for all $x \in X$. To see this, first we argue that if $|\mathbf{a}| \neq n$ then $\mathbb{P}(\mathbf{x} = \mathbf{a}) = 0$. For this it suffices to show that the matrix $L_{\mathbf{a}\cup[n]}$ is never invertible whenever $|\mathbf{a}| \neq n$. Indeed if

$$L_{\mathbf{a}\cup[n]} \begin{bmatrix} v \\ w \end{bmatrix} = 0,$$

with $v \in \mathbb{R}^n$, $w \in \mathbb{R}^{|\mathbf{a}|}$, then

$$\Phi_{[n],\mathbf{a}} w = 0, \qquad \Psi^*_{\mathbf{a},[n]} v = 0,$$

and one of these linear equations are under-determined whenever $|\mathbf{a}| \neq n$. Moreover when $|\mathbf{a}| = n$, with $\mathbf{a} = \{a_1, \ldots, a_n\}$, then the determinant of $L_{\mathbf{a}\cup[n]}$ is calculated as

$$\begin{aligned}\det L_{\mathbf{a}\cup[n]} &= (-1)^n \ \det \begin{bmatrix} \Phi_{[n],\mathbf{a}} & 0 \\ 0 & \left(\Psi_{[n],\mathbf{a}}\right)^* \end{bmatrix} \\ &= (-1)^n \ \det \left[\phi_i(a_j)\right]_{i,j=1}^n \det \left[\psi_i(a_j)\right]_{i,j=1}^n,\end{aligned}$$

which yields (3.15) after a normalization. Recall that the correlation kernel $\hat{K}$ is given by (3.14). According to our calculation, $\bar{K}$ is also given by (3.21). It is instructive to see that (3.14) yields (3.21). For this, we first need to invert

$$\mathbb{1}_X + L = \begin{bmatrix} 0 & \Phi \\ \Psi^* & \mathbb{1}_X \end{bmatrix}. \tag{3.22}$$

Observe that if $\Phi\Psi^* =: C = \left[c_{ij}\right]_{i,j=1}^n$, then

$$c_{ij} = \sum_{x\in X} \phi_i(x)\psi_j(x) = \langle \phi_i, \psi_j \rangle,$$

which is exactly what we had in part **(iv)**. To invert $\mathbb{1}_X + L$, observe

$$\left(\mathbb{1}_X + L\right) \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a' \\ b' \end{bmatrix},$$

means $\Phi b = a'$ and $\Psi^* a + b = b'$. Multiplying the second equation by Φ implies $Ca + a' = \Phi b'$. This in turn yields $a = C^{-1}\Phi b' - C^{-1}a'$. From this, we can readily deduce

$$\left(\mathbb{1}_X + L\right)^{-1} = \begin{bmatrix} -C^{-1} & C^{-1}\Phi \\ \Psi^* C^{-1} & \mathbb{1}_X - \Psi^* C^{-1}\Phi \end{bmatrix}. \tag{3.23}$$

As a result

$$\hat{K} = \mathbb{1}_X - \left(\left(\mathbb{1}_X + L\right)^{-1}\right)_X = \mathbb{1}_X - \left(\mathbb{1}_X - \Psi^* C^{-1}\Phi\right) = \Psi^* C^{-1}\Phi,$$

which is exactly what we had in (3.21). Observe that if $\widehat{\Phi} = C^{-1}\Phi$ with

$$\Phi = \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_n \end{bmatrix}, \qquad \widehat{\Phi} = \begin{bmatrix} \hat{\phi}_1 \\ \vdots \\ \hat{\phi}_n \end{bmatrix},$$

then

$$\widehat{\Phi}\Psi^* = \mathbb{1}_{[n]}, \qquad \hat{\phi}_i \in \ span\{\phi_1, \ldots, \phi_n\},$$

for every $i \in [n]$.

(vi) As our next example, we discuss a determinantal process that was studied by Eynard-Mehta. Assume $X = X_1 \sqcup \cdots \sqcup X_N$, and set

$$\mathcal{X}(n) = \left\{ \mathbf{x} : \ \left| \mathbf{x} \cap X_i \right| = n \quad \text{for} \quad i = 1, \ldots, N \right\}.$$

Given functions $\phi_i, \psi_i : X_1 \to \mathbb{R}, \ i = 1, \ldots, n$, and

$$W^i : X_i \times X_{i+1} \to \mathbb{R}, \qquad i = 1, \ldots, n-1,$$

we consider a probability measure on $\mathcal{X}(n)$ such that

$$\mathbb{P}\left(\mathbf{x} \cap X_1 = \left\{ z_1^1, \ldots, z_n^1 \right\}, \ldots, \mathbf{x} \cap X_N = \left\{ z_1^N, \ldots, z_n^N \right\}\right),$$

equals

$$Z^{-1} \ \det \left[\phi_j(z_i^1) \right]_{i,j=1}^n \ \prod_{k=1}^{N-1} \ \det \left[W^k(z_i^k, z_j^{k+1})) \right]_{i,j=1}^n \ \ \det \left[\psi_j(z_i^N) \right]_{i,j=1}^n. \tag{3.24}$$

Here Z is the normalizing constant, and we are assuming that the above expression is non-negative. The interpretation is that the collection the sequence $z^1, \ldots, z^N$ is described by an initial law for z^1, a Markovian kernel $W^i\left(z^i, z^{i+1}\right)$, and a final (conditional) law for the last state z^N. Note that $\mathbb{P}$ is as part **(iii)** or **(iv)** when $N = 1$. We now follow Borodin-Rains [BR] to show that $\mathbb{P}$ is determinantal and it can be formulated as in part **(ii)**. As in **(iv)**, we set $\hat{X} = \{1, \ldots, n\} \cup X$, and consider a matrix $L : \hat{X} \times \hat{X} \to \mathbb{R}$ defined by

$$L(a,b) = \begin{cases} \phi_a(b) & \text{if} \ \ a \in [n], \ b \in X_1, \\ -W^k(a,b) & \text{if} \ \ a \in X_k, \ b \in X_{k+1}, \ k = 1, \ldots, N-1, \\ \psi_b(a) & \text{if} \ \ b \in [n], \ a \in X_N, \\ 0 & \text{otherwise.} \end{cases}$$

Expressing L as a matrix, we have

$$L = \begin{bmatrix} 0 & \Phi & 0 & 0 & \ldots & 0 \\ 0 & 0 & -W^1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & -W^{N-1} \\ \Psi^* & 0 & 0 & 0 & \ldots & 0 \end{bmatrix}.$$

Given $\mathbf{a}_1 \subseteq X_1, \ldots, \mathbf{a}_N \subseteq X_N$,

$$L_{\mathbf{a}_1 \cup \cdots \cup \mathbf{a}_N \cup [n]} = \begin{bmatrix} 0 & \Phi_{[n],\mathbf{a}_1} & 0 & 0 & \ldots & 0 \\ 0 & 0 & -W^1_{\mathbf{a}_1,\mathbf{a}_2} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & -W^{N-1}_{\mathbf{a}_{N-1},\mathbf{a}_N} \\ \left(\Psi_{[n],\mathbf{a}_N}\right)^* & 0 & 0 & 0 & \ldots & 0 \end{bmatrix}.$$

Let us consider a point process $\mathbf{x}$ in X with

$$\mathbb{P}\left(\mathbf{x} \cap X_1 = \mathbf{a}_1, \ldots, \mathbf{x} \cap X_N = \mathbf{a}_N\right) = \det\left(\mathbb{1}_X + L\right)^{-1} \det L_{\mathbf{a}_1 \cup \cdots \cup \mathbf{a}_N \cup [n]},$$

as in part **(ii)**. We claim that this is the same point process we defined by (3.24). To see this, first we argue that if $|\mathbf{a}_i| \neq n$ for some $i = 1, \ldots, N$, then $\mathbb{P}\big(\mathbf{x} = \mathbf{a}_1 \cup \cdots \cup \mathbf{a}_N\big) = 0$. For this it suffices to show that if $\mathbf{a} = \mathbf{a}_1 \cup \cdots \cup \mathbf{a}_N$, the then the matrix $L_{\mathbf{a} \cup [n]}$ is never invertible unless $|\mathbf{a}_1| = \cdots = |\mathbf{a}_N| = n$. Indeed if

$$L_{\mathbf{a} \cup [n]} \begin{bmatrix} v \\ w \\ e^1 \\ \vdots \\ e^{N-1} \end{bmatrix} = 0,$$

with $v \in \mathbb{R}^n$, $w \in \mathbb{R}^{|\mathbf{a}_N|}$, and $e^i \in \mathbb{R}^{|\mathbf{a}_i|}$, for $i = 1, \ldots, N-1$, then

$$\Phi_{[n],\mathbf{a}_1} w = 0, \qquad \Psi^*_{\mathbf{a}_N,[n]} v = 0, \qquad W^k_{\mathbf{a}_k,\mathbf{a}_{k+1}} e^k = 0, \tag{3.25}$$

for $k = 1, \ldots, N-1$. Clearly if $|\mathbf{a}_{k+1}| > |\mathbf{a}_k|$, then the last equation in (3.25) is under-determined and has non-zero solution. So for the invertibility of $L_{\mathbf{a} \cup [n]}$ we must have

$$|\mathbf{a}_1| \geq |\mathbf{a}_2| \geq \cdots \geq |\mathbf{a}_N|.$$

On the other hand, if $|\mathbf{a}_1| > n$, the first equation in (3.25) is under-determined. Hence the invertibility of $L_{\mathbf{a} \cup [n]}$ forces $n \geq |\mathbf{a}_1|$. Finally if $|\mathbf{a}_N| < n$, then the second equation in (3.25) is under-determined. Thus the invertibility of $L_{\mathbf{a} \cup [n]}$ implies

$$|\mathbf{a}_1| = |\mathbf{a}_2| = \cdots = |\mathbf{a}_N| = n.$$

Under this assumption, let us write $\mathbf{a}_i = \left\{a^i_1, \ldots, a^n_n\right\}$. Then the determinant of $L_{\mathbf{a} \cup [n]}$ is calculated as

$$\det L_{\mathbf{a}\cup[n]} = (-1)^{nN+(N-1)} \det \begin{bmatrix} \Phi_{[n],\mathbf{a}_1} & 0 & 0 & \dots & 0 & \\ 0 & W^1_{\mathbf{a}_1,\mathbf{a}_2} & 0 & \dots & 0 & \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & W^{N-1}_{\mathbf{a}_{N-1},\mathbf{a}_N} & 0 \\ 0 & 0 & 0 & \dots & 0 & \left(\Psi_{[n],\mathbf{a}_N}\right)^* \end{bmatrix}$$

$$= (-1)^{n(N+1)-1} \ \det\left[\phi_i(a^1_j)\right]^n_{i,j=1} \prod_{k=1}^{N-1} \det\left[W^k(a^k_i, a^{k-1}_j)\right]^n_{i,j=1} \det\left[\psi_i(a^N_j)\right]^n_{i,j=1},$$

which yields (3.25) after a normalization. Observe that if

$$M := \Phi W^1 \dots W^{N-1}\Psi^*,$$

then the normalizing constant Z in (3.24) is simply $\det M$.

To find the correlation kernel $\hat{K}$, we first need to invert

$$\mathbb{1}_X + L = \begin{bmatrix} 0 & \Phi & 0 & 0 & \dots & 0 & 0 \\ 0 & \mathbb{1}_{X_1} & -W^1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \mathbb{1}_{X_{N-1}} & -W^{N-1} \\ \Psi^* & 0 & 0 & 0 & \dots & 0 & \mathbb{1}_{X_N}. \end{bmatrix}.$$

To invert this matrix, we first write it as

$$\mathbb{1}_X + L = \begin{bmatrix} 0 & A \\ B^* & D \end{bmatrix}, \tag{3.26}$$

where

$$A = \begin{bmatrix} \Phi & 0 & \dots & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & \dots & 0 & \Psi \end{bmatrix},$$

$$D = \begin{bmatrix} \mathbb{1}_{X_1} & -W^1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mathbb{1}_{X_{N-1}} & -W^{N-1} \\ 0 & 0 & 0 & \dots & 0 & \mathbb{1}_{X_N} \end{bmatrix}. \tag{3.27}$$

Note that if we replace D with the identity matrix, then (3.25) becomes (3.22) and we already know that the inverse is given by (3.23). On the other hand,

$$\begin{bmatrix} \mathbb{1}_{[n]} & 0 \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} 0 & A \\ B^* & D \end{bmatrix} = \begin{bmatrix} 0 & A \\ D^{-1}B^* & \mathbb{1}_X \end{bmatrix}$$

is of the form (3.22) with inverse

$$\begin{bmatrix} -C^{-1} & C^{-1}A \\ D^{-1}B^*C^{-1} & \mathbb{1}_X - D^{-1}B^*C^{-1}A \end{bmatrix},$$

with $C = AD^{-1}B^*$. This in turn implies

$$\begin{aligned} \begin{bmatrix} 0 & A \\ B^* & D \end{bmatrix}^{-1} &= \begin{bmatrix} -C^{-1} & C^{-1}A \\ D^{-1}B^*C^{-1} & \mathbb{1}_X - D^{-1}B^*C^{-1}A \end{bmatrix} \begin{bmatrix} \mathbb{1} & 0 \\ 0 & D^{-1} \end{bmatrix} \\ &= \begin{bmatrix} -C^{-1} & C^{-1}AD^{-1} \\ D^{-1}B^*C^{-1} & D^{-1} - D^{-1}B^*C^{-1}AD^{-1} \end{bmatrix}. \end{aligned} \tag{3.28}$$

As a result

$$\widehat{K} = \mathbb{1}_X - \left(\left(\mathbb{1}_X + L \right)^{-1} \right)_X = \mathbb{1}_X - D^{-1} + D^{-1}B^*C^{-1}AD^{-1}. \tag{3.29}$$

For this, we need to find D^{-1}. Observe that since $D = \mathbb{1}_X - E$ with $E^N = 0$, we have

$$D^{-1} = \mathbb{1}_X + E + E^2 + \cdots + E^{N-1}.$$

We may write $E = \left[\delta_{i+1,j} W^i \right]_{i,j=1}^N$, with the entry (i,j) representing a matrix of size $|X_i| \times |X_j|$. We can then write

$$E^r = \left[\delta_{i+r,j} W_{[i,i+r)} \right]_{i,j=1}^N, \qquad \text{where} \qquad W_{[i,i+r)} = W^i W^{i+1} \dots W^{i+r-1}.$$

This leads to

$$D^{-1} = \begin{bmatrix} \mathbb{1}_{X_1} & W_{[1,2)} & W_{[1,3)} & \dots & W_{[1,N-1)} & W_{[1,N)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \mathbb{1}_{X_{N-1}} & W_{[N-1,N)} \\ 0 & 0 & 0 & \dots & 0 & \mathbb{1}_{X_N} \end{bmatrix} =: \mathbb{1}_X + \widehat{W}. \tag{3.30}$$

Moreover

$$C = AD^{-1}B^* = \begin{bmatrix} \Phi & \Phi W_{[1,2)} & \dots & \Phi W_{[1,N)} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \Psi^* \end{bmatrix} = \Phi W_{[1,N)} \Psi^* = M$$

$$B^*C^{-1}A = B^*M^{-1}A = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \Psi^* \end{bmatrix} \begin{bmatrix} M^{-1}\Phi & 0 & \dots & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \\ \Psi^* M^{-1} \Phi & 0 & \dots & 0 \end{bmatrix},$$

where the last matrix is regarded as $N \times N$ matrix with (i,j)-th block of size $|X_i| \times |X_j|$. If we write $D^{-1} = \big[R_{ij}\big]_{i,j=1}^N$ with again (i,j)-th block of size $|X_i| \times |X_j|$, we learn that the (i,j)-th block of $D^{-1}B^*M^{-1}AD^{-1}$ of the form

$$R_{iN}\Psi^*M^{-1}\Phi R_{1j} = W_{[i,N)}\Psi^*M^{-1}\Phi W_{[1,j)},$$

with the convention that $W_{[i,i)} = 1\!\!1_{X_i}$. Finally

$$\widehat{K} = D^{-1}B^*M^{-1}AD^{-1} - \widehat{W}.$$

Its (i,j)-th block is given by

$$\widehat{K}_{ij} = W_{[i,N)}\Psi^*M^{-1}\Phi W_{[1,j)} - W_{[i,j)}1\!\!1(i<j).$$

We note that for every $i \in [N]$, the point process $\mathbf{z}^i = \mathbf{x} \cap X_i$ is also a determinantal process with correlation kernel

$$\widehat{K}_{ii} = W_{[i,N)}\Psi^*M^{-1}\Phi W_{[1.i)},$$

with $M = \Phi W_{[1.N)}\Psi^*$. Writing

$$\Phi_j = \Phi W_{[1,j)}, \qquad \Psi_i^* = W_{[i,N)}\Psi^*,$$

we realize that $M = \Phi_i\Psi_i^*$, and

$$\widehat{K}_{ii} = \Psi_i^*M^{-1}\Phi_i.$$

Hence the law of $\mathbf{z}^i$ is of the type that appeared in part **(i)**. More generally,

$$\widehat{K}_{ij} = \Psi_i^*M^{-1}\Phi_j - W_{[i,j)}1\!\!1(i<j).$$

Assume that W^i is invertible for all i and set

$$W_{[i,j)} := W_{[j,i)}^{-1},$$

for $i > j$ so that $W_{[i,j)}W_{[j,i)} = 1\!\!1$. We then always have

$$\widehat{K}_{ij} = W_{[i,j)}\big(\widehat{K}_{jj} - 1\!\!1(i<j)\big).$$

(vii) Let us assume that $n = 1$ in **(vii)**, $W^i = P$, and $X_i = X$ for all i. The configuration $\mathbf{x} = (x_1, \dots, x_N) \in X^N$ is roughly the state of Markov chain with Markovian kernel $P(x,y)$, initial distribution related to ϕ, and some conditioning at time N that is related to ψ:

$$\mathbb{P}(\mathbf{x}) = Z^{-1}\phi(x_1)\psi(x_N)\prod_{i=1}^{N-1} P(x_i, x_{i+1}).$$

We assume that $P, \phi, \psi \geq 0$, and

$$\sum_{y \in X} P(x, y) = 1, \qquad \sum_{y \in X} \phi(y) = 1,$$

for every x. Then the (i, j)-th block of the correlation function $\widehat{\mathcal{K}}$ is now given by

$$\begin{aligned}
\widehat{K}_{ij}(x, y) &= M^{-1} \sum_{a, b \in X} P^{N-i}(x, a)\psi(a)\phi(b)P^{j-1}(b, y) - P^{j-i}(x, y)\mathbb{1}(i < j) \\
&= M^{-1}\big(P^{N-i}\Psi^*\big)(x)\ \big(\Phi P^{j-1}\big)(y) - P^{j-i}(x, y)\mathbb{1}(i < j),
\end{aligned}$$

where $M = Z$ is the normalizing constant:

$$M = Z = \Phi P^{N-1}\Psi^* = \sum_{a, b \in X} P^{N-1}(a, b)\phi(a)\psi(b).$$

Note that if P is invertible, then

$$\widehat{K}_{ij} = P^{j-i}\big(\widehat{K}_{jj} - \mathbb{1}(i < j)\big).$$

To see this, write $f_i = P^{N-i}\Psi^*$ and $g_j = \Phi P^{j-1}$, so that

$$\begin{aligned}
\widehat{K}_{ij} &= M^{-1}\ f_i \otimes g_j - P^{j-i}\mathbb{1}(i < j) = P^{j-i}\left[M^{-1}\ P^{i-j}\big(f_i \otimes g_j\big) - \mathbb{1}(i < j)\right] \\
&= P^{j-i}\left[M^{-1}\ \big(P^{i-j} f_i \otimes g_j\big) - \mathbb{1}(i < j)\right].
\end{aligned}$$

On the other hand,

$$P^{i-j} f_i = P^{i-j} P^{N-i}\Psi^* = P^{N-j}\Psi^* = f_j,$$

as desired.

We now consider two special cases. For our first case, we assume that $\psi = 1$, so that $\mathbb{P}$ is simply the law of a Markov chain of size N with kernel P and initial law ϕ. In this case $P^r\Psi^* = 1$, $M = 1$, and $\widehat{K}_{ij}$ simplifies to

$$\widehat{K}_{ij}(x, y) = \big(\phi P^{j-1}\big)(y) - P^{j-i}(x, y)\mathbb{1}(i < j).$$

For our second special case, we choose $\phi(x) = \mathbb{1}(x = \bar{a})$ and $\psi(y) = \mathbb{1}(y = \bar{b})$. Then $\mathbb{P}$ represents a Markov chain that is conditioned to start from $\bar{a}$ initially, and arrive at $\bar{b}$ at time N. Now the normalizing constant is

$$M = Z = P^{N-1}(\bar{a}, \bar{b}),$$

and $\widehat{K}_{ij}$ takes the form

$$\widehat{K}_{ij}(x, y) = Z^{-1}\ P^{N-i}(x, \bar{b})\ P^{j-1}(\bar{a}, y) - P^{j-i}(x, y)\mathbb{1}(i < j).$$

(viii) We describe a determinantal process that is very much related to TASEP (see Chapter 5) and was studied by Borodin et al. [BFPS]. We assume there are countable sets $X_1, \dots, X_N$ and a collection of functions

$$W^k : X_k \times X_{k+1} \to \mathbb{R}, \qquad k = 1, \dots, N-1,$$
$$\gamma_i : X_i \to \mathbb{R}, \qquad \psi_i : X_N \to \mathbb{R}, \quad i = 1, \dots, N.$$

Consider a point process $\mathbf{z}$ that lives in the set

$$X = X_1 \sqcup \cdots \sqcup X_N,$$

such that $\mathbf{z} \cap X_i = z^i$ with $z^i = \{z_1^i, \dots, z_i^i\}$, $|z^i| = i$ and

$$\mathbb{P}(\mathbf{z}) = Z^{-1}\ \gamma_1(z_1^1) \prod_{k=1}^{N-1} \det \Phi_k(z^k, z^{k+1})\ \det \left[\psi_j(z_i^N)\right]_{i,j=1}^N, \tag{3.31}$$

where Z is the normalizing constant, and $\Phi_k(z^k, z^{k+1})$ is a $(k+1) \times (k+1)$ matrix that we obtain from the matrix $\left[W^k(z_i^k, z_j^{k+1})\right]_{i \in [k], j \in [k+1]}$ by adding a row (say as the last row) of the form

$$\gamma_{k+1}(z^{k+1}) := \left[\gamma_{k+1}(z_1^{k+1}), \dots, \gamma_{k+1}(z_{k+1}^{k+1})\right].$$

We now claim that the point process given by (3.28) can be recast as part **(ii)**. For this, we set $\hat{X} = [N] \sqcup X$ and define $L : \hat{X} \times \hat{X} \to \mathbb{R}$ as follows

$$L(a,b) = \begin{cases} \delta_{ai}\ \gamma_i(b) & \text{if } \ a \in [N],\ b \in X_i,\ i = 1, \dots, N \\ -W^k(a,b) & \text{if } \ a \in X_k,\ b \in X_{k+1},\ k = 1, \dots, N-1, \\ \psi_b(a) & \text{if } \ b \in [N],\ a \in X_N, \\ 0 & \text{otherwise.} \end{cases}$$

Expressing L as a matrix, we have

$$L = \begin{bmatrix} 0 & E^1 & E^2 & E^3 & \dots & E^N \\ 0 & 0 & -W^1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -W^{N-1} \\ \Psi^* & 0 & 0 & 0 & \dots & 0 \end{bmatrix},$$

where $\Psi = \left[\psi_i(x)\right]_{i \in [N], x \in X_N}$, and E_i has only its i-th row nonzero. Given $\mathbf{a}^1 \subseteq X_1, \dots, \mathbf{a}^N \subseteq X_N$,

$$L_{\mathbf{a}^1 \cup \cdots \cup \mathbf{a}^N \cup [N]} = \begin{bmatrix} 0 & E^1_{[N],\mathbf{a}^1} & E^1_{[N],\mathbf{a}^2} & E^3_{[N],\mathbf{a}^3} & \dots & E^N_{[N],\mathbf{a}^N} \\ 0 & 0 & -W^1_{\mathbf{a}^1,\mathbf{a}^2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -W^{N-1}_{\mathbf{a}^{N-1},\mathbf{a}^N} \\ \left(\Psi_{[N],\mathbf{a}^N}\right)^* & 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Let us consider a point process $\mathbf{z}$ in X with

$$\mathbb{P}'\left(\mathbf{z}\cap X_1=\mathbf{a}^1,\dots,\mathbf{z}\cap X_N=\mathbf{a}^N\right)=\det\left(\mathbb{1}_X+L\right)^{-1}\ \det L_{\mathbf{a}^1\cup\dots\cup\mathbf{a}^N\cup[N]},\tag{3.32}$$

as in part **(ii)**. We claim that this is the same point process we defined by (3.31). To see this, first we argue that if $|\mathbf{a}^i|\neq i$ for some $i=1,\dots,N$, then $\mathbb{P}\big(\mathbf{a}^1\cup\dots\cup\mathbf{a}^N\big)=0$. For this it suffices to show that if $\mathbf{a}=\mathbf{a}^1\cup\dots\cup\mathbf{a}^N$, then the matrix $L_{\mathbf{a}\cup[N]}$ is never invertible unless $|\mathbf{a}^i|=i$ for all $i=1,\dots,N$. Indeed if

$$L_{\mathbf{a}\cup[N]}\begin{bmatrix} v\\ e^1\\ \vdots\\ e^N\end{bmatrix}=0,$$

with $v\in\mathbb{R}^N$, and $e^i\in\mathbb{R}^{|\mathbf{a}^i|}$, for $i=1,\dots,N$, then

$$\sum_{i=1}^N E^i_{[N],\mathbf{a}^i}e^i=0,,\qquad W^k_{\mathbf{a}^k,\mathbf{a}^{k+1}}e^{k+1}=0\qquad \Psi^*_{\mathbf{a}^N,[N]}v=0,\tag{3.33}$$

for $k=1,\dots,N-1$. Note that the last equation is under-determined unless $|\mathbf{a}^N|\geq N$. If we write

$$c^i=\big(\gamma_i(b):\ b\in\mathbf{a}^i\big)\in\mathbb{R}^{|\mathbf{a}^i|},$$

then the first equation in (3.33) means that $c^i\cdot e^i=0$ for $i=1,\dots,N$. For $i=1$, the equation $c^1\cdot e^1=0$ is under-determined unless $|\mathbf{a}^1|\leq 1$. Moreover, for $k=1,\dots,N-1$, pair of equations

$$c^{k+1}\cdot\mathbf{a}^{k+1}=0,\qquad W^k_{\mathbf{a}^k,\mathbf{a}^{k+1}}e^{k+1}=0,$$

is a system of $|\mathbf{a}^k|+1$ equations, and this system is under-determined unless $|\mathbf{a}^{k+1}|\leq|\mathbf{a}^k|+1$. From this and $|\mathbf{a}^1|\leq 1$ we deduce $|\mathbf{a}^k|\leq k$ for $k=1,\dots,N$. This, and $|\mathbf{a}^N|\geq N$ yields $|\mathbf{a}^N|=N$. From this and $|\mathbf{a}^{k+1}|\leq|\mathbf{a}^k|+1$, and a backward induction (starting from $k=N$) we can readily show that $|\mathbf{a}^k|=k$ for $k=1,\dots,N$. In summary, $\mathbb{P}'(\mathbf{z})=0$ unless $|\mathbf{z}\cap X_k|=|\mathbf{a}^k|=k$. Assuming this, we now examine $\det L_{\mathbf{z}\cup[N]}$. Let us write $\mathbf{a}^k=\big(z_1^k,\dots,z_k^k\big)$. Observe that $E^1=\gamma_1(z_1^1)$. Also observe that the second row is the vector $\big(0,0,\gamma_2(z_1^2),\gamma_2(z_2^2),0,\dots,0\big)$. We place this row below $-W^2_{\mathbf{a}^1,\mathbf{a}^2}$ and drop the minus sign. This action turn the matrix $-W^2$ to $\Phi_2\big(\mathbf{a}^1,\mathbf{a}^2\big)$. Similarly place the k-th below $-W^k_{\mathbf{a}^k,\mathbf{a}^{k+1}}$ and drop the minus sign. From all these actions we deduce

$$\det L_{\mathbf{a}^1\cup\dots\cup\mathbf{a}^N\cup[N]}=\pm\det\begin{bmatrix} 0 & \gamma_1(z_1^1) & 0 & 0 & \dots & 0\\ 0 & 0 & \Phi_2\big(\mathbf{a}^1,\mathbf{a}^2\big) & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & \dots & \Phi_{N-1}\big(\mathbf{a}^{N-1},\mathbf{a}^N\big)\\ \big(\Psi_{[N],\mathbf{a}^N}\big)^* & 0 & 0 & 0 & \dots & 0\end{bmatrix}.$$

This matches $\mathbb{P}$ except for the normalizing constant. To find the correlation kernel $\hat{K}$, we first need to invert

$$\mathbb{1}_X + L = \begin{bmatrix} 0 & E^1 & E^2 & E^3 & \dots & E^{N-1} & E^N \\ 0 & \mathbb{1}_{X_1} & -W^1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \mathbb{1}_{X_{N-1}} & -W^{N-1} \\ \Psi^* & 0 & 0 & 0 & \dots & 0 & \mathbb{1}_{X_N} \end{bmatrix} = \begin{bmatrix} 0 & A \\ B^* & D \end{bmatrix},$$

where D is as in (3.27), and

$$A = \begin{bmatrix} E^1 & E^2 & \dots & E^N \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & \dots & 0 & \Psi \end{bmatrix}.$$

Then $\left(\mathbb{1}_X + L\right)^{-1}$ is given by (3.28) with $C = AD^{-1}B^*$. This in turn implies

$$\widehat{K} = \mathbb{1}_X - D^{-1} + D^{-1}B^*C^{-1}AD^{-1}.$$

as in (3.29) with D^{-1} as in (3.30). Observe

$$\begin{aligned} AD^{-1} &= \begin{bmatrix} E^1 & E^1 W_{[1,2)} + E^2 & \dots & \sum_{i=1}^{N-1} E^i W_{[i,N)} + E^N \end{bmatrix} \\ &=: \begin{bmatrix} \Lambda_1 & \Lambda_2 & \dots & \Lambda_N \end{bmatrix} := \Lambda, \\ D^{-1}B^* &= \begin{bmatrix} W_{[1,N)}\Psi^* \\ W_{[2,N)}\Psi^* \\ \vdots \\ W_{[N-1,N)}\Psi^* \\ \Psi^* \end{bmatrix} = \begin{bmatrix} W_{[1,N)} \\ W_{[2,N)} \\ \vdots \\ W_{[N-1,N)} \\ W_{[N,N)} \end{bmatrix} \Psi^* =: \Gamma\Psi^*, \end{aligned}$$

where we use the convention $W_{[i,i)} = \mathbb{1}_{X_i}$. On the other hand,

$$C = AD^{-1}B^* = \Lambda B^* = \Lambda_N \Psi^*. \tag{3.34}$$

As a result,

$$\widehat{K} = \Gamma\Psi^* C^{-1}\Lambda - \widehat{W}, \qquad K_{ij} = W_{[i,N)}\Psi^* C^{-1}\Lambda_j - \mathbb{1}(i<j)W_{[i,j)}. \tag{3.35}$$

where $K_{ij} : X_i \times X_j \to \mathbb{R}$ denotes the (i,j)-th block of $\hat{K}$. Here $\Gamma : X \times X_N \to \mathbb{R}$, $C : [N] \times [N] \to \mathbb{R}$ is an $N \times N$ matrix, $\Lambda_i : [N] \times X_i \to \mathbb{R}$, and $\Lambda : [N] \times X \to \mathbb{R}$, so that the right-hand side of the first display is from $X \times X$ to $\mathbb{R}$. In practice, the main challenge comes from inverting C (compare with part **(vi)** and **(v)**), or more specifically, evaluating

$$\Phi_j := C^{-1}\Lambda_j : [N] \times X_j \to \mathbb{R}.$$

Note that for sure we have

$$\Phi_N \Psi^* = C^{-1}\Lambda_N \Psi^* = C^{-1}C = 1\!\!1_{[N]}.$$

In other words Φ_N is a left inverse of Ψ^*. $\square$

We continue with some general facts about determinantal processes. The following result is due to Shirai and Takahashi [TH]:

Proposition 3.2 *Assume that $K : X \times X \to \mathbb{R}$ is symmetric with $0 \le K \le 1$. Then there is a unique determinantal process with correlation kernel K.*

Proof *(Step 1.)* Let $M, M' : A \times A \to \mathbb{R}$ be two symmetric matrices such that $MM' = M'M$, and $M, M' \ge 0$. We then claim that for every $\mathbf{a} \subseteq A$,

$$\det\big(\chi_{A\setminus\mathbf{a}} M' + \chi_{\mathbf{a}} M\big) \ge 0. \tag{3.36}$$

By continuity, it suffices to establish (3.36) when $M > 0$. Indeed

$$\det\big(\chi_{A\setminus\mathbf{a}} M' + \chi_{\mathbf{a}} M\big) = \det M \ \det\big(\chi_{A\setminus\mathbf{a}} M'M^{-1} + \chi_{\mathbf{a}} 1\!\!1_A\big) = \det M \ \det\big(M'M^{-1}\big)_{A\setminus\mathbf{a}}.$$

On the other hand, since $M' \ge 0, M > 0$, and

$$M'M^{-1} = M^{-1/2}M'M^{-1/2} \ge 0,$$

we deduce that $\det\big(M'M^{-1}\big)_{A\setminus\mathbf{a}}$, which in turn implies (3.36).

(Step 2.) Fix a nonempty finite set $A \subseteq X$. Motivated by (3.10), we define $\mathbb{P}_A : 2^A \to \mathbb{R}$ by the formula

$$\mathbb{P}_A(\mathbf{a}) = \det\big(\chi_{A\setminus\mathbf{a}}(1\!\!1 - K)_A + \chi_{\mathbf{a}} K_A\big).$$

From (3.36) we learn that $\mathbb{P}_A(\mathbf{a}) \ge 0$. Moreover, by (3.4),

$$\sum_{\mathbf{a}\subseteq A} \mathbb{P}_A(\mathbf{a}) = \det\big((1\!\!1 - K)_A + K_A\big) = 1.$$

As a result, $\mathbb{P}_A$ is a probability measure. On the other hand, we may use (3.10) to assert that for any $\mathbf{c} \subseteq A$,

$$\begin{aligned}\mathbb{P}_A\big(\mathbf{a} : \ \mathbf{c} \subseteq \mathbf{a}\big) &= \sum_{\mathbf{c}\subseteq\mathbf{a}} \sum_{\mathbf{a}\subseteq\mathbf{b}\subseteq A} (-1)^{|\mathbf{b}|-|\mathbf{a}|} K_{\mathbf{b}} = \sum_{\mathbf{c}\subseteq\mathbf{b}\subseteq A} (-1)^{|\mathbf{b}|-|\mathbf{c}|} K_{\mathbf{b}} \sum_{\mathbf{a}} (-1)^{|\mathbf{c}|-|\mathbf{a}|} 1\!\!1\big(\mathbf{c} \subseteq \mathbf{a} \subseteq \mathbf{b}\big) \\ &= \sum_{\mathbf{c}\subseteq\mathbf{b}\subseteq A} (-1)^{|\mathbf{b}|-|\mathbf{c}|} K_{\mathbf{b}} \sum_{\mathbf{a}'\subseteq\mathbf{b}\setminus\mathbf{c}} (-1)^{|\mathbf{a}'|} = \sum_{\mathbf{c}\subseteq\mathbf{b}\subseteq A} (-1)^{|\mathbf{b}|-|\mathbf{c}|} K_{\mathbf{b}} \ 1\!\!1\big(\mathbf{b} \setminus \mathbf{c} = \emptyset\big) = K_{\mathbf{c}}.\end{aligned}$$

Hence $\mathbb{P}_A$ is the law of a determinantal process with correlation kernel K_A.

(Step 3.) We are done if we can verify the consistency for the family $\big(\mathbb{P}_A: \ A \subseteq X\big)$. Indeed if $\mathbf{a} \subseteq A \subset A'$, then the restriction of $\mathbb{P}_{A'}$ to A yields a determinantal process with kernel K_A which coincide with $\mathbb{P}_A$. $\square$

In fact Proposition 3.2 is closely related to Example 3.2**(i)**. Given a determinantal process with correlation kernel K, we may wonder whether or not it is as in Example 3.2**(i)** for some $L : X \times X \to \mathbb{R}$. In view of (3.12), we may try to find L for which (3.12) holds:

$$L = (\mathbb{1} - K)^{-1} - \mathbb{1} = (\mathbb{1} - K)^{-1}K. \tag{3.37}$$

This is certainly well-defined if $\mathbb{1} - K$ is invertible. In particular whenever $K < \mathbb{1}$ holds. If we assume that $0 < K < \mathbb{1}$, then the corresponding determinantal process is a Gibbs measure (possibly with long correlation) as was observed by Shirai and Takahashi [TH]. Roughly speaking, we need to calculate

$$\frac{\mathbb{P}\big(\mathbf{x} = \mathbf{a} \cup \{y\}\big)}{\mathbb{P}(\mathbf{x} = \mathbf{a})} = \lim_{Y \to X} \frac{\mathbb{P}\big(\mathbf{x} \cap Y = (\mathbf{a} \cup \{y\}) \cap Y\big)}{\mathbb{P}(\mathbf{x} \cap Y = \mathbf{a} \cap Y)},$$

where Y is a finite subset of X that increases. $\square$

When K is symmetric and $0 \leq K \leq 1$, then we may use the eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots$ and the corresponding eigenfunctions $\phi_1, \phi_2, \dots$ to express

$$\begin{aligned} K(x,y) &= \sum_{i=1}^n \lambda_i \phi_i(x)\phi_i(y) = \sum_{i=1}^n \lambda_i \big(\phi_i \otimes \phi_i\big)(x,y), \\ &\sum_{x \in X} \phi_i(x)\phi_j(x) = \mathbb{1}(i = j), \end{aligned} \tag{3.38}$$

where $n = |X|$ is the size of the space X. It was observe by Hough et al. [HKPV] that the total number of particles have a simple description in terms of the eigenvalues: If we write $\zeta_1, \zeta_2, \dots,$ for a sequence of Bernoulli random variables (i.e., with values in $\{0,1\}$) with $\mathbb{P}(\zeta_i = 1) = \lambda_i$, then $N = |\mathbf{x}|$ has the same distribution as $\sum_i \zeta_i$. In particular, if K is the correlation kernel for a determinantal process such that N is fixed and deterministic, then we much have that there are exactly N many eigenvalues 1, and the remaining eigenvalues are 0. In this case, (3.38) becomes

$$K(x,y) = \sum_{i=1}^N \phi_i(x)\phi_i(y), \qquad \sum_{x \in X} \phi_i(x)\phi_j(x) = \mathbb{1}(i = j), \quad i, j \in \{1, \dots, N\}. \tag{3.39}$$

In short

$$K = \Phi^*\Phi, \qquad \Phi\Phi^* = \mathbb{1}_{[N]},$$

where $\Phi = \big[\phi_i(x)\big]_{(i,x)\in[N]\times X}$. This means that the kernel K is associated with the projection operator onto the span of the eigenfunctions $\phi_1, \dots, \phi_N$, and

$$\mathbb{P}(\mathbf{x}) = \Big(\det\big[\phi_i(x_j)\big]_{i,j=1}^N\Big)^2.$$

The projection operator K is uniquely determined from its range $\Pi_N = span\{\phi_1, \dots, \phi_N\}$. In other words, if X is a discrete set and $L^2(X)$ is the set of functions $f : X \to \mathbb{R}$ and equipped with the inner product

$$\langle f, g\rangle = \sum_x f(x)g(x),$$

then there is a one-to-one correspondence between determinantal point processes with symmetric kernel of exactly N particles, and N-dimensional linear subspaces of $L^2(X)$. Given such a subspace, any orthonormal basis $\{\phi_1, \dots, \phi_N\}$, yields a representation as in (3.39). Let us write $\mathbb{P}_\Pi$ for the corresponding probability measure and $\mathbf{x} = \{x_1, \dots, x_N\}$ for the corresponding determinantal process. It is worth mentioning that $\mathbf{x} = (x_1, \dots, x_N)$-coordinate of the N-vector $\phi_1 \wedge \cdots \wedge \phi_N$ is exactly

$$\big(\phi_1 \wedge \cdots \wedge \phi_N\big)(\mathbf{x}) = \det\big[\phi_i(x_j)\big].$$

Hence $\mathbb{P}_\Pi(\mathbf{x}) = \big(\phi_1 \wedge \cdots \wedge \phi_N\big)(\mathbf{x})^2$.

Proposition 3.3 *Given distinct point* $\mathbf{a} := \{a_1, \dots, a_k\} \subset X$*, with* $0 < k < N$*, and an* N*-dimensional linear subspace* $\Pi = \Pi_N = \big\{\phi_1, \dots, \phi_N\big\}$*, we have*

$$\mathbb{P}_\Pi\big(\cdot \mid \mathbf{a} \subset \mathbf{x}\big) = \mathbb{P}_{\widehat{\Pi}}, \tag{3.40}$$

where $\widehat{\Pi}$ *is the orthogonal complement of*

$$\Pi_k = span\big\{K(\cdot, a_1), \dots, K(\cdot, a_k)\big\} =: \big\{\gamma_1, \dots, \gamma_k\big\},$$

with K *as in* (3.39).

Proof Given $\mathbf{x}$, let us write

$$\gamma_i = K(\cdot, x_i) = K\delta_{x_i},$$

where $\delta_a(b) = 1\!\!1(a = b)$. Obviously

$$\langle \gamma_i, \gamma_j\rangle = \langle \delta_{x_i}, \Pi\delta_{x_j}\rangle = K(x_i, x_j).$$

From this and Gram's identity (A.6),

$$\mathbb{P}_\Pi(\mathbf{x}) = \det\big[K(x_i, x_j)\big]_{i,j=1}^N = \det\big[\langle\gamma_i, \gamma_j\rangle\big]_{i,j=1}^N = \big|\gamma_1 \wedge \cdots \wedge \gamma_N\big|^2 = \big|\Pi\delta_{x_1} \wedge \cdots \wedge \Pi\delta_{x_N}\big|^2.$$

Let $\Pi_{N-1} = \Pi_N \cap \gamma_N^{\perp}$. Also write K_{N-1} for the orthogonal projection onto Π_{N-1}. Set $\zeta_i = K_{N-1}\gamma_i$. Evidently,

$$\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_N = \gamma_1 \wedge \zeta_2 \wedge \cdots \wedge \zeta_N.$$

From this and Gram's identity, we deduce

$$\left|\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_N\right|^2 = \left|\gamma_1 \wedge \zeta_2 \wedge \cdots \wedge \zeta_N\right|^2 = \det \begin{bmatrix} A & 0 \\ 0 & |\gamma_1|^2 \end{bmatrix} = |\gamma_1|^2 \det A,$$

where $A = \left[\langle \zeta_i, \zeta_j \rangle\right]_{i,j=1}^{N-1}$. On the other hand, since

$$\zeta_i = K_{N-1}\gamma_i = K_{N-1}K_N\delta_{x_i} = K_{N-1}\delta_{x_i},$$

we learn

$$\det A = \det \left[K_{N-1}(x_i, x_j)\right]_{i,j=1}^{N-1}$$

As we fix x_N and vary $x_{N-1}, \dots, x_1$, we deduce that the process $\{x_1, \dots, x_{N-1}\}$ is a point process associated with the linear subspace $\Pi_{N-1} = \Pi_{N-1}(x_N)$. In summary

$$\mathbb{P}_\Pi(\mathbf{x}) = |\gamma_1|^2 \, \mathbb{P}_{\Pi_{N-1}(x_N)}\left(\{x_1, \dots, x_{N-1}\}\right) = \mathbb{P}_\Pi\big(x_N \in \mathbf{x}\big)\mathbb{P}_{\Pi_{N-1}(x_N)}\left(\{x_1, \dots, x_{N-1}\}\right). \tag{3.41}$$

This implies (3.40) when $k = 1$. The general case can be established by an induction on k. $\square$

So far we have assumed that K is symmetric. For a non-symmetric K, we may use left and right eigenfunctions to guarantee *biorthogonality*. More precisely, we search for a collection of pairs $\big((\phi_i, \psi_i) : \ i = 1, \dots, n\big)$ such that

$$K\phi_i = \lambda_i \phi_i, \qquad \phi_i K = \lambda_i \psi_i.$$

The point is that if $\lambda_i \neq \lambda_j$, then $\langle \phi_i, \psi_j \rangle = 0$. We may try to find a representations of K as

$$\begin{aligned} K &= \sum_{i=1}^n \lambda_i\big(\phi_i \otimes \psi_i\big), \\ &\sum_{x \in X} \phi_i(x)\psi_j(x) = \mathbb{1}(i = j). \end{aligned} \tag{3.42}$$

In the case of a constant size N for the configuration, the analog of (3.39) is

$$K(x, y) = \sum_{i=1}^N \phi_i(x)\psi_i(y), \qquad \sum_{x \in X} \phi_i(x)\psi_j(x) = \mathbb{1}(i = j), \quad i, j \in \{1, \dots, N\}. \tag{3.43}$$

This is exactly our example Example 3,2**(iii)**. The operator K is a projection onto Π_N, the span of $\phi_1, \dots, \phi_N$. Let us also define $\Pi'_N := span\{\psi_1, \dots, \psi_N\}$: For every $f \in L^2(X)$,

$$\begin{aligned} Kf &\in \Pi_N, & (f - Kf) &\perp \Pi'_N, \\ fK &\in \Pi'_N, & (f - fK) &\perp \Pi_N. \end{aligned}$$

In fact the operator $K = K_{\Pi,\Pi'}$ is uniquely defined by the the property:

$$f \in L^2(X) \quad \Longrightarrow \quad Kf \in \Pi_N, \quad fK \in \Pi'_N,$$

(Note that the second condition is equivalent to $(f - Kf) \perp \Pi'_N$.) As a consequence if

$$\Pi \subset \widehat{\Pi}, \quad \Pi' \subset \widehat{\Pi}' \quad \Longrightarrow \quad K_{\widehat{\Pi},\widehat{\Pi}'}\, K_{\Pi,\Pi'} = K_{\Pi,\Pi'}\, K_{\widehat{\Pi},\widehat{\Pi}'} = K_{\Pi,\Pi'}. \tag{3.44}$$

and we write K and $\hat{K}$ for the corresponding projection, then

Observe

$$\begin{aligned} (KK^*)(x,y) = \sum_z K(x,z)K(z,y) &= \sum_{i,j=1}^N \sum_z \phi_i(x)\psi_i(z)\phi_j(z)\psi_j(y) \\ &= \sum_{i=1}^N \phi_i(x)\psi_i(y) = K(x,y). \end{aligned}$$

Also observe

$$\sum_x K(x,x) = N.$$

In fact the operator K is determined uniquely by the two N-dimensional subspaces Π_N and Π'_N. Once they are given, we then select a basis $\{\phi_1, \dots, \phi_N\}$ for Π_N, and find $\psi_1, \dots, \psi_N \in \Pi'_N$ such that the second equation in (3.40) is true. This specifies $\psi_1, \dots, \psi_N$ uniquely. We write $\mathbb{P}_{\Pi,\Pi'}$ for the corresponding determinantal process.

The following construction of the corresponding point process $\mathbf{x}$ gives a Gibbsian flavor to such determinantal process:

Step 1. Pick a point $x = x_N$ according to the probability measure $N^{-1}K(x,x)$.

Step 2. Take the complement of $\gamma_n(x) = K(x, x_n)$ in Π_N:

$$\Pi_{N-1} = \big\{ f \in \Pi_N : \ f - \gamma_b \in \Pi'_N \big\}.$$

Define K_{N-1} as the ψ-projection onto Π_{N-1}.

Step 3. Pick x_{N-1} according to $(N-1)^{-1}K_{N-1}(x,x)$.

Step 4. Continue inductively to construct the sequence $x_N, \dots, x_1$. □

The above algorithm produces the desired determinantal process because of the following generalization of Proposition 3.3.

Proposition 3.4 *Given distinct point* $\mathbf{a} := \{a_1, \ldots, a_k\} \subset X$*, with* $0 < k < N$*, and an* N*-dimensional linear subspaces*

$$\Pi = \Pi_N = \big\{\phi_1, \ldots, \phi_N\big\}, \qquad \Pi' = \Pi'_N = \big\{\psi_1, \ldots, \psi_N\big\}.$$

we have

$$\mathbb{P}_\Pi\big(\cdot \ \mid \mathbf{a} \subset \mathbf{x}\big) = \mathbb{P}_{(\widehat{\Pi}, \widehat{\Pi}')}, \tag{3.45}$$

where $\widehat{\Pi}$ *and is a suitable complement of*

$$\Pi_k = span\big\{K(\cdot, a_1), \ldots, K(\cdot, a_k)\big\} =: \big\{\gamma_1, \ldots, \gamma_k\big\},$$

with K *as in* (3.39).

Proof Given $\mathbf{x}$, let us define $\gamma_i = \delta_{x_i} K, \gamma'_i = K\delta_{x_i}$. We certainly have

$$\langle \gamma_i, \gamma'_j \rangle = \sum_x \left(\sum_{r=1}^N \phi_r(x_i)\psi_r(x) \right) \left(\sum_{r'=1}^N \phi_{r'}(x)\psi_{r'}(x_j) \right) = K(x_i, x_j).$$

From this and Gram's identity (A.6),

$$\mathbb{P}_\Pi(\mathbf{x}) = \det\big[K(x_i, x_j)\big]_{i,j=1}^N = \det\big[\langle \gamma_i, \gamma'_j \rangle\big]_{i,j=1}^N = \langle \gamma_1 \wedge \cdots \wedge \gamma_N, \gamma'_1 \wedge \cdots \wedge \gamma'_N \rangle.$$

Set

$$\Pi_{N-1} = \Pi_N \ \cap (\gamma'_N)^\perp, \qquad \Pi'_{N-1} = \Pi'_N \ \cap \gamma_N^\perp.$$

We also define K_{N-1} for the projection associated with the pair $\big(\Pi_{N-1}, \Pi'_{N-1}\big)$. We now set

$$\zeta_i = K_{N-1}\gamma_i = K_{N-1}K_N\delta_{x_i} = K_{N-1}\delta_{x_i}, \qquad \zeta'_i = \gamma'_i K_{N-1} = \gamma'_i K_{N-1}K_N = \delta_{x_i}K_{N-1},$$

where we have used (3.44) for the third equalities. Evidently,

$$\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_N = \gamma_1 \wedge \zeta_2 \wedge \cdots \wedge \zeta_N, \qquad \gamma'_1 \wedge \gamma'_2 \wedge \cdots \wedge \gamma'_N = \gamma'_1 \wedge \zeta'_2 \wedge \cdots \wedge \zeta'_N.$$

From this and Gram's identity, we deduce

$$\begin{aligned} \langle \gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_N, \gamma'_1 \wedge \gamma'_2 \wedge \cdots \wedge \gamma'_N \rangle &= \langle \gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_N, \gamma'_1 \wedge \zeta'_2 \wedge \cdots \wedge \zeta'_N \rangle \\ &= \det \begin{bmatrix} \langle \gamma_1, \gamma'_1 \rangle & 0 \\ 0 & A \end{bmatrix} = |\gamma_1|^2 \det A, \end{aligned}$$

where $A = \big[\langle \zeta_i, \zeta'_j \rangle\big]_{i,j=1}^{N-1}$. On the other hand,

$$\det A = \det\big[K_{N-1}(x_i, x_j)\big]_{i,j=1}^{N-1}$$

As we fix x_N and vary $x_{N-1}, \dots, x_1$, we deduce that the process $\{x_1, \dots, x_{N-1}\}$ is a point process associated with the pair of linear subspaces $\big(\Pi_{N-1}, \Pi'_{N-1}\big) = \big(\Pi_{N-1}(x_N), \Pi'_{N-1}(x_N)\big)$. In summary

$$\begin{aligned} \mathbb{P}_{\Pi_N,\Pi'_N}(\mathbf{x}) &= \langle \gamma_1, \gamma'_1\rangle \ \mathbb{P}_{\Pi_{N-1}(x_N),\Pi'_{N-1}(x_N)}\left(\{x_1, \dots, x_{N-1}\}\right) \\ &= \mathbb{P}_{\Pi_N,\Pi'_N}\big(x_N \in \mathbf{x}\big) \ \mathbb{P}_{\Pi_{N-1}(x_N),\Pi'_{N-1}(x_N)}\left(\{x_1, \dots, x_{N-1}\}\right). \end{aligned} \tag{3.46}$$

This implies (3.44) when $k = 1$. The general case can be established by an induction on k. $\square$

Exercise

(i) Verify (3.3).

(ii) In Example 3.2(vi) consider the following scenario:

- $X_i = 2\mathbb{Z}$ is the set of even integers for i even, and $X_i = 2\mathbb{Z}+1$ is the set of odd integers for i odd.
- $\mathcal{X}_i = \big\{(z_1, \dots, z_n) \in X_i^n : \ z_1 < \dots < z_n\big\}$.
- All $W^k = P$ are equal with $P(a, b) = p_-1\!\!1(b = a - 1) + p_+1\!\!1(b = a + 1)$.
- $\phi_i(x) = 1\!\!1(x = a_i)$ and $\psi_j(y) = 1\!\!1(y = b_j)$ for some $(a_1, \dots, a_n) \in \mathcal{X}_1$ and $(b_1, \dots, b_n) \in \mathcal{X}_N$.

Given $\mathbf{z} = \big(\mathbf{z}^1, \dots, \mathbf{z}^N\big)$, with $\mathbf{z}^i = (z^i_1, \dots, z^i_n) \in \mathcal{X}_i,\ i \in [N]$, $\mathbf{z}^1 = \mathbf{a}, \mathbf{z}^N = \mathbf{b}$, set

$$\mathbb{P}(\mathbf{z}) = Z^{-1} \prod_{i=1}^{N-1} \det \left[W(z^i_j, z^{i+1}_k)\right]_{j,k=1}^N.$$

(1) Show that if $\det \left[P(z^i_j, z^{i+1}_k)\right]_{j,k=1}^N \neq 0$, then

$$\det \left[P(z^i_j, z^{i+1}_k)\right]_{j,k=1}^N = \prod_{j=1}^{n} P(z^i_j, z^{i+1}_j),$$

and $\mathbb{P}$ is the law of n walks that are conditioned on non-intersecting, and $\mathbf{z}^1 = \mathbf{a}, \mathbf{z}^N = \mathbf{b}$.
(2) Show that the corresponding M is $\left[P^{N-1}(a_i, b_j)\right]_{i,j=1}^N$.
(3) Show that for each $i \in [N]$, the point process $\mathbf{z}^i$ is determinantal with correlation kernel

$$\hat{K}_i(x, y) = \sum_{r,s=1}^{N} P^{N-i}(x, b_r) M^{-1}(b_r, a_s) P^{i-1}(a_s, y).$$

(4) Conclude that the point process $\mathbf{z}^i$ is of the same type that appeared in Example 3.2**(v)** for $\Phi = \left[P^{N-i}(x, b_r)\right]_{r=1}^{N}$ and $\Psi = \left[P^{i-1}(a_s, y)\right]_{s=1}^{N}$.

(iii) Let X be a discrete set. Assume that $x_1, x_2, \ldots, x_n, \ldots$ is a Markov chain in X with initial distribution π, and transition matrix $P : X \times X \to \mathbb{R}$. Assume that this Markov chain does not visit any point twice so that the matrix

$$Q = \sum_{k=1}^{\infty} P^k,$$

is finite. Let $\mathbf{x} = \{x_n : \ n \in \mathbb{N}\}$ be a realization of the above Markov chain. Express the correlation function of $\mathbf{x}$ in terms of Q and show that $\mathbf{x}$ is determinantal with correlation kernel

$$K(x, y) = \pi(x) + (\pi Q)(x) - Q(y, x).$$

$\square$

4 SEP and Bethe Ansatz

Totally Asymmetric Exclusion Process (TASEP) is a particle system on $\mathbb{Z}$ that belongs to KPZ universality class. It is an exactly solvable model and has been used to obtain various information about solutions to KPZ equation. For a finite TASEP with exactly N particles, we may choose the state space

$$E_N = \{\mathbf{x} = (x_1, \dots, x_N) \in \mathbb{Z}^N : \ x_1 < \cdots < x_N\}.$$

The process $\mathbf{x}(t)$ is a Markov process with the infinitesimal generator

$$\mathcal{L}f(\mathbf{x}) = \sum_i \mathbb{1}\big(\mathbf{x}_i \in E_N\big)\big(f(\mathbf{x}_i) - f(\mathbf{x})\big),$$

where $\mathbf{x}_i$ denotes the state we obtain from $\mathbf{x}$ by moving the i-the particle from its location x_i to the new location x_i+1. It is also useful to think of x_i as the height above i so that the lattice function $i \mapsto x_i$ is a strictly increasing function. The height differences $\zeta_i(t) := x_{i+1}(t) - x_i(t)$ evolve as a Markov process and is an example of a family of particle systems known as Zero Range Processes. TASEP is exactly solvable because there is a rather simple explicit formula for its tansition probability that was derived by Schütz. This derivation is based on a useful trick known as *Bethe Ansatz* that by initiated by Bethe for some classical models in quantum mechanics. To explain this, let us write $P(\mathbf{x}, t; \mathbf{y}) = P(\mathbf{x}, t)$ for the probability of $\mathbf{x}(t) = \mathbf{x}$, conditioned that $\mathbf{x}(0) = \mathbf{y}$. Then P, as a function of $(\mathbf{x}, t)$ solves the forward equation

$$P_t(\mathbf{x}, t) = \mathcal{L}^* P(\mathbf{x}, t) = \sum_i \mathbb{1}\big(\mathbf{x}^i \in E_N\big)\big(P(\mathbf{x}^i, t) - P(\mathbf{x}, t)\big), \tag{4.1}$$

where $\mathbf{x}^i$ denotes the state we obtain from $\mathbf{x}$ by moving the i-th particle from its location x_i to its new location $x_i - 1$. The proof of (4.1) is a consequence of the identity

$$\sum_{\mathbf{x} \in E_N} \sum_i \mathbb{1}\big(x_i + 1 < x_{i+1}\big) f(\mathbf{x}_i) P(\mathbf{x}) = \sum_{\mathbf{x} \in E_N} \sum_i \mathbb{1}\big(x_{i-1} < x_i - 1\big) f(\mathbf{x}) P(\mathbf{x}^i).$$

Let us define

$$Df(x) = f(x-1) - f(x).$$

We also write $D_i f(\mathbf{x})$ when we apply D on the i-th variable x_i. According to Bethe Ansatz, there is an extension $Q : \mathbb{Z}^N \times [0, \infty) \to \mathbb{R}$ of $P : E_N \times [0, \infty) \to \mathbb{R}$, such that Q solves the free equation

$$Q_t = \sum_{i=1}^N D_i Q := \mathbf{D}Q, \tag{4.2}$$

subject to the boundary equation

$$\mathbf{x} \in E_N, \quad \mathbf{x}^i \notin E_N \implies D_i Q(\mathbf{x}, t) = 0, \tag{4.3}$$

for every $i \in \{2, \dots, N\}$. In other words,

$$Q(\dots, x_i - 1, x_i, \dots, t) = Q(\dots, x_i, x_i, \dots, t),$$

whenever $(\dots, x_i - 1, x_i, \dots) \in E_N$. The point is that when the boundary condition (4.3) is satisfied, it is harmless to add

$$1\!\!1\big(x_i = x_{i+1} - 1\big)\big(Q(\mathbf{x}_{i+1}, t) - Q(\mathbf{x}, t)\big),$$

to the right-hand side of (4.1), because it is 0 when $\mathbf{x} \in E_N$. Indeed if Q satisfies (4.2) and (4.3), then P, the restriction of Q to E_N, satisfies (4.1). Indeed for every $\mathbf{x} \in E_N$,

$$\begin{aligned} P_t(\mathbf{x}, t) = Q_t(\mathbf{x}, t) = \mathbf{D}Q(\mathbf{x}, t) &= \sum_{i=1}^N D_i Q(\mathbf{x}, t) \\ &= \sum_{i=1}^N 1\!\!1\big(\mathbf{x}^i \in E_N\big)) D_i Q(\mathbf{x}, t) + \sum_{i=1}^N 1\!\!1\big(\mathbf{x}^i \notin E_N\big)) D_i Q(\mathbf{x}, t) \\ &= \sum_{i=1}^N 1\!\!1\big(\mathbf{x}^i \in E_N\big)) D_i P(\mathbf{x}, t) = \mathcal{L}^* P(\mathbf{x}, t). \end{aligned}$$

Here we used (4.3) to assert that each summand in the second sum on second line is zero.

In summary, we now need to solve the free equation (4.2) for the price of some boundary conditions given by (4.3). Schütz found an explicit formula for (4.2)-(4.3). We arrive at this formula in 3 stages:

(i) We first ignore the boundary equation and derive an explicit formula for solutions to the free equation (4.2).

(ii) We replace the boundary condition (4.3) with a simpler boundary equation, namely we kill $x(t)$ as it exits E_N. This has the same flavor as Example 1.3 and we derive an explicit formula for its solution.

(iii) What we really have in (4.3) is a Neumann-type boundary condition. After all by suppressing jumps that would take $\mathbf{x}$ outside E_N, the configuration stays inside E_N for all time. We modify our formula in (ii) to satisfy the requirement (4.3).

Remark 4.1 It is worth mentioning that when we have to solve $u_t = \mathcal{L}u$ in a domain U with some boundary condition, we first find eigenvalues and eigenfunctions of $\mathcal{L}$ in U and use them to to write down an expression for u. When U is bounded (or the dimension is finite), then we only need to consider the point spectrum of $\mathcal{L}$. When $\mathcal{L}$ is symmetric, we have a discrete set of eigenvalues $\lambda_1 \le \dots \le \lambda_k \le \dots$, and the corresponding eigenfunctions can be chosen to form an orthonormal basis. However, since the set E_N is unbounded, we are really dealing with the continuous spectrum. $\square$

(i) We first solve the free equation:

$$\begin{cases} p_t(\mathbf{x},t) = \mathbf{D}p(\mathbf{x},t), & t>0, \\ p(\mathbf{x},0) = 1\!\!1(\mathbf{x}=\mathbf{y}); \end{cases} \tag{4.4}$$

The solution of (4.4) is simply given by

$$p(\mathbf{x},t) = \prod_{i=1}^{d} p(x_i - y_i, t), \tag{4.5}$$

where $p(x,t)$ solves

$$\begin{cases} p_t(x,t) = p(x-1,t) - p(x,t), & t>0, \\ p(x,0) = 1\!\!1(x=0). \end{cases} \tag{4.6}$$

We already know what the solution is because $x_1(t)$ is simply a Poisson process:

$$p(x,t) = \frac{t^x}{x!}\, e^{-t} 1\!\!1(x \ge 0).$$

We may also solve (4.6) with the aid of Fourier series: Indeed if

$$\varphi(z,t) = \sum_{x=-\infty}^{\infty} p(x,t) z^x,$$

then

$$\varphi_t(z,t) = (z-1)\varphi(z,t), \qquad \varphi(z,0) = 1,$$

which leads to the identity $\varphi(z,t) = e^{t(z-1)}$. As a result,

$$p(x,t) = (2\pi)^{-1} \int_0^{2\pi} e^{-ix\theta + t(e^{i\theta}-1)}\, d\theta = (2\pi i)^{-1} \oint_{|z|=1} z^{-x-1} e^{t(z-1)}\, dz,$$

where the last integral is a contour integration over the unit circle $|z|=1$. This circle may be replaced with any positively oriented contour γ about the origin. In summary,

$$p(z,t) = (2\pi i)^{-1} \oint_\gamma z^{-x-1} e^{t(z-1)}\, dz. \tag{4.7}$$

solves (4.6) for any positive contour γ with 0 inside γ. More generally, if

$$\begin{cases} p_t(x,t) = p(x-1,t) - p(x,t), & t > 0, \\ p(x,0) = p^0(z), \end{cases} \tag{4.8}$$

then

$$p(z,t) = (2\pi i)^{-1} \oint_\gamma z^{-x-1} e^{t(z-1)} \varphi^0(z)\, dz, \tag{4.9}$$

where

$$\varphi^0(z) = \sum_{x\in\mathbb{Z}} p^0(x) z^x.$$

Alternatively, we may solve (4.9) by separation of variables. For any $z \in \mathbb{C} \setminus \{0\}$, the function $x \mapsto z^{-x}$ is an eigenfunction of D associated with the eigenvalue $z-1$. Similarly the function

$$\prod_{i=1}^N z_i^{x_i},$$

is an eigenfunction for the operator $\mathbf{D} = \sum_i D_i$, associated with the eigenvalue

$$\sum_{i=1}^N (z_i - 1).$$

Hence $z^{-x}e^{t(z-1)}$ solves (4.8) with initial condition z^{-x}. On the other hand the eigenfunctions satisfy

$$(2\pi i)^{-1} \oint_\gamma z^{-x-1}\, dz = \mathbb{1}(x=0),$$

which explains why (4.7) is true. $\square$

(ii) We next search for a solution R of (4.2) for which the boundary condition (4.3) is replaced with zero Dirichlet boundary condition:

$$\mathbf{x} \in E_N, \quad x_{i-1} = x_i \implies R(\mathbf{x},t) = 0. \tag{4.10}$$

Recall that by Example 2.2**(i)**, we may interpret R as

$$R(\mathbf{x},t) = \mathbb{P}^{\mathbf{a}(0)=\mathbf{y}}\big(\mathbf{a}(s) \in E_N \text{ for all } s \in [0,t],\ \mathbf{a}(t) = \mathbf{x}\big),$$

where $\mathbf{a}(\cdot)$ is the random walk generated by the generator $\mathbf{D}$.

For a start, let us assume that $N = 2$. The solution we found in **(i)**, namely $p(x_1, x_2, t) = p(x_1 - y_1, t)p(x_2 - y_2, t)$, does not satisfy the boundary condition (4.9) because

$$p(x_1, x_1, t) = p(x_1 - y_1, t)p(x_1 - y_2, t) \neq 0.$$

Now imagine that we can find a function $q(x_1, x_2, t)$ such that solves (4.2), and have the following initial and boundary conditions:

$$q(x_1, x_1, t) = p(x_1, x_1, t), \qquad q(x_1, x_2, 0) = 0,$$

whenever $x_1 < x_2$, and $t \geq 0$. Then

$$R(x_1, x_2, t) = p(x_1, x_2, t) - q(x_1, x_2, t),$$

does the job. In fact for q we may choose

$$q(x_1, x_2, t) = p(x_1, x_2, t; y_2, y_1) = p(x_1 - y_2, t)p(x_2 - y_1, t),$$

because $y_1 < y_2$ which implies that $(x_1, x_2) \neq (y_1, y_2)$, whenever $x_1 < x_2$. In summary, when $N = 2$, we simply have

$$R(x_1, x_2, t) = p(x_1 - y_1, t)p(x_2 - y_2, t) - p(x_1 - y_2, t)p(x_2 - y_1, t) = \det \left[p(x_i - y_j, t)\right]_{i,j=1}^{3}.$$

Now it is easy to guess the form of R for general N:

$$R(\mathbf{x}, t) = \sum_{\sigma} \varepsilon(\sigma)p(\mathbf{x}, t; \sigma\mathbf{y}) = \det \left[p(x_i - y_j, t)\right]_{i,j=1}^{N},$$

where the summation is over the permutations of $\{1, \ldots, N\}$, and $\sigma\mathbf{y} = \big(y_{\sigma(1)}, \ldots, y_{\sigma(N)}\big)$. Indeed R solves the free equation (4.2) because each $p(\mathbf{x}, t; \sigma\mathbf{y})$ is a solution; it satisfies the initial condition because when $x \in E_N$ and σ is not identity, then $\sigma\mathbf{y} \notin E_N$, and we have

$$p(\mathbf{x}, 0; \sigma\mathbf{y}) = \mathbb{1}\big(\mathbf{x} = \sigma\mathbf{y}\big) = 0;$$

and it satisfies the boundary condition because when $x_i = x_{i+1}$, we have two equal columns in the matrix $\det \left[p(x_i - y_j, t)\right]_{i,j=1}^{N}$.

The formula we have obtained for R is due to *Karlin and McGregor*. It is worth mentioning that there is nothing special about random walk in our formula for R; we could have replaced $p(x - y, t)$ with any kernel $p(x, y, t)$ of a Markov process in $\mathbb{Z}$ and derive the above formula for the probability of non intersection up to time t.

We may use our representation (4.7) to write

$$R(\mathbf{x}, t) = (2\pi i)^{-N} \oint_{\gamma_1} \cdots \oint_{\gamma_N} \det \left[z_j^{-x_i + y_j - 1}\right]_{i,j=1}^{N} e^{t \sum_{j=1}^{N}(z_j - 1)}\, dz_1 \ldots dz_N.$$

Note that we could have used $z_i^{-x_i+y_j-1}$ instead of $z_j^{-x_i+y_j-1}$. However the function

$$\det\left[z_j^{-x_i+y_j-1}\right]_{i,j=1}^N,$$

has the advantage to be an eigenfunction of $\mathbf{D}$ in domain E_N with 0 boundary condition. The corresponding eigenvalue is again $\sum_i(z_i-1)$. $\square$

(iii) We now turn to the equation (4.2) with boundary condition (4.3). We first focus on the case $N=2$. If we try $p(x_1,x_2,t)$, it fails the boundary equation as before. We may search for a solution $Q(x_1,x_2,t)$ of the form

$$p(x_1-y_1,t)p(x_2-y_2,t)-q^-(x_1-y_2,t)q^+(x_2-y_1,t)=\det\begin{bmatrix}p(x_1-y_1,t) & q^-(x_1-y_2,t)\\ q^+(x_2-y_1,t) & p(x_2-y_2,t)\end{bmatrix},$$

for function $q^\pm$ with the following properties:

- $q^\pm$ solves (4.7).
- Either $q^-(x_1-y_2,0)=0$, or $q^+(x_2-y_1,0)=0$, whenever $x_1<x_2$.
- $Q(x,x,t)=Q(x,x+1,t)$.

The latter means

$$\det\begin{bmatrix}p(x-y_1,t) & q^-(x-y_2,t)\\ q^+(x-y_1,t) & p(x-y_2,t)\end{bmatrix}=\det\begin{bmatrix}p(x-y_1,t) & q^-(x-y_2,t)\\ q^+(x+1-y_1,t) & p(x+1-y_2,t)\end{bmatrix}.$$

Equivalently

$$\det\begin{bmatrix}p(x-y_1,t) & q^-(x-y_2,t)\\ q^+(x+1-y_1,t)-q^+(x-y_1,t) & p(x+1-y_2,t)-p(x-y_2,t)\end{bmatrix}=0.$$

We may achieve this by choosing $q^\pm$ so that

$$p(a,t)=-(q^+(a+1,t)-q^+(a,t)),\qquad q^-(a,t)=-(p(a+1,t)-p(a,t)).$$

(The minus sign in the definition of $q^\pm$ is selected to avoid a minus sign in (4.14) below.) We may define $\big(p_n(x,t):\ n\in\mathbb{Z}\big)$ by the requirements:

$$p_0=p,\qquad p_n(x,t)=-\big(p_{n+1}(x+1,t)-p_{n+1}(x,t)\big). \tag{4.11}$$

In terms of p_n's we have $q^\pm=p_{\pm1}$, and

$$Q(x_1,x_2,t)=\det\begin{bmatrix}p_0(x_1-y_1,t) & p_{-1}(x_1-y_2,t)\\ p_1(x_2-y_1,t) & p_0(x_2-y_2,t)\end{bmatrix}. \tag{4.12}$$

Starting from $p_0 = p$, and using (4.11), we certainly have

$$p_{-n}(x,t) = (2\pi i)^{-1} \oint_\gamma z^{-x-1}\big(1-z^{-1}\big)^n e^{t(z-1)}\, dz, \tag{4.13}$$

for $n \geq 0$. (Here γ is any positive contour that includes 0 and 1.) Moreover, once p_n is determined for $n \geq 0$, then we may define p_{n+1} by

$$p_{n+1}(x,t) = \sum_{y=x}^{\infty} p_n(y,t), \tag{4.14}$$

This and (4.7) yield

$$p_n(x,t) = (2\pi i)^{-1} \oint_\gamma z^{-x-1}\big(1-z^{-1}\big)^{-n} e^{t(z-1)}\, dz, \tag{4.15}$$

for $n \geq 0$. In comparison with (4.14) we deduce that (4.15) is true for all $n \in \mathbb{Z}$.

Motivated by (4.13), we set

$$Q(\mathbf{x},t) = \det\big[p_{i-j}(x_i - y_j, t)\big]_{i,j=1}^N. \tag{4.16}$$

We may use our representation (4.7) to write

$$Q(\mathbf{x},t) = (2\pi i)^{-N} \oint_{\gamma_1} \dots \oint_{\gamma_N} \det\left[z_j^{-x_i+y_j-1}\big(1-z_j^{-1}\big)^{j-i}\right]_{i,j=1}^N e^{t\sum_{j=1}^N (z_j-1)} \prod_{j=1}^N dz_j. \tag{4.17}$$

Observe that the function

$$w(\mathbf{x}) = w(\mathbf{x};\mathbf{y},\mathbf{z}) = \det\left[z_j^{-x_i+y_j-1}\big(1-z_j^{-1}\big)^{j-i}\right]_{i,j=1}^N,$$

is an eigenfunction of $\mathbf{D}$ in domain E_N with Neumann boundary condition (4.3). The corresponding eigenvalue is again $\sum_j (z_j - 1)$.

Theorem 4.1 *(Schütz) Let $\mathbf{x}(t)$ be the standard TASEP. Then for every $\mathbf{y}, \mathbf{x} \in E_N$,*

$$\mathbb{P}\big(\mathbf{x}(t) = \mathbf{x} | \mathbf{x}(0) = \mathbf{y}\big) = Q(\mathbf{x}, t; \mathbf{y}).$$

Proof *(Step 1.)* To show that $Q(\mathbf{x},t) = Q(\mathbf{x},t;\mathbf{y})$ solves (4.2), it simply use the fact that that p_n satisfies (4.5). For the boundary condition, take any $\mathbf{x}$ with $x_{i-1} + 1 = x_i$. Then

$$\begin{aligned} D_i Q(\mathbf{x},t) &= \det\Big[\dots \quad \big[p_{i-j-1}(x_{i-1} - y_j, t)\big]_{j=1}^N \quad \big[p_{i-j}(x_i - y_j - 1, t) - p_{i-j}(x_i - y_j, t)\big]_{j=1}^N \quad \dots\Big] \\ &= \det\Big[\dots \quad \big[p_{i-j-1}(x_{i-1} - y_j, t)\big]_{j=1}^N \quad \big[p_{i-j-1}(x_i - y_j - 1, t)\big]_{j=1}^N \quad \dots\Big] \\ &= \det\Big[\dots, \quad \big[p_{i-j-1}(x_{i-1} - y_j, t)\big]_{j=1}^N \quad \big[p_{i-j-1}(x_{i-1} - y_j, t)\big]_{j=1}^N \quad \dots\Big] = 0, \end{aligned}$$

as desired. Alternatively, we may directly verify the boundary conditions for the eigenfunction w: If $x_{i-1}+1=x_i$, then $D_i w(\mathbf{x})$ equals

$$\det\Big[\dots \quad \Big[z_j^{-x_{i-1}+y_j-1}\big(1-z_j^{-1}\big)^{j-i+1}\Big]_{j=1}^N \quad \Big[\big(z_j^{-x_i+y_j}-z_j^{-x_i+y_j-1}\big)\big(1-z_j^{-1}\big)^{j-i}\Big]_{j=1}^N \quad \dots\Big]$$
$$=\det\Big[\dots \quad \Big[z_j^{-x_{i-1}+y_j-1}\big(1-z_j^{-1}\big)^{j-i+1}\Big]_{j=1}^N \quad \Big[z_j^{-x_i+y_j}\big(1-z_j^{-1}\big)^{j-i+1}\Big]_{j=1}^N \quad \dots\Big]=0.$$

(Step3.) For the initial condition, we need figure out how each p_n behaves initially. From $p_0(x,0)=\mathbb{1}(x=0)$, (4.11), and an induction on n, it is not hard to show

$$x<-n \quad \text{or} \quad x>0 \quad \Longrightarrow \quad p_{-n}(x,0)=0, \qquad \text{and} \qquad p_{-n}(-n,0)=(-1)^n, \tag{4.18}$$

for every $n\ge 0$. On the other hand, with the aid of (4.14) and induction on n we can show

$$x>0 \quad \Longrightarrow \quad p_n(x,0)=0, \tag{4.19}$$

for every $n\ge 0$.

We wish to show that if $Q(\mathbf{x},0)\ne 0$, then $\mathbf{x}=\mathbf{y}$ and $Q(\mathbf{y},0)=1$. If $x_1-y_1>0$, then $x_i-y_1>0$ for all i, and this implies the first column is 0 by (4.18), contradicting $Q(\mathbf{x},0)\ne 0$. On the other hand, if $x_1-y_1\le -1$, then $x_1-y_i\le -i$, which in turn implies that the first row is zero by (4.18), contradicting again $Q(\mathbf{x},0)\ne 0$. As a result, $x_1=y_1$. We also have $p_i(x_i-y_1,0)=0$ for $i>1$. This implies

$$Q(\mathbf{x},0)=\det\big[p_{i-j}(x_i-y_j,0)\big]_{i,j=2}^N.$$

We are now in a position to apply the above argument to $x_2<\dots<x_N$ and $y_2<\dots<y_N$ to deduce that $x_2=y_2$. Continuing this manner, we deduce that $\mathbf{x}=\mathbf{y}$. □

We now turn to general SEP. Recall that the jump rates to the right and left are given by λ and $1-\lambda$. Let us define an operator $\widehat{D}:\mathbb{Z}^{\mathbb{R}}\to\mathbb{Z}^{\mathbb{R}}$ by

$$\widehat{D}f(x)=(1-\lambda)\big(f(x+1)-f(x)\big)+\lambda\big(f(x-1)-f(x)\big).$$

This operator is the adjoint of the underlying random walk in SEP. We also write $\widehat{D}_i f(\mathbf{x})$ when we apply D on the i-th variable x_i. Set

$$\widehat{\mathbf{D}}Q=\sum_{i=1}^N \widehat{D}_i.$$

As before let us write $P(\mathbf{x},t;\mathbf{y})$ for the probability of $\mathbf{x}(t)=\mathbf{x}$ provided that $\mathbf{x}(0)=\mathbf{y}$. We wish to find a function $Q(\mathbf{x},t;\mathbf{y})$ that extends P to $\mathbb{Z}^N$. As before, Q solves the free equation

$$Q_t=\widehat{\mathbf{D}}Q, \tag{4.20}$$

such that if $\mathbf{x} \in E_N$, and $x_{i-1} + 1 = x_i$, for some $i \in \{2, \dots, N\}$, then

$$\lambda Q(\dots, x_{i-1}, x_i - 1, t) + (1-\lambda) Q(\dots, x_{i-1} + 1, x_i, t) - Q(\dots, x_{i-1}, x_i, t) = 0. \tag{4.21}$$

We wish to derive an explicit formula for Q. We prepare for this derivation in three steps:

(i) For $z \in \mathbb{C} \setminus \{0\}$, the operator $\widehat{D}$ has an eigenfunction of the form z^x corresponding to the eigenvalue $e(z) = \lambda z + (1-\lambda) z^{-1} - 1$. The equation

$$\begin{cases} p_t(x,t) = \widehat{D} p(x,t), & t > 0, \\ p(x,0) = \mathbb{1}(x = 0). \end{cases} \tag{4.22}$$

has an explicit solution of the form

$$p(z,t) = (2\pi i)^{-1} \oint_\gamma z^{-x-1} e^{te(z)}\, dz. \tag{4.23}$$

solves (4.22) for any positive contour γ that encloses 0.

(ii) We make an ansatz that the solution Q of (4.20), with boundary condition (4.21) takes the form

$$(2\pi i)^{-N} \sum_\sigma \oint_\gamma \dots \oint_\gamma \sum_\sigma A(\sigma, \mathbf{z}) \prod_{i=1}^N z_{\sigma(i)}^{-x_i + y_{\sigma(i)} - 1} \quad \prod_{j=1}^N e^{e(z_j)t} dz_j, \tag{4.24}$$

where $\mathbf{z} = (z_1, \dots, z_N)$, and the summation is over the permutations of $\{1, \dots, N\}$. In the case of TASEP, we simply have $e(z) = z - 1$ and

$$A(\sigma, \mathbf{z}) = \varepsilon(\sigma) \prod_{i=1}^N \left(1 - z_{\sigma(i)}^{-1}\right)^{\sigma(i) - i}.$$

We wish to find $A(\sigma, \mathbf{z})$ for general SEP. We start with the case $N = 2$. There are two permutations $id = (1,2)$, and $\sigma = (2,1)$, assuming that $A(id, \mathbf{z}) = 1$, and simply writing $A(\sigma, \mathbf{z}) = A(\mathbf{z}) = A$, our candidate for the eigenfunction reads as

$$\begin{aligned} z_1^{-x_1 + y_1 - 1} z_2^{-x_2 + y_2 - 1} + A z_1^{-x_2 + y_1 - 1} z_2^{-x_1 + y_2 - 1} &= z_1^{y_1 - 1} z_2^{y_2 - 1} \left(z_1^{-x_1} z_2^{-x_2} + A z_1^{-x_2} z_2^{-x_1}\right) \\ &=: z_1^{y_1 - 1} z_2^{y_2 - 1}\ w(x_1, x_2). \end{aligned}$$

We would like to choose A so that the eigenfunction w satisfies the boundary condition:

$$\begin{aligned} 0 =& (1-\lambda) w(x+1, x+1) + \lambda w(x,x) - w(x, x+1) \\ =& (1-\lambda)\left(z_1^{-x-1} z_2^{-x-1} + A z_1^{-x-1} z_2^{-x-1}\right) + \lambda\left(z_1^{-x} z_2^{-x} + A z_1^{-x} z_2^{-x}\right) \\ & - z_1^{-x} z_2^{-x-1} - A z_1^{-x-1} z_2^{-x} \\ =& z_1^{-x-1} z_2^{-x-1} \left(\lambda(A+1) z_1 z_2 + (1-\lambda)(A+1) - A z_2 - z_1\right). \end{aligned}$$

This leads to the choice

$$A = -\frac{(1-\lambda) - z_1 + \lambda z_1 z_2}{(1-\lambda) - z_2 + \lambda z_1 z_2},$$

which coincides with what we had when $\lambda = 1$. We also note that the corresponding solution Q now takes the form

$$Q(x_1, x_2, t) = p(x_1 - y_1, t)p(x_2 - y_2, t) - q(x_1 - y_2, x_2 - y_1, t),$$

with q a solution to the free equation that has no longer a product form.

(iii) We now turn to the general case. Our candidate for the eigenfunction is now

$$G(\mathbf{x}) = G(\mathbf{x}; \mathbf{y}, \mathbf{z}) = \sum_{\sigma} A(\sigma, \mathbf{z}) \prod_{i=1}^{N} z_{\sigma(i)}^{-x_i + y_{\sigma(i)} - 1}.$$

We wish to choose the coefficients $A(\sigma, \mathbf{z})$ so that the boundary condition

$$\lambda G(\dots, x_i, x_i, \dots) + (1-\lambda) G(\dots, x_i + 1, x_i + 1, \dots) - G(\dots, x_i, x_i + 1, \dots) = 0. \tag{4.25}$$

Here $x_i + 1 = x_{i+1}$, and (4.26) holds if

$$\begin{aligned} 0 =& (1-\lambda)\big(A_1 z_{\sigma(i)}^{-x-1} z_{\sigma(i+1)}^{-x-1} + A_2 z_{\sigma(i+1)}^{-x-1} z_{\sigma(i)}^{-x-1}\big) + \lambda\big(A_1 z_{\sigma(i)}^{-x} z_{\sigma(i+1)}^{-x} + A_2 z_{\sigma(i+1)}^{-x} z_{\sigma(i)}^{-x}\big) \\ &- A_1 z_{\sigma(i)}^{-x} z_{\sigma(i+1)}^{-x-1} - A_2 z_{\sigma(i+1)}^{-x-1} z_{\sigma(i)}^{-x} \\ =& z_{\sigma(i)}^{-x-1} z_{\sigma(i+1)}^{-x-1} \big(\lambda(A_1 + A_2) z_{\sigma(i)} z_{\sigma(i+1)} + (1-\lambda)(A_1 + A_2) - A_1 z_{\sigma(i+1)} - A_2 z_{\sigma(i)}\big), \end{aligned}$$

where $A_1 = A(\sigma, \mathbf{z})$, and $A_2 = A(\tau_i \sigma, \mathbf{z})$, with $\tau_i \sigma$ is the permutation we get from σ by interchanging $\sigma(i)$ with $\sigma(i+1)$. This equation means

$$\frac{A_2}{A_1} = -\frac{\lambda - z_{\sigma(i+1)} + (1-\lambda) z_{\sigma(i)} z_{\sigma(i+1)}}{\lambda - z_{\sigma(i)} + (1-\lambda) z_{\sigma(i)} z_{\sigma(i+1)}}.$$

To satisfy this, we may choose

$$A(\sigma, \mathbf{z}) = \varepsilon(\sigma) \frac{\prod_{i<j} \big(\lambda - z_{\sigma(i)} + (1-\lambda) z_{\sigma(i)} z_{\sigma(j)}\big)}{\prod_{i<j} \big(\lambda - z_i + (1-\lambda) z_i z_j\big)}. \tag{4.26}$$

Theorem 4.2 *(Tracy-Widom) The function Q given by* (4.24) *with A as in* (4.26) *satisfies* (4.20) *and* (4.21).

Exercise

(i) Define $h : \mathbb{R}^N \to \mathbb{R}$ by

$$h(\mathbf{x}) = \prod_{1 \le i < j \le N} (x_j - x_i) = \det \left[x_i^{j-1} \right]_{i,j=1}^{N},$$

write $h_k(\mathbf{x})$ for the corresponding h where instead of $x_1, \dots, x_N$, we use $x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_N$. Show

$$h(\mathbf{x}) = (-1)^N \sum_{k=1}^{N} (-1)^k x_k^{N-1} h_k(\mathbf{x}), \qquad \sum_{k=1}^{N} (-1)^k x_k^r h_k(\mathbf{x}) = 0,$$

for $r = 0, 1, \dots, N-2$. Use this and an induction on N to show that $\Delta h = 0$ and $\mathbf{D}h = 0$. When $N = 2$, determine the processes associated with $\mathbf{D}^h$ and Δ^h.

(ii) Given $\alpha \in (0,1)$, define a discrete time Markov chain on $\mathbb{Z}^N$ such that $\big(x_i(t) : \ i \in \{1, \dots, N\}\big)$ are independent, and

$$\mathbb{P}\big(x_i(1) = x | x_i(0) = y\big) = (1 - \alpha) \mathbb{1}(x = y) + \alpha \mathbb{1}(x = y + 1).$$

Write

$$Tf(\mathbf{y}) = \mathbb{E}^{\mathbf{x}(0) = \mathbf{y}} f(\mathbf{x}(1)).$$

Show that $Th = h$ for h as in part (**i**). $\square$

5 TASEP as a Determinantal Process

In Chapter 4 we learned that the distribution of $\mathbf{x}(t)$ can be expressed as a determinant in the case of TASEP. We now would like to show that in fact TASEP is a *determinantal process.* This means that even

$$\mathbb{P}\big\{\{a_1, \ldots, a_k\} \subset \{x_1(t), \ldots, x_N(t)\}\big\},$$

can be expressed as determinant for any $a_1, \ldots, a_k \in \mathbb{Z}$ with $a_1 < \cdots < a_k$. Our interest in such probabilities stems from the fact that we are ultimately looking for the certain scaling behavior of $\mathbf{x}$ as $N \to \infty$. To achieve this, we need some preparations. Let us relabel our particles from right to left; we set

$$\hat{x}_i = x_{N+1-i}, \qquad \hat{y}_i = y_{N+1-i}.$$

This new labeling is particularly advantageous if we have infinitely many particles that are bounded above and we wish to approximate it by a finite configuration. Our first result is due to Borodin er al. [BFPS] and can be established with the aid of determinantal process we studied in Example 3.2**(viii)**. In Theorem 5.1 we show that the point process $\hat{\mathbf{x}}$ is determinantal. Note that $\hat{\mathbf{x}}$ is a point process of size N in $\mathbb{Z}$. Once a set $\mathbf{a} \subset \mathbb{Z}$ with $|\mathbf{a}| = N$ is specified, we label the points of $\mathbf{a}$ in a decreasing order to recover $\hat{x}_1 > \hat{x}_2 > \cdots > \hat{x}_N$. However we would rather think of $\hat{\mathbf{x}}$ as a point process in $[N] \times \mathbb{Z}$. The point is that when we evaluate

$$\mathbb{P}\left(\big\{(n_1, a_1), \ldots, (n_k, a_k)\big\} \subset \big\{(1, x_1(t)), \ldots, (N, x_N(t))\big\}\right),$$

we have more information by specifying the labels of our k particles.

Theorem 5.1 *For each t, the law of the process $\hat{\mathbf{x}}(t)$ of a TASEP is determinantal with correlation kernel $\mathcal{K}(t) = \mathcal{K} : \big([N] \times \mathbb{Z}\big)^2 \to \mathbb{R}$, given by*

$$\mathcal{K}\big((i, x), (j, y)\big) = -\phi^{j-i}(x, y)\mathbb{1}(i < j) + \sum_{k=1}^{j} \psi_k^i(x)\varphi_k^j(y),$$

where $\phi(x, y) = \mathbb{1}(x > y)$, $\psi_k^i(x) = (-1)^{k-i} p_{k-i}(x - \hat{y}_k, t)$, and $\varphi_k^j(\cdot)$ is a polynomial of degree at most j such that

$$\sum_{x \in \mathbb{Z}} \varphi_k^j(x)\psi_\ell^j(x) = \mathbb{1}(k = \ell),$$

for all $k, \ell \in [j]$ and every $j \in \mathbb{N}$.

It is worth mentioning that the very form of $\mathcal{K}$ is compatible with our expectation as formulated in (3.46). We first use an idea of Sasamoto to rewrite Schütz' formula in a more suggestive way. The only ingredient for Sasamoto's derivation is (4.14). Let us write

$$\hat{E}_N = \big\{(z_1, \ldots, z_N) \in \mathbb{Z}^N : \ z_1 > \cdots > z_N\big\}.$$

By slight abuse of notation, we write $P(\hat{\mathbf{x}},t)$ for $P(\mathbf{x},t)$ as we switched to $\hat{\mathbf{x}}$.

Recall

$$P(\hat{\mathbf{x}},t) = \det \begin{bmatrix} p_0\big(\hat{x}_N - \hat{y}_N, t\big) & \dots & p_{-N+1}\big(\hat{x}_N - \hat{y}_1, t\big) \\ \vdots & \ddots & \vdots \\ p_{N-1}\big(\hat{x}_1 - \hat{y}_N, t\big) & \dots & p_0\big(\hat{x}_1 - \hat{y}_1, t\big) \end{bmatrix}$$

by applying (4.14) twice to the last row, and once to the penultimate row, we obtain

$$P(\hat{\mathbf{x}},t) = \sum_{z_3^3 \geq z_2^2 \geq z_1^1} \sum_{z_2^3 \geq z_1^2} \det \begin{bmatrix} p_0\big(z_1^N - \hat{y}_N, t\big) & \dots & p_{-N+1}\big(z_1^N - \hat{y}_1, t\big) \\ \vdots & \ddots & \vdots \\ p_{N-3}\big(z_1^3 - \hat{y}_N, t\big) & \dots & p_{-2}\big(z_1^3 - \hat{y}_1, t\big) \\ p_{N-3}\big(z_2^3 - \hat{y}_N, t\big) & \dots & p_{-2}\big(z_2^3 - \hat{y}_1, t\big) \\ p_{N-3}\big(z_3^3 - \hat{y}_N, t\big) & \dots & p_{-2}\big(z_3^3 - \hat{y}_1, t\big). \end{bmatrix} \tag{5.1}$$

Here the first sum is over (z_2^2, z_3^3), the second sum is over z_2^3, and we have written z_N^i for $\hat{x}_i$. Note that if $z_2^2 \leq z_2^3$, then we have

$$z_1^1 \leq z_2^2 \leq z_2^3, \quad z_1^1 \leq z_2^2 \leq z_3^3,$$

which is symmetric in (z_2^3, z_3^3). This means that we can swap z_2^3 with z_3^3 and the set triplet (z_2^2, z_2^3, z_3^3) on which the summation is performed is not changed. However swapping z_2^3 with z_3^3 is equivalent to swapping the last row with the penultimate row, which result in changing sign of the corresponding determinant. From this we learn that the contribution of those triplets (z_2^2, z_2^3, z_3^3) with $z_2^2 \leq z_2^3$ to the sum is zero. As a result, we may restrict the summation in (5.1) to those triplets (z_2^2, z_2^3, z_3^3) such that

$$z_3^3 \geq z_2^2 \geq z_1^1, \quad z_2^2 > z_2^3 \geq z_1^2. \tag{5.2}$$

When $N = 3$, we now have a representation of the form

$$P(\hat{\mathbf{x}},t) = \sum_{\mathbf{z} \in GT_3(\mathbf{x})} \det \begin{bmatrix} p_0\big(z_1^3 - \hat{y}_3, t\big) & p_{-1}\big(z_1^3 - \hat{y}_2, t\big) & p_{-2}\big(z_1^3 - \hat{y}_1, t\big) \\ p_0\big(z_2^3 - \hat{y}_3, t\big) & p_{-1}\big(z_1^3 - \hat{y}_2, t\big) & p_{-2}\big(z_2^3 - \hat{y}_1, t\big) \\ p_0\big(z_3^3 - \hat{y}_3, t\big) & p_{-1}\big(z_1^3 - \hat{y}_3, t\big) & p_{-2}\big(z_3^3 - \hat{y}_1, t\big). \end{bmatrix}$$

Here $GT_3(\mathbf{x})$ is the set of (z_2^2, z_2^3, z_3^3) such that (4.2) holds, and $z_3^i = \hat{x}_i$ for $i = 1, 2, 3$. The inequalities in (5.2) are related to the celebrated *Gelfand-Tsetlin pattern.*

More generally, we define

$$GT_N = \Big\{ \big(z_i^j : \ 1 \leq i \leq j \leq N\big) \in \mathbb{Z}^{\frac{N(N+1)}{2}} : \ z_i^n < z_i^{n-1} \leq z_{i+1}^n \ \text{ for } \ (i,n) \text{ with } \ 1 \leq i < n \leq N \Big\}$$

For $N > 3$, we repeat the above procedure to obtain

$$P(\hat{\mathbf{x}},t) = \sum_{\mathbf{z} \in GT_N(\hat{\mathbf{x}})} \det \begin{bmatrix} p_0\big(z_1^N - \hat{y}_N, t\big) & \dots & p_{-N+1}\big(z_1^N - \hat{y}_1, t\big) \\ \vdots & \ddots & \vdots \\ p_0\big(z_{N-1}^N - \hat{y}_N, t\big) & \dots & p_{-N+1}\big(z_{N-1}^N - \hat{y}_1, t\big) \\ p_0\big(z_N^N - \hat{y}_N, t\big) & \dots & p_{-N+1}\big(z_N^N - \hat{y}_1, t\big), \end{bmatrix}$$

where the summation is over $\big(z_i^j:\ 1 < i \le j \le N\big)$ and

$$GT_N(\hat{\mathbf{x}}) = \big\{\mathbf{z} \in GT_N:\ z_1^i = \hat{x}_i \quad \text{for } i = 1, \dots, N\big\}.$$

In short,

$$P(\hat{\mathbf{x}}, t) = \sum_{\mathbf{z} \in GT_N(\mathbf{x})} \det\Big[p_{1-j}\big(z_i^N - y_j, t\big)\Big]_{i,j=1}^N. \tag{5.3}$$

Note that if $z^k = \big(z_1^k, \dots, z_k^k\big)$, then $z^k \in E_k$. The configurations z^{k-1} and z^k are *interlaced*: For $i \in \{1, \dots, k-1\}$, we have $z_i^k < z_i^{k-1} \le z_{i+1}^k$. If this is the case, we simply write $z^{k-1} \prec z^k$.

Remark 5.1(i) As our first reaction to (5.3), we may wonder whether or not there is Markov process $\mathbf{z}(t)$ such that its marginal $\hat{\mathbf{x}}(t) = \big(z_1^1(t), \dots, z_1^N(t)\big)$ is a TASEP. The most natural candidate for the evolution of the triangular array $\mathbf{z}(t)$ is as follows: Each z_j^i has a rate one Poisson clock for its jumping times to the right, with these clocks all independent. However the jump of z_i^k is suppressed whenever $z_i^k + 1 = z_i^{k-1}$, and when the jump of z_i^k is materialized, all particles with

$$z_{i+r}^{k+r} = \cdots = z_{i+1}^{k+1} = z_i^k,$$

are pushed to jump as well. For example the generator for $N = 2$ looks like

$$\begin{aligned} \mathcal{L}F\big(z_1^2, z_1^1, z_2^2\big) =& 1\!\!1\big(z_1^2 + 1 < z_1^1\big)\left[F\big(z_1^2 + 1, z_1^1, z_2^2\big) - F\big(z_1^2, z_1^1, z_2^2\big)\right] \\ &+ 1\!\!1\big(z_1^1 < z_2^2\big)\left[F\big(z_1^2, z_1^1 + 1, z_2^2\big) - F\big(z_1^2, z_1^1, z_2^2\big)\right] \\ &+ 1\!\!1\big(z_1^1 = z_2^2\big)\left[F\big(z_1^2, z_1^1 + 1, z_2^2 + 1\big) - F\big(z_1^2, z_1^1, z_2^2\big)\right] \\ &+ \left[F\big(z_1^2, z_1^1, z_2^2 + 1\big) - F\big(z_1^2, z_1^1, z_2^2\big)\right]. \end{aligned}$$

The triangular array process $\mathbf{z}(t) = \big(z_j^k(t):\ 1 \le j \le k,\ 1 \le k \le N\big)$ enjoys the following properties:

- The left side $\big(z_1^i:\ i = 1, \dots, N\big)$ is evolved as a TASEP.
- The right side $\big(z_i^i:\ i = 1, \dots, N\big)$ is evolved as a pushed collection of random walks. When a particle z_i^i is jumping to the right, any other particle $z_j^j, j > i$, that shares the same site as z_i^i jumps with it.

(ii) Given $\mathbf{a} = (a_1, \dots, a_N) \in E_k$, we can show

$$\sharp\big\{\mathbf{z} \in GT_N:\ z_i^N = a_i \quad \text{for } i = 1, \dots N\big\} = \left(\prod_{j=1}^{N-1} j!\right)^{-1} \Delta_N(\mathbf{a}), \tag{5.4}$$

where

$$\Delta_N(\mathbf{a}) = \prod_{1\le i<j\le N} (a_i - a_j)\ 1\!\!1\big(\mathbf{a} \in E_N\big).$$

It turns out that the function $h = \Delta_N$ satisfies $\mathbf{D}h = 0$. The restriction of Δ_N to E_N is a positive "harmonic" function with zero boundary condition. This function is very much related to the (Martin) boundary point ∞ of E_N. The Δ_N-Doob transform of $\mathbb{D}_N$ (with generator $\mathbb{D}_N^{\Delta_N}$ is the Markov process we get from the independent walks that are conditioned on never meeting in finite time (reaching the boundary ∞ at time ∞).

(iii) Borodin and Ferrari show that if $\mathbf{z}$ initially starts from a *packed* configuration (i.e., $z_i^j = -j-1+i, 1 \le i \le j \le N$), the dynamics of its marginal $\mathbf{x} = z_1 = (z_1^1, \dots, z_1^N)$ is the D_N-Doob transform of free particles. This Markov process is known as *Charlier process*. The same applies if we take any horizontal line $z^k = \big(z_i^k :\ i = 1, \dots, k\big)$; it is Δ_k-Doob transform of the free walk in E_k. Moreover, the pair $\big(z^k, z^{k-1}\big)$ is an example of intertwined processes. In fact once the law of $z^k(t)$ is known, then we can determine the law of $z^{k-1}(t)$ in a Markovian fashion. The Markov kernel is given by

$$\Lambda_{k-1}^k\big(z^k, z^{k-1}\big) = (k-1)!\frac{\Delta_{k-1}\big(z^{k-1}\big)}{\Delta_k\big(z^k\big)} 1\!\!1\big(z^{k-1} \prec z^k\big).$$

Let us define

$$\big(T_{k-1}^k f\big)\big(z^k\big) = \sum_{z^{k-1}} f\big(z^{k-1}\big)\Lambda_{k-1}^k\big(z^k, z^{k-1}\big).$$

We also write $\mathbf{D}_k$ for the generator of the free motion in E_k and $z^k(i)$ for the configuration we get from z^k by moving the i-th particle to the right. For $z^k \in E_k$,

$$\begin{aligned}
\big(\mathbf{D}_k^{\Delta_k}\ T_{k-1}^k f\big)\big(z^k\big) =& \sum_{i=1}^{k} \sum_{z^{k-1}} f\big(z^{k-1}\big)\frac{\Delta_k\big(z^k(i)\big)}{\Delta_k\big(z^k\big)} \left(\Lambda_{k-1}^k\big(z^k(i), z^{k-1}\big) - \Lambda_{k-1}^k\big(z^k, z^{k-1}\big)\right)\\
=&(k-1)!\sum_{i=1}^{k} \sum_{z^{k-1}} f\big(z^{k-1}\big)\frac{\Delta_{k-1}\big(z^{k-1}\big)}{\Delta_k\big(z^k\big)} \left(1\!\!1\big(z^{k-1} \prec z^k(i)\big) - \Lambda_{k-1}^k\big(z^k, z^{k-1}\big)\right)\\
\big(T_{k-1}^k\ \mathbf{D}_{k-1}^{\Delta_{k-1}} f\big)\big(z^k\big) =& \sum_{z^{k-1}} \sum_{i=1}^{k-1} \left(f\big(z^{k-1}(i)\big) - f\big(z^{k-1}\big)\right) \frac{\Delta_{k-1}\big(z^{k-1}(i)\big)}{\Delta_{k-1}\big(z^{k-1}\big)}\Lambda_{k-1}^k\big(z^k, z^{k-1}\big)\\
=&(k-1)!\sum_{z^{k-1}} \sum_{i=1}^{k-1} \left(f\big(z^{k-1}(i)\big) - f\big(z^{k-1}\big)\right) \frac{\Delta_{k-1}\big(z^{k-1}(i)\big)}{\Delta_k\big(z^k\big)} 1\!\!1\big(z^{k-1} \prec z^k\big)
\end{aligned}$$

□

To establish Theorem 5.1, it suffice to show that the process $\mathbf{z}$ is determinantal.

Theorem 5.2 *The point process* $\mathbf{z} = (z^1, \ldots, z^N)$ *is determinantal in* $X = X_1 \sqcup \cdots \sqcup X_N$ *with* $X_1 = \cdots = X_N = \mathbb{Z}$ *and each* z^i *a point process in* X_i *for* $i \in [N]$*. The correlation kernel* $\mathcal{K} : X \times X \to \mathbb{R}$ *has* (i,j)*-th block* $K_{ij} : X_i \times X_j \to \mathbb{R}$ *of the form*

$$K_{ij} = -\phi^{j-i} \mathbb{1}(i < j) + \Psi_{i,j}^* \Phi_j, \tag{5.5}$$

with $\Phi_j, \Psi_{i,j} : [j] \times \mathbb{Z} \to \mathbb{R}$*, where*

$$\Psi_{i,j}(k,x) = \psi_k^i(x), \qquad \Phi_j(k,x) = \varphi_k^i(x),$$

with ψ *and* φ *as in Theorem 5.1.*

Proof Observe

$$\mathbb{1}\big(z^k \prec z^{k+1}\big) = \det\Big[\mathbb{1}\big(z_i^k > z_j^{k-1}\big)\Big]_{i,j=1}^{k},$$

provided that $z_1^{k+1} < z_1^k$, $z^k \in E_k$, and we set $z_{k+1}^k = \infty$. Using this, we can show

$$\mathbb{1}\big(\mathbf{z} \in GT_N\big) = \prod_{k=1}^{N-1} \det\Big[\mathbb{1}\big(z_i^k > z_j^{k+1}\big)\Big]_{i,j=1}^{k+1}, \tag{5.6}$$

provided that $z_1^N < \cdots < z_1^1$.

If we write $\Lambda_N(\hat{\mathbf{x}})$ for the set of triangular array $\big(z_i^j :\ 1 \le i \le j \le N\big)$ such that $z_1^j = \hat{x}_j$ for $j = 1, \ldots, N$, then by (5.6),

$$P(\hat{\mathbf{x}}, t) = \sum_{\mathbf{z} \in \Lambda_N(\hat{\mathbf{x}})} \prod_{k=1}^{N-1} \det\Big[\mathbb{1}\big(z_i^k > z_j^{k+1}\big)\Big]_{i,j=1}^{k+1} \det\Big[p_{1-j}\big(z_i^N - y_j, t\big)\Big]_{i,j=1}^{N}, \tag{5.7}$$

where $\Lambda_N(\hat{\mathbf{x}})$ is the set $\mathbf{z} \in \mathbb{Z}^{\frac{N(N+1)}{2}}$ such that $z_1^i = \hat{x}_i$ for $i = 1, \ldots, N$. Observe

$$\begin{aligned}
\det\Big[p_{1-j}\big(z_i^N - y_j, t\big)\Big]_{i,j=1}^{N} &= (-1)^{N(N-1)/2} \det\Big[(-1)^{1-j} p_{1-j}\big(z_i^N - y_j, t\big)\Big]_{i,j=1}^{N} \\
&= (-1)^{N(N-1)/2 + \lfloor N/2 \rfloor} \det\Big[(-1)^{N-j} p_{j-N}\big(z_i^N - y_{N+1-j}, t\big)\Big]_{i,j=1}^{N} \\
&= \det\Big[(-1)^{j-N} p_{j-N}\big(z_i^N - \hat{y}_j, t\big)\Big]_{i,j=1}^{N},
\end{aligned}$$

because $N(N-1)/2 + \lfloor N/2 \rfloor$ is always even. From this and (5.7) we deduce

$$P(\hat{\mathbf{x}}, t) = \sum_{\mathbf{z} \in \Lambda_N(\hat{\mathbf{x}})} \prod_{k=1}^{N-1} \det\Big[\phi\big(z_i^k, z_j^{k+1}\big)\Big]_{i,j=1}^{k+1} \det\Big[\psi_j\big(z_i^N\big)\Big]_{i,j=1}^{N}, \tag{5.8}$$

where

$$\phi(x,y) = \varphi(x-y) = \mathbb{1}(x>y), \qquad \psi_j(x) = (-1)^{j-N} p_{j-N}(x-\hat{y}_j, t).$$

This is very much an example of the point process we examined in Example 3.2**(viii)** with

$$X_1 = \cdots = X_N = \mathbb{Z}, \qquad W^k(x,y) = \phi(x,y), \qquad \gamma_i(x) = 1.$$

Recall

$$p_n(x,t) = \frac{1}{2\pi i}\oint_\gamma z^{-x-1}\left(1-z^{-1}\right)^{-n} e^{t(z-1)}\,dz$$

with γ any positive contour including 0 and 1. In fact for $n \le 0$, $z=1$ is no longer a pole, and we may choose γ any positive contour that includes 0. Recall that for all n, we have

$$p_{n+1}(x,t) = \sum_{y \ge x} p_n(y,t).$$

However, for $n \le 0$, we also have

$$p_{n+1}(x,t) = -\sum_{y<x} p_n(y,t). \tag{5.9}$$

Regarding $p_n = p_n(\cdot, t) : \mathbb{Z} \to \mathbb{R}$ as a function on $\mathbb{Z}$, (5.9) means

$$\phi p_n = -p_{n+1}, \qquad \phi^{-1} p_{n+1} = -p_n,$$

where $\phi^{-1} = \mathbb{D}_+$ is simply the operator

$$\mathbb{D}_+ f(x) = f(x+1) - f(x).$$

To take advantage of this, let us write

$$\psi_j^k(x) := (-1)^{j-k} p_{j-k}(x-\hat{y}_j, t), \qquad \Psi_k := \left[\psi_j^k(x)\right]_{j\in[k], x\in\mathbb{Z}}.$$

Clearly,

$$\Psi_{i,k}^* := \phi^{k-i}\Psi_k^* = \left[\psi_j^i(x)\right]_{x\in\mathbb{Z}, j\in[k]}. \tag{5.10}$$

As we demonstrated in Chapter 3, the probability measure P is determinantal with correlation kernel that is given by $\mathcal{K}$ as in (3.35). In other words

$$\mathcal{K} = \left[K_{ij}\right]_{i,j=1}^N = \Gamma\Psi^* C^{-1}\Lambda - \widehat{W}, \tag{5.11}$$

where $K_{i,j} : X_i \times X_j \to \mathbb{R}$ is the (i,j)-th block of the matrix K. From (3.34),

$$K_{ij} = -\mathbb{1}(i<j)\phi^{j-i} + \phi^{N-i}\Psi^* C^{-1}\Lambda_j. \tag{5.12}$$

with $\Psi = \Psi_N = \Psi_{N,N}$, $C = C_N = \Lambda_N \Psi^*$, and

$$\Lambda_j = \sum_{k=1}^{j} E^k \phi^{j-k}.$$

Here we have used the fact that $W_{[i,j)} = \phi^{j-i}$ because all W^k's are equal to ϕ. Note that by (5.10),

$$\phi^{N-i}\Psi^* = \phi^{N-i}\Psi_N^* = \Psi_{i,N}^*. \tag{5.13}$$

On the other-hand, for $\varphi(x) = 1\!\!1(x \geq 1)$, we can inductively derive

$$\varphi^{*k}(x) = \frac{(x-1)\dots(x-k+1)}{(k-1)!}\ 1\!\!1(x \geq k),$$

because

$$\sum_{y=k}^{x-1} (y-1)\dots(y-k+1) = k^{-1}\ (x-1)\dots(x-k),$$

that can be verified by induction on x. From this we deduce

$$\phi^k(x,y) = \binom{x-y-1}{k-1}\ 1\!\!1(x \geq y+k). \tag{5.14}$$

We now turn to Λ_j. First observe

$$E^k(i,x) = \delta_{ik} 1\!\!1(a_k > x) = \delta_{ik}\phi(a_k, x),$$

where $a_k = z_{k+1}^k$. Even though $a_k = \infty$, we would rather think of it as a fixed finite point. After all we only need to calculate $\det L_{[N]\sqcup \mathbf{z}}$ for a finite configuration $\mathbf{z}$; for such a finite configuration any sufficiently large a_k can serve as z_{k+1}^k. After this interpretation, we take $n \in [N]$ and $x \in \mathbb{Z}$, and observe

$$\begin{aligned} \Lambda_j(n,x) &= \left(\sum_{k=1}^{j} E^k \phi^{j-k}\right)(n,x) = \sum_{k=1}^{j}\sum_{y} \delta_{nk}\ \phi(a_k,y)\phi^{j-k}(y,x) \\ &= 1\!\!1(1 \leq n \leq j)\ \phi^{j-n+1}(a_n,x). \end{aligned} \tag{5.15}$$

Observe that each $\Lambda_j(n,\cdot)$ is a polynomial of degree $j-n$.

If we define $\widehat{\Lambda}_j : [j] \times \mathbb{Z} \to \mathbb{R}$ by $\widehat{\Lambda}_j(n,x) = \phi^{j-n+1}(a_n,x)$, then

$$\Lambda_j = \begin{bmatrix} \widehat{\Lambda}_j \\ 0 \end{bmatrix}.$$

This implies

$$\begin{aligned}\big(\Lambda_j\Psi^*_{j,N}\big)(n,m) &= \mathbb{1}(1\le n\le j)\ \sum_y \phi^{j-n+1}(a_n,y)\Psi^*_{j,N}(y,m)\\ &= \mathbb{1}(1\le n\le j)\Psi^*_{n-1,N}(a_n,m)\\ &= \mathbb{1}(1\le n\le j)(-1)^{m-n+1}p_{m-n+1}(a_n-\hat y_m,t).\end{aligned}$$

In particular

$$C = C_N = \big[(-1)^{m-n+1}p_{m-n+1}(a_n-\hat y_m,t)\big]_{n,m=1}^N. \tag{5.16}$$

Recall

$$p_0(x,t)=\frac{t^x}{x!}\mathbb{1}(x\ge 0),\qquad p_{k-1}(x,t)=p_k(x,t)-p_k(x+1,t).$$

From this we learn

$$k\le 0 \qquad\Longrightarrow\qquad p_k(\infty,t)=0.$$

Since we may send a_n to infinity, we learn that the matrix C_N is upper-triangular. From this and Cramer's formula for C_N^{-1} we learn that C_N^{-1} is also upper-triangular.

$$C=\begin{bmatrix} C_{11} & C_{12}\\ 0 & C_{22},\end{bmatrix},\qquad C^{-1}=\begin{bmatrix} R_{11} & R_{12}\\ 0 & R_{22},\end{bmatrix}$$

with R_{11} and C_{11} matrices of size $j\times j$, and R_{22} and C_{22} matrices of size $(N-j)\times(N-j)$. As a result $R_{11}=C_{11}^{-1}$, and

$$C^{-1}\Lambda_j=\begin{bmatrix} R_{11} & R_{12}\\ 0 & R_{22}\end{bmatrix}\begin{bmatrix}\widehat\Lambda_j\\ 0\end{bmatrix}=\begin{bmatrix} C_{11}^{-1}\ \widehat\Lambda_j\\ 0\end{bmatrix}.$$

We may write C_j for C_{11}. Indeed if we switch from N to j, then the matrix C_{11} is the corresponding C-matrix. From all this we learn

$$\phi^{N-i}\Psi^*C_N^{-1}\Lambda_j=\Psi^*_{i,N}\begin{bmatrix} C_j^{-1}\ \widehat\Lambda_j\\ 0\end{bmatrix},$$

with $C_j^{-1}\ \widehat\Lambda_j:[j]\times\mathbb{Z}\to\mathbb{R}$. Observe

$$\Psi^*_{i,N}=\begin{bmatrix}\Psi^*_{i,j} & *\end{bmatrix},$$

with $\Psi_{i,j}:\mathbb{Z}\times[j]\to\mathbb{R}$. From this we deduce

$$K_{ij}=-\mathbb{1}(i<j)\phi^{j-i}+\Psi^*_{i,j}C_j^{-1}\ \widehat\Lambda_j. \tag{5.17}$$

If we write

$$\Phi_j := C_j^{-1}\,\widehat{\Lambda}_j, \tag{5.18}$$

then (5.17) can be written as

$$K_{ij} = -\mathbb{1}(i<j)\phi^{j-i} + \phi^{j-i}\Psi_j^*\Phi_j, \tag{5.19}$$

with Φ satisfying

$$\Phi_j\Psi_j^* = \mathbb{1}_{[j]}.$$

We set

$$\varphi_k^j(x) := \Phi_j(k,x).$$

From (5.15) and (5.18) we learn that each $\varphi_k(x;j)$ is a polynomial of degree at most $j-k$ because the matrix C_j^{-1} is upper triangular.We may write

$$\big(\Psi_{i,j}^*\Phi_j\big)(x,y) = \sum_{k=1}^{j}\Psi_{i,j}(k,x)\Phi_j(k,y) = \sum_{k=1}^{j}\psi_k^i(x)\varphi_k^j(y).$$

In summary

$$K_{ij}(x,y) = -\mathbb{1}(i<j)\phi^{j-i}(x,y) + \sum_{k=1}^{j}\psi_k^i(x)\varphi_k^j(y), \tag{5.20}$$

with

$$\sum_{x\in\mathbb{Z}}\varphi_k^n(x)\psi_\ell^n(x) = \delta_{k\ell}, \tag{5.21}$$

for every $k,\ell\in[n]$ and $n\in\mathbb{N}$. □

We now introduce some notations that would help us to simplify(5.19). Let us write

$$\mathbb{D}_\pm f(x) = \pm\big(f(x\pm 1) - f(x)\big).$$

Observe that $\phi^{-1} = \mathbb{D}_+$, $\mathbb{D} = -\mathbb{D}_-$, and recall

$$\psi_k^n(x) = (-1)^{k-n}p_{k-n}(x-\hat{y}_k,t).$$

Now writing $P_k(x,y;t) = p_k(x-y,t)$ and regard it as a kernel/operator, we have

$$(-1)^r\phi^{-r}P_k = P_{k-r}.$$

Hence

$$\psi_k^n(x) = \big(\phi^{k-n}P_0\big)\big(x,\hat{y}_k\big).$$

On the other hand,

$$P_0\big(x,\hat{y}_k\big) = \big(e^{t\mathbb{D}}\delta_{\hat{y}_k}\big)(x) = \big(e^{t\mathbb{D}}\delta_{\hat{y}_k}\big)(x).$$

As a result,

$$\psi_k^n = \phi^{k-n} e^{t\mathbb{D}} \delta_{\hat{y}_k} = e^{t\mathbb{D}} \phi^{k-n} \delta_{\hat{y}_k}, \tag{5.22}$$

because kernels associated with ϕ^{k-n} and $e^{t\mathbb{D}}$ depend on $x-y$ and convolution is commutative. This suggests defining $\Theta_n : \mathbb{Z} \times [n] \to \mathbb{R}$ such that $\Psi_n^* = e^{t\mathbb{D}}\Theta_n$, and

$$\Theta_n = \Big[\phi^{k-n}(x,\hat{y}_k)\Big]_{x\in\mathbb{Z},k\in[n]}.$$

Recall that $\phi^{-1} = \mathbb{D}_+$. We wish to find Φ_n that is a left inverse of Ψ^*. For this, we find H_n so that $\Phi_n = H_n e^{-t\mathbb{D}}$, with $H_n = \big[h_k^n\big]_{k\in[n],x\in\mathbb{Z}}$;

$$\varphi_k^n(x) = \sum_y h_k^n(y) e^{-t\mathbb{D}}(y,x), \qquad h_k^n(x) = \sum_y \varphi_k(y;n) e^{t\mathbb{D}_-}(y,x). \tag{5.23}$$

This means that now H_n satisfies $H_n\Theta_n = \mathbb{1}_{[n]}$, or more explicitly,

$$\delta_{k\ell} = \sum_x h_k^n(x)\phi^{\ell-n}(x,\hat{y}_\ell) = \big(h_k^n\phi^{\ell-n}\big)(\hat{y}_\ell).$$

We note that the operator $e^{-t\mathbb{D}} = e^{t\mathbb{D}_-}$, the inverse of $e^{t\mathbb{D}}$ is well-defined:

$$e^{t\mathbb{D}}(x,y) = e^t\, \frac{t^{x-y}}{(x-y)!}\, \mathbb{1}(x \geq y) =: R_t(x-y), \qquad e^{-t\mathbb{D}}(x,y) = e^{-t}\, \frac{(-t)^{x-y}}{(x-y)!}\, \mathbb{1}(x \geq y).$$

Indeed,

$$\big(R_t * R_{-t}\big)(x) = \sum_{y=0}^{x} \frac{t^{x-y}}{(x-y)!}\frac{(-t)^y}{y!} = \mathbb{1}(x=0) + \mathbb{1}(x>0)(x!)^{-1}(t-t)^x = \mathbb{1}(x=0).$$

We wish to find $\big(h_k^n :\ k\in[n]\big)$ with the following properties:

(1) $\big(h_k^n\phi^{\ell-n}\big)(\hat{y}_\ell) = \mathbb{1}(k=\ell)$;

(2) h_k^n is a polynomial of degree at most $n-k$.

Note that since $\big(\phi^{-1}\big)^* = -\mathbb{D}_- = \mathbb{D}$ is a discrete differentiation, the requirement (2) is equivalent to the assertion that $\mathbb{D}^{n-k}h_k^n$ is a constant. This constant must be 1 because $\big(\mathbb{D}^{n-k}h_k^n\big)\big(\hat{y}_k\big) = 1$, by (1). In other words,

$$\mathbb{D}^{n-k}h_k^n = 1,$$

for $k \in [n]$. To determine h_k^n, we use the requirement (1). More precisely, we fix k and n, and determine uniquely the polynomial h_k^n of degree $n-k$ such that the following $n-k$ conditions are satisfied:

$$\mathbb{D}^{n-k} h_k^n = 1, \quad \big(\mathbb{D}^{n-k-1} h_k^n\big)(\hat{y}_{k+1}) = 0, \quad \big(\mathbb{D}^{n-k-2} h_k^n\big)(\hat{y}_{k+2}) = 0, \ \dots, \ h_k^n(\hat{y}_n) = 0. \tag{5.24}$$

This can be achieved inductively in the following fashion: If for some integer $\ell \in [k,n)$, the function $g_\ell := \mathbb{D}^{n-\ell} h_k^n$ is determined, then we find $g_{\ell+1} := \mathbb{D}^{n-\ell-1} h_k^n$ by solving

$$\mathbb{D} g_{\ell+1} = g_\ell, \qquad g_{\ell+1}(\hat{y}_{\ell+1}) = 0.$$

(The second condition also follows from (1) because $\ell + 1 > k$.) We may display the dependence of h_k^n on $\hat{\mathbf{y}}$ by writing $h_k^n\big(\cdot;\hat{\mathbf{y}}_{k+1}^n\big)$, where $\hat{\mathbf{y}}_{k+1}^n = \big(\hat{y}_{k+1},\dots,\hat{y}_n\big)$. Because of this, we may also write $V_j\big(x,y;\hat{\mathbf{y}}_{k+1}^n\big)$ or simply $V_j(x,y;\hat{\mathbf{y}})$ for $V_j(x,y)$. Note that $h_k^n = g_n$ and

$$h_k^n(\hat{y}_n) = 0, \tag{5.25}$$

provided that $k < n$. By (5.24), we also know

$$h_n^n = 1. \tag{5.26}$$

We can now express our correlation kernel in terms of Θ and H:

$$K_{ij} = K_{ij}(t) = \phi^{j-i}\big(K_{jj} - \mathbb{1}(i<j)\big), \qquad K_{jj} = e^{t\mathbb{D}}\, V_j e^{-t\mathbb{D}}.$$

where

$$V_j(x,y) = \big(\Theta_j H_j\big)(x,y) = \sum_{k=1}^{j} \phi^{k-j}(x,\hat{y}_k) h_k^j(y).$$

Observe that since ϕ and $e^{t\mathbb{D}}$ commute, we always have

$$K_{ij}(t) = e^{t\mathbb{D}}\, V_{ij} e^{-t\mathbb{D}}. \tag{5.27}$$

Evidently

$$\frac{dK_{ij}(t)}{dt} = \frac{d}{dt}\left(e^{t\mathbb{D}} V_{ij} e^{-t\mathbb{D}}\right) = e^{t\mathbb{D}}\big(\mathbb{D}V_{ij} - V_{ij}\mathbb{D}\big)e^{-t\mathbb{D}}.$$

In other words,

$$\frac{dK_{ij}}{dt} = \mathbb{D}K_{ij} - K_{ij}\mathbb{D}. \tag{5.28}$$

Since $\mathbb{D}$ and ϕ^k commute, we have

$$\mathbb{D}V_{ij} - V_{ij}\mathbb{D} = \phi^{j-i}\big(\mathbb{D}V_j - V_j\mathbb{D}\big).$$

By the definition of h_k^j,

$$\mathbb{D}h_k^j\big(y;\hat{y}_{k+1},\dots,\hat{y}_j\big)=h_k^{j-1}\big(y;\hat{y}_{k+1},\dots,\hat{y}_{j-1}\big).$$

Since $\big(\phi^{-1}\big)^*=\mathbb{D}$, We have

$$\phi V_j\phi^{-1}=V_{j-1}.$$

We now study $\mathcal{V}=\mathcal{K}(0)$, which has

$$V_{ij}=K_{ij}(0)=\phi^{j-i}\big(K_{jj}-1\!\!1(i<j)\big)=\phi^{j-i}\big(V_j-1\!\!1(i<j)\big),$$

for its (i,j)-th block. Also note that for $j\le n$,

$$V_{ij}=\phi^{n-i}V_n\phi^{j-n}-1\!\!1(i<j)\phi^{j-i}.$$

This means that $\mathcal{V}$ and $\mathcal{K}$ are determined from V_N.

Needless to say that $\mathcal{V}$ should serve as a correlation kernel for a point process that is concentrated on a single configuration, namely $\hat{\mathbf{y}}$. We will verify this directly.

Theorem 5.3 **(i)** *The operator $\mathcal{V}$ is the correlation function of the trivial point process that is concentrated on the single configuration $\hat{\mathbf{y}}$.*

(ii) *For each r, the operator $\partial_r V_N$ is rank-one. Moreover*

$$\hat{y}_r+1=\hat{y}_{r-1},\quad 1\le r<N\qquad\Longrightarrow\qquad V_N(x,y;\hat{\mathbf{y}}_r)=V_N(x,y;\hat{\mathbf{y}}).\tag{5.29}$$

(iii) *The function $V_N(x,y;\hat{\mathbf{y}})$ satisfies $\mathbf{D}_{\mathbf{y}}V_N=[\mathbb{D},V_N]$.*

As a preparation, we explore some of the properties of the polynomials h_k^j. We define

$$\tau_a f(x)=f(x+a),\qquad \hat{\tau}_a F\big(\hat{\mathbf{y}}\big)=F\big(\tau_a\hat{\mathbf{y}}\big),$$

where $\tau_a\hat{\mathbf{y}}=\hat{\mathbf{y}}-a=(\hat{y}_1-a,\dots,\hat{y}_N-a)$. We also set

$$\hat{h}_k^j(\hat{\mathbf{y}})=\hat{h}_k^j\big(\hat{y}_{k+1},\dots,\hat{y}_j\big):=h_k^j(0;\hat{\mathbf{y}}).$$

To ease the notation, we also define

$$\partial_r F\big(\hat{\mathbf{y}}\big)=F\big(\hat{\mathbf{y}}_r\big)-F\big(\hat{\mathbf{y}}\big).$$

Proposition 5.1 *The following statements are true:*

(i) $\mathbb{D}^r h_k^j\big(y;\hat{y}_{k+1},\dots,\hat{y}_j\big)=h_k^{j-r}\big(y;\hat{y}_{k+1},\dots,\hat{y}_{j-r}\big)$*, for* $r\in\{1,\dots,j-k\}$.

(ii) $h_k^j(y;\hat{\mathbf{y}})=\hat{h}_k^j\big(\hat{\tau}_y\hat{\mathbf{y}}\big)$.

(iii) *We have*

$$\partial_r h_k^j(y;\hat{\mathbf{y}}) = \begin{cases} h_k^{r-1}(\hat{y}_r+1;\hat{y}_{k+1},\dots,\hat{y}_{r-1})h_r^j(y;\hat{y}_{r+1},\dots,\hat{y}_j) & k+1<r<j, \\ -h_{k+1}^j(y;\hat{y}_{k+2},\dots,\hat{y}_j) & k+1=r, \\ h_k^{j-1}(\hat{y}_j+1;\hat{y}_{k+1},\dots,\hat{y}_{j-1}) & r=j. \end{cases} \tag{5.30}$$

(iv) h_k^j *satisfies the Neumann boundary conditions: If* $\hat{y}_{r-1}=\hat{y}_r+1$, *and* $r>k+1$, *then* $\partial_r h_k^j=0$.

(v) *We have the following (anti)duality relationship*

$$\mathbf{D}_{\hat{\mathbf{y}}} h_k^j(y;\hat{\mathbf{y}}) = h_k^j(y;\hat{\mathbf{y}}) - h_k^j(y+1;\hat{\mathbf{y}}). \tag{5.31}$$

Proof (i) and **(ii)** follow from the definition and the elementary identity $\tau_a\mathbb{D}=\mathbb{D}\tau_a$.

(iii) We only verify (5.30) when $k+1<r<j$. Set $g(y)=h_k^j(y;\hat{\mathbf{y}}_r)-h_k^j(y;\hat{\mathbf{y}})$. We certainly have

$$\mathbb{D}^{j-k}g=0,\quad \mathbb{D}^{j-k-1}g\big(\hat{y}_{k+1}\big)=\mathbb{D}^{j-k-2}g\big(\hat{y}_{k+2}\big)=\cdots=\mathbb{D}^{j-r+1}g\big(\hat{y}_{r-1}\big)=0.$$

This inductively implies

$$\mathbb{D}^{j-k}g=\mathbb{D}^{j-k-1}g=\mathbb{D}^{j-k-2}g=\cdots=\mathbb{D}^{j-r+1}g=0.$$

Hence $c=\mathbb{D}^{j-r}g$ is a constant, and $g'=c^{-1}g$ satisfies

$$\mathbb{D}^{j-r}g'=1,\quad \mathbb{D}^{j-r-1}g'\big(\hat{y}_{r+1}\big)=\cdots=\mathbb{D}g'\big(\hat{y}_{j-1}\big)=g'\big(\hat{y}_j\big).$$

Hence $g'=h_r^j$. On the other hand,

$$\begin{aligned} c &= h_k^r(y;\hat{\mathbf{y}}_r)-h_k^r(y;\hat{\mathbf{y}}) = h_k^r(\hat{y}_r;\hat{\mathbf{y}}_r)-h_k^r(\hat{y}_r;\hat{\mathbf{y}}) = h_k^r(\hat{y}_r;\hat{\mathbf{y}}_r) \\ &= h_k^r(\hat{y}_r;\hat{\mathbf{y}}_r)-h_k^r(\hat{y}_r+1;\hat{\mathbf{y}}_r) = \mathbb{D}h_k^r(\hat{y}_r+1;\hat{\mathbf{y}}_r) = h_k^{r-1}(\hat{y}_r+1;\hat{\mathbf{y}}_r) \\ &= h_k^{r-1}(\hat{y}_r+1;\hat{\mathbf{y}}), \end{aligned}$$

as desired.

(iv) Clearly, if $\hat{y}_{r-1}=\hat{y}_r+1$, and $r>k+1$, then

$$h_k^{r-1}(\hat{y}_r+1;\hat{\mathbf{y}}) = h_k^{r-1}(\hat{y}_{r-1};\hat{y}_{k+1},\dots,\hat{y}_{r-1})=0,$$

by the definition h_k^{r-1}. This and (5.30) imply $\partial_r h_k^j=0$.

(v) Define

$$T_r\hat{\mathbf{y}}=\big(\hat{y}_{k+1}-1,\dots,\hat{y}_r-1,\hat{y}_{r+1},\dots,\hat{y}_j\big).$$

Note that $\big(T_r\hat{\mathbf{y}}\big)_r = T_{r-1}\hat{\mathbf{y}}$. Hence by (5.30) and part **(ii)**,

$$\begin{aligned} h_k^j\big(y;T_{r-1}\hat{\mathbf{y}}\big) - h_k^j\big(y;T_r\hat{\mathbf{y}}\big) &= h_k^{r-1}\big(\hat{y}_r;T_r\hat{\mathbf{y}}\big)h_r^j(y;\hat{\mathbf{y}}) \\ &= \hat{h}_k^{r-1}(\hat{y}_{k+1}-\hat{y}_r-1,\dots,\hat{y}_{r-1}-\hat{y}_r-1)h_r^j(y;\hat{\mathbf{y}}) \\ &= h_k^{r-1}\big(\hat{y}_r+1;\hat{\mathbf{y}}\big)h_r^j(y;\hat{\mathbf{y}}) = \partial_r h_k^j\big(y;\hat{\mathbf{y}}\big). \end{aligned}$$

By summing over $r=k+1,\dots,j$, we arrive at

$$\mathbf{D}h_k^j(y;\hat{\mathbf{y}}) = h_k^j(y;\hat{\mathbf{y}}) - h_k^j(y;\hat{\tau}_1\hat{\mathbf{y}}), \tag{5.32}$$

which is equivalent to (5.31). □

Proof of Theorem 5.3(i) *(Step 1)*The main ingredient for the proof is the following:

$$\mathcal{V}\big((i,x),(j,y)\big) = V_{ij}(x,y) = \begin{cases} \phi^{j-i}(x,\hat{y}_j)\mathbb{1}(i\ge j), & y=\hat{y}_j; \\ -\phi^{j-i}(x,y)\mathbb{1}(i<j), & x<\hat{y}_i. \end{cases} \tag{5.33}$$

First note that by (5.26)

$$V_j(x,y) = \sum_{k=1}^{j}\phi^{k-j}(x,\hat{y}_k)h_k^j(y) = \sum_{k=1}^{j-1}\phi^{k-j}(x,\hat{y}_k)h_k^j(y) + \mathbb{1}(x=\hat{y}_j). \tag{5.34}$$

Let us write $(\tau f)(x) = f(x+1)$ so that $\phi^{-1} = \mathbb{D}_+ = \tau - \mathbb{1}$. We certainly have

$$\phi^{-\ell}(x,y) = \sum_{s=0}^{\ell}(-1)^{k-s}\binom{\ell}{s}\tau^s(x,y) = \sum_{s=0}^{\ell}(-1)^{k-s}\binom{\ell}{s}\mathbb{1}(y=x+s).$$

From this and (5.14) we learn

$$r\in\mathbb{Z}\ ,\ x-a<r \quad\Longrightarrow\quad \phi^r(x,a)=0. \tag{5.35}$$

We claim that $V_j(x,y)=0$ for $x<\hat{y}_j$. To see this observe that if $x<\hat{y}_j$, then for every $k\in[j]$,

$$x-\hat{y}_k = x-\hat{y}_j+\hat{y}_j-\hat{y}_k < k-j,$$

which leads to $\phi^{k-j}(x,\hat{y}_k)=0$ by (5.34). This and (5.34) imply that $V_j(x,y)=0$. Moreover, using (5.14), and

$$\phi^{j-i}V_j(x,y) = \sum_a \phi^{j-i}(x,a)V_j(a,y) = \sum_{a\ge\hat{y}_j}\phi^{j-i}(x,a)V_j(a,y),$$

we deduce

$$x < \hat{y}_i \quad \implies \quad \phi^{j-i} V_j(x, y) = 0, \tag{5.36}$$

because $x < \hat{y}_i$ and $a \geq \hat{y}_j$ imply $x - a < \hat{y}_i - \hat{y}_j \leq j - i$.

Using (5.25) and (5.34),

$$V_j(x, \hat{y}_j) = 1\!\!1(x = \hat{y}_j).$$

This and (5.34) lead to

$$V_j(x, y) = 1\!\!1(x \geq \hat{y}_j \neq y) \sum_{k=1}^{j-1} \phi^{k-j}(x, \hat{y}_k) h_k^j(y) + 1\!\!1(x = \hat{y}_j), \tag{5.37}$$

$$\big(\phi^{j-i} V_j\big)(x, \hat{y}_j) = \sum_a \phi^{j-i}(x, a) V_j(a, \hat{y}_j) = \phi^{j-i}(x, \hat{y}_j).$$

As a consequence,

$$V_{ij}(x, \hat{y}_j) = \big(\phi^{j-i} V_j\big)(x, \hat{y}_j) - \phi^{j-i}(x, \hat{y}_j) 1\!\!1(i < j) = \phi^{j-i}(x, \hat{y}_j) 1\!\!1(i \geq j). \tag{5.38}$$

(Step 2.) Given $a_1, \dots, a_N \in \mathbb{Z}$, let us define $u : [N] \times \mathbb{Z} \to \mathbb{R}$ by $u(i, x) = 1\!\!1(x \leq a_i)$. We use Proposition 3.1**(iii)** to assert

$$\mathbb{P}\big(\hat{x}_1 > a_1, \dots, \hat{x}_N > a_N\big) = \det\big(1\!\!1 - \widehat{\mathcal{K}}\big), \tag{5.39}$$

where

$$\widehat{\mathcal{K}}\big((i, x), (j, y)\big) = \mathcal{K}\big((i, x), (j, y)\big)(0) u^{1/2}(i, x) u^{1/2}(j, y) = K_{ij}(0)(x, y) 1\!\!1(x \leq a_i, y \leq a_j).$$

Note that since the sequence $\big(\hat{x}_i : \ i \in [N]\big)$ is decreasing , we may replace the sequence $(a_i : \ i \in [N])$ with $(a_i' : \ i \in [N])$, for $a_i' = \max(a_j : j \geq i)$. Hence, we may assume that $a_1 > \cdots > a_N$ without loss of generality. Under this assumption, we wish to show that the right-hand side of (5.36) is

$$1\!\!1\big(\hat{y}_1 > a_1, \dots, \hat{y}_N > a_N\big) =: 1\!\!1\big(\hat{\mathbf{y}} \in \mathbf{a}\big).$$

To prove this, first assume $\hat{\mathbf{y}} \notin \mathbf{a}$. This means that $\hat{y}_j \leq a_j$ for some $j \in [N]$. Without loss of generality, we assume that j is the largest such index. Under such circumstances, the $(j, \hat{y}_j)$-th column of $\widehat{\mathcal{K}}$ is the vector

$$\big(V_{ij}(x, \hat{y}_j) 1\!\!1(x \leq a_i) : \ i \in [N], x \in \mathbb{Z}\big) = \big(\phi^{j-i}(x, \hat{y}_j) 1\!\!1(i \geq j, x \leq a_i) : \ i \in [N], x \in \mathbb{Z}\big).$$

Observe that if $\phi^{j-i}(x, \hat{y}_j) \neq 0$, then

$$\hat{y}_i \leq \hat{y}_j + j - i \leq x.$$

by (5.35). As a result

$$\phi^{j-i}(x,\hat{y}_j)\mathbb{1}(i \geq j, x \leq a_i) \neq 0 \qquad \Longrightarrow \; i \geq j, \; \hat{y}_i \leq a_i.$$

But since j is the largest possible index for which $\hat{y}_j \leq a_j$, we learn

$$\begin{aligned}\big(V_{ij}(x,\hat{y}_j)\mathbb{1}(x \leq a_i): \; i \in [N], x \in \mathbb{Z}\big) &= \big(\phi^{j-i}(x,\hat{y}_j)\mathbb{1}(i=j, x \leq a_i): \; i \in [N], x \in \mathbb{Z}\big) \\ &= \big(\mathbb{1}(x=\hat{y}_j, \; i=j): \; i \in [N], x \in \mathbb{Z}\big).\end{aligned}$$

As a result, the matrix $\mathbb{1} - \widehat{\mathcal{K}}$ has a 0 column. In particular,

$$\hat{\mathbf{y}} \notin \mathbf{a} \qquad \Longrightarrow \qquad \det\big(\mathbb{1} - \widehat{\mathcal{K}}\big) = 0.$$

We now turn to the case $\hat{\mathbf{y}} \in \mathbf{a}$, which means that $\hat{y}_i > a_i$ for all $i \in [N]$. Let us write $\widehat{V}_{ij}$ for the (i,j)-th block of $\widehat{\mathcal{K}}$. Then by (5.36),

$$\begin{aligned}\widehat{V}_{ij}(x,y) = V_{ij}(x,y)\mathbb{1}(x \leq a_i, y \leq a_j) &= \big(\phi^{j-i}V_j - \phi^{j-i}\mathbb{1}(i<j)\big)(x,y)\mathbb{1}(x \leq a_i, y \leq a_j) \\ &= -\phi^{j-i}(x,y)\mathbb{1}(i<j)\mathbb{1}(x \leq a_i, y \leq a_j) \\ &= -\phi^{j-i}(x,y)\mathbb{1}(i<j)\mathbb{1}(x \leq a_i, y \leq a_j, x-y>0).\end{aligned}$$

As a consequence, the matrix $\widehat{\mathcal{K}}$ is strictly lower triangular. This implies that $\det(\mathbb{1} - \widehat{\mathcal{K}}) = 1$, as desired.

Proof of (ii) Clearly if $r > j$, then $\partial_r V_j = 0$. On the other hand, if $r \leq j$, then by Proposition 5.1**(iii)**,

$$\begin{aligned}\partial_r V_j\big(\hat{\mathbf{y}}\big)(x,y) &= \sum_{k=1}^{r-1} \phi^{k-j}(x,\hat{y}_k)\partial_r h_k^j\big(y;\hat{\mathbf{y}}\big) + \big(\phi^{r-j}(x,\hat{y}_r+1) - \phi^{r-j}(x,\hat{y}_r)\big)h_r^j(y;\hat{\mathbf{y}}) \\ &= \left(\sum_{k=1}^{r-1} \phi^{k-j}(x,\hat{y}_k)h_k^{r-1}(\hat{y}_r+1;\hat{\mathbf{y}})\right) h_r^j(y;\hat{\mathbf{y}}) \\ &\qquad + \big(\phi^{r-j}(x,\hat{y}_r+1) - \phi^{r-j}(x,\hat{y}_r)\big)h_r^j(y;\hat{\mathbf{y}}) \\ &=: f_r^j\big(x;\hat{\mathbf{y}}\big)h_r^j\big(y;\hat{\mathbf{y}}\big),\end{aligned}$$

where

$$f_r^j\big(x;\hat{\mathbf{y}}\big) = \phi^{r-j}(x,\hat{y}_r+1) - \phi^{r-j}(x,\hat{y}_r) + \sum_{k=1}^{r-1} \phi^{k-j}(x,\hat{y}_k)h_k^{r-1}(\hat{y}_r+1;\hat{\mathbf{y}}).$$

Note that if $\hat{y}_r + 1 = \hat{y}_{r-1}$, then $h_k^{r-1}(\hat{y}_r+1;\hat{\mathbf{y}}) = h_k^{r-1}(\hat{y}_{r-1};\hat{\mathbf{y}}) = 0$, whenever $k < r-1$. Thus,

$$\hat{y}_r + 1 = \hat{y}_{r-1} \;\text{ and }\; 1 < r \leq j \qquad \Longrightarrow \qquad V_j\big(\hat{\mathbf{y}}_r\big) - V_j\big(\hat{\mathbf{y}}\big) = 0,$$

as desired.

Proof of (iii) Since $\mathbb{D}(x,y) = \mathbb{1}(y = x-1) - \mathbb{1}(y = x) = \mathbb{1}(x = y+1) - \mathbb{1}(y = x)$,

$$\big(\mathbb{D}V_j - V_j\mathbb{D}\big)(x,y) = V_j(x-1,y) - V_j(x,y) - \big(V_j(x,y+1) - V_j(x,y)\big).$$

Note

$$\begin{aligned}
V_j(x-1,y) - V_j(x,y) &= \sum_{r=1}^{j} \big(\phi^{r-j}(x-1,\hat{y}_k) - \phi^{r-j}(x,\hat{y}_k)\big) h_r^j(y) \\
&= \sum_{r=1}^{j} \big(\phi^{r-j}(x,\hat{y}_r+1) - \phi^{r-j}(x,\hat{y}_r+1)\big) h_r^j(y), \\
V_j(x,y+1) - V_j(x,y) &= \sum_{k=1}^{j} \phi^{k-j}(x,\hat{y}_k)\big(h_k^j\big(y+1;\hat{\mathbf{y}}\big) - h_k^j\big(y;\hat{\mathbf{y}}\big)\big).
\end{aligned}$$

On the other hand, $\mathbf{D}_{\hat{\mathbf{y}}}V_j = \Omega_1 + \Omega_2$, with

$$\begin{aligned}
\Omega_1 =& \sum_{r=1}^{j}\sum_{k=1}^{r-1} \phi^{k-j}(x,\hat{y}_k)\left(h_k^j\big(y;\hat{\mathbf{y}}_r\big) - h_k^j(y;\hat{\mathbf{y}})\right) \\
=& \sum_{k=1}^{j-1} \phi^{k-j}(x,\hat{y}_k) \sum_{r=k+1}^{j} \left(h_k^j\big(y;\hat{\mathbf{y}}_r\big) - h_k^j(y;\hat{\mathbf{y}})\right) \\
=& -\sum_{k=1}^{j-1} \phi^{k-j}(x,\hat{y}_k)\big(h_k^j\big(y+1;\pi_{k+1}^j\hat{\mathbf{y}}\big) - h_k^j\big(y;\pi_{k+1}^j\hat{\mathbf{y}}\big)\big) \\
=& -\big(V_j(x,y+1) - V_j(x,y)\big), \\
\Omega_2 =& \sum_{r=1}^{j} \big(\phi^{r-j}(x,\hat{y}_r+1) - \phi^{r-j}(x,\hat{y}_r)\big) h_r^j(y;\hat{\mathbf{y}}) \\
=& V_j(x-1,y) - V_j(x,y).
\end{aligned}$$

Here we used (5.31) for the fourth equality. This completes the proof. $\square$

If we set

$$F(\hat{\mathbf{y}},t) = \mathbb{P}(\hat{\mathbf{x}}(t) > \mathbf{a} \mid \hat{\mathbf{x}}(0) = \hat{\mathbf{y}}),$$

then

$$\frac{dF}{dt}(\hat{\mathbf{y}},t) = \mathcal{L}F(\hat{\mathbf{y}},t), \qquad F(\hat{\mathbf{y}},0) = \mathbb{1}\big(\hat{\mathbf{y}} > \mathbf{a}\big). \tag{5.40}$$

On the other hand, we have a candidate for F with the aid of our determinantal formula:

$$F(\hat{\mathbf{y}},t) == \det\big(\mathbb{1}-\widehat{\mathcal{K}}(\hat{\mathbf{y}},t)\big), \tag{5.41}$$

where

$$\widehat{\mathcal{K}}(\hat{\mathbf{y}},t)=\mathcal{K}(\hat{\mathbf{y}},t)\chi=e^{t\mathbb{D}}\mathcal{V}(\hat{\mathbf{y}})e^{-t\mathbb{D}}\chi,$$

where $\chi\big((i,x),(j,y)\big)=\mathbb{1}\big(x\ge a_i, y\ge a_j\big)$. It is instructive to verify this directly.

Theorem 5.4 *The function F, given by* (5.41) *satisfies* (5.40).

Proof Set $G(\hat{\mathbf{y}},t)=\big(\mathbb{1}-\widehat{\mathcal{K}}\big)^{-1}$. From the elementary identity

$$\det(A+\delta B)=\det A\big(1+\delta\ tr\big(A^{-1}B\big)+O(\delta^2)\big),$$

we deduce

$$\begin{aligned} F_t(\hat{\mathbf{y}},t)&=-F(\hat{\mathbf{y}})\ tr\left(G(\hat{\mathbf{y}},t)\frac{d}{dt}\widehat{\mathcal{K}}(\hat{\mathbf{y}},t)\right)\\ &=-F(\hat{\mathbf{y}})\ tr\left(G(\hat{\mathbf{y}},t)\big(\mathbb{D}\mathcal{K}(\hat{\mathbf{y}},t)-\mathcal{K}(\hat{\mathbf{y}},t)\mathbb{D}\big)\chi\right). \end{aligned} \tag{5.42}$$

On the other hand, by Theorem 5.3**(ii)**, the kernel $\mathcal{K}(\hat{\mathbf{y}},t)$ satisfies the same boundary condition as in (5.29) because $\mathcal{V}$ satisfies (5.29). This in turn implies the function F also satisfies the same boundary condition. Hence we can replace $\mathcal{L}$ with $\mathbf{D}=\sum_r \mathbb{D}_r^+$ in (5.40), where $\mathbb{D}_r^+$ acts on x_r only. Let us write

$$\mathcal{K}\big((i,x),(j,y);\hat{\mathbf{y}}_r,t\big)-\mathcal{K}\big((i,x),(j,y);\hat{\mathbf{y}},t\big)=a^r((i,x);\hat{\mathbf{y}},t)\otimes b^r((j,y);\hat{\mathbf{y}},t),$$

because $\mathbb{D}_r^+\mathcal{K}$ is a rank-one matrix. Observe that for a matrix A and vectors $\mathbf{a},\mathbf{b}$, we always have

$$\begin{aligned} \det(A+\mathbf{a}\otimes\mathbf{b})&=\det A\ \det\big(\mathbb{1}+A^{-1}\mathbf{a}\otimes\mathbf{b}\big)=\det A\ \det\big(\mathbb{1}+\big(A^{-1}\mathbf{a}\big)\otimes\mathbf{b}\big)\\ &=\det A\sum_{\mathbf{c}}\det\big(\big(A^{-1}\mathbf{a}\big)\otimes\mathbf{b}\big)_{\mathbf{c}}=\det A\sum_{|\mathbf{c}|=1}\det\big(\big(A^{-1}\mathbf{a}\big)\otimes\mathbf{b}\big)_{\mathbf{c}}\\ &=(\det A)\ \big(A^{-1}\mathbf{a}\cdot\mathbf{b}\big)=(\det A)\ tr\big(A^{-1}\mathbf{a}\otimes\mathbf{b}\big). \end{aligned}$$

As a result

$$\begin{aligned} \mathcal{L}F(\hat{\mathbf{y}},t)=\mathbf{D}F(\hat{\mathbf{y}},t)&=\sum_r\Big[\det\Big(\mathbb{1}-\widehat{\mathcal{K}}(\hat{\mathbf{y}},t)-\big(\widehat{\mathcal{K}}(\hat{\mathbf{y}}_r,t)-\widehat{\mathcal{K}}(\hat{\mathbf{y}},t)\big)\Big)-\det\Big(\mathbb{1}-\widehat{\mathcal{K}}(\hat{\mathbf{y}},t)\Big)\Big]\\ &=\sum_r\Big[\det\Big(\mathbb{1}-\widehat{\mathcal{K}}(\hat{\mathbf{y}},t)-a^r(\hat{\mathbf{y}}_r,t)\otimes b^r(\hat{\mathbf{y}}_r,t)\ \chi\Big)-\det\Big(\mathbb{1}-\widehat{\mathcal{K}}(\hat{\mathbf{y}},t)\Big)\Big]\\ &=-F(\hat{\mathbf{y}})\sum_r tr\big(G(\hat{\mathbf{y}},t)\ a^r(\hat{\mathbf{y}}_r,t)\otimes b^r(\hat{\mathbf{y}}_r,t)\ \chi\big)=-F(\hat{\mathbf{y}})\ tr\big(G(\hat{\mathbf{y}},t)\mathbf{D}\widehat{\mathcal{K}}(\hat{\mathbf{y}},\mathbf{t})\big). \end{aligned}$$

We are done if we can show

$$tr\Big(G(\hat{\mathbf{y}},t)\big(\mathbb{D}\widehat{\mathcal{K}}(\hat{\mathbf{y}},t)-\widehat{\mathcal{K}}(\hat{\mathbf{y}},t)\mathbb{D}\big)\Big)==tr\big(G(\hat{\mathbf{y}},t)\mathbf{D}\widehat{\mathcal{K}}(\hat{\mathbf{y}},\mathbf{t})\big). \tag{5.43}$$

This is an immediate consequence of Theorem 5.3. □

Remark 5.2 From the proof of Theorem 5.4, we learn this: If $\mathcal{K}(x,y;\hat{\mathbf{y}},t)=\mathcal{K}(\hat{\mathbf{y}},t)$ satisfies

- $\mathcal{K}_t(\hat{\mathbf{y}},t)=\mathbf{D}\mathcal{K}(\hat{\mathbf{y}},\mathbf{t})$.
- $D_i\mathcal{K}(\hat{\mathbf{y}},t)$ is a rank-one matrix for each i,

then F given by (5.41) satisfies (5.40). □

Example 5.1 When $\hat{y}_i=i+r$ for $i=k,\dots,j$, then

$$h_k^j(y;\hat{\mathbf{y}})=h_k^j(y;\hat{y}_{k+1},\dots,\hat{y}_j)=(j-k)!^{-1}(y-\hat{y}_{k+1})(y-\hat{y}_{k+2})\dots(y-\hat{y}_j).$$

Indeed $h_k^{k+1}(y;\hat{\mathbf{y}})=1$, and

$$\begin{aligned}\mathbb{D}h_k^j(y;\hat{\mathbf{y}})&=h_k^j(y-1;\hat{\mathbf{y}})-h_k^j(y;\hat{\mathbf{y}})\\&=(\hat{y}_j-\hat{y}_k)(j-k)!^{-1}(y-\hat{y}_{k+1})(y-\hat{y}_{k+2})\dots(y-\hat{y}_{j-1})\\&=h_k^{j-1}(y;\hat{\mathbf{y}}),\end{aligned}$$

which implies our claim. □

Remark 5.3 We now try to justify the form of the kernel $\mathcal{V}$ or $\mathcal{K}$. Our discussion in Chapter 4 suggests a representation of the form (3.43) for the correlation kernel when the total number of particles is fixed and equals N. In other setting we are cosidering an extended kernel $\mathcal{V}:\hat{X}^2\to\mathbb{R}$, where $\hat{X}=[N]\times X$, $X=\mathbb{Z}$, and

$$V_{ij}(x,y)=\mathcal{V}\big((i,x),(j,y)\big)=\sum_{k=1}^N\phi_k(i,x)h_k(j,y)=\sum_{k=1}^N\phi_k^i(x)h_k^j(y).$$

However what we really have is of the form

$$V_{ij}(x,y)=\mathcal{V}\big((i,x),(j,y)\big)=\sum_{k=1}^N\phi_k^i(x)h_k^j(y)-\mathbb{1}(i<j)\phi^{j-i}(x,y),$$

where $\phi_k^i(x)=\phi^{k-i}(x,\hat{y}_k)$. To understand what is is going on, observe

$$\begin{aligned}V_{ij}\phi_r^j&=\phi_r^i-\mathbb{1}(i<j)\phi^{j-i}\phi_r^j=\phi_r^i-\mathbb{1}(i<j)\phi_r^i=\mathbb{1}(j\le i)\phi_r^i,\\h_r^iV_{ij}&=h_r^j-\mathbb{1}(i<j)h_r^i\phi^{j-i}=h_r^j-\mathbb{1}(i<j)h_r^j=\mathbb{1}(j\le i)h_r^j.\end{aligned}$$

For the second equality on the second line we have used Proposition 5.1**(i)**: If $i < j$, then

$$h_r^j \phi^{i-j} = \left(\phi^*\right)^{i-j} h_r^j = \mathbb{D}^{j-i} h_r^j = h_r^i, \qquad \text{or} \qquad h_r^i \phi^{j-i} = h_r^j.$$

Exercise

(i) Derive (5.6). $\square$

6 Scaling Limits for TASEP and KPZ Fixed Point

In Section 1.4 of Introduction, we formulated a scaling limit (1.22) that should be for any stochastic growth model in dimension 2. The determinantal formula of Chapter 5 for TASEP allows us to establish this scaling limit for TASEP. In fact since the determinantal formulation of Chapter 5 is for the particle location (as opposed to the height function), it is more convenient to establish such a scaling limit for particle locations. It is worth mentioning that the particle system $\mathbf{x}(t)$ is an example of a Zero Range Process (ZRP). The relationship between h and x is that the map $h \mapsto x(h)$ and $x \mapsto h(x)$ is

$$h(x(i)) - h(x(0)) = i.$$

From this, it is not hard to see that if

$$u(x,t) = \lim_{\varepsilon \to 0} \varepsilon h\left(\left[\frac{x}{\varepsilon}\right], \frac{t}{\varepsilon}\right), \qquad v(a,t) = \lim_{\varepsilon \to 0} \varepsilon x\left(\left[\frac{a}{\varepsilon}\right], \frac{t}{\varepsilon}\right),$$

then $u\big(v(a,t),t\big) = a$. As for the macroscopic equation, if

$$u_t = H(u_x), \qquad v_t = \hat{H}(v_a),$$

then using $H(p) = -p(1-p)$, $u_x v_a = 1$, and $u_t + u_x v_t = 0$, we obtain

$$\hat{H}(\rho) = 1 - \rho^{-1}.$$

Note,

$$\hat{H}'(\rho) = \rho^{-2}, \quad \hat{L}(\rho) := \rho\hat{H}'(\rho) - \hat{H}(\rho) = 2\rho^{-1} - 1, \quad \hat{H}''(\rho) = -2\rho^{-3}.$$

In particular, for $\rho = 2$, we have $\hat{L}(2) = 0$, $\hat{H}'(2) = 4^{-1}$, and $\hat{H}''(2) = -4^{-1}$. According to our formulation (1.22) we expect

$$\bar{h}(a,t) := \lim_{\varepsilon \to 0} X^{\varepsilon}(a,t) := \lim_{\varepsilon \to 0} \varepsilon^{-1/2}\left(\varepsilon x\left(\left[\frac{a}{\varepsilon} - \frac{t}{4\varepsilon^{3/2}}\right], \frac{t}{\varepsilon^{3/2}}\right) - 2a\right), \tag{6.1}$$

to be a solution to the KPZ fixed point

$$\bar{h}_t + 8^{-1}\, \bar{h}_a^2 = 0. \tag{6.2}$$

For each $\rho \geq 1$ ZRP has an invariant measure that can be interpreted as a random walk with geometric jump law. In other words we have an i.i.d sequence $\big(x_{i+1} - x_i :\ i \in \mathbb{Z}\big)$ with

$$\nu_\rho\big(x_{i+1} - x_i = k\big) = \rho^{-1}\big(1 - \rho^{-1}\big)^{k-1}\ 1\!\!1(k \geq 1).$$

This invariant measure is particularly simple when $\rho = 2$. In this case, we think of $(x_i :\ i \in \mathbb{Z})$ as a Markov process process with

$$\nu\big(\hat{x}_{n+1} = y \bigm| \hat{x}_n = x\big) = \nu\big(x_{n+1} = x \bigm| x_n = y\big) = 2^{y-x}\mathbb{1}(x > y). \tag{6.3}$$

Recall that by Remark **3.1(ii)**, we may conjugate a correlation kernel of a determinantal process by a function $\lambda(x)$ to obtain another correlation kernel for the same process. The form of (6.3) (keeping in mind that for our scaling formulation we have chosen $\rho = 2$) suggests that we conjugate our kernel of Theorem 5.1 by the function $\lambda(x) = 2^x$. Note that if K_1 and K_2 are two operators associated with kernels K_1 and K_2, then

$$\big(K_1 K_2\big)^\lambda = K_1^\lambda\ K_2^\lambda.$$

Hence conjugating the kernel K with λ is equivalent to conjugating both ϕ and Ψ. We write $\widehat{\phi}$ for the λ-conjugation of ϕ. Observe

$$\widehat{\phi}(x, y) = 2^{y-x}\mathbb{1}(x > y),$$

which is (6.3) and can be regarded as the jump kernel of a walk. In view of the scaling formulation (6.1), we wish to rewrite the correlation kernel in terms of macroscopic time T, macroscopic locations X and Y, and macroscopic labels a and b. This means that in Theorem 5.1, we have

$$\begin{aligned} &t = \varepsilon^{-3/2}T, \quad i = \big[\varepsilon^{-1}a - 4^{-1}\varepsilon^{-3/2}T\big], \quad j = \big[\varepsilon^{-1}b - 4^{-1}\varepsilon^{-3/2}T\big], \\ &x = 2\varepsilon^{-1}a + \varepsilon^{-1/2}X, \qquad y = 2\varepsilon^{-1}b + \varepsilon^{-1/2}Y. \end{aligned} \tag{6.4}$$

Let us first examine the first term on the right-hand side of (5.5): Since $j > i$ means that $b > a$, and $j - i = \varepsilon^{-1}(b - a)$, we wish to analyze $\widehat{\phi}^k$, for $k = r\varepsilon^{-1}, r > 0$ and after a change of variable $(x, y) \mapsto (X, Y)$ as in (6.4). Note that if $(\theta_i :\ i \in \mathbb{N})$ is a sequence of iid positive random variables with $\theta_i = n \in \mathbb{N}$ occurring with probability 2^{-n}, then

$$\theta_1 + \cdots + \theta_{[r\varepsilon^{-1}]} = 2r\varepsilon^{-1} + 2^{1/2}\varepsilon^{-1/2}B(r) + o\big(\varepsilon^{-1/2}\big),$$

where $B(r)$ is a standard Brownian motion. (This is a consequence of the classical Donsker Invariance Principle, though we only need CLT for (6.5) below.) Now take any bounded continuous $f : \mathbb{R} \to \mathbb{R}$. We certainly have

$$\varepsilon^{1/2}\sum_{y\in\mathbb{Z}} \varepsilon^{-1/2}\widehat{\phi}^{\varepsilon^{-1}(b-a)}\big(2\varepsilon^{-1}a + \varepsilon^{-1/2}X, y\big) f\left(\varepsilon^{1/2}\big(y - 2\varepsilon^{-1}b\big)\right) = \mathbb{E}f\big(X + 2^{1/2}B(b-a)\big) + o(1),$$

in small ε limit; the left-hand side is the expected value of $f(Y)$ for a walk that starts at time i from the location $x = 2\varepsilon^{-1}a + \varepsilon^{-1/2}X$, and lands at $y = 2\varepsilon^{-1}b + \varepsilon^{-1/2}Y$ at time j. Our convergence is an immediate consequence of CLT. In summary, weakly

$$\lim_{\varepsilon\to 0} \varepsilon^{-1/2}\ \widehat{\phi}^{\varepsilon^{-1}(b-a)}\big(2\varepsilon^{-1}a + \varepsilon^{-1/2}X, 2\varepsilon^{-1}b + \varepsilon^{-1/2}Y\big) = e^{(b-a)\partial^2}(X, Y), \tag{6.5}$$

with
$$e^{(b-a)\partial^2}(X,Y) = (4\pi(b-a))^{-1/2}\exp\big(-(X-Y)^2/(4t)\big).$$
The convergence (6.5) is also true locally uniformly in (X,Y). To see this, recall that explicitely
$$\widehat{\phi}^k(x,y) = 2^{x-y}\binom{x-y-1}{k-1}\mathbb{1}(x>y+k).$$
We may use this and Stirling's formula to establish (6.5).

We now turn our attention to the operator/matrix
$$\Psi_j = \big[\psi_k(x;j)\big]_{k\in[j],x\in\mathbb{Z}} = \big[e^{-t\mathbb{D}_-}\phi^{k-j}\big(x,\delta_{\hat{y}_k}\big)\big]_{k\in[j],x\in\mathbb{Z}}.$$
After conjugation with 2^x we get
$$\widehat{\Psi}_j = \Big[e^{-t\widehat{\mathbb{D}}_-}\widehat{\phi}^{k-j}\big(x,\delta_{\hat{y}_k}\big)\Big]_{k\in[j],x\in\mathbb{Z}},$$
with $\widehat{\mathbb{D}}_-$ the conjugation of $\mathbb{D}_-$. Observe that $\phi^{-1}=\mathbb{D}_+$ and $\widehat{\phi}^{-1}$ is $\widehat{\mathbb{D}}_+$, the conjugation of $\mathbb{D}_+$. Indeed
$$\big(\widehat{\mathbb{D}}_\pm f\big)(x) = \pm 2^{-x}\big(f(x\pm 1)2^{x\pm 1} - f(x)2^x\big) = \pm\big(2^{\pm}f(x\pm 1) - f(x)\big).$$
Hence
$$\widehat{\mathbb{D}}_+ = id + 2\mathbb{D}_+, \qquad \widehat{\mathbb{D}}_- = -2^{-1}\big(\mathbb{D}_- + id\big).$$
Our strategy for the derivation of (6.1) is as follows:

Theorem 6.1 *Given macroscopic parameters (X,Y,a,b,T), define microscopic parameters (x,y,i,j,t) as in (6.1). Then*

$$\lim_{\varepsilon\to 0}\varepsilon^{-1/2}\mathcal{K}\big((i,x),(j,y);t\big) = \lim_{\varepsilon\to 0}\varepsilon^{-1/2}K_{ij}(x,y;t) =: \bar{K}_{a,b}(X,Y;T), \tag{6.6}$$

where $\bar{K}$ will be defined below.

To get a feel for $\bar{K}$, write
$$\begin{aligned}K_{ij} &= e^{t\mathbb{D}}V_{ij}e^{-t\mathbb{D}} = e^{t\mathbb{D}}\phi^{j-i}\big(V_j - \mathbb{1}(i<j)\big)e^{-t\mathbb{D}}\\ &= e^{t\mathbb{D}}\phi^{-i}\big(\phi^{-j}V_j\phi^j\big)\phi^{-j}e^{-t\mathbb{D}} - \mathbb{1}(i<j)\phi^{j-i}\end{aligned}$$
We already know the limit of the last time by (6.5). We next examine the operator $e^{t\widehat{\mathbb{D}}_-}\widehat{\phi}^{-i}$.

Proposition 6.1 *Fot $T > 0$, we have*

$$\lim_{\varepsilon\to 0}\varepsilon^{-1/2}\left(e^{-\frac{T}{2\varepsilon^{3/2}}\mathbb{D}_-}\ \widehat{\phi}^{\frac{a}{\varepsilon}-\frac{T}{4\varepsilon^{3/2}}}\right)\left(\frac{X}{\varepsilon^{1/2}},\frac{2a}{\varepsilon}+\frac{Y}{\varepsilon^{1/2}}\right)=e^{a\partial^2+\frac{T}{6}\partial^3}(X,Y). \tag{6.7}$$

If we write $\bar{V}_b$ for the contribution coming from $(\phi^{-j}V_j\phi^j$, we end up with a candidate for $\bar{K}$ of the form

$$\bar{K}_{a,b}=e^{a\partial^2-\frac{T}{6}\partial^3}\bar{V}e^{b\partial^2+\frac{T}{6}\partial^3}-1\!\!1(a<b)e^{(b-a)a\partial^2}. \tag{6.8}$$

Before embarking on the proof we make some conventions and comments, and give a heuristic proof of (6.7).

(i) So far we have used the same notation for an operator an its kernel. This is a extension of the common convention that we identify a linear operator with its matrix representation in finite dimension. We now push this convention further in the case of a convolution. More precisely, when the kernel $A(x,y)$ of an operator depends on $x-y$, then we write $A(x,y)=A(x-y)$. Our convention allows us to write

$$\text{Discrete Setting:}\quad (Af)(x)=\sum_{y\in\mathbb{Z}}A(x,y)f(y)=\sum_{y\in\mathbb{Z}}A(y)f(x-y)=(A*f)(x),$$

$$\text{Continuous Setting:}\quad (Af)(x)=\int A(x,y)f(y)\ dy=\int A(y)f(x-y)\ dy=(A*f)(x),$$

for a convolution operator. The next thing to address is the effect of our spatial rescaling. Given a kernel $A(x,y)=A(x-y)$, let us write

$$\big(S^\varepsilon A\big)(X,Y)=\varepsilon^{-1/2}A\big(\varepsilon^{-1/2}X,\varepsilon^{-1/2}Y\big).$$

Observe that in continuous setting

$$\big(S^\varepsilon A\big)f(X)=\int\varepsilon^{-1/2}A\big(\varepsilon^{-1/2}Y\big)f(X-Y)\ dY=\int A(y)f\big(X-\varepsilon^{1/2}y\big)\ dy.$$

(ii) Next we clarify what we really mean by (6.6) because both $\mathbb{D}_\pm$ and $\widehat{\phi}$ are operators on $\mathbb{Z}$. As in (4.9), we may represent the kernel $\phi^k(x,y)=\phi^{*k}(x-y)$ by a contour integral. Here we also write $\phi(x)=1\!\!1(x>0)$ and ϕ^{*k} we mean the k-fold convolution of ϕ by itself. Note that since

$$\sum_{y\in\mathbb{Z}}\phi(y)z^y=z(1-z)^{-1},$$

we have

$$\sum_{y\in\mathbb{Z}}\phi^{*k}(y)z^y=z^k(1-z)^{-k}.$$

By Fourier inversion

$$\phi^{*k}(x) = \frac{1}{2\pi i} \oint_{C_1(0)} z^{-x-1} z^k (1-z)^{-k} \, dz,$$

where $C_1(0)$ denotes the circle of radius 1 about the origin. So far we know this is true for $k \in \mathbb{N}$. But we also have

$$\sum_y \phi^{-1}(y) z^y = z^{-1}(1-z) = z^{-1} - 1,$$

where $\phi^{-1}(y) = \mathbb{1}(y=-1) - \mathbb{1}(y=0)$ is the kernel of the operator $\phi^{-1} = \mathbb{D}_+$. Hence it is true for $k \in \mathbb{Z}$. (Note that z^{-x} is an eigenfunction of the operator ϕ^k, with the corresponding eigenvalue $z^k(1-z)^{-k}$.) When k is negative, $z = 1$ is no longer a pole and we can deform the circle C_1 to any positive contour that includes 0. In other words, for any integer $k < 0$,

$$\phi^{*k}(x) = \frac{1}{2\pi i} \oint_\gamma z^{-x-1} \big(z^{-1} - 1\big)^{-k} \, dz,$$

or equivalently

$$\phi^k(x,y) = \frac{1}{2\pi i} \oint_\gamma z^{-(x-y)-1} \big(z^{-1} - 1\big)^{-k} \, dz.$$

We may also conjugate our operators with $\lambda(x) = 2^x$. The outcome is

$$\widehat{\phi}^k(x,y) = \frac{1}{2\pi i} \oint_\gamma 2^{x-y} z^{-(x-y)-1} \big(z^{-1} - 1\big)^{-k} \, dz, \tag{6.9}$$

for any integer $k < 0$. In the same manner we arrive at

$$\left(e^{-\frac{t}{2}(id+\mathbb{D}_-)} \ \widehat{\phi}^k\right)(x,y) = \frac{1}{2\pi i} \oint_\gamma 2^{x-y} z^{-(x-y)-1} z^k (1-z)^{-k} e^{t(z-1)} \, dz. \tag{6.10}$$

Equivalently

$$\left(e^{-\frac{t}{2}\mathbb{D}_-} \ \widehat{\phi}^k\right)(x,y) = \frac{1}{2\pi i} \oint_\gamma 2^{x-y} z^{-(x-y)-1} z^k (1-z)^{-k} e^{t(z-1/2)} \, dz. \tag{6.11}$$

(iii) We now provide a heuristic proof of (6.6) when $a = 0$. Observe that since

$$\widehat{\phi}^{-1} = id + 2\mathbb{D}_+,$$

we may formally write

$$e^{-2^{-1}\varepsilon^{-3/2}T\mathbb{D}_-} \ \widehat{\phi}^{-4^{-1}\varepsilon^{-3/2}T} = \exp\left[2^{-1}\varepsilon^{-3/2} T \left(-\mathbb{D}_- + \frac{1}{2}\log\big(id + 2\mathbb{D}_+\big)\right)\right].$$

We now apply S^ε to both sides. This turns the operator $\mathbb{D}_\pm$ to $\mathbb{D}^\varepsilon_\pm = S^\varepsilon \mathbb{D}_\pm$ with

$$\big(\mathbb{D}^\varepsilon_\pm f\big)(X) = \pm\big(f\big(X \pm \varepsilon^{-1/2}\big) - f(X)\big).$$

Since these operators are of order $O(\varepsilon^{1/2})$, we may write

$$-\mathbb{D}^\varepsilon_- + \frac{1}{2}\log\big(id + 2\mathbb{D}^\varepsilon_+\big) = \mathbb{D}^\varepsilon_+ - \mathbb{D}^\varepsilon_- - \big(\mathbb{D}^\varepsilon_+\big)^2 + \frac{4}{3}\big(\mathbb{D}^\varepsilon_+\big)^3 + O\big(\varepsilon^2\big).$$

On the other hand, the expression

$$\left(\mathbb{D}^\varepsilon_+ - \mathbb{D}^\varepsilon_- - \big(\mathbb{D}^\varepsilon_+\big)^2 + \frac{4}{3}\big(\mathbb{D}^\varepsilon_+\big)^3\right) f(X),$$

equals to

$$\begin{aligned} f\big(X+\varepsilon^{1/2}\big) + f\big(X-\varepsilon^{1/2}\big) - 2f(X) - \big[f\big(X+2\varepsilon^{1/2}\big) - 2f\big(X+\varepsilon^{1/2}\big) + f(X)\big]& \\ + \frac{4}{3}\big[f\big(X+3\varepsilon^{1/2}\big) - 3f\big(X+2\varepsilon^{1/2}\big) + 3f\big(X+\varepsilon^{1/2}\big) - f(X)\big]& \\ =\varepsilon f''(X) - \big(\varepsilon f''(X) + \varepsilon^{3/2} f'''(X)\big) + \frac{4}{3}\varepsilon^{3/2} f'''(X) + O\big(\varepsilon^2\big)& \\ =3^{-1}\varepsilon^{3/2} f'''(X) + O\big(\varepsilon^2\big).& \end{aligned}$$

(iv) The kernel of the right-hand side of (6.6) can be expressed as the *Airy function*. Originally the Airy function was defined as a solution of the simple ODE $y''(x) = xy(x)$. We may attempt to find a solution by Fourier Transform, or more generally as

$$y(x) = \int_\gamma f(z) e^{-xz}\, dz,$$

for a suitable of a function f and a path $\gamma : (a_-, a_+) \to \mathbb{C}$. Observe

$$\begin{aligned} y''(x) - xy(x) &= \int_\gamma z^2 f(z) e^{-xz}\, dz + \int_\gamma f(z)\, de^{-xz} \\ &= f(\gamma(a_-))e^{-x\gamma(a_-)} - f(\gamma(a_+))e^{-x\gamma(a_+)} + \int_\gamma \big(z^2 f(z) - f'(z)\big)e^{-xz}\, dz, \end{aligned}$$

which is zero if $f'(z) = z^2 f(z)$ and $f(\gamma(a_\pm))e^{-x\gamma(a_\pm)} = 0$. We choose $f(z) = z^3/3$, and $a_\pm$ suitable points at ∞ so that

$$y(x) = \int_\gamma e^{z^3/3 - zx}\, dz.$$

For this we require

$$\lim_{z\to a_\pm}\left|e^{z^3/3-zx}\right|=0.$$

By choosing $a_\pm$ different points at ∞, we may get different solutions. Indeed if $z=re^{i\theta}$, then

$$\left|e^{z^3/3-zx}\right|=e^{r^3\cos(3\theta)-xr\cos\theta}\to 0,$$

provided that $r\to\infty$ and $\cos(3\theta)$ remains negative. A simple choice would be this: take γ the union two half lines emanating from the origin that make angles $\pm\pi/3$ for the positive x-axis. We orient this γ as $\infty e^{-i\pi/3}\to 0\to\infty e^{i\pi/3}$. Given such a path γ, and after a renormalization, we define the Airy function by

$$Ai(x)=\frac{1}{2\pi i}\int_\gamma e^{z^3/3-zx}\ dz.$$

In fact we may even choose $\theta=3\pi/2$, though we will not have a convergence of the integral and we need to take an improper integral. When $\theta=\pi/2$, the integral is over the imaginary axis and we arrive at

$$\begin{aligned}Ai(x)&=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{z^3/3-zx}\ dz=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(r^3/3+rx)}\ dr=\frac{1}{\pi}\int_0^\infty \cos\left(r^3/3+rx\right)\ dr\\&:=\lim_{a\to\infty}\frac{1}{\pi}\int_0^a\cos\left(r^3/3+rx\right)\ dr.\end{aligned}$$

Airy function may be used to determine the kernel of the operator $e^{t\partial^3}$: The function

$$u(x,t)=\frac{1}{2\pi i}\int_\gamma e^{tz^3/3-zx}f(z)\ dz,$$

solves the PDE $u_t=3^{-1}u_{xxx}$ for any bounded measurable function f with the initial condition

$$u(x,0)=\frac{1}{2\pi i}\int_\gamma e^{-zx}f(z)\ dz$$

If we choose γ to be the imaginary axis, we obtain

$$u(x,0)=\int_{-\infty}^{\infty}e^{-2\pi ix\xi}f(i\xi)\ d\xi=\int_{-\infty}^{\infty}e^{-2\pi ix\xi}g(\xi)\ d\xi=\hat g(x).$$

Hence by choosing

$$g(\xi)=\hat u(\xi,0)=\int_{-\infty}^{\infty}e^{-2\pi ix\xi}u(x,0)\ dx,$$

we have the following expression: If $u^0(x) = u(x,0)$, then

$$u(x,t) = e^{t\partial^3/3} u^0(x) = \big(A^t * u^0\big)(x),$$

with

$$A^t(x) = \frac{1}{t^{1/3}} Ai\left(\frac{x}{t^{1/3}}\right) = \frac{1}{2\pi i}\int_\gamma e^{tz^3/3 - zx}\, dz.$$

$\square$

Proof of Proposition 6.1 Since all the operators that appear on the left-hand side of (6.6) are convolution operators, we may set $X = 0$ in (6.6) and evaluate the following limit

$$\lim_{\varepsilon\to 0} X_\varepsilon = \lim_{\varepsilon\to 0} \frac{1}{2\pi i}\oint_\gamma \varepsilon^{-1/2} 2^{-x} z^{-x-1} z^k (1-z)^{-k} e^{t(z-1/2)}\, dz = \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\oint_\gamma \varepsilon^{-1/2} e^{F(z)}\, dz$$

with γ a circle about the origin of radius $1/2$, and

$$\begin{aligned} x &= \frac{2a}{\varepsilon} + \frac{Y}{\varepsilon^{1/2}}, \qquad t = \frac{T}{2\varepsilon^{3/2}}, \qquad k = \frac{a}{\varepsilon} - \frac{T}{4\varepsilon^{3/2}}, \\ F(z) &= -x\log 2 + (k-x-1)\log z - k\log(1-z) + t(z-1/2). \end{aligned}$$

We now change variables $z = 2^{-1}\big(1 - w_\varepsilon\big) := 2^{-1}\big(1 - \varepsilon^{1/2} w\big)$ to write

$$X_\varepsilon = -\frac{1}{2\pi i}\oint_{\gamma'} \frac{1}{2} e^{G(w)}\, dw,$$

where γ' is a positively oriented circle of center $\varepsilon^{-1/2}$ and radius $\varepsilon^{-1/2}$, and

$$\begin{aligned} G(w) &= \log 2 + (k-x-1)\log\big(1-w_\varepsilon\big) - k\log\big(1+w_\varepsilon\big) - 2^{-1} t w_\varepsilon \\ &= \log 2 - \log\big(1-w_\varepsilon\big) - \frac{Y}{\varepsilon^{1/2}}\log\big(1-w_\varepsilon\big) - \frac{a}{\varepsilon}\log\big(1-w_\varepsilon^2\big) \\ &\quad + \frac{T}{4\varepsilon^{3/2}}\log\frac{1+w_\varepsilon}{1-w_\varepsilon} - \frac{T}{2\varepsilon^{3/2}} w_\varepsilon \\ &= \log 2 + w_\varepsilon + O\big(\varepsilon w\big) + Yw + O\big(\varepsilon^{1/2} w\big) + aw^2 + O\big(\varepsilon w^4\big) + Tw^3/6 + O\big(\varepsilon^{1/2} w^4\big) \\ &=: \log 2 + H_\varepsilon(w) = \log 2 + Yw + aw^2 + Tw^3/6 + O\big(\varepsilon^{1/2}(w^4+1)\big), \end{aligned}$$

because

$$\log\frac{1+w_\varepsilon}{1-w_\varepsilon} - 2w_\varepsilon = \frac{2}{3} w_\varepsilon^3 + O\big(w_\varepsilon^4\big) = \frac{2}{3}\varepsilon^{3/2} w + O\big(\varepsilon^2 w^4\big).$$

As a result,

$$X_\varepsilon = -\frac{1}{2\pi i}\oint_{\gamma'} e^{H_\varepsilon(w)}\, dw.$$

for a function $H_\varepsilon(w)$ such that for each $w \in \mathbb{C}$,

$$\lim_{\varepsilon\to 0} H_\varepsilon(w) = Yw + aw^2 + Tw^3/6.$$

We now write $\gamma' = \gamma' \cup \gamma''$, where γ' is the set $w = re^{i\theta} \in \gamma$ such that $\theta \notin [-\pi/6, \pi/6]$. After reversing the orientation in γ' and deforming γ' to two lines through the origin with angles $\pm\pi/3$ that lie inside γ', it is not hard to show

$$\lim_{\varepsilon\to 0} \frac{-1}{2\pi i} \oint_{\gamma'} e^{H_\varepsilon(w)}\, dw = \frac{1}{2\pi i} \oint_C e^{Yw+aw^2+Tw^3/6}\, dw,$$

with C the union of oriented half lines $\big(\infty e^{-i\pi/3}, 0\big]$ and $\big[0, \infty e^{i\pi/3}\big)$. To complete the proof, it remains to show

$$\lim_{\varepsilon\to 0} \frac{1}{2\pi i} \oint_{\gamma''} e^{H_\varepsilon(w)}\, dw = 0. \tag{6.12}$$

For $w = re^{i\theta} \in \gamma''$, we have $\theta \in [-\pi/6, \pi/6]$. Since for $z = 2^{-1}e^{i\alpha}$,

$$\varepsilon^{1/2} re^{i\theta} = \varepsilon^{1/2} w = 1 - 2z = 1 - e^{i\alpha} = (1 - \cos\alpha) - i \sin\alpha,$$

we deduce

$$\oint_{\gamma''} e^{H_\varepsilon(w)}\, dw = \varepsilon^{1/2} \oint_{\hat\gamma} e^{F(z)}\, dz,$$

where $\hat\gamma$ represents the part of the circle γ for which

$$\tan\big(\pi/2 - \theta\big) = \cot\theta = -\frac{1-\cos\alpha}{\sin\alpha} = -2\frac{\sin^2(\alpha/2)}{\sin\alpha} = -\tan(\alpha/2),$$

for some $\theta \in [-\pi/6, \pi/6]$. Hence $\arg z = \alpha \in [2\pi/3, 4\pi/3]$, on $\hat\gamma$. On the other hand,

$$\begin{aligned}
\big|e^{F(z)}\big| = \Re F(z) &= -x\log 2 - (k - x - 1)\log 2 - k\log|1 - z| + 2^{-1}t\big(\cos\alpha - 1\big)\\
&= \log 2 - k\big[\log 2 + 2^{-1}\log(5/4 - \cos\alpha)\big] + 2^{-1}t\big(\cos\alpha - 1\big)\\
&= \log 2 - 2^{-1}k\log(5 - 4\cos\alpha) + 2^{-1}t\big(\cos\alpha - 1\big)\\
&= \log 2 - 2^{-1}k\log\big(1 + 4(1-\cos\alpha)\big) + 2^{-1}t\big(\cos\alpha - 1\big)\\
&= \log 2 - \big(2k + 2^{-1}t\big)(1 - \cos\alpha) + k\zeta\big(\cos\alpha - 1\big),
\end{aligned}$$

where

$$\zeta(a) = 2a - 2^{-1}\log(1 + 4a),$$

which is a positive function if $a > 0$. On the other hand $2k + 2^{-1}t = 2a\varepsilon^{-1}$, and for $T > 0$, $-k$ is positive and of order $\varepsilon^{-3/2}$. As a result, there exists a positive constant c_0 such that

$$\big|e^{F(z)}\big| = \log 2 - 2a\varepsilon^{-1}(1 - \cos\alpha) + k\zeta\big(\cos\alpha - 1\big) \ge -c_0\varepsilon^{-3/2},$$

for small ε. This completes the proof of (6.12). $\square$

Exercise

(i) Suppose that η is a white noise in $\mathbb{R}^d$. Show that its Fourier transform $\hat{\eta}$ is also a white noise. When $\eta : \mathbb{T}^d \to \mathbb{R}$ is a white noise on the d-dimensional torus $\mathbb{T}^d$, describe its Fourier expansion.

(ii) Consider the PDE $u_t = u_{xxx}$. Show that if u is initially a white, then it is a white noise at later times.

(iii) Use

$$\varphi^{*k}(x) = \frac{1}{2\pi i} \oint_{C_r(0)} z^{-x-1}\big(z^{-1} - 1\big)^{-k}\, dz,$$

to show that if $k = -n$ for some $n \in \mathbb{N}$, then

$$\varphi^{*k}(x) = 1\!\!1(k \leq x \leq 0)(-1)^{k-x}\binom{-k}{-x}.$$

$\square$

7 Multiline and Multiclass Processes

Definition 7.1(i) Let us write Ω for the set of discrete subsets of $\mathbb{R} \times (0,\infty)$ that are unbounded in both coordinates. We also write X for unbounded discrete subsets of $\mathbb{R}$ such that if $(a,s),(b,t)$ are two distinct points in $\boldsymbol{\omega} \in \Omega$, then $a \neq b$ and $s \neq t$. We write $\mathbf{x} = \big(x_i : i \in \mathbb{Z}\big)$ for members of X, and assume

$$x_i < x_{i+1}, \qquad \lim_{i \to \pm\infty} x_i = \pm\infty.$$

for every $i \in \mathbb{Z}$. We write $\mathcal{X}$ for the set of non-crossing paths

$$\mathbf{x}(\cdot) = \big(x_i(\cdot) : \ i \in \mathbb{Z}\big) : [0,\infty) \to X,$$

such that each $x_i(\cdot)$ is an up-left left-continuous path which stays constant in between jumps to the left.

(ii) We define $\Phi^- : X \times \Omega \to \mathcal{X}$ with the following rule: If $\Phi^-(\mathbf{x},\boldsymbol{\omega}) = \mathbf{x}(\cdot)$, then each time an $\omega = (a,t) \in \boldsymbol{\omega}$ point shows up between $x_{i-1}(t)$ and $x_i(t)$, then $x_i(t)$ jumps to a, i.e., $x_i(t+) = a$. We also write $\Phi^-(t)(\mathbf{x},\boldsymbol{\omega}) = \Phi^-(\mathbf{x},\boldsymbol{\omega})(t) = \mathbf{x}(t)$. Think of $\mathbf{x}(t)$ as a point process in $\mathbb{R}$ with initial configuration $\mathbf{x} = \mathbf{x}(0)$. Note that each x_i path has an $\boldsymbol{\omega}$ point on every of its L corners. We also define $\Lambda^- : X \times \Omega \to \Omega$, where $\Lambda^-(\mathbf{x},\boldsymbol{\omega}) = \hat{\boldsymbol{\omega}}$ is the set of all $\neg$ corners:

$$\hat{\boldsymbol{\omega}} = \big\{(x_i(t),t) : \ \text{either} \ \ (x_i(t),t) \in \boldsymbol{\omega}, \ \ \text{or} \ \ (x_i(t+),t) \in \boldsymbol{\omega}, \ t \in \mathbb{R}^+, \ i \in \mathbb{Z}\big\}.$$

Note that by convention, if $(x_i(t),t) = \omega \in \boldsymbol{\omega}$, then $x_i(t+) = x_i(t)$ and we do include such a point in $\hat{\omega}$. Similarly, we define $\Phi^+ : X \times \Omega \to \mathcal{X}$ with the following rule: If $\Phi^+(\mathbf{x},\boldsymbol{\omega}) = \mathbf{x}(\cdot)$, then each time an $\omega = (a,t) \in \boldsymbol{\omega}$ point shows up between $x_i(t)$ and $x_{i+1}(t)$, then $x_i(t)$ jumps to a, i.e., $x_i(t+) = a$. In the same fashion, we define Λ^+.

(iii) We define $\Phi_n^- : X^n \times \Omega \to \mathcal{X}^n$ in the following manner:

$$\Phi_n^-\big(\mathbf{x}^1,\dots,\mathbf{x}^n,\omega\big) = \big(\mathbf{x}^1(\cdot),\dots,\mathbf{x}^n(\cdot)\big),$$

means that inductively,

$$\begin{aligned} \boldsymbol{\omega}_n &= \boldsymbol{\omega}, & \mathbf{x}^n(\cdot) &= \Phi^-\big(\mathbf{x}^n,\boldsymbol{\omega}_n\big), \\ \boldsymbol{\omega}_{n-1} &= \Lambda\big(\mathbf{x}^n,\boldsymbol{\omega}_n\big), & \mathbf{x}^{n-1}(\cdot) &= \Phi^-\big(\mathbf{x}^{n-1},\boldsymbol{\omega}_{n-1}\big), \\ &\ \vdots & &\ \vdots \\ \boldsymbol{\omega}_1 &= \Lambda\big(\mathbf{x}^2,\boldsymbol{\omega}_2\big), & \mathbf{x}^1(\cdot) &= \Phi^-\big(\mathbf{x}^1,\boldsymbol{\omega}_1\big). \end{aligned}$$

We think of $\overrightarrow{\mathbf{x}}(t) = \big(\mathbf{x}^1(t),\dots,\mathbf{x}^n(t)\big)$ as a *multiline* process with initial condition

$$\big(\mathbf{x}^1(0),\dots,\mathbf{x}^n(0)\big) = \overrightarrow{\mathbf{x}} = \big(\mathbf{x}^1,\dots,\mathbf{x}^n\big).$$

In the same fashion, we define Φ_n^+.

(iv) The process $\mathbf{x}(t) = \mathbf{x}^{\pm}(t) = \Phi_t^{\pm}(\mathbf{x}, \boldsymbol{\omega})$ is random once we put a probability measure $\mathbb{P}$ on Ω. If $\mathbb{P}$ is a Poisson point process of intensity one, then $\Phi^-(\mathbf{x}, \omega)$ is HAD process as was defined in Chapter 1. We refer to $\Phi^+(\mathbf{x}, \omega)$ as the *reverse HAD process.* More generally, we refer to the multiline $\Phi_n^-(\mathbf{x}, \omega)$ as PSF process after the work of Prähofer-Sphon [PS] and P. L. Ferrari [F]. We refer to $\Phi_n^+(\mathbf{x}, \omega)$ as the *reverse PSF process.* In our presentation, we have followed P.A. Ferrari and Martin [FM2]. $\square$

The multiline process can be used to explore the monotonicity properties of HAD process. PSF process is a Markov process with the generator

$$\mathcal{A}^- F\big(\overrightarrow{\mathbf{x}}\big) = \int_{\mathbb{R}} \big(F\big(J_a^- \overrightarrow{\mathbf{x}}\big) - F\big(\overrightarrow{\mathbf{x}}\big)\big)\ da,$$

where $J_a^- \overrightarrow{\mathbf{x}}$ is the configuration we obtain from $\overrightarrow{\mathbf{x}}$ by performing n many particle movements:

- For the first movement, find $x_{i_n}^n \in \mathbf{x}^n$ such that $a \in \big(x_{i_n-1}^n, x_{i_n}^n\big)$. Move $x_{i_n}^n$ to a.
- For the second movement, find $x_{i_{n-1}}^{n-1} \in \mathbf{x}^{n-1}$ such that $x_{i_n}^n \in \big(x_{i_{n-1}-1}^{n-1}, x_{i_{n-1}}^{n-1}\big]$. Move $x_{i_{n-1}}^{n-1}$ to $x_{i_n}^n$.
- Inductively repeat the above operation n times till a particle $x_{i_1}^1 \in \mathbf{x}^1$ such that $x_{i_2}^2 \in \big(x_{i_1-1}^1, x_{i_1}^1\big]$ is moved to $x_{i_2}^2$.

Similarly the reverse PSF process has a generator of the form

$$\mathcal{A}^+ F\big(\overrightarrow{\mathbf{y}}\big) = \int_{\mathbb{R}} \big(F\big(J_b^+ \overrightarrow{\mathbf{y}}\big) - F\big(\overrightarrow{\mathbf{y}}\big)\big)\ db,$$

where $J_b^+ \overrightarrow{\mathbf{y}}$ is the configuration we obtain from $\overrightarrow{\mathbf{y}}$ by performing n many particle movements:

- For the first movement, find $y_{i_1}^1 \in \mathbf{y}^1$ such that $b \in \big(y_{i_1}^1, y_{i_1+1}^1\big)$. Move $y_{i_1}^1$ to b.
- For the second movement, find $y_{i_2}^2 \in \mathbf{y}^2$ such that $y_{i_1}^1 \in \big[y_{i_2}^2, y_{i_2+1}^2\big)$. Move $y_{i_2}^2$ to $y_{i_1}^1$.
- Inductively repeat the above operation n times till a particle $y_{i_n}^n \in \mathbf{y}^n$ such that $y_{i_{n-1}}^{n-1} \in \big[y_{i_n}^n, y_{i_n+1}^n\big)$ is moved to $y_{i_{n-1}}^{n-1}$.

The following result of Cator and Groeneboom [CG] supports the relevance of the multiline process:

Theorem 7.1 . *Assume that* $\mathbf{x}$ *is independent of* $\boldsymbol{\omega}$*, and that* $\mathbf{x}$ *is a Poisson point process of intensity* $\rho > 0$*, and* ω *is a Poisson point process of intensity one. Then* $\hat{\boldsymbol{\omega}} = \Lambda^{\pm}(\mathbf{x}, \boldsymbol{\omega})$ *is also a Poisson point process of intensity one.*

Now imagine that $\overrightarrow{\mathbf{x}}$ and $\boldsymbol{\omega}$ are independent with $\boldsymbol{\omega}$ distributed according to a Poisson point process of intensity one, and $\overrightarrow{\mathbf{x}}$ selected according to

$$\nu^{\rho_1,\dots,\rho_n} = \nu^{\rho_1} \times \ldots \times \nu^{\rho_n}, \qquad \rho_1,\dots,\rho_n > 0,$$

where ν^ρ represents a Poisson point process in $\mathbb{R}$ of intensity ρ. Then by Theorem 7.1, each $\mathbf{x}^i(t)$ is a stationary HAD process for $i = 1,\dots,n$. P. A. Ferrari and Martin [FM2] show that in fact $\nu^{\rho_1,\dots,\rho_n}$ is an invariant measure for the process $\overrightarrow{\mathbf{x}}(t)$.

Theorem 7.2 *The adjoint $\mathcal{A}^{-*}$ of the operator $\mathcal{A}^-$ with respect to the measure $\nu^{\rho_1,\dots,\rho_n}$ is $\mathcal{A}^+$, the generator of the reverse PSF process. In particular, the measure $\nu^{\rho_1,\dots,\rho_n}$ is invariant for PSF process for every $\rho_1,\dots,\rho_n > 0$.*

Proof Let us write ν for $\nu^{\rho_1,\dots,\rho_n}$. We wish to show

$$\int G\big(\overrightarrow{\mathbf{x}}\big)\mathcal{A}^- F\big(\overrightarrow{\mathbf{x}}\big)\ \nu\big(d\overrightarrow{\mathbf{x}}\big) = \int F\big(\overrightarrow{\mathbf{y}}\big)\mathcal{A}^+ G\big(\overrightarrow{\mathbf{y}}\big)\ \nu\big(d\overrightarrow{\mathbf{y}}\big).$$

for bounded continuous $F, G : X^n \to \mathbb{R}$. For this, it suffice to show

$$\int_{\mathbb{R}}\int G\big(\overrightarrow{\mathbf{x}}\big)F\big(J_a^-\overrightarrow{\mathbf{x}}\big)\ \nu\big(d\overrightarrow{\mathbf{x}}\big)da = \int_{\mathbb{R}}\int F\big(\overrightarrow{\mathbf{y}}\big)G\big(J_b^+\overrightarrow{\mathbf{y}}\big)\ \nu\big(d\overrightarrow{\mathbf{x}}\big)db.$$

This is basically achieved by making a change of variable $J_a^-\overrightarrow{\mathbf{x}} = \overrightarrow{\mathbf{y}}$. We may express the left-hand side as a sum over integrals of the form

$$\int_{\mathbb{R}} da \int_{\mathbb{R}^n} dx^n_{i_n}\dots dx^1_{i_1}\ 1\!\!1\big(a =: x^{n+1}_{i_{n+1}} < x^n_{i_n} < x^{n-1}_{i_{n-1}} < \cdots < x^1_{i_1}\big)$$
$$\prod_{k=1}^n \rho_k \exp\big(\rho_k\big(x^{k+1}_{i_{k+1}} - x^k_{i_k}\big)\big)\int \hat\nu\big(d\overrightarrow{\mathbf{x}}\big)\ G\big(\overrightarrow{\mathbf{x}}\big)F\big(\overrightarrow{\mathbf{x}}^a\big),$$

where $\hat\nu$ is the measure ν conditioned that $\mathbf{x}^k$ has a particle at location $x^k_{i_k}$, and no particle in the interval $\big(x^{k+1}_{i_{k+1}}, x^k_{i_k}\big)$. Now as we make a change of variable $\overrightarrow{\mathbf{x}}^a = \overrightarrow{\mathbf{y}}$, and rename

$$a = x^{n+1}_{i_{n+1}}, x^n_{i_n}, x^{n-1}_{i_{n-1}}, \dots, x^1_{i_1}, \qquad \text{as} \qquad y^n_{j_n}, y^{n-1}_{j_{n-1}}, \dots, y^1_{j_1}, y^0_{j_0} =: b,$$

we arrive at

$$\int_{\mathbb{R}} db \int_{\mathbb{R}^n} dy^n_{j_n}\dots dy^1_{j_1}\ 1\!\!1\big(y^n_{j_n} < y^{n-1}_{j_{n-1}} < \cdots < y^1_{j_1} < y^0_{j_0} = b\big)$$
$$\prod_{k=1}^n \rho_k \exp\big(\rho_k\big(y^k_{j_k} - y^{k-1}_{j_{k-1}}\big)\big)\int \hat\nu'\big(d\overrightarrow{\mathbf{y}}\big)\ G\big(\overrightarrow{\mathbf{y}}_b\big)F\big(\overrightarrow{\mathbf{y}}\big),$$

where $\hat{\nu}'$ is the measure ν conditioned that $\mathbf{y}^k$ has a particle at location $y^k_{j_k}$ and no particle in the interval $\big(y^{k-1}_{j_{k-1}}, y^k_{j_k}\big)$. We are done. $\square$

We next construct a multiclass processes. The idea of the multiclass process is related to the monotonicity of the operator Φ^-:

Proposition 7.1 *If* $\mathbf{x} \subseteq \mathbf{y}$*, then* $\Phi(t)(\mathbf{x},\boldsymbol{\omega}) \subseteq \Phi(t)(\mathbf{y},\boldsymbol{\omega})$.

The proof of this Proposition is elementary and omitted. This Proposition allows us to define the multiclass process associated with HAD process.

Definition 7.2(i) We write

$$X_n = \big\{\big(\mathbf{x}^1,\dots,\mathbf{x}^n\big) \in X^n :\ \mathbf{x}^1 \subset \mathbf{x}^2 \subset \cdots \subset \mathbf{x}^n\big\}.$$

We define $R : X_n \to X^n$ by $R\big(\mathbf{x}^1,\dots,\mathbf{x}^n\big) = \big(\mathbf{z}^1,\dots,\mathbf{z}^n\big)$, where $\mathbf{z}^1 = \mathbf{x}^1$, and $\mathbf{z}^k = \mathbf{x}^k \setminus \mathbf{x}^{k-1}$ for $k = 2,\dots,n$.

(ii) We define $\widehat{\Phi}_n : X_n \times \Omega \to \mathcal{X}_n$, by

$$\widehat{\Phi}^{\pm}_n\big(\mathbf{x}^1,\dots,\mathbf{x}^n,\boldsymbol{\omega}\big) = \big(\Phi\big(\mathbf{x}^1,\boldsymbol{\omega}\big),\dots,\Phi\big(\mathbf{x}^n,\boldsymbol{\omega}\big)\big).$$

(iii) We define the multiclass operator $\Psi_n : X^n \times \Omega \to \mathcal{X}^n$, by

$$\Psi^{\pm}_n\big(\overrightarrow{\mathbf{z}},\boldsymbol{\omega}\big) = R\widehat{\Phi}^{\pm}_n\big(R^{-1}\overrightarrow{\mathbf{z}},\boldsymbol{\omega}\big).$$

We refer to

$$\overrightarrow{\mathbf{z}}(t) = \Psi^{\pm}_n\big(\overrightarrow{\mathbf{z}},\boldsymbol{\omega}\big)(t) = R\Psi^{\pm}_n(t)\big(R^{-1}\overrightarrow{\mathbf{z}},\boldsymbol{\omega}\big),$$

as the multiclass process. $\square$

Remark 7.1 For multiclass process, it is more convenient to consider the state space

$$Z = \big\{\overrightarrow{\mathbf{z}} = (\mathbf{x},\mathbf{m}) :\ \mathbf{x} \in X, \quad \mathbf{m} \in (0,\infty)^{\mathbb{Z}}\big\}.$$

In other words the i-th particle of coordinates $\bar{z}_i = (x_i, m_i)$ has a location $x_i \in \mathbb{R}$ and a class $m_i > 0$, where particles of lower classes have higher priority for jumping. We then define $\overrightarrow{\mathbf{z}}(t) = \mathbf{\Phi}^-\big(\overrightarrow{\mathbf{z}},\boldsymbol{\omega}\big)(t)$ to be a trajectory of a multiclass process. What we have in mind is that if we set

$$\mathbf{x}^m = \big\{x_i :\ (x_i, m_i) \in \overrightarrow{\mathbf{z}}(t),\ m_i \le m\big\},$$

then $\mathbf{x}^m(t) = \Phi^-\big(\mathbf{x}^m(0),\boldsymbol{\omega}\big)$ evolves as HAD process for every $m > 0$. We may write

$$R^{-1}\overrightarrow{\mathbf{z}} = \big(\mathbf{x}^m :\ m > 0\big).$$

For $\omega = (y_0, t) \in \boldsymbol{\omega}$, a particle of coordinate y_0 shows up at time t. When this happen we consider all particles that are to the right of y_0. For example, if $\overrightarrow{\mathbf{z}} = (\mathbf{x}, \mathbf{m})$ with $\mathbf{x} = (x_i : i \in \mathbb{Z})$, and $x_{i-1} < y_0 < x_i$, then we can find a unique sequence $i_1 = i < i_2 < \cdots < i_k$ such that the following conditions hold: $m_{i_1} > m_{i_2} > \cdots > m_{i_k}$, and if $j \in (i_r, i_{r+1})$, then $m_j \geq m_{i_r}$. We now change the configuration $\overrightarrow{\mathbf{z}}(t)$ to $K_{y_0}\overrightarrow{\mathbf{z}}$ that is defined by the following recipe: Replace $(x_{i_1}, m_{i_1}), (x_{i_2}, m_{i_2}) \dots, (x_{i_k}, m_{i_k})$ with $(y_0, m_{i_k}), (x_{i_2}, m_{i_1}) \dots, (x_{i_k}, m_{i_{k-1}})$. In other words, if we write $y_j = x_{i_j}$, $n_j = m_{i_j}$, then

$$(y_1, y_2, \dots, y_k) \ \to (y_0, y_2, \dots, y_k), \qquad (m_1, m_2, \dots, m_k) \ \to (m_k, m_1, \dots, m_{k-1}).$$

$\square$

Using an idea of Angle [A], we construct invariant measure for multiclass process. For our presentation we follow [FM2] to use queuing interpretation for Angle *collapsing process.*

Definition 7.3(i) We first define collapsing process for periodic configuration. Write $X(per)$ for set of 1-periodic $\mathbf{x} \in X$. In other words, $\mathbf{x} \in X(per)$ means that $\tau_1\mathbf{x} = \mathbf{x}$ where $\tau_1\mathbf{x}$ denotes the set we get from $\mathbf{x}$ by adding 1 to elements of $\mathbf{x}$. Now given two $\mathbf{x}, \mathbf{y} \in X(per)$ with

$$\big|\mathbf{x} \cap [0,1)\big| \leq \big|\mathbf{y} \cap [0,1)\big|,$$

we define $D^+(\mathbf{x}, \mathbf{y}) \in X(per)$ to be the unique configuration $\mathbf{z}$ with the following two properties:

$$\mathbf{z} \subseteq \mathbf{y}, \qquad \big|\mathbf{x} \cap [0,1)\big| = \big|\mathbf{z} \cap [0,1)\big|, \tag{7.1}$$

and for every $a \in \mathbf{z}$, there exists $b \leq a$ such that

$$\big|[b,a] \cap \mathbf{x}\big| \geq \big|[b,a] \cap \mathbf{y}\big|.$$

Similarly, we define $D^-(\mathbf{x}, \mathbf{y}) \in X(per)$ to be the unique configuration $\mathbf{z}$ for which (7.1) is true, and and for every $a \in \mathbf{z}$, there exists $b \leq a$ such that

$$\big|[a,b] \cap \mathbf{x}\big| \geq \big|[a,b] \cap \mathbf{y}\big|.$$

(ii) Let $\mathbf{x}, \mathbf{y} \in X$ be two stationary processes with intensities m and m'. Assume that $m < m'$. We may approximate $\mathbf{x}$ and $\mathbf{y}$ with ℓ-periodic configurations $\mathbf{x}_\ell$ and $\mathbf{y}_\ell$. By Ergodic Theorem we have

$$\big|\mathbf{x}_\ell \cap [0,\ell)\big| < \big|\mathbf{y}_\ell \cap [0,\ell)\big|,$$

for sufficiently large ℓ, almost surely. We then define $\mathbf{z}_\ell = D(\mathbf{x}_\ell, \mathbf{y}_\ell)$. Finally we set

$$\mathbf{z} = D^+(\mathbf{x}, \mathbf{y}) = \lim_{\ell \to \infty} D^+(\mathbf{x}_\ell, \mathbf{y}_\ell),$$

with satisfies the following two properties: $\mathbf{z} \subseteq \mathbf{y}$, and for every $a \in \mathbf{z}$, there exists $b \leq a$ such that

$$\big|[b,a] \cap \mathbf{x}\big| \geq \big|[b,a] \cap \mathbf{y}\big|. \tag{7.2}$$

We write $D_2 : X^2 \to X_2$ for

$$D_2^+(\mathbf{x},\mathbf{y}) = \big(D(\mathbf{x},\mathbf{y}),\mathbf{y}\big).$$

We define D^- and D_2^- is a similar fashion.

(iii) Given $\overrightarrow{\mathbf{x}} = \big(\mathbf{x}^1,\dots,\mathbf{x}^n\big)$, we define

$$D_n^{\pm}\big(\overrightarrow{\mathbf{x}}\big) = \overrightarrow{\mathbf{z}} = \big(\mathbf{x}^1,\dots,\mathbf{x}^n\big) \in X_n,$$

in the following manner:

$$\begin{aligned} \mathbf{z}^n = \mathbf{x}^n, \quad \mathbf{z}^{n-1} &= D^{\pm}\big(\mathbf{x}^{n-1},\mathbf{x}^n\big), \\ \mathbf{z}^{n-2} &= D^{\pm}\big(\mathbf{x}^{n-2},\mathbf{x}^{n-1},\mathbf{x}^n\big) := D^{\pm}\big(D^{\pm}\big(\mathbf{x}^{n-2},\mathbf{x}^{n-1}\big),\mathbf{x}^n\big), \\ \vdots \qquad & \qquad \vdots \qquad\qquad \vdots \\ \mathbf{z}^1 &= D^{\pm}\big(\mathbf{x}^1,\dots,\mathbf{x}^n\big) := D^{\pm}\big(D\big(\mathbf{x}^1,\dots,\mathbf{x}^{n-1}\big),\mathbf{x}^n\big). \end{aligned}$$

□

Remark 7.2(i) The following interpretation of $\mathbf{z} = D^+(\mathbf{x},\mathbf{y})$ was given in [PM]. We think of $\mathbf{x}$ as the set of times at which a costumer arrive at a queue. We think of $\mathbf{y}$ as the set of times at which service is provided to costumers. The set $\mathbf{z}$ is the departure times at which a costumer is received service and departs the queue, whereas $\mathbf{y} \setminus \mathbf{z}$ is the set of unused service times. When inequality (7.2) is true, then for sure b is a departure time because the number of costumers arriving during $[a,b]$ is at least the number of services available during the same period. Alternatively, we may use the points of $\mathbf{x}$ and $\mathbf{y}$ to define a process $q : \mathbb{R} \to \mathbb{N}^*$, which represents the length of the queue. More precisely at each occurrence of $x \in \mathbf{x}$, the function q increases by 1 i.e., $q(x+) = q(x) + 1$, and at each occurrence of $\mathbf{y}$, the function q decreases by 1, provided that $q(x) > 0$. When $\mathbf{x}$ and $\mathbf{y}$ are Poisson point processes of intensity ρ_1 and ρ_2, then q is a birth-death process with the birth and death rates ρ_1 and ρ_2 respectively. We may then define $D^+(\mathbf{x},\mathbf{y})$ to be the set at which q decreases. A similar interpretation can be given for D^-.

It is worth mentioning that D is monotone in the first argument:

$$\mathbf{x} \subseteq \mathbf{x}' \quad \Longrightarrow \quad D^{\pm}(\mathbf{x}',\mathbf{y}) \subseteq D^{\pm}(\mathbf{x},\mathbf{y}). \tag{7.3}$$

(ii) More generally we have a queuing interpretation for $D_n^+\big(\overrightarrow{\mathbf{x}}\big) = \overrightarrow{\mathbf{z}}$ that involves $n-1$ many queues. For this, we provide a queuing interpretation for $D^+\big(\mathbf{x}^1,\mathbf{x}^2,\dots,\mathbf{x}^n\big)$. We think of points in $\mathbf{x}^1$ as the arrival times for costumers, and $\mathbf{x}^2, \dots, \mathbf{x}^n$ as the services times for queues $1, 2, \dots, n-1$, respectively. As a costumer departs the first queue, it enters the second queue to receive service some time later at a moment in $\mathbf{x}^3$. This process continues until the costumer receives service in the $n-1$-th queue. The departure times from the last queue are the members of the set $\mathbf{z}^n = D^+\big(\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^n\big)$. $\square$

Our next result is due to P. A. Ferrari and James; it establishes a connection between the multiline and multiclass processes.

Theorem 7.3 *We have* $\hat{\Phi}^{\pm}\big(D_n^{\pm}\big(\overrightarrow{\mathbf{x}}\big), \boldsymbol{\omega}\big) = D_n^{\pm}\big(\Phi^{\pm}\big(\overrightarrow{\mathbf{x}}, \boldsymbol{\omega}\big)\big)$. *In particular, for every* $\rho_1, \dots, \rho_n$, *the measure* $D_n^{\pm}\big(\nu^{\rho_1, \dots, \rho_n}\big)$ *is invariant for the multi-class process.*

Proof Note that $\hat{\Phi}^{\pm}$ preserves the order but the multiline operator $\Phi^{\pm}$ does not. We basically need to prove this: If $\overrightarrow{\mathbf{x}} \in X_n$, and $y \in \mathbb{R}$, then

$$D_n^- J_y^- \overrightarrow{\mathbf{x}} = L_y^- \overrightarrow{\mathbf{x}}. \tag{7.4}$$

In other words, for a sequence of ordered $\overrightarrow{\mathbf{x}} = \big(\mathbf{x}^1, \dots, \mathbf{x}^n\big)$, we perform a multi-jump as a Poisson point (y, t) occurs at y. This multi-jump may mess up the order, i.e., $J_y^- \overrightarrow{\mathbf{x}}$ may not be in X_n. We restore the order by applying D_n^- to $J_y^- \overrightarrow{\mathbf{x}}$. The outcome coincide with the multiclass jump L_y^- that is related to K_y of Remark 7.1:

$$L_y^- \overrightarrow{\mathbf{x}} = R^{-1} K_y R \overrightarrow{\mathbf{x}}.$$

We first verify (7.4) when $n = 2$. Recall that $\mathbf{x}^1 \subseteq \mathbf{x}^2$. Choose $a_1 \in \mathbf{x}^1$ and $a_2 \in \mathbf{x}^2$ so that $y \le a_2 \le a_1$ and $(y, a_i) \cap \mathbf{x}^i = \emptyset$ for $i = 1, 2$. There are two cases to consider:

(i) (*Case* $a = a_1 = a_2$) We have $J_y^- \overrightarrow{\mathbf{x}} = \big(\mathbf{x}^1, \big(\mathbf{x}^2 \setminus \{a\}\big) \cup \{y\}\big) = \big(\mathbf{x}^1, \hat{\mathbf{x}}^2\big)$, and after matching a_1 of $\mathbf{x}^1$ with y of $\hat{\mathbf{x}}^2$, we arrive at $D_2\big(\mathbf{x}^1, \hat{\mathbf{x}}^2\big) = \big(\hat{\mathbf{x}}^1, \hat{\mathbf{x}}^2\big)$, where $\hat{\mathbf{x}}^1 = \big(\mathbf{x}^1 \setminus \{a\}\big) \cup \{y\}$. Evidently $L_y^- \overrightarrow{\mathbf{x}} = \big(\hat{\mathbf{x}}^1, \hat{\mathbf{x}}^2\big)$.

(ii) (*Case* $a_1 > a_2$) We have $J_y^- \overrightarrow{\mathbf{x}} = \big(\hat{\mathbf{x}}^1, \hat{\mathbf{x}}^2\big)$, where $\hat{\mathbf{x}}^2 = \big(\mathbf{x}^2 \setminus \{a_2\}\big) \cup \{y\}$, and $\hat{\mathbf{x}}^1 = \big(\mathbf{x}^1 \setminus \{a_1\}\big) \cup \{a_2\}$. After matching a_2 of $\hat{\mathbf{x}}^1$ with y of $\hat{\mathbf{x}}^2$, we arrive at $D_2\big(\hat{\mathbf{x}}^1, \hat{\mathbf{x}}^2\big) = \big(\mathbf{y}, \hat{\mathbf{x}}^2\big)$, where $\mathbf{y} = \big(\mathbf{x}^1 \setminus \{a_1\}\big) \cup \{y\}$. Evidently $L_y^- \overrightarrow{\mathbf{x}} = \big(\mathbf{y}, \hat{\mathbf{x}}^2\big)$.

We now turn to the general n. Recall that $\mathbf{x}^1 \subseteq \dots \subseteq \mathbf{x}^n$. Choose $a_i \in \mathbf{x}^i$ so that $y = y_0 \le a_n \le a_{n-1} \le \dots \le a_1$ and $(y, a_i) \cap \mathbf{x}^i = \emptyset$ for $i = 1, \dots, n$. Choose $1 \le n_1 < n_2 < \dots < n_k = n$ so that

$$\begin{aligned} y_0 < y_1 :&= a_{n_k} = \dots = a_{n_{k-1}-1} < y_2 := a_{n_{k-1}} = \dots = a_{n_{k-2}-1} < y_3 := a_{n_{k-2}} \\ &= \dots < y_k := a_{n_1} = \dots = a_1. \end{aligned}$$

Note that $J_y^- \overrightarrow{\mathbf{x}} = \big(\hat{\mathbf{x}}^1, \ldots, \hat{\mathbf{x}}^n\big)$, where

$$\begin{array}{rclcl}
\hat{\mathbf{x}}^{n_k} &=& \big(\mathbf{x}^{n_k} \setminus \{y_1\}\big) \cup \{y_0\}, & \ldots & \hat{\mathbf{x}}^{n_{k-1}-1} = \mathbf{x}^{n_{k-1}-1}, \\
\hat{\mathbf{x}}^{n_{k-1}} &=& \big(\mathbf{x}^{n_{k-1}} \setminus \{y_2\}\big) \cup \{y_1\}, & \ldots & \hat{\mathbf{x}}^{n_{k-2}-1} = \mathbf{x}^{n_{k-2}-1}, \\
& \vdots & & \vdots & \vdots \\
\hat{\mathbf{x}}^{n_1} &=& \big(\mathbf{x}^{n_1} \setminus \{y_k\}\big) \cup \{y_{k-1}\}, & \ldots & \ldots .
\end{array}$$

After matching the points

$$y_k \in \hat{\mathbf{x}}^1,\ \ldots,\ y_1 \in \hat{\mathbf{x}}^{n_{k-1}},\ \ldots,\ y_1 \in \hat{\mathbf{x}}^{n-1},\ y_0 \in \hat{\mathbf{x}}^n,$$

we arrive at

$$D_n\big(\hat{\mathbf{x}}^1, \ldots, \hat{\mathbf{x}}^n\big) = L_y^- \overrightarrow{\mathbf{x}}.$$

$\square$

A Exterior Algebra and Cauchy-Binet Formula

Given a vector space V, its r-fold exterior power $\wedge^r V$ is a vector space consisting of

$$\wedge^r V = \big\{v_1 \wedge \cdots \wedge v_r : \ v_1, \ldots, v_r \in V\big\}.$$

What we have in mind is that the r-vector $v_1 \wedge \cdots \wedge v_r$ represents the r-dimensional linear subspace that is spanned by vectors $v_1, \ldots, v_r$. By convention, $v_1 \wedge \cdots \wedge v_r = 0$ if $v_1, \ldots, v_r$ are not linearly independent. The wedge product is characterized by two properties: it is multilinear and alternative. By the former we mean that for all scalers c and c',

$$(cv_1 + c'v_1') \wedge v_2 \wedge \cdots \wedge v_r = c(v_1 \wedge v_2 \wedge \cdots \wedge v_r) + c'(v_1' \wedge v_2 \wedge \cdots \wedge v_r).$$

By the latter we mean that interchanging two vectors in $a = v_1 \wedge \cdots \wedge v_r$ changes the sign of a. If $\{e_1, \ldots, e_d\}$ is a basis for V, then

$$\big\{e_{i_1,i_2,\ldots,i_r} := e_{i_1} \wedge \cdots \wedge e_{i_r} : \ i_r < \cdots < i_r\big\}\,,$$

is a basis for $\wedge^r V$. In particular $\dim \wedge^r V = \binom{d}{r}$. If

$$v_j = \sum_{i=1}^{d} v_j^i e_i,$$

for coefficients $v_j^i \in \mathbb{R}$, and $j = 1, \ldots, r$, then

$$v_1 \wedge \cdots \wedge v_r = \sum_{1 \le i_1 < \cdots < i_r \le d} v^{i_1,\ldots,i_r} \ e_{i_1} \wedge \cdots \wedge e_{i_r},$$

where

$$v^{i_1,\ldots,i_r} = \sum_{\sigma \in S_r} \varepsilon(\sigma) v_1^{i_{\sigma(1)}} \ldots v_r^{i_{\sigma(r)}} = \det \big[v_j^{i_k}\big]_{j,k=1}^r.$$

In fact we may use (A.1) as our definition of the wedge product. To have a compact notation, write

$$M = \big[v_j^i\big], \qquad i \in I = \{1, \ldots, d\}, \ j \in J = \{1, \ldots, r\}.$$

Let us write $\hat{I}_r$ for the set of subsets $\mathbf{a} = \{i_1, \ldots, i_r\} \subseteq I$ with $i_1 < \cdots < i_r$. We then set

$$M_{\mathbf{a},J} = \big[v_j^{i_k}\big]_{j,k=1}^r, \qquad e_{\mathbf{a}} = e_{i_1} \wedge \cdots \wedge e_{i_r},$$

for any $\mathbf{a} \in I_k$. Then (A.1) can be written as

$$v_1 \wedge \cdots \wedge v_r = \sum_{\mathbf{a} \in \hat{I}_k} \det M_{\mathbf{a},J} \ e_{\mathbf{a}}. \tag{A.1}$$

To appreciate the role of exterior algebra in differential geometry, observe that if $P(v_1,\dots,v_k)$ denotes a parallelepiped formed from vectors $v_1,\dots,v_r$, then $\det M_{\mathbf{a},J}$ is the signed r-dimensional volume of the projection of $\det M_{\mathbf{a},J}$ on the linear span of $e_{i_1},\dots,e_{i_r}$. As a consequence

$$|v_1\wedge\dots\wedge v_r| := \left[\sum_{\mathbf{a}\in\hat I_k}\big(\det M_{\mathbf{a},J}\big)^2\right]^{1/2}, \tag{A.2}$$

is nothing other than the r-dimensional volume of $P(v_1,\dots,v_k)$.

Let V and V' be two vector spaces and assume that $A:V\to V'$ is a linear transformation. We define

$$\wedge^r A:\wedge^r V\to\wedge^r V',$$

by

$$\big(\wedge^r A\big)(v_1\wedge\dots\wedge v_r) = (Av_1)\wedge\dots\wedge(Av_r).$$

We continue with a list of straightforward properties of r-vectors.

Proposition A.1 **(i)** *Let $A:V'\to V$ be a linear transformation. Assume that $\{e_1,\dots,e_d\}$ and $\{e'_1,\dots,e'_{d'}\}$ are bases for V and V' respectively. If A is represented by a $d\times d'$ matrix with respect to the above bases, then the transformation $\wedge^r A$ is represented by a $\binom{d}{r}\times\binom{d'}{r}$ matrix we obtain by taking the determinants of all $r\times r$ submatrices of A.*

(iii) *If V,V',V'' are three vector spaces and $A:V'\to V$, $B:V''\to V'$, are linear, then $\wedge^r\big(A\circ B\big)=\big(\wedge^r A\circ\wedge^r B\big)$. If A is invertible, then $\wedge^r A^{-1}=\big(\wedge^r A\big)^{-1}$. If V and V' are inner product spaces and $A^*:V'\to V$ is the transpose of A, then $\wedge^r A^*=\big(\wedge^r A\big)^*$.*

Proof(i) If we write

$$a^j = Ae'_j = \sum_{i=1}^d a_{ij}e_i,$$

then

$$\big(\Lambda^r A\big)\big(e'_{j_1}\wedge\dots\wedge e'_{j_r}\big) = (Ae'_{j_1})\wedge\dots\wedge(Ae'_{j_r}) = a^{j_1}\wedge\dots\wedge a^{j_r}.$$

Given two sets $\mathbf{a}=\{i_1,\dots,i_r\}$ and $\mathbf{b}=\{j_1,\dots,j_r\}$, with $i_1<\dots<i_r$ and $j_1<\dots<j_r$, we write

$$A_{\mathbf{a},\mathbf{b}} = \big[a_{i_s j_t}\big]_{s,t=1}^r.$$

We now use (A.1) to write

$$\big(\Lambda^r A\big)e'_{\mathbf{b}} = \sum_{\mathbf{a}\in\hat I_r}\det A_{\mathbf{a},\mathbf{b}}\ e_{\mathbf{a}}, \tag{A.3}$$

as desired.

(ii) The proof follows from the definition of $\Lambda^r A$. $\square$

Given a $d \times d'$ matrix A, we may write

$$\Lambda^r A = \big[\det A_{\mathbf{ab}}\big]_{(\mathbf{a},\mathbf{b})\in \hat{I}_r \times \hat{I}'_r}, \tag{A.4}$$

where $\hat{I}_r$ (respectively $\hat{I}'_r$) denotes the set of subsets $\mathbf{a}$ of $I = \{1, \dots, d\}$ (respectively $I' = \{1, \dots, d'\}$) with $|\mathbf{a}| = r$. On account of Proposition A1, we have

$$\big(\Lambda^r(AB)\big)_{\mathbf{ab}} = \det(AB)_{\mathbf{ab}} = \sum_{\mathbf{c}\in \hat{I}'_r} \det A_{\mathbf{ac}} \ \det B_{\mathbf{cb}}, \quad (\mathbf{a},\mathbf{b}) \in \hat{I}_r \times \hat{I}''_r, \tag{A.5}$$

for any matrices A and B of sizes $d \times d'$ and $d' \times d''$. This identity is known as *Cauchy-Binet Formula.*

Example A1(i) Assume that $v_1, \dots, v_r \in \mathbb{R}^d$ with $r \in \{1, \dots, d\}$. Let A be a $d \times r$ matrix with rows $v_1, \dots, v_r$. Then $\Lambda^r A$ is a column vector in $\mathbb{R}^{\binom{d}{r}}$ that is exactly the expression (A.1) or (A.2), if we identify $\Lambda^r \mathbb{R}^d$ with $\mathbb{R}^{\binom{d}{r}}$. The identity

$$\Lambda^r(A^*A) = \big(\Lambda^r A^*\big)\big(\Lambda^r A\big) = \big(\Lambda^r A\big)^*\big(\Lambda^r A\big),$$

means

$$\det\big[v_i \cdot v_j\big]_{i,j=1}^r = \sum_{\mathbf{a}\in \hat{I}_r} \det A^2_{\mathbf{a},J} = \big|v_1 \wedge \cdots \wedge v_r\big|^2, \tag{A.6}$$

by (A.1). Here $J = \{1, \dots, r\}$.

(ii) We now derive a generalization of Cramer's formula for the inverse of a square matrix. Let A be a $d \times d$ matrix. Set $I = \{1, \dots k\}$ and assume that $1 \le r < k$. We claim that for any $\mathbf{b}, \mathbf{c} \in \hat{I}_r$,

$$\det\big(A^{-1}\big)_{\mathbf{bc}} = (-1)^{\zeta(\mathbf{b})+\zeta(\mathbf{c})}\ (\det A)^{-1} \det\big(A_{\mathbf{c}^c\mathbf{b}^c}\big), \tag{A.7}$$

where $\mathbf{b}^c = I \setminus \mathbf{b}$, $\mathbf{c}^c = I \setminus \mathbf{c}$ and

$$\zeta(\mathbf{b}) = \sum_{i\in\mathbf{b}} i.$$

This is the classical Cramer's formula when $r = 1$. To prove (A.7), write $\mathbf{b} = \{i_1, \dots, i_r\}$, and $\mathbf{c} = \{j_1, \dots, j_r\}$ with $i_1 < \cdots < i_r$ and $j_1 < \cdots < j_r$. Observe that by (A.3)

$$\big(\Lambda^r\big(A^{-1}\big)e_{\mathbf{c}}\big) \wedge e_{\mathbf{b}^c} = \sum_{\mathbf{a}\in \hat{I}_r} \det\big(A^{-1}\big)_{\mathbf{ac}} e_{\mathbf{a}} \wedge e_{\mathbf{b}^c} = \det\big(A^{-1}\big)_{\mathbf{bc}}\ e_{\mathbf{b}} \wedge e_{\mathbf{b}^c}.$$

We now apply $\Lambda^d A$ to both sides. The left-side yields

$$\big[(\Lambda^r A)\Lambda^r\big(A^{-1}\big)e_{\mathbf{c}}\big] \wedge \big(\Lambda^{d-r}A\big)e_{\mathbf{b}^c} = e_{\mathbf{c}} \wedge \big(\Lambda^{d-r}A\big)e_{\mathbf{b}^c} = e_{\mathbf{c}} \wedge \sum_{\mathbf{a}\in\hat{I}_r} \big(\det A_{\mathbf{a}\mathbf{b}^c}\big)e_{\mathbf{a}}$$
$$= \big(\det A_{\mathbf{c}^c\mathbf{b}^c}\big)e_{\mathbf{c}} \wedge e_{\mathbf{c}^c}.$$

The right-hand side yields

$$\det\big(A^{-1}\big)_{\mathbf{bc}}(\det A)\ e_{\mathbf{b}} \wedge e_{\mathbf{b}^c}.$$

From this we deduce

$$\big(\det A_{\mathbf{c}^c\mathbf{b}^c}\big) = \pm\det\big(A^{-1}\big)_{\mathbf{bc}}(\det A).$$

To figure out the sign, observe that in the expression

$$e_{\mathbf{b}} \wedge e_{\mathbf{b}^c} = e_{i_1} \wedge \cdots \wedge e_{i_r} \wedge e_{\mathbf{b}^c},$$

we need $i_r - r$ adjacent swapping to move e_{i_r} to its original place in $e_1 \wedge \cdots \wedge e_d$. Hence

$$e_{\mathbf{b}} \wedge e_{\mathbf{b}^c} = (-1)^{\eta(\mathbf{b})}e_1 \wedge \cdots \wedge e_d$$

for

$$\eta(\mathbf{b}) = i_1 + \cdots + i_r - (1 + \cdots + r).$$

Similarly

$$e_{\mathbf{c}} \wedge e_{\mathbf{c}^c} = (-1)^{\eta(\mathbf{c})}e_1 \wedge \cdots \wedge e_d.$$

This completes the proof of (A.7).

(iii) Let $A : \big(\mathbf{a} \cup \{y\}\big)^2 \to \mathbb{R}$ be a symmetric matrix. Then

$$\text{(A.8)} \qquad \frac{\det A}{\det A_{\mathbf{a}}} = A(y,y) - \sum_{a,b\in\mathbf{a}} (A_{\mathbf{a}})^{-1}(b.a)A(a,y)A(y,b).$$

Label points in $\mathbf{a} \cup \{y\}$ so that y is labeled $|\mathbf{a}| + 1$. By expanding the determinant with respect to the last row we obtain

$$\det A = A(y,y)\det A_{\mathbf{a}} + \sum_{b\in\mathbf{a}}(-1)^{b+|\mathbf{a}|+1}A(y,b)\det A^{1b},$$

where A^{1b} is the matrix that we obtain from A by deleting the last row and the b-th column. We now expand $\det A^{1b}$ with respect to the last column:

$$\det A^{1b} = \sum_{a\in\mathbf{a}}(-1)^{a+|\mathbf{a}|}A(a,y)\det A^{ab}_{\mathbf{a}},$$

where $A_{\mathbf{a}}^{ab}$ is the matrix that we obtain from $A_{\mathbf{a}}$ by deleting the a-th row and the b-th column. As a result

$$\det A = A(y,y)\det A_{\mathbf{a}} - \sum_{a,b\in\mathbf{a}} (-1)^{a+b} A(a,y)A(y,b)\det A_{\mathbf{a}}^{ab}.$$

This and Cramer's rule imply (A.8). □

If $\langle\cdot,\cdot\rangle$ is an inner product on the vector space V, then we equip $\wedge^r V$ with the inner product

$$\langle v_1\wedge\cdots\wedge v_r, v_1'\wedge\cdots\wedge v_r'\rangle = \det\left[\langle v_i, v_j'\rangle\right]_{i,j=1}^r.$$

By Example A1, the quantity

$$\|v_1\wedge\cdots\wedge v_r\|^2 = \langle v_1\wedge\cdots\wedge v_r,\ v_1\wedge\cdots\wedge v_r\rangle = \det\left[\langle v_i, v_j\rangle\right]_{i,j=1}^r,$$

represents the r-dimensional volume of the parallelepiped generated by vectors $v_1,\dots,v_r$.

Proposition A.2 **(i)** *If $\langle\cdot,\cdot\rangle$ is an inner product on the vector space V, and $\{e_1,\dots,e_d\}$ is an orthonormal basis for V, then the set $\left\{e_{i_1}\wedge\cdots\wedge e_{i_r}:\ 1\le i_1<\cdots<i_r\right\}$ is an orthonormal basis for $\wedge^r V$.*

(ii) *Suppose that V is an inner product space of dimension d, and $A: V\to V$ is a symmetric linear transformation. If $\{e_1,\dots,e_d\}$ is an orthonormal basis consisting of eigenvectors, associated with eigenvalues $\lambda_1\le\cdots\le\lambda_d$, then the set $\left\{e_{i_1}\wedge\cdots\wedge e_{i_r}:\ 1\le i_1<\cdots< i_r\right\}$ is an orthonormal basis consisting of eigenvectors of $\wedge^r A$ associated with eigenvalues $\left\{l_{i_1}\dots l_{i_r}:\ 1\le i_1<\cdots<i_r\right\}$.*

We end this chapter with a useful matrix identity.

Proposition A.3 *Consider the matrices $A\in Mat(d\times d), D\in Mat(d'\times d'), B\in Mat(d\times d'), C\in Mat(d'\times d)$, and set*

$$E = \begin{bmatrix} A & B\\ C & D\end{bmatrix}.$$

If E is symmetric and positive definite, so are A and D. If E, A and D are invertible, so are $G = A - BD^{-1}C$ and $H = D - CA^{-1}B$. Moreover

$$E^{-1} = \begin{bmatrix} G^{-1} & -G^{-1}BD^{-1}\\ -H^{-1}CA^{-1} & H^{-1}\end{bmatrix}. \tag{A.9}$$

Proof Note

$$E\begin{bmatrix} x\\ y\end{bmatrix} = \begin{bmatrix} x'\\ y'\end{bmatrix} \qquad \Longrightarrow \qquad Ax + By = x', \quad Cx + Dy = y'.$$

In particular, $y = D^{-1}y' - D^{-1}Cx$, which in turn implies that $Gx + BD^{-1}y' = x'$. This implies the entries for the first row in E^{-1}. n the same fashion, we derive $CA^{-1}x' + Hy = y'$, as desired. The first claim follows from

$$E \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = Aa \cdot a + 2Bb \cdot a + Db \cdot b.$$

For the second claim observe

$$E \begin{bmatrix} a \\ b \end{bmatrix} = 0 \qquad \Longrightarrow \qquad Ga = Hb = 0.$$

□

B Trace Class Operators and Fredholm Determinant

In this chapter we review some basic facts about bounded operator on Hilbert spaces.

(i) Let $\mathcal{H}$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. For a bounded linear operator $\mathcal{K} : \mathcal{H} \to \mathcal{H}$, we write $\|\mathcal{K}\|$ for its norm:

$$\|\mathcal{K}\| = \sup_{\|f\|=1} \|K(f)\|.$$

We write $\mathcal{B}(\mathcal{H})$ for the set of such operators which is a *Banach algebra* with respect to the above norm. We say that $\mathcal{K}$ is of *finite rank* if the dimension of its range is finite. We write $\mathcal{B}_0(\mathcal{H})$ for the set of finite rank operators. The topological closure of $\mathcal{B}_0(\mathcal{H})$ in $\mathcal{B}(\mathcal{H})$ is denoted by $\mathcal{B}_c(\mathcal{H})$. This set coincides with the set of *compact* operators. In other words a linear operator $\mathcal{K} \in \mathcal{B}_c(\mathcal{H})$ iff it maps bounded closed sets onto compact subsets of $\mathcal{H}$. One can show that if $\mathcal{K} \in \mathcal{B}_c(\mathcal{H})$, then $\mathcal{K}^* \in \mathcal{B}_c(\mathcal{H})$. The spectrum of a compact operator is countable with the only possible accumulation point at 0. Any non-zero point in the spectrum is an eigenvalue of finite multiplicity.

(ii) Fix an orthonormal basis $\{e_n : \ n \in I\}$ of $\mathcal{H}$. We write $\mathcal{B}_2(\mathcal{H})$ for the set of *Hilbert-Schmidt* operators: $\mathcal{K} \in \mathcal{B}_2(\mathcal{H})$ iff

$$\|\mathcal{K}\|_2^2 := \sum_{i \in I} \|\mathcal{K}e_i\|^2 = \sum_{i \in I} \langle (\mathcal{K}^*\mathcal{K})e_i, e_i \rangle < \infty.$$

Note

$$\|\mathcal{K}x\|^2 = \left\| \sum_n \langle x, e_n \rangle \mathcal{K}e_n \right\|^2 \le \left(\sum_n |\langle x, e_n \rangle| \, \|\mathcal{K}e_n\| \right)^2 \le \|x\|^2 \|\mathcal{K}\|_2^2.$$

Hence $\|\mathcal{K}\| \leq \|\mathcal{K}\|_2$. In fact we may define an inner product on $\mathcal{B}_2(\mathcal{H})$ by

$$\langle \mathcal{K}, \mathcal{K}' \rangle = \sum_{i \in I} \langle \mathcal{K} e_i, \mathcal{K}' e_i \rangle < \infty.$$

One can show that this inner product is independent of the choice of the orthonormal basis. Let us assume that $\mathcal{H}$ is infinite dimensional and write $\mathbb{N}$ for I. If we set

$$\mathcal{K}_n(f) = \sum_{i=1}^{n} \langle \mathcal{K}(f), e_i \rangle e_i,$$

then $\mathcal{K}_n \in \mathcal{B}_0(\mathcal{H})$, and

$$\left\| \mathcal{K} - \mathcal{K}_n \right\|_2^2 = \sum_{i > n} \|\mathcal{K} e_i\|^2 \to 0,$$

as $n \to \infty$. This implies that $\mathcal{B}_2(\mathcal{H}) \subset \mathcal{B}_0(\mathcal{H})$ and that $\mathcal{B}_0(\mathcal{H})$ is dense in $\mathcal{B}_2(\mathcal{H})$ with respect to the Hilbert-Schmidt topology.

Note that for $\mathcal{K} \in \mathcal{B}_2(\mathcal{H})$, we know that $\mathcal{K}^*\mathcal{K}$ is a symmetric compact operator. Hence by Spectral Theorem, there exists an orthonormal basis $\{e_n : n \in I\}$ consisting of eigenvectors of $|\mathcal{K}|^2 := \mathcal{K}^*\mathcal{K}$. In other words if $r_1 \geq r_2 \geq \cdots \geq 0$ are the eigenvalues of $|\mathcal{K}|$ (they are called the singular values of $\mathcal{K}$), then $|\mathcal{K}|^2 e_n = r_n^2 e_n$, and

$$\|\mathcal{K}\|_2^2 = \sum_n r_n^2.$$

If $\mathcal{K}$ is symmetric as well and $\lambda_1, \lambda_2, \ldots$ are its eigenvalues, then $\|\mathcal{K}\|_2^2 = \sum_n \lambda_n^2$. In this case, we may apply the Spectral Theorem to write

$$\mathcal{K} = \sum_n \lambda_n e_n \otimes e_n. \tag{B.1}$$

Here $\{e_n\}_n$ is an orthonormal basis with $\mathcal{K} e_n = \lambda_n e_n$, $e_n \otimes e_n$ is defined by $(e_n \otimes e_n)x = \langle x, e_n \rangle e_n$, and the convergence occurs with respect to the Hilbert-Schmidt norm:

$$\left\| \mathcal{K} - \sum_{n \leq N} \lambda_n e_n \otimes e_n \right\|_2^2 = \sum_{n > N} |\lambda_n\|^2 \to 0,$$

as $N \to \infty$.

(iii) For our purposes, we need to make sense of $\det(id + \mathcal{K})$, known as the *Fredholm determinant* of an operator $\mathcal{K}$. This is possible for any *trace class* operator: an operator such that

$$\|\mathcal{K}\|_1 = \sum_{n \in I} \langle (\mathcal{K}^*\mathcal{K})^{1/2} e_n, e_n \rangle < \infty.$$

The set of trace class operators is denote by $\mathcal{B}_1(\mathcal{H})$. Evidently any trace class operator is Hilbert-Schmidt. Indeed in terms of the singular values of $\mathcal{K}$, we have

$$\|\mathcal{K}\|_1 = \sum_{n\in I} \langle (\mathcal{K}^*\mathcal{K})^{1/2} e_n, e_n\rangle = \sum_{n\in I} r_n,$$

provided that $\{e_n : n \in I\}$ consists of eigenvectors of $|\mathcal{K}|$. If $\mathcal{K}$ is self-adjoint and positive, then eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots$ of $\mathcal{K}$ are positive, and we obtain

$$\|\mathcal{K}\|_1 = \sum_{n\in\mathbb{N}} \lambda_n < \infty.$$

Then for the determinant of $\mathbb{1} + \mathcal{K}$ we have a natural candidate

$$\det(\mathbb{1} + \mathcal{K}) = \prod_{n\in\mathbb{N}} (1 + \lambda_n) \leq e^{\sum_n \lambda_n}, \tag{B.2}$$

that is finite.

(iv) As a classical example, let μ be a σ-finite measure on X and choose $\mathcal{H} = L^2(\mu)$. In particular $\mathcal{H} = \ell^2(X)$ when X is countable and μ is the counting measure. Given any $K : X \times X \to \mathbb{R}$ with $K \in L^2(\mu \times \mu)$, define the operator

$$\mathcal{K}f(x) = \int K(x,y) f(y)\ \mu(dy).$$

The operator $\mathcal{K}$ is Hilbert-Schmidt because for any orthonormal basis $\{\phi_n :\ n \in \mathbb{N}\}$,

$$\begin{aligned}\|\mathcal{K}\|_2^2 &= \sum_n \int \left(\int K(x,y)\phi_n(y)\ \mu(dy)\right)^2\ \mu(dx) \\ &= \int \sum_n \langle K(x,\cdot), \phi_n\rangle^2\ \mu(dx) = \int \|K(x,\cdot)\|^2_{L^2(\mu)}\ \mu(dx) = \|K\|^2_{L^2(\mu\times\mu)}.\end{aligned}$$

If $K(x,y) = K(x,y)$ (or in the case of a complex-valued K, assume $K(x,y) = \overline{K(y,x)}$), the operator $\mathcal{K}$ is compact and symmetric. By Spectral Theorem, we can find an orthonormal basis $\{\phi_n :\ n \in \mathbb{N}\}$, consisting of eigenfunctions of $\mathcal{K}$: $\mathcal{K}\phi_n = \lambda_n \phi_n$. By Spectral Theorem

$$\begin{aligned}\mathcal{K}f(x) &= \int K(x,y) f(y)\ \mu(dy) = \lim_{N\to\infty} \sum_{n\leq N} \left(\int \phi_n(y) f(y)\ \mu(dy)\right) (\mathcal{K}\phi_n)(x) \\ &= \lim_{N\to\infty} \int K_N(x,y) f(y)\ \mu(dy),\end{aligned}$$

where $K_N(x,y)=\sum_{n\le N}\lambda_n\phi_n(x)\phi_n(y)$. On the other hand

$$\lim_{N\to\infty} K_N(x,y)=\hat{K}(x,y):=\sum_n \lambda_n\phi_n(x)\phi_n(y),$$

with the convergence occurring in $L^2(\mu\times\mu)$. Indeed if we set

$$\big(\phi_n\otimes\phi_n\big)(x,y)=\phi_n(x)\phi_n(y),$$

$\big(\phi_n\otimes\phi_n:\ n\in\mathbb{N}\big)$ consists of mutually orthogonal functions in $L^2(\mu\times\mu)$, that allows us to make sense of $\hat{K}$ because

$$\|\hat{K}\|^2_{L^2(\mu\times\mu)}=\sum_n|\lambda_n|^2<\infty,$$

and

$$\|\hat{K}-K_N\|^2_{L^2(\mu\times\mu)}=\sum_{n>N}|\lambda_n|^2\to 0,$$

as $N\to\infty$. Since

$$\mathcal{K}f(x)=\int K(x,y)f(y)\ \mu(dy)=\int \hat{K}(x,y)f(y)\ \mu(dy),$$

for every $f\in L^2(\mu)$, we deduce that $K=\hat{K}$ $\mu\times\mu$-almost everywhere. In summary,

$$K(x,y)=\sum_n \lambda_n\phi_n(x)\phi_n(y), \tag{B.3}$$

$\mu\times\mu$-almost everywhere, with right-hand side converging in $L^2(\mu\times\mu)$.

(v) If X is countable and μ is the counting measure, then a symmetric kernel $K:X\times X\to\mathbb{R}$ is Hilbert-Schmidt iff

$$\sum_{x,y}|K(x,y)|^2<\infty. \tag{B.4}$$

Moreover, if $\{\phi_n:\ n\in\mathbb{N}\}$ is an orthonormal basis consisting of eigenfunctions of the associated $\mathcal{K}$, then for every $x,y\in X$, we have (B.3). Assume that the operator $\mathcal{K}$ is non-negative so that the eigenvalues satisfy $\lambda_n\ge 0$. Then using (B.3),

$$\sum_{x\in X}K(x,x)^2=\sum_{x\in X}\sum_n\lambda_n\phi_n(x)=\sum_n\lambda_n.$$

Hence such $\mathcal{K}$ is in the trace class iff

$$\sum_{x\in X}K(x,x)^2<\infty. \tag{B.5}$$

References

[A] O. Angel, The stationary measure of a 2-type totally asymmetric exclusion process, Journal of Combinatorial Theory, Series A **113** (2006), 625-635.

[BFPS] A. Borodin, P. L. Ferrari, M. Prähofer, and T. Sasamoto, Fluctuation properties of the TASEP with periodic initial configuration, J. Stat. Phys. **129** (2007), 1055-1080.

[BR] A. Borodin, and E. M. Rains, Eynard-Mehta Theorem, Schur Process, and their Pfaffian Analogs, Journal of Statistical Physics, **121** (2005), 291317.

[CG] E. Cator and P. Groeneboom, Hammersleys process with sources and sinks, Ann. Probab. **33** (2005), 879-903.

[CQR] I. Corwin, J. Quastel, and D. Remenik, Renormalization fixed point of the KPZ universality class. J. of Stat. Phys. **160** (2015), 815-834.

[FM1] P. A. Ferrari, J. B. Martin, Stationary distributions of multi-type totally asymmetric exclusion processes. Ann. Probab. **35** (2007), no. 3, 807-832.

[FM2] P. A. Ferrari, J. B. Martin, Multiclass Hammersley-Aldous-Diaconis process and multiclass-customer queues, Ann. Inst. Henri Poincare Probab. Stat. **45** (2009), no. 1, 250-265.

[F] P. L. Ferrari, Polynuclear growth on a flat substrate and edge scaling of GOE eigenvalues. Comm. Math. Phys. **252** (2004), no. 1-3, 77-109.

[MQR] K. Matetski, J. Quastel, and D. Remenik, The KPZ fixed point, preprint.

bibitem[PS]PS M. Prähofer, H. Spohn, Scale invariance of the PNG droplet and the Airy process. J. Statist. Phys. **108** (2002), no. 5-6, 1071-1106.

[Sa] T. Sasamoto, Spatial correlations of the 1D KPZ surface on a flat substrate, Journal of Physics A: Mathematical and General **38** (2005), no. 33, L549.

[Sc] G. M. Schütz, Exact solution of the master equation for the asymmetric exclusion process, J. Stat. Phys. **88** (1997), no. 1-2, 427-445.

[ST] T. Shirai and Y. Takahashi, Random point fields associated with certain Fredholm determinants II: Fermion shifts and their ergodic and Gibbs properties, Ann. Probab. **31** (2003), 1533-1564.

[R] F. Rezakhanlou, Stochastic Solutions to Hamilton-Jacobi Equation. LNM.