A Prelude to the Theory of Random Walks in Random Environments

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May 25, 2009

1 Introduction

A random walk on a lattice is one of the simplest and most fundamental models in the probability theory. For a random walk, a probability density $p : \mathbb{Z}^d \to [0, 1]$ with $\sum_z p(z) = 1$ is given. The set of such probability densities is denoted by Γ. Given $p \in \Gamma$, we define a random walk $(X_n : n = 0, 1, \dots)$ with the following rules: If $X_n = a$, then $X_{n+1} = a+z$ with probability $p(z)$. Put differently, for each initial position $a \in \mathbb{Z}^d$, we construct a probability measure P^a on the space of sequences $(x_n : n \in \mathbb{N})$ such that $P^a(x_0 = a) = 1$ and if $z_n = x_{n+1} - x_n$, then the sequence $(z_n : n \in \mathbb{N})$ consists of independent random variables with each z_n distributed according to $p(\cdot) \in \Gamma$. For simplicity, let us assume that $p(\cdot)$ is of finite range, i.e., $p(z) = 0$ for $|z| > R_0$ for some R_0. The space of such $p(\cdot)$ is denoted by Γ_0. We now review some basic facts about random walks with $p \in \Gamma_0$:

(i) Law of Large Numbers (LLN). There exists an asymptotic velocity. That is,

$$\frac{1}{n} X_{[nt]} = x + \bar{v}t + o(1) \tag{1.1}$$

with probability one with respect to $P^{[xn]}$, where

$$\bar{v} = \sum_z p(z)z. \tag{1.2}$$

(ii) Central Limit Theorem (CLT). There exists a Gaussian correction to (1.1), namely

$$\frac{1}{n} X_{[nt]} = x + \bar{v}t + \frac{1}{\sqrt{n}} B(t) + o\left(\frac{1}{\sqrt{n}}\right) \tag{1.3}$$

with $B(\cdot)$ a diffusion with covariance

$$E(B(t)\cdot a)^2 = t\sum_z p(z)(z\cdot a)^2. \tag{1.4}$$

(iiii) Large Deviation Principle (LDP). The probability of large deviations from the mean is exponentially small with a precisely defined exponential decay rate. More precisely, for every bounded continuous function $f:\mathbb{R}^d\to\mathbb{R}$,

$$\lim_{n\to\infty}\frac{1}{n}\log E^{[xn]}\exp\left(nf\left(\frac{1}{n}X_{[nt]}\right)\right) = \sup_v(f(x+vt)-t\bar{L}(v)) \tag{1.5}$$

where $\bar{L}$ is the Legendre transform of $\bar{H}$ with $\bar{H}$ given by

$$\bar{H}(P) = \log\sum_z e^{z\cdot P}p(z). \tag{1.6}$$

Let us make some comments about (1.5) and (1.6). First observe that if we denote the right-hand side of (1.5) by $\bar{u}(x,t)$, then $\bar{u}$ solves a Hamilton–Jacobi PDE of the form

$$\bar{u}_t = \bar{H}(\bar{u}_x) \tag{1.7}$$

subject to the initial condition $\bar{u}(x,0)=f(x)$. The right-hand side of (1.5) is known as Hopf–Lax–Oleinik Formula and is valid for any Hamilton–Jacobi PDE with convex Hamiltonian function $\bar{H}$.

As our second comment, let us mention that a large-deviation principle is informally stated as

$$P^{[xn]}\left(\frac{1}{n}X_{[nt]}\approx x+tv\right)\approx e^{-nt\bar{L}(v)}. \tag{1.8}$$

The equivalence of (1.8) to the statement (1.1) is the celebrated Varadhan's lemma. Here is the meaning of (1.8): Imagine that the velocity of the walk is near v for some $v\neq\bar{v}$. By LLN, this would happen with a probability that goes to 0. This happens exponentially fast with an exponential rate given by $t\bar{L}(v)$. Since the approximation on the left-hand side of (1.8) is of the form $X_{[nt]}\approx nx+(nt)v$, the right-hand side is of the form $\exp(-(nt)\bar{L}(v))$. It is not hard to justify the variational form of $\bar{u}(x,t)$. It has to do with the elementary principle that a sum of exponentials is dominated by the term of the largest exponent, i.e.,

$$\lim_{n\to\infty}\frac{1}{n}\log(e^{\lambda_1 n}+\cdots+e^{\lambda_k n}) = \sup_{1\le j\le k}\lambda_j.$$

We can also explain the relationship between (1.5) and (1.6). First observee that if we allow a linear function $f(x) = P \cdot x$ in (1.5), then the left-hand side equals

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \log E^{[xn]} \exp(P \cdot X_{[nt]}) &= t \lim_{m\to\infty} \frac{1}{m} \log E^0 \exp\left(P \cdot \frac{[xm]}{t} + P \cdot X_m\right) \\
&= t \lim_{m\to\infty} \frac{1}{m} \log E^0 \exp(P \cdot z_0 + \cdots + P \cdot z_{m-1}) + P \cdot x \\
&= t \lim_{m\to\infty} \frac{1}{m} \log[\exp(\bar{H}(P))]^m + P \cdot x \\
&= P \cdot x + t\bar{H}(P).
\end{aligned}$$

This equals the right-hand side of (1.5) if $\bar{L}$ is the Legendre transform of $\bar{H}$.

So far we have discussed a *homogeneous* random walk because the jump rate $p(z)$ is independent of the position of the walk. More generally we may look at an *inhomogeneous* walk with the jump rate from a to $a + z$ given by $p_a(z)$ where $p_a \in \Gamma_0$ for each $a \in \mathbb{Z}^d$. In other words, a collection

$$\omega = (p_a(\cdot) : a \in \mathbb{Z}^d) \in \Omega = \Gamma_0^{\mathbb{Z}^d}$$

is given and for any such ω, we define a probability measure P_ω^a on the space of sequences $\mathbf{x} = (x_n : n \in \mathbb{N}) \in (\mathbb{Z}^d)^{\mathbb{N}}$, such that $P_\omega^a(x_0 = a) = 1$, and the law of $x_{n+1} - x_n$ conditioned on $x_n = b$ is given by p_b.

Naturally we would like to address the questions of LLN, CLT and LDP for an inhomogeneous random walk, but these questions would not have any reasonable answer unless some regularity or pattern is assumed about the sequence $\omega \in \Omega$. The type of condition we assume in this article is that ω itself is selected randomly and the inhomogeneity is stochastically homogeneous. That is, the law of ω is stationary with respect to the lattice translation. More precisely, we have a probability measure $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{P}$ invariant under the translation $(\tau_a : a \in \mathbb{Z}^d)$ where $\tau_a\omega = (p_{a+b} : b \in \mathbb{Z}^d)$ for $\omega = (p_b : b \in \mathbb{Z}^d)$. We also assume that $\mathbb{P}$ is ergodic in the sense that if a measurable set $A \in \mathcal{F}$ is invariant under all translations, i.e., $\tau_a A = A$ for all $a \in \mathbb{Z}^d$, then $\mathbb{P}(A) = 0$ or 1. Of course we put the product topology on $\Omega = \Gamma_0^{\mathbb{Z}^d}$ with each Γ_0 equipped with its standard topology. Also $\mathcal{F}$ is the Borel σ-algebra of Ω. We regard each $\omega \in \Omega$ as a realization of an *environment*, and P_ω is the law of the corresponding random walk. We will be interested in two types of questions for such a walk. As the first type, we will be interested in the walk P_ω for almost all realizations of ω with respect to $\mathbb{P}$. The probability measure P_ω is called a *quenched law* and the corresponding walk is referred to as a *quenched walk*. As the second type, we will be interested in the averaged law

$$\int_\Omega P_\omega(d\mathbf{x})\mathbb{P}(d\omega) =: \hat{\mathbb{P}}(d\mathbf{x})$$

which is known as the *annealed law*. Note that the quenched law is still the law of a Markov chain and the main challenge for such a law comes from its inhomogeneity. On the other hand the annealed law is the law of a highly non-Markovian walk as we will see in the proceeding sections.

It is convenient to write $p_0(\omega, z)$ for the probability of jumping from the origin to z for a given $\omega = (p_a(\cdot) : a \in \mathbb{Z}^d) \in \Omega$. With such a notation, the probability of jumping from a to $a + z$ is now given by $p_0(\tau_a\omega, z)$. As a result, the corresponding walk $X_n = X_n^\omega$ jumps to $X_n + z$ with probability $p_0(\tau_{X_n}\omega, z)$. Note that X_n is random for a given realization of ω. Sometimes we write ω' for the randomness of the walk so that $X_n = X_n^\omega(\omega')$. These two layers of randomness makes the analysis of such walks rather complex. Following an idea of Kozlov, it is often useful to combine the two randomnesses into one by defining a Markov chain on Ω with the following recipe: the state of chain at time n is $\omega_n = \tau_{X_n}\omega$ where X_n is the walk associated with ω, which starts from the origin. More precisely if $\omega_n = \alpha$, then α changes to $\tau_z\alpha$ with probability $p_0(\alpha, z)$. With this idea, we have been able to produce a Markov chain ω_n which has all the information about the original chain X_n. But now the state space has changed from $\mathbb{Z}^d$ to Ω which is far more complicated. The interpretation of ω_n is that each time the walk makes a new jump, we shift the environment so that the walker is always at the origin. In other words, we are looking at the environment from the current position of the walker. That is, we are taking the point of view of the walker to study the environment.

Let us see how by taking the point of view of the walker we can calculate the velocity of the walker. First recall that if $Z_n = X_n - X_{n-1}$, then

$$\begin{aligned} E_\omega^0 X_n = E_\omega^0 \sum_{j=1}^n Z_n &= \sum_{j=1}^n \sum_z z p_0(\tau_{X_j}\omega, z) \\ &= \sum_{j=1}^n \sum_z z p_0(\omega_j, z). \end{aligned}$$

Now if $\mathbb{Q}$ is an ergodic invariant measure for the chain ω_n, then for $\mathbb{Q}$-almost all ω,

$$\bar{v} = \frac{1}{n} E_\omega^0 X_n \to \int \sum_z z p_0(\omega, z) \mathbb{Q}(d\omega). \tag{1.9}$$

This suggests studying the invariant measures for the chain ω_n. Once these invariant measures are known, then the velocity $\bar{v}$ for the walker can be evaluated by (1.9). As we will see later, studying the invariant measures for ω_n is a formidable task. For example, it is not known in general whether or not there is an invariant measure which is absolutely continuous with respect to $\mathbb{P}$. If such an invariant measure exists, then by a result of Kozlov, we also have $\mathbb{P} \ll \mathbb{Q}$ and $\mathbb{Q}$ is the unique invariant measure. This question is rather well-understood when $d = 1$ and only nearest neighbor jumps are allowed. We will discuss this in Section 2.

We end this introduction with an overview of some of the known results and an outline of the rest of the paper.

The quenched LDP in the case of $d = 1$ with nearest-neighbor jumps and independent environment was established by Greven and den Hollander [GH]. Section 2 and 3 are devoted to LLN and LDP for this case. The extension to the general environment was achieved by Comets, Gantert and Zeitouni [CGZ]. The first quenched LDP in $d \geq 2$ for independent environment was carried out by Zerner [Z] provided that the *nestling* condition is satisfied, i.e., the convex hull of the support of the law $\sum_z zp_0(\omega, z)$ contains the origin. In Section 3 we discuss a formula of Rosenbluth [R] for the quenched LDP rate function. Varadhan [V] proved quenched LDP for general stationary ergodic environment in all dimension. He also established the quenched LDP for an independent environment under some ellipticity conditions. Section 5 is devoted to Varadhan's treatment of anealed LDP.

2 RWRE in dimension 1 with nearest neighbor jumps, LLN

In this section we study RWRE when the dimension is one and only the jumps to the adjacent sites are allowed. As it turns out, many of the questions we discussed in Section 1 have been settled successfully in this case. However, many of the arguments used to treat this case are not applicable for general RWRE.

In spite of significant simplification, it is instructive to understand the case of nearest neighbor RWRE in dimension one first and use it as a model to compare with when we discuss the general case in the proceeding sections.

Because of nearest neighbor jumps, we only need to know $p_0(\omega, 1)$ because $p_0(\omega, -1) = 1 - p_0(\omega, 1)$ and $p_0(\omega, z) = 0$ for $z \neq 1, -1$. Let us simply write $\omega)$ for $p_0(\omega, 1)$ and observe that the associated walk X_n jumps to $X_n + 1$ with probability $q(\tau_{X_n}\omega)$ and to $X_n - 1$ with probability $1 - q(\tau_{X_n}\omega)$. The corresponding chain evolves by a simple rule: $\omega_{n+1} = T(\omega_n)$ with probability $q(\omega_n)$ and $\omega_{n+1} = T^{-1}(\omega_n)$ with probability $1 - q(\omega_n)$, where $T = \tau_1$. The following theorem is due to Alili [A].

Theorem 2.1 *If $\int \log \frac{1-q}{q} d\mathbb{P} \neq 0$, then the chain ω_n has an invariant measure $\mathbb{Q}$ which is absolutely continuous with respect to $\mathbb{P}$.*

We will give a proof of Theorem 2.1 shortly and in the process we find an explicit formula for the invariant measure $\mathbb{Q}$. However $\mathbb{Q}$ would not be a probability measure unless we make a more stringent assumption. Also using $\mathbb{Q}$ we find an explicit expression for the average velocity $\bar{v}$. In the case of an independent environment,

$$\bar{v} = \begin{cases} \frac{1-\bar{\gamma}}{1+\bar{\gamma}} & \text{if } \bar{\gamma} = \int \frac{1-q}{q} d\mathbb{P} < 1, \\ \frac{\bar{\eta}-1}{\bar{\eta}+1} & \text{if } \bar{\eta} = \int \frac{q}{1-q} d\mathbb{P} < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

The formula (2.1) is due to Solomon [S] who proved LLN for RWRE when in addition the environment consists of independent jump probabilities.

Proof of Theorem 2.1. If $\mathbb{Q}$ is invariant, then

$$\int [q(\omega)J(T(\omega)) + (1-q(\omega))J(T^{-1}(\omega))]\mathbb{Q}(d\omega) = \int Jd\mathbb{Q} \tag{2.2}$$

for every measurable bounded function J. If $d\mathbb{Q} = \rho d\mathbb{P}$, then (2.2) is equivalent to

$$q(T^{-1}(\omega))\rho(T^{-1}(\omega)) + (1-q(T(\omega)))\rho(T(\omega)) = \rho(\omega). \tag{2.3}$$

So the question is whether (2.3) has a solution for a probability density ρ. In some sense, (2.3) is a "second order equation" because it involves both T and T^{-1}. In fact if we set $\eta = \frac{q}{1-q}$ and $R = (1-q)\rho$, then (2.3) says

$$R \circ T^2 - \left(\frac{1}{1-q}R\right) \circ T + \eta R = 0. \tag{2.4}$$

We now define the "first order" operator $\mathcal{A}R = R \circ T - \eta R$ to write (2.4) as

$$(\mathcal{A}R) \circ T - \mathcal{A}R = 0. \tag{2.5}$$

Since $\mathbb{P}$ is T-ergodic, the only solution of (2.5) is $\mathcal{A}R \equiv c$ for a constant c. We now have to solve the first order equation

$$R \circ T - \eta R \equiv c. \tag{2.6}$$

It is not hard to show that if $c = 0$, then the only solution to (2.6) is $R = 0$. For a nontrivial solution, we need to consider the case $c \neq 0$. For such c, we only need to solve (2.6) for one choice of c; for any other choice, we multipy R by a suitable constant. We may consider the choice $c = -1$, so that the equation (2.6) reads as $R = \gamma(R \circ T + 1)$ with $\gamma = \frac{1-q}{q}$. To find a solution, start from some R_0 and define $R_{n+1} = \gamma(R_n \circ T + 1)$, which means that

$$R_n = \sum_{j=0}^{n-1} \prod_{r=0}^{j} \gamma \circ T^r + \left(\prod_{r=0}^{n-1} \gamma \circ T^r\right) R_0 \circ T^n.$$

From this we guess that

$$R = \sum_{j=0}^{\infty} \prod_{r=0}^{j} \gamma \circ T^r$$

is a solution. As a result

$$\rho = \rho^{+} = (1+\gamma)\sum_{j=1}^{\infty}\prod_{r=1}^{j}\gamma \circ T^{r}. \tag{2.7}$$

But now we have to make sure that the right-hand side of (2.7) is convergent. For this we need the condition $\int \log \frac{1-q}{q} d\mathbb{P} =: \zeta < 0$. This is because

$$\frac{1}{j}\log\prod_{1}^{j}\gamma\circ T^{r} = \frac{1}{j}\sum_{1}^{j}\log\gamma\circ T^{r} \to \zeta$$

$\mathbb{P}$-almost surely by Birkhoff's ergodic theorem. This implies an exponential decay for the j-th term of the right-hand side of (2.7).

If instead $\zeta > 0$, in the above argument we replace the role of T with T^{-1}. This time

$$\rho = \rho^{-} = (1+\eta)\sum_{j=1}^{\infty}\prod_{r=1}^{j}\eta\circ T^{-r}$$

is a solution. □

Note that the invariant measure we constructed in Theorem 2.1 is not necessarily a finite measure. However if $\int \rho^{-} d\mathbb{P}$ or $\int \rho^{+} d\mathbb{P}$ is finite, then we can turn it into a probability density by setting $\hat{\rho}^{\pm} = \frac{1}{Z^{\pm}}\rho^{\pm}$, with $Z^{\pm} = \int \rho^{\pm} d\mathbb{P}$. In the case of independent environment, $Z^{\pm}$ can be calculated explicitly. For example, if $\bar{\gamma} = \int \gamma d\mathbb{P} < 1$, then

$$Z^{+} = (1+\bar{\gamma})\sum_{1}^{\infty}\bar{\gamma}^{j} = \bar{\gamma}\frac{1+\bar{\gamma}}{1-\bar{\gamma}},$$

and

$$\begin{aligned}\bar{v} &= \frac{1}{Z^{+}}\int (2q-1)\rho^{+} d\mathbb{P} = \frac{1}{Z^{+}}\int \frac{1-\gamma}{1+\gamma}\rho^{+} d\mathbb{P} \\ &= \frac{1-\bar{\gamma}}{Z^{+}}\sum_{1}^{\infty}\bar{\gamma}^{j} = \frac{1-\bar{\gamma}}{1+\bar{\gamma}} > 0,\end{aligned}$$

as we claimed earlier in (2.1). Similarly, if $\bar{\eta} = \int \eta d\mathbb{P} < 1$, then $\bar{v} = \frac{\bar{\eta}-1}{\bar{\eta}+1} < 0$. In the remaining cases,

$$\frac{1}{\bar{\eta}} = \left(\int \gamma^{-1} d\mathbb{P}\right)^{-1} \leq 1 \leq \int \gamma d\mathbb{P},$$

the velocity is 0 as was shown by Solomon.

3 RWRE in dimension 1 with nearest neighbor jumps, LDP

Based on our discussion in Section 1, the LDP for RWRE comes in two flavors; quenched and annealed. The former is stated a

$$\lim_{n\to\infty}\frac{1}{n}u([xn],[tn];\omega)=\sup_v(f(x+vt)-t\bar{L}(v)), \tag{3.1}$$

for almost all $\mathbb{P}-\omega$ realizations, where

$$u(a,n;\omega)=\log E^a_\omega\exp\left(nf\left(\frac{1}{n}X_n\right)\right). \tag{3.2}$$

The latter means

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{E}\exp(u([xn],[tn];\omega))=\sup_v(f(x+vt)-t\hat{L}(v)). \tag{3.3}$$

In some sense the quenched LDP is really a LDP for P_ω but a LLN for the ω-variable. On the other hand, the annealed LDP is a LDP for the annealed law $\int P_\omega\mathbb{P}(dw)=\hat{\mathbb{P}}$, because the left-hand side of (3.3) equals

$$\lim_{n\to\infty}\frac{1}{n}\log\hat{\mathbb{E}}\exp\left(nf\left(\frac{1}{n}X_n\right)\right).$$

There is yet another interpretation for both (3.1) and (3.2). If we denote the right-hand side of (3.1) by $\bar{u}(x,t)$, then again

$$\begin{cases}\bar{u}_t=\bar{H}(\bar{u}_x),\\ \bar{u}(x,0)=f(x)\end{cases} \tag{3.4}$$

where $\bar{H}$ is the Legendre transform of $\bar{L}$. In some sense, $u(a,n;\omega)$ solve a discrete Hamilton–Jacobi–Bellman equation and (3.2) can be recast as a homogenization problem for such equation. To explain this further, define an operator $\mathcal{H}(\cdot;\omega):L(\mathbb{Z}^d)\to L(\mathbb{Z}^d)$

$$\begin{aligned}\mathcal{H}(f;\omega)(a)&=\log E^a_\omega\exp(f(X_1)-f(X_0))\\ &=\log\sum_z\exp(f(a+z)-f(a))p_0(\tau_a\omega,z)\end{aligned} \tag{3.5}$$

where $L(\mathbb{Z}^d)$ consists of functions $f:\mathbb{Z}^d\to\mathbb{R}$. Then u solves

$$u(a,n+1;\omega)-u(a,n;\omega)=\mathcal{H}(u(\cdot,n;\omega);\omega).$$

Now (3.1) says that a *homogenization* occurs, i.e., $\frac{1}{n}u([xn],[tn];\omega) \to \bar{u}(x,t)$ as $n \to \infty$, with $\bar{u}$ independent of ω, solving the homogeneous Hamilton–Jacobi equation (3.4). A LDP for this homogenization requires a calculation of the sort

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E} \exp(\lambda u([xn],[tn];\omega))$$

for every $\lambda \in \mathbb{R}$. This for $\lambda = 1$ is the annealed LDP (3.2).

In the case of $d = 1$ with nearest neighbor jumps both the quenched and the annealed LDP were established in Comets et al. [CGZ]. For the rest of this section, we offer an alternative approach to understanding the quenched LDP which, in spirit is very close to the recent work of Yilmaz [Y]. Our presentation of quenched LDP would hopefully help the reader to appreciate Rosenbluth's formula for the quenched LD rate function that will be discussed in Section 4.

Recall that in the homogeneous random walk, we observed that if $f(x) = x \cdot P$, then the corresponding

$$u(a,n) = \log E^a \exp(P \cdot X_n) = n\bar{H}(P) + P \cdot a.$$

This ultimately has to do with the fact that $\bar{u}(x,t) = x \cdot P + t\bar{H}(P)$ solves the HJ equation (1.7).

In the case of the inhomogeneous random walk, we may wonder whether or not a suitable function would play the role of $x \cdot P$. In this case, we need to add a corrector to the linear function $x \cdot P$. More precisely, we search for a function $F(x,\omega;P)$ such that its discrete derivative is of the form

$$F(x+z,\omega;P) - F(x,\omega;P) = z \cdot P + g(\tau_x\omega, z; P), \tag{3.6}$$

asymptotically

$$\frac{1}{n}F([x_n],\omega;P) = x \cdot P + o(1), \tag{3.7}$$

as $n \to \infty$, and

$$\mathcal{H}(F(\cdot,\omega;P);\omega) \equiv \bar{H}(P). \tag{3.8}$$

In other words, once a function F with (3.7) and (3.8) is found such that $\mathcal{H}(F(\cdot,\omega;P);\omega)$ is a constant, then the constant is the number $\bar{H}(P)$ we are looking for. Indeed for such a function F we have that

$$u(a,n,\omega;P) := \log E^a \exp(F(X_n,\omega;P)) = F(a,\omega;P) + n\bar{H}(P),$$

and

$$\lim_{n\to\infty} \frac{1}{n} u([xn],[tn],\omega;P) = x \cdot P + t\bar{H}(P).$$

Such functions $F(\cdot,\omega;P)$ can be constructed explicitly in the case of $d=1$ with nearest neighbor jumps. To see this, observe that we want to find $g(\omega,1)$ and $g(\omega,-1)$ such that

$$q(\omega)e^{g(\omega,1)+P}+(1-q(\omega))e^{g(\omega,-1)-P}=e^{\bar{H}(P)}, \tag{3.9}$$

by (3.8). But we need to solve (3.6) for F once g is determined. This forces a compatibility condition on g; we must have

$$g(\omega,1;P)+g(\tau_1\omega,-1;P)=0.$$

If we write $g(\omega)$ for $g(\omega,1;P)$, then $g(\tau_{-1}\omega,-1;P)=-g(\tau_{-1}\omega)=-g(T^{-1}(\omega))$. So (3.9) becomes

$$q(\omega)e^{g(\omega)+P}+(1-q(\omega))e^{-g(T^{-1}(\omega))-P}=e^{\bar{H}(P)}. \tag{3.10}$$

In fact the function $F(x,\omega;P)$ is simply given by

$$F(x,\omega;P)=x\cdot P+\sum_{j=0}^{x-1}g(\tau_j\omega),$$

for x a positive integer. Now (3.7) is satisfied if $\mathbb{E}g=0$. This suggests setting $h=g+P$ so that (3.10) now reads

$$q(\omega)e^{h(\omega)}+(1-q(\omega))e^{-h(T^{-1}(\omega))}=\lambda, \tag{3.11}$$

with $\lambda=e^{\bar{H}(P)}$ with $P=\mathbb{E}h$. So, we want to find functions $h(\omega)$ such that $\lambda=qe^h+(1-q)e^{-h\circ T^{-1}}$ is a constant and once this is achieved, we set $\bar{H}(P)=\log\lambda$ for $P=\mathbb{E}h$, so $\mathbb{E}h$ is $\bar{H}^{-1}(\log\lambda)$. Indeed if we set $\sigma_l=\inf\{k:X_k=l\}$, and

$$h(\omega)=-\log E^0_\omega e^{r\sigma_1}1\!\!1(\sigma_1<\infty),$$

then (3.11) is satisfied for $\bar{H}(P)=-r$, and the corresponding P is

$$-P^-(r)=P=-\mathbb{E}\log E^0_\omega e^{r\sigma_1}1\!\!1(\sigma_1<\infty).$$

Note however that $h(\omega)=\infty$ if r is too large. In fact there exists a critical $r_c^->0$ such that $h(\omega)<\infty$ if and only if $r\le r_c^-$. We set $P_c^-=P(r_c)$. In summary, for $P\in[-P_c^-,0]$, we have that $\bar{H}(P)=-r^-(P)$ where $r^-(P)$ is the inverse of $-P^-(r)$.

If

$$h'(\omega)=\log E^0_\omega e^{r\sigma_{-1}}1\!\!1(\sigma_{-1}<\infty),$$

then

$$q(\omega)e^{h'(T(\omega))}+(1-q(\omega))e^{-h'(\omega)}=e^{-r}$$

and $h(\omega)=h'\circ T$ satisfies

$$q(\omega)e^{h(\omega)}+(1-q(\omega))e^{-h(T^{-1}(\omega))}=e^{\bar{H}(P)}$$

where $\bar{H}(P)=r$, and

$$P^+(r)=P=\mathbb{E}\log E^0_\omega e^{r\sigma_{-1}}1\!\!1(\sigma_{-1}<\infty).$$

4 Quenched LDP

In the case of $d = 1$ with nearest neighbor jumps, we were able to solve (3.8) for a function F satisfying (3.6) and (3.7). It seems unlikely that such a solution can be found in general. Instead, we replace (3.8) with

$$\mathcal{H}(F(\cdot, \omega); \omega) \leq \text{constant} \tag{4.1}$$

and try to optimize the outcome. First we try to formulate (3.6) more carefully. We define a set of functions g for which (3.6) has a solution for F. That is, the set of gradient-type functions. Since we will be dealing with the limit points of gradient-type functions, perhaps we should look at those functions with 0 curl. More precisely let $\mathcal{F}_0$ denote the set of functions $g : \Omega \to \mathbb{R}$ such that $\mathbb{E}g = 0$, and for any loop $x_0 = x, x_1, x_2, \dots, x_{k-1}, x_k = x$, we have

$$g(\tau_{x_0}\omega, x_1 - x_0) + g(\tau_{x_1}\omega, x_2 - x_1) + \cdots + g(\tau_{x_{n-1}}\omega, x_n - x_{n-1}) = 0.$$

Note that the constant in (4.1) can be chosen to be $\operatorname{esssup}_\omega \mathcal{H}(F(\cdot, \omega); \omega)$. Here is the Rosenbluth's formula for $\bar{H}$:

$$\bar{H}(P) = \inf_{g \in \mathcal{F}_0} \operatorname*{esssup}_{\omega} \log \sum_z \exp(z \cdot P + g(\omega, z)) p_0(\omega, z), \tag{4.2}$$

where the essential supremum is taken with respect to $\mathbb{P}$. The formula (4.2) was derived by Rosenbluth [R] under the assumption that for some $\alpha > 0$

$$\int |\log p_0(\omega, z)|^{d+\alpha} d\mathbb{P} < \infty.$$

The way to think about $\mathcal{F}$ is that $\mathcal{F}$ is the space of closed 1-forms. In fact the space of 1-form is defined by

$$\mathcal{F}_1(\Omega) = \{f = (f(\omega, z) : z \in R_0) : f(\cdot, z) : \Omega \to \mathbb{R} \text{ is bounded measurable for each } z \in R_0\}$$

where $R_0 \subseteq \mathbb{Z}^d$ is chosen so that if $z \notin R_0$, then $p_0(\omega, z) = 0$. Now for 1-form f, we define

$$\mathcal{H}(f) = \log \sum_z e^{f(\omega, z)} p_0(\omega, z).$$

If $L(\Omega)$ is the space of bounded measurable functions, then $\mathcal{H} : \mathcal{F}_1 \to L(\Omega)$. Recall that $\mathcal{F}_0$ is the set of closed 1-form of 0 average. Note that the space $\mathcal{F}_1$ contains constant 1-forms

$$e_P = (P \cdot z : z \in R_0).$$

Now we say two 1-forms f and f' are equivalent if $f - f' \in \mathcal{F}_0$, i.e., $f - f'$ is a We can now write

$$\bar{H}(P) = \inf_{f \sim e_P} \operatorname*{esssup}_{\omega} \mathcal{H}(f),$$

where the essential supremum is taken with respect to $\mathbb{P}$. More generally, for every 1-form f, define

$$\bar{\mathcal{H}}(f) = \inf_{f' \sim f} \operatorname*{esssup}_{\omega} \mathcal{H}(f').$$

We have the following generalization of (3.1): for every 1-form f,

$$\lim_{n\to\infty} \frac{1}{n} \log E^0_\omega \exp\left(\sum_{j=0}^{n-1} f(\tau_{X_j}\omega, X_{j+1} - X_j)\right) = \bar{H}(f). \tag{4.3}$$

This formula was established by Yilmaz [Y]. Note that if $f = e_P$, then $\bar{\mathcal{H}}(e_P) = \bar{H}(P) = \sup_v(P \cdot v - \bar{L}(v))$, which is what we get by (3.1) because $\sum_{j=0}^{n-1} f(\tau_{X_j}\omega, X_{j+1} - X_j) = P \cdot X_n$ in this case.

5 Annealed LDP

We now turn to the annealed large deviations for a RWRE. Recall that the quenched LDP is really a LDP for $\mathbb{P}_\omega$ and a LLN for the ω variable. That is why we have a quenched LDP under a rather mild condition on the environment measure $\mathbb{P}$. For anneald LDP however, we need to select a tractable law for the environment beacuse we are seeking for a LDP for the annealed law $\hat{\mathbb{P}}$. Pick a probability measure β on the set Γ_0 and set $\mathbb{P}$ to be the product of β to obtain a law on $\Omega = \Gamma_0^{\mathbb{Z}^d}$. The annealed measure $\hat{\mathbb{P}} = \int P^\omega \ \mathbb{P}(d\omega)$ has a simple description. Let us write $Z(n) = X(n+1) - X(n)$ for the jump the walk performs at time n. We certainly have

$$P_\omega(X(1) = x_1, \dots, X(n) = x_n) = \prod_{z,x \in \mathbb{Z}^d} p_x(z)^{N_{x,z}(n)},$$

where $\omega = (p_x : a \in \mathbb{Z}^d)$ and

$$N_{x,z}(n) = \#\{i \in \{0, 1, \dots, n-1\} : x_i = x, x_{i+1} - x_i = z\}.$$

Hence

$$\hat{\mathbb{P}}(X(1) = x_1, \dots, X(n) = x_n) = \prod_{x \in \mathbb{Z}^d} \int \prod_z p(z)^{N_{x,z}(n)} \beta(dp).$$

Evidently $\hat{\mathbb{P}}$ is the law of a non-Markovian walk in $\mathbb{Z}^d$. Varadhan in [V] establishes the annealed large deviations principle under a suitable ellipticity condition on β. The method relies on the fact that the environment seen from the walker is a Markov process for which Donsker–Varadhan Theory may apply if we have enough control on the transition probabilities. If we set

$$W_n = (0 - X(n), X(1) - X(n), \dots, X(n-1) - X(n), X(n) - X(n)) = (s_{-n}, \dots, s_{-1}, s_0 = 0)$$

for the chain seen from the location $X(n)$, then we obtain a walk of length n that ends at 0. The space of such walks is denoted by $\mathbf{W}_n$. Under the law $\hat{\mathbb{P}}$, the sequence $W_1, W_2, \dots$ is a Markov chain with the following rule:

$$\hat{\mathbb{P}}(W_{n+1} = T_z W_n \mid W_n) = \frac{\hat{\mathbb{P}}(T_z W_n)}{\hat{\mathbb{P}}(W_n)} = \frac{\int_{\Gamma_0} p(z) \prod_a p(a)^{N_{0,a}} \beta(dp)}{\int_{\Gamma_0} \prod_a p(a)^{N_{0,a}} \beta(dp)}, \tag{5.1}$$

where $N_{0,a} = N_{0,a}(W_n)$ is the number of jumps of size a from 0 for the walk W_n. Here $T_z W_n$ denotes a walk of size $n+1$, which is formed by translating the walk W_n by $-z$ so that it ends at $-z$ instead of 0, and then making the new jump of size z so that it ends at 0. We wish to establish a large deviation principle for the Markov chain with transition probability $\mathbf{q}(W, z)$ given by (5.1) where $W = W_n \in \bigcup_{m=0}^{\infty} \mathbf{W}_m$ and z is the jump size. We assume that with probability one, the support of $p_0(\cdot)$ is contained in the set $D = \{z : |z| \le C_0\}$. Naturally q extends to those infinite walks $W \in \mathbf{W}_\infty$ with $N_{0,a} < \infty$ for every $a \in D$. If we let $\mathbf{W}_\infty^{tr}$ denote the set of transient walks, then the expression $\mathbf{q}(W, z) = \mathbf{q}(W, T_z W)$ given by (5.1) defines the transition probability for a Markov chain in $\mathbf{W}_\infty^{tr}$. Donsker–Varadhan Theory suggests that the empirical measure

$$\frac{1}{n} \sum_{m=0}^{n-1} \delta_{W_n}$$

satisfies a large deviation principle with a rate function

$$I(\mu) = \int_{\mathbf{W}_\infty^{tr}} \mathbf{q}_\mu(W, z) \log \frac{\mathbf{q}_\mu(W, z)}{\mathbf{q}(W, z)} \mu(dW)$$

where μ is any T-invariant measure on $\mathbf{W}_\infty^{tr}$ and $\mathbf{q}_\mu(W, z)$ is the conditional probability of a jump of size z, given the past history. We then use the contraction principle to come up with a candidate for the large deviation rate function

$$\hat{H}(v) = \inf \left\{ I(\mu) : \int z_0 \mu(dW) = v \right\}$$

where z_0 denotes the jump of a walk W from the origin. Even though we have been able to state our problem as a LDP for a Markov chain, many difficulties arise because the state space is rather large and the transition probabilities are not continuous with respect to any natural topology we may choose. We refer the reader to [V] as how these issues are handeled.

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