METASTABILITY OF ZERO RANGE PROCESSES VIA POISSON EQUATIONS

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ABSTRACT. We prove the metastability of zero range processes on a finite set with an approach using the Poisson equation. Certain zero range processes on a finite set exhibit condensation. Most of the time, nearly all particles of the zero range process are at one single site. The site of condensate asymptotically behaves as a Markov chain. This is proven in [4] for the reversible case, [14] for the totally asymmetric case, and [18] for the non-reversible case. In these articles, the martingale approach is used and precise estimates of capacities are needed. We take an approach using solutions of Poisson equations. We circumvent precise estimates of capacities and prove the metastability for both reversible and non-reversible cases.

1. INTRODUCTION

Metastability is a dynamical phenomenon of some non-linear system with temporal random forces (noises). Metastability can be seen as first-order phase transition. We refer to monographs [9, 17] for an overview on metastability.

Some zero range processes exhibit condensation in the physics literature, which means above the critical density, as the number of particles increases to the infinity, a finite fraction of particles gather at a single site in the steady state. We refer to [11] for the review of condensation.

The site of condensate of the zero range process follows a Markov chain asymptotically after suitable time rescaling. This phenomenon is proved in [4, 14, 18] by Beltran, Landim and Seo, using the martingale approach. We refer to [5] for review of the martingale approach and differences between this approach, the pathwise approach [10], and the potential theoretic approach [7, 8]. Also we refer to [15] for some review and recent progress.

We prove metastabilty of condensed zero range processes on a finite set with an approach using solutions of Poisson equations. The model is the same as one in [4, 14, 18]. We assume that the invariant measure of underlying random walk is the uniform measure for simplification. We anticipate that our approach can be applied for the case of the general invariant measure with little modification. We refer to the Section 8 of [15] for introduction to this approach.

First we get an estimate on the solutions of Poisson equations and obtain asymptotic mean jump rates from the estimate. At the beginning, we investigate the properties of solutions of speeded-up Poisson equations $-\theta_N L_N F_N(\eta) = h_N(\eta)$ in the Section 4. Then we get asymptotic mean jump rates of the zero range process in the Section 5 in the following way. We multiply an auxiliary function to the

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Poisson equation and integrate the equation with respect to the unique invariant measure of the zero range process. Using several estimates, approximation and manipulation, we get asymptotics for the solutions of the Poisson equations. From asymptotic values of the solutions, we obtain the asymptotic mean jump rates.

Second we prove that the site of condensate follows a Markov chain asymptotically in Section 6. The asymptotic mean jump rates of the zero range process become the jump rates of the asymptotic Markov chain. We show tightness and convergence of stochastic processes using properties and estimate of the solutions of the Poisson equations in the Sections 4, 5 and martingale problems for Markov processes.

The first advantage of our method is that we circumvent sharp estimates of capacities. The martingale approach needs precise estimates of capacities. Getting sharp estimates are challenging, especially for the non-reversible case. It requires delicate construction of approximating objects. We use an auxiliary function, which is similar to the approximating function for the reversible case in [4]. The auxiliary function is simpler than approximating objects for the non-reversible case. Handling the auxiliary function and the solution of the Poisson equation is easier than handling approximating objects for the non-reversible case.

Also getting asymptotic mean jump rates is direct in this article, and not from capacities of the zero range process. For the reversible case, mean jump rates can be expressed in terms of capacities(Lemma 6.8 in [2]). But for non-reversible case, we don't have direct relation between mean jump rates and capacities. The collapsed chain is introduced in [3] as a tool for getting asymptotic mean jump rates. Also a general method is established in [18].

The method of using the Poisson equations have been applied for other models, but not for interacting particle systems such as the condensing zero range process in this article. This method is applied for elliptic operators on $\mathbb{R}^d$ of the form $L_N f = e^{NV}\nabla\cdot\left(e^{-NV}a\nabla f\right)$ in [12, 19], and one-dimensional diffusions with periodic boundary conditions in [16]. We refer to the Section 8 of [15].

We expect that this method can be applied for the case of the zero range process when the numbers of sites and particles of zero range process increases to infinity with a fixed ratio of numbers of sites and particles. The metastability of this model is proven in [1] for a parameter $\alpha > 20$. We hope to be able to use this method for small α.

Organization of the article. In Section 2, we introduce definitions, notations, and statements that we use in this article. In Section 3, we states main result of this article. In Section 4, we state and prove the properties of the solution of the Poisson equation. In Section 5, we estimate asymptotic mean jump rate for the zero range process. In Section 6, we prove main result using outcomes in previous sections.

2. Zero range processes

Definitions and notations in this section are similar to [4]. We assumed that the uniform measure is an invariant measure for the underlying random walk of the zero range process for making calculation simpler.

2.1. Underlying Random Walk.

Define $S := \{1, 2, ..., L\}$, where L is a fixed natural number larger than 1. For $x, y \in S$, let $r(x, y)$ be the jump rate for a random walk on S. Assume that this random walk is irreducible and has the uniform invariant measure on S.

2.2. **Definition of Zero Range Process.** For $S_0 \subset S$, an integer $N \geq 1$, define

$$E_{N,S_0} := \left\{ \eta \in \mathbb{N}_0^{S_0} : \sum_{x \in S_0} \eta_x = N \right\}.$$

Let $E_N = E_{N,S}$. Let α be a real number larger than 1.

Define a function $g : \mathbb{N}_0 \to \mathbb{R}$ by

$$g(0) = 0,\ g(1) = 1, \text{ and } g(n) = \frac{a(n)}{a(n-1)} \text{ for } n \geq 2, \text{ where } a(n) = n^\alpha.$$

For $x, y \in S$, we define a function $\sigma^{xy} : E_N \to E_N$ by the following way. For $x \neq y$, $\eta \in E_N$ with $\eta_x \geq 1$, define $\sigma^{xy}\eta \in E_N$ by

$$(\sigma^{xy}\eta)_z = \begin{cases} \eta_x - 1 & \text{for } z = x \\ \eta_y + 1 & \text{for } z = y \\ \eta_z & \text{otherwise}. \end{cases}$$

If $\eta_x = 0$ or $x = y$, then define $\sigma^{xy}\eta := \eta$. $\sigma^{xy}\eta$ is the configuration obtained from η by moving a particle from x to y.

The zero range process is a jump-type Markov process on $E_{N,S}$, whose infinitesimal generator is given by

$$(L_N F)(\eta) := \sum_{z,w \in S} g(\eta_z) r(z,w) \left(F(\sigma^{zw}\eta) - F(\eta) \right),$$

where F is a function from E_N to $\mathbb{R}$.

The interpretation for the zero range process is that we have N many particles that are scattered on a periodic lattice with L sites. Each particle performs a random walk with jump rate r, and the jump probabilities are adjusted by certain rules that depend on the number of particles of the departing site. To experience a condensation phenomenon, we choose $g(n)$ to be a decreasing function of $n \geq 2$ so that the particles tend to pile up at a site.

For a function F from E_N to $\mathbb{R}$, define the Dirichlet form associated the generator L_N by

$$D_N(F) := -\sum_{\eta \in E_N} F(\eta) (L_N F)(\eta) \mu(\eta).$$

2.3. **The Invariant Measure for the Zero Range Process.** This zero range process defined in the previous section has a unique invariant measure μ_N given by

$$\mu_N(\eta) = \frac{N^\alpha}{Z_{N,S}} \prod_{x \in S} \frac{1}{a(\eta_x)} = \frac{N^\alpha}{Z_{N,S}} \frac{1}{a(\eta)}, \quad \eta \in E_N,$$

where $a(\eta) = \prod_{x \in S} a(\eta_x)$ and $Z_{N,S}$ is the normalizing constant. Also define $\Gamma(\alpha) := \sum_{i=0}^{\infty} \frac{1}{a(i)}$ and $Z_S := L\,\Gamma(\alpha)^{L-1}$

Fix a sequence of integers $(\ell_N : N \geq 1)$ with $1 << \ell_N << N$. For $x \in S$, define

$$\mathcal{E}_N^x := \{\eta \in E_N : \eta_x \geq N - \ell_N\}.$$

Let $\mathcal{E}_N := \bigcup_{x \in S} \mathcal{E}_N^x$ and $\Delta_N := E_N \setminus \left(\bigcup_{x \in S} \mathcal{E}_N^x \right)$.

We omit the subscript N when there's no confusion.

The following propositions hold.

Proposition 2.1. *For every* $L \geq 2$,
$\lim_{N\to\infty} Z_{N,S} = Z_S$.

Proof. See the proof of Proposition 2.1 in Section 3 of [4]. □

Proposition 2.2. $\lim_{N\to\infty} \mu_N(\Delta_N) = 0$.

Proof. See the derivation of the equation (3.2) in [4] □

Proposition 2.3. $\lim_{N\to\infty} \mu_N(\mathcal{E}_N^x) = \frac{1}{L}$ *for all* $x \in S$.

Proof. By the definition of μ_N, $\mu_N(\mathcal{E}_N^x)$'s are the same for all $x \in S$. By Proposition 2.2, we get $\lim_{N\to\infty} \mu_N(\mathcal{E}_N^x) = \frac{1}{L}$. □

2.4. **Potential Theory.** In this subsection, we define the capacity for a Markov process. Consider a Markov process on a state space U. Let L be the infinitesimal generator of the Markov process. Refer to the Chapter 7 of [9] for the details.

Let $A, B \subset U$ be two non-empty disjoint subset. Consider the following Dirichlet problem

$$\begin{cases} (-Lh)(x) & = 0, \quad x \in U \setminus (A \cup B), \\ h(x) & = 1, \quad x \in A, \\ h(x) & = 0, \quad x \in B. \end{cases}$$

The harmonic function solves the previous problem is denoted by $h_{A,B}$, which is called the *equilibrium potential*.

Define

$$e_{A,B}(x) := (-Lh_{A,B})(x), \qquad x \in A.$$

This function is called the *equilibrium measure* on A.

Let ν is the unique ergodic invariant measure. The *capacity of the pair* A, B is defined by

$$\text{cap}(A, B) := \sum_{x \in A} \nu(x) e_{A,B}(x).$$

Consider the underlying random walk of the zero range process in this article. Denote the capacity of the pair $A, B \subset S$ for the underlying random walk by $\text{cap}_S(A, B)$. When $A = \{x\}, B = \{y\}$, denote $\text{cap}_S(A, B)$ by $\text{cap}_S(x, y)$.

3. Main result

3.1. **Metastability of the Zero Range Process.** For stating main result, We define the trace process for the zero range process.

Define $\mathcal{T}_t^A(\eta.)$ be the time spent by the zero range process $\{\eta^N(t) : t \geq 0\}$ on the set $A \subset E_N$ in the time interval $[0, t]$;

$$\mathcal{T}_t^A := \int_0^t \mathbf{1}\{\eta^N(s) \in A\}\, ds.$$

Define $\mathcal{S}_t^A$ be as the generalized inverse of $\mathcal{T}_t^A$;

$$\mathcal{S}_t^A := \sup\{s \geq 0 : \mathcal{T}_s^A(\eta.) \leq t\}.$$

For a subset A of E_N, the trace process $\{\eta^{N,A}(t) : t \geq 0\}$ is defined by $\eta^{N,A}(t) := \eta^N(\mathcal{S}_t^A)$, which is a strong Markov process with the state space A.

Define $\eta^{\mathcal{E}_N}(t) := \eta^{N,\mathcal{E}_N}(t)$. Let a projection function $\Psi_N : \mathcal{E}_N \to S$, $\Psi_N(\eta) := \sum_{x \in S} x\, \mathbf{1}\{\eta \in \mathcal{E}_N^x\}$. Define $X_t^N := \Psi_N(\eta^{\mathcal{E}_N}(t))$.

Let the speed-up constants $\theta_N := N^{1+\alpha}$, $N \geq 1$. Let $I_\alpha := \int_0^1 u^\alpha (1-u)^\alpha \, du$.

Define a Markov process $(Y_t : t \geq 0)$ on S by the generator $\mathfrak{L}$ which is given by

$$\mathfrak{L}f(x) = \frac{L}{\Gamma(\alpha) I_\alpha} \sum_{y \in S} \mathrm{cap}_S(x,y) \, \left(f(y) - f(x)\right), \text{ for } x \in S.$$

Let $\mathbb{P}_x$ be the probability measure on the path space $D(\mathbb{R}_+, S)$ induced by $\mathfrak{L}$ starting at $x \in S$. Similarly let $\mathbb{P}_{\xi_N}^N$ be the probability measure on the path space $D(\mathbb{R}_+, E_N)$ induced by L_N starting at $\xi_N \in E_N$.

We impose a condition on ℓ_N, which is

$$\lim_{N \to \infty} \frac{\ell_N^{1+\alpha(L-1)}}{N^{1+\alpha}} = 0. \tag{3.1}$$

Then the following propositions hold.

Proposition 3.1. *Fix $x \in S$. For any sequences $\xi_N \in \mathcal{E}_N^x$, $N \geq 1$, the sequence of laws of stochastic processes $(X_{\theta_N t} : t \geq 0)$ under $\mathbb{P}_{\xi_N}^N$ is tight.*

The proof of the Proposition 3.1 is in the Section 6.

Theorem 3.2. *The sequence of laws of stochastic processes $(X_{\theta_N t} : t \geq 0)$ in Proposition 3.1 converges to $\mathbb{P}_x$ as $N \to \infty$.*

The proof of the Theorem 3.2 is in the Section 6.

Theorem 3.3. *Let ν_N be a probability measure on E_N, absolutely continuous with respect to μ_N. Denote $\nu_N = f_N \mu_N$. Assume $\left(\|f_N\|_{L^2(\mu_N)} : N \geq 1\right)$ is bounded. Let $\mathbb{P}_{\nu_N}^N$ be the measure on the path space $D(\mathbb{R}_+, E_N)$ induced by L_N with the initial distribution ν_N. Then for every $T > 0$,*

$$\lim_{N \to \infty} \mathbb{E}^{\mathbb{P}_{\nu_N}^N} \left[\int_0^T 1 \left\{ \eta^N \left(N^{1+\alpha} s \right) \in \Delta_N \right\} ds \right] = 0$$

The proof of the Theorem 3.3 is in the Section 6.

The Theorem 3.3 holds when $\nu_N = \delta_{\eta_N}$, where $\eta_N \in \mathcal{E}_N^x$ for fixed $x \in S$. For the proof of this general case, refer to [2, 3].

4. Properties of the solution of Poisson equation

We consider the solutions of the speeded-up Poisson equations.

The sequence of functions $(F_N^{a,b} : N \geq 1)$ is defined by

$$-\theta_N L_N F_N^{a,b}(\eta) = \mathbf{1}\{\eta \in \mathcal{E}_N^a\} - \mathbf{1}\{\eta \in \mathcal{E}_N^b\} = h_N^{a,b}(\eta) \tag{4.1}$$

$$\int_{E_{N,S}} F_N^{a,b}(\eta) \, d\mu = 0. \tag{4.2}$$

Denote $F_N^{a,b}$ by F_N or F and $h_N^{a,b}$ by h_N or h when there's no confusion.
We state and prove the following proposition.

Proposition 4.1. *The function $F_N^{a,b}$ defined above satisfies the followings*
(1) $\min_{E_{N,S}} F_N^{a,b} = \min_{\mathcal{E}_N^b} F_N^{a,b}$ *and* $\max_{E_{N,S}} F_N^{a,b} = \max_{\mathcal{E}_N^a} F_N^{a,b}$.
(2) $\sup_N \theta_N D_N(F_N^{a,b}) < \infty$.
(3) Let $x \in S$. For any $\eta_1^N, \eta_2^N \in \mathcal{E}_N^x$, $|F_N^{a,b}(\eta_1^N) - F_N^{a,b}(\eta_2^N)| \to 0$ as $N \to \infty$.

Proof. Let $\mathcal{E}^+ = \mathcal{E}^a$, $\mathcal{E}^- = \mathcal{E}^b$.
(1) To see this, set

$$M^+ = \big\{\bar\eta \in E_{N,S} :\ F(\eta) = \max_{E_{N,S}} F\big\}, \qquad M^- = \big\{\bar\eta \in E_{N,S} :\ F(\eta) = \min_{E_{N,S}} F\big\}.$$

We wish to show $M^\pm \cap \mathcal{E}^\pm \neq \emptyset$. Suppose for example that $M^+ \cap \mathcal{E}^+ = \emptyset$. For every $\eta \in M^+$, we have $-L_N F_N(\eta) \geq 0$. From the right hand side of the equation (4.1), we can see $-L_N F_N(\eta) = 0$ and $\eta \in (\mathcal{E}^+ \cup \mathcal{E}^-)^{\complement}$. Since the maximum of F is attained at η, we learn

$$\eta \in M^+,\ \eta_x > 0,\ r(x,y) > 0 \implies \sigma^{xy}\eta \in M^+.$$

By irreducibility of r, we can start from some $\hat\eta \in M^+$ and reach a configuration on the boundary of $\mathcal{E}^+$ by applying the operation $\eta \to \sigma^{xy}\eta$ finitely many times. This contradicts $M^+ \cap \mathcal{E}^+ = \emptyset$. The proof of $M^- \cap \mathcal{E}^- \neq \emptyset$ is identical.

(2-1) First consider the case of reversible process.
Multiplying F to the equation (4.1) and integrating in $d\mu$ on E_N, we get

$$\begin{aligned}\theta_N D_N(F) &= \int_{\mathcal{E}^+} F(\eta)\, d\mu - \int_{\mathcal{E}^-} F(\eta)\, d\mu \\ &= \sum_{\eta\in\mathcal{E}^+} F(\eta)\,\mu(\eta) - \sum_{\eta\in\mathcal{E}^-} F(\eta)\,\mu(\eta)\,.\end{aligned}$$

It suffices to show that there exist a constant $C > 0$ satisfying

$$\theta_N D_N(F) \geq C\left(\sum_{\mathcal{E}^+} F(\eta)\,\mu(\eta) - \sum_{\mathcal{E}^-} F(\eta)\,\mu(\eta)\right)^2.$$

By definition,

$$\theta_N D_N(F_N) = \frac{N^{1+\alpha}}{2} \sum_{z,w\in S}\sum_{\eta\in E_N} \mu_N(\eta)\, r(z,w)\, g(\eta_z)\,\{F(\sigma^{zw}\eta) - F(\eta)\}^2\,.$$

By the change of variable $\xi = \eta - \mathfrak{d}_z$,
$\frac{N^{1+\alpha}}{2}\sum_{z,w\in S}\sum_{\eta\in E_N} \mu_N(\eta)\, r(z,w)\, g(\eta_z)\,\{F(\sigma^{zw}\eta) - F(\eta)\}^2$
$= \frac{N^{1+\alpha}}{2}\sum_{z,w\in S}\sum_{\xi\in E_{N-1}} \frac{N^\alpha}{Z_{N,S}}\,\frac{1}{a(\xi)}\, r(z,w)\,\{F(\xi+\mathfrak{d}_w) - F(\xi+\mathfrak{d}_z)\}^2$
We can easily find a constant $c_1 = c_1(a,b) > 0$ such that
$\frac{1}{2}\sum_{z,w\in S} r(z,w)\,\{f(w) - f(z)\}^2 \geq c_1\,(f(a) - f(b))^2$ for every function $f : S \to \mathbb{R}$.
Fix a configuration $\xi \in E_{N-1}$ and use the above inequality, then we get
$\frac{N^{1+\alpha}}{2}\sum_{z,w\in S}\sum_{\xi\in E_{N-1}} \frac{N^\alpha}{Z_{N,S}}\,\frac{1}{a(\xi)}\, r(z,w)\,\{F(\xi+\mathfrak{d}_w) - F(\xi+\mathfrak{d}_z)\}^2$

$\geq \frac{c_1 N^{1+2\alpha}}{Z_{N,S}} \sum_{\xi\in E_{N-1}} \frac{1}{a(\xi)} \{F(\xi+\mathfrak{d}_a) - F(\xi+\mathfrak{d}_b)\}^2.$

Let $\hat{\xi}$ be the restriction of ξ to sites $z \neq a, b$. the previous expression is equal or larger than

$$\frac{c_1 N}{Z_{N,S}} \sum_{\xi\in E_{N-1}} \frac{1}{a(\hat{\xi})} \{F(\xi+\mathfrak{d}_a) - F(\xi+\mathfrak{d}_b)\}^2$$
$$\geq \frac{c_1 N}{Z_{N,S}} \sum_{k=0}^{\ell} \sum_{\hat{\xi}\in E_{k,S\setminus\{a,b\}}} \sum_{\xi_a+\xi_b\leq N-1-k} \frac{1}{a(\hat{\xi})} \{F(\xi+\mathfrak{d}_a) - F(\xi+\mathfrak{d}_b)\}^2$$

Let $\eta \in \mathcal{E}^+$. Define a map σ on configurations that swaps η_a with η_b. Then $\sigma(\eta) \in \mathcal{E}^-$. Let $\hat{\eta}$ be the restriction of η to sites $z \neq a, b$. Let $\hat{S} = S\setminus\{a, b\}$. Let us write $\eta = (\hat{\eta}; \eta_a, \eta_b)$. We can change $\eta = (\hat{\eta}; N-k-i, i) \in \mathcal{E}^+$ to $\sigma(\eta) = (\hat{\eta}; i, N-k-i) \in \mathcal{E}^-$ by operations that move a particle on the site a to the site b , where $|\hat{\eta}| = k$. We will use the Cauchy-Schwarz inequalities.

The previous expression equals

$$\frac{c_1 N}{Z_{N,S}} \sum_{k=0}^{\ell} \sum_{\hat{\eta}\in E_{k,\hat{S}}} \frac{1}{a(\hat{\eta})} \sum_{j=0}^{N-k-1} \left(F(\hat{\eta}; N-k-1-j, j+1) - F(\hat{\eta}; N-k-j, j)\right)^2$$
$$\geq \frac{c_1 N}{Z_{N,S}} \sum_{k=0}^{\ell} \sum_{\hat{\eta}\in E_{k,\hat{S}}} \frac{1}{a(\hat{\eta})} \sum_{j=0}^{N-k-1} \left(F(\hat{\eta}; N-k-1-j, j+1) - F(\hat{\eta}; N-k-j, j)\right)^2$$
$$\geq \frac{c_1}{Z_{N,S}} \sum_{k=0}^{\ell} \sum_{\hat{\eta}\in E_{k,\hat{S}}} \frac{1}{a(\hat{\eta})} \frac{1}{\left(\sum_{i=0}^{\infty} \frac{1}{a(i)}\right)^2} N\times$$
$$\left(\sum_{j=0}^{\ell-k-1} \left(\sum_{i=0}^{j} \frac{1}{a(i)}\right)^2 \left(F(\hat{\eta}; N-k-1-j, j+1) - F(\hat{\eta}; N-k-j, j)\right)^2\right.$$
$$+ \sum_{j=\ell-k}^{N-\ell-1} \left(\sum_{i=0}^{\ell-k} \frac{1}{a(i)}\right)^2 \left(F(\hat{\eta}; N-k-1-j, j+1) - F(\hat{\eta}; N-k-j, j)\right)^2 +$$
$$\left.\sum_{j=N-\ell}^{N-k-1} \left(\sum_{i=0}^{N-k-1-j} \frac{1}{a(i)}\right)^2 \left(F(\hat{\eta}; N-k-1-j, j+1) - F(\hat{\eta}; N-k-j, j)\right)^2\right)$$
$$\geq \frac{c_1}{\Gamma(\alpha)^2 Z_{N,S}} \sum_{k=0}^{\ell} \sum_{\hat{\eta}\in E_{k,\hat{S}}} \frac{1}{a(\hat{\eta})} \times$$
$$\left(\sum_{j=0}^{\ell-k-1} \left(\sum_{i=0}^{j} \frac{1}{a(i)}\right) \left(F(\hat{\eta}; N-k-j, j) - F(\hat{\eta}; N-k-1-j, j)\right)\right.$$
$$+ \sum_{j=\ell-k}^{N-\ell-1} \left(\sum_{i=0}^{\ell-k} \frac{1}{a(i)}\right)^2 \left(F(\hat{\eta}; N-k-1-j, j) - F(\hat{\eta}; N-k-1-j, j)\right) +$$
$$\left.\sum_{j=N-\ell}^{N-k-1} \left(\sum_{i=0}^{N-k-1-j} \frac{1}{a(i)}\right)^2 \left(F(\hat{\eta}; N-k-1-j, j) - F(\hat{\eta}; N-k-1-j, j)\right)\right)^2$$

by Cauchy-Schwarz inequality.

$$= \frac{c_1}{\Gamma(\alpha)^2 Z_{N,S}} \sum_{k=0}^{\ell} \sum_{\hat{\eta}\in E_{k,\hat{S}}} \frac{1}{a(\hat{\eta})} \left(\sum_{i=0}^{\ell-k} \frac{1}{a(i)} \left(F(\hat{\eta}; N-k-i, i) - F(\hat{\eta}; i, N-k-i)\right)\right)^2$$

$\geq \frac{c_1}{\Gamma(\alpha)^2 Z_{N,S}} \frac{1}{\sum_{k=0}^{\ell} \sum_{\hat\eta \in E_{k,\hat S}} \frac{1}{a(\hat\eta)}} \times$

$$\left(\sum_{k=0}^{\ell} \sum_{\hat\eta \in E_{k,S\setminus\{a,b\}}} \frac{1}{a(\hat\eta)} \left(\sum_{i=0}^{\ell-k} \frac{1}{a(i)} \left(F(\hat\eta; N-k-i,\, i) - F(\hat\eta; i,\, N-k-i)\right)\right)\right)^2$$

by Cauchy-Schwarz inequality.

$\geq \frac{c_1}{\Gamma(\alpha)^2 Z_{N,S}} \frac{1}{\Gamma(\alpha)^{L-2}} \times$

$$\left(\sum_{k=0}^{\ell} \sum_{\hat\eta \in E_{k,\hat S}} \sum_{i=0}^{\ell-k} \frac{1}{a(\hat\eta) a(i)} \left(F(\hat\eta; N-k-i,\, i) - F(\hat\eta; i,\, N-k-i)\right)\right)^2 .$$

For $\eta = (\hat\eta; N-k-i,\, i) \in \mathcal{E}^+,\ \mu(\eta) = \mu(\sigma(\eta)) = \frac{N^\alpha}{Z_{N,S}} \frac{1}{a(\hat\eta)a(i)a(N-k-i)} = \frac{1}{Z_{N,S}} \frac{1}{a(\hat\eta)a(i)} \frac{N^\alpha}{a(N-k-i)} \leq \frac{2^\alpha}{Z_{N,S}} \frac{1}{a(\hat\eta)a(i)}$.

So the previous expression is equal or larger than

$$\frac{c_1 Z_{N,S}}{4^\alpha \Gamma(\alpha)^L} \left(\sum_{k=0}^{\ell} \sum_{\hat\eta \in E_{k,\hat S}} \sum_{i=0}^{\ell-k} \left(F(\hat\eta; N-k-i,\, i)\, \mu(\hat\eta; N-k-i,\, i) - F(\hat\eta; i,\, N-k-i)\, \mu(\hat\eta; i,\, N-k-i)\right)\right)^2$$

$= \frac{c_1 Z_{N,S}}{4^\alpha \Gamma(\alpha)^L} \left(\sum_{\eta \in \mathcal{E}^+} \left(F(\eta)\, \mu(\eta) - F(\sigma(\eta))\, \mu(\sigma(\eta))\right)\right)^2$

$= \frac{c_1 Z_{N,S}}{4^\alpha \Gamma(\alpha)^L} \left(\sum_{\eta \in \mathcal{E}^+} F(\eta)\, \mu(\eta) - \sum_{\eta \in \mathcal{E}^-} F(\eta)\, \mu(\eta)\right)^2.$

Since $Z_{N,S}$ is uniformly bounded in N by the Proposition 2.1, this proves (2) for the non-reversible case.

(2-2) Assume that the process is non-reversible. We write $S_N = (L_N + L_N^*)/2$ for the symmetric part of L_N. Note the jump rates of underlying random walks for L_N, L_N^*, and S_N are respectively $r(x,y)$, $r(y,x)$ and $\bar r(x,y) = (r(x,y)+r(y,x))/2$. We have $\theta_N D_N(G) = N^{1+\alpha} \int_{E_N} G(-L_N G)\ d\mu = N^{1+\alpha} \int_{E_N} G(-S_N G)\ d\mu$. Recall $h(\eta) = h^{a,b}(\eta) = \mathbf{1}\{\eta \in \mathcal{E}^a\} - \mathbf{1}\{\eta \in \mathcal{E}^b\}$ as the equation (4.1). We note that if

$$\hat c_N = \hat c = \max_G \left\{ \int_{E_N} Gh\ d\mu - \frac{1}{2}\theta_N D_N(G) \right\} = \frac{1}{2} \max_G \left\{ \frac{\left[\int_{E_N} Gh\ d\mu\right]^2}{\theta_N D_N(G)} \right\},$$

then

$$\hat c = \frac{1}{2} \int \bar F h\ d\mu = \frac{1}{2} \theta_N D_N(\bar F)$$

with $\bar F$ solving $-\theta_N S_N \bar F = h$. Since we have the uniform bound on $\theta_N D_N(F)$ for the reversible case, we know $\sup_N \hat c_N < \infty$.

Note that if we choose $F = F_N$ for G, we get

$$\frac{1}{2}\theta_N D_N(F) = \int Fh\ d\mu - \frac{1}{2}\theta_N D_N(F) \leq \hat c.$$

This gives a uniform bound on $\theta_N D_N(F)$ for the non-reversible case.

(3) Since we have a uniform bound on $\theta_N D_N(F)$,

$$N^{1+2\alpha}\sum_{\zeta\in E_{N-1}}\sum_{z,w} r(x,y)\frac{1}{a(\zeta)}\big[F(\zeta+\mathfrak{d}_z)-F(\zeta+\mathfrak{d}_w)\big]^2\le\bar{c}$$

for some constant $\bar{c}$.

For $\eta=\zeta+\mathfrak{d}_z\in\mathcal{E}_N^x$ we know that $\frac{1}{a(\zeta)}\ge\left(\frac{\ell_N}{L-1}\right)^{-\alpha(L-1)}N^{-\alpha}$ and $\min_{r(z,w)\neq 0} r(z,w)>0$. Hence

$$\sum_{\zeta\in E_{N-1}}\sum_{\substack{z,w\in S\\ r(z,w)\neq 0}}\big[F(\zeta+\mathfrak{d}_z)-F(\zeta+\mathfrak{d}_w)\big]^2\le c_0\ell_N^{\alpha(L-1)}N^{-\alpha-1}$$

for some constant c_0.

It takes $O(\ell_N)$ jumps to go from any configuration to any other configuration in $\mathcal{E}_N^x$. So for $\eta^1,\eta^2\in\mathcal{E}_N^x$

$$\big[F(\eta^1)-F(\eta^2)\big]^2\le c_1\ell_N\ell_N^{\alpha(L-1)}N^{-\alpha-1}=c_1\ell_N^{1+\alpha(L-1)}N^{-\alpha-1},$$

which converges to 0 since we have the condition $\frac{\ell_N^{1+\alpha(L-1)}}{N^{\alpha+1}}\to 0$ as $N\to\infty$, which is (3.1). □

5. Estimate on mean jump rates

In this section, we prove the Proposition 5.1.

Define the function $f_{a,b}:S\to\mathbb{R}$ for $a\neq b\in S$ by

$$-\mathfrak{L}f_{a,b}(x)=\mathbf{1}\{x=a\}-\mathbf{1}\{x=b\},\quad\text{for all } x\in S \tag{5.1}$$

and

$$\sum_{x\in S}f_{a,b}(x)=0. \tag{5.2}$$

Proposition 5.1. *Fix $x\in S$. For any sequence $(\eta^N\in\mathcal{E}_N^x:N\ge 1)$,*

$$\lim_{N\to\infty}F_N^{a,b}(\eta^N)=f_{a,b}(x).$$

We prove this proposition in the following subsections. To prove this proposition, we will define a function H_N on E_N and multiply H_N to the equation (4.1). Then we get

$$\int_{E_N}-\theta_N L_N F_N\,H_N\,d\mu_N=\int_{E_N}h_N H_N\,d\mu_N.$$

From this equation, we will get the estimate.

5.1. Proof of Proposition 5.1 for The Reversible Case.

First consider the reversible case.

We define the function $H_N^\epsilon(\eta)=H_N(\eta)=H(\eta)$ on E_N.

Fix small $0<\epsilon<\frac{1}{12}$. Let $\mathscr{D}:=\{u\in\mathbb{R}_+^S:\sum_{x\in S}u_x=1\}$. Let $0<\delta<1$ and $x\in S$. Let$\mathscr{D}_\delta^x:=\{u\in\mathscr{D}:u_x>1-\delta\}$ and $\mathscr{L}_\delta^{xy}:=\{u\in\mathscr{D}:u_x+u_y\ge 1-\delta\}$.

Define$\mathscr{K}_y^x=\mathscr{K}_y^x(\epsilon):=\mathscr{L}_\epsilon^{xy}\setminus\mathscr{D}_{3\epsilon}^x$, $y\neq x$.

There exists a smooth partition of unity

$$\Theta_y^x:\mathscr{D}\to[0,1]\,,\quad y\in S\setminus\{x\}\,,$$

such that $\sum_{y\in S\setminus\{x\}}\Theta^x_y(u)=1$ for all u in $\mathscr{D}$, and $\Theta^x_y(u)=1$ for all u in $\mathscr{K}^x_y$ and $y\in S\setminus\{x\}$.

Let $\hat{H}:[0,1]\to[0,1]$ be the smooth function given by

$$\hat{H}(t) \;:=\; \frac{1}{I_\alpha}\int_0^{\phi(t)} u^\alpha(1-u)^\alpha\,du\;,$$

where I_α is the constant defined above and $\phi:[0,1]\to[0,1]$ is a piecewise linear function whose graph connects $(0,0)$, $(3\epsilon,0)$, $(1-3\epsilon,1)$, $(1,1)$.

Let $\bar{L}$ be the infinitesimal generator of the underlying random walk.

Fix $x\in S$. For $y\neq x$, define $H_{xy}(\eta)=\hat{H}(\frac{\eta_x}{N}+\min\{\frac{J_{xy}\cdot\eta-\eta_x}{N},\epsilon\}),\quad \eta\in E_N$,

where $J_{xy}:S\to[0,1]$ solves $\begin{cases}\bar{L}J_{xy}(z) &=0,\quad z\neq x,y\\ J_{xy}(x) &=1\\ J_{xy}(y) &=0\end{cases}$ and $J\cdot\eta=\sum_z J_z\eta_z$, the dot product where $J_z=J(z)$ for $z\in S$.

Let $H=H_x:E_N\to\mathbb{R}$ be given by $H_x(\eta):=\sum_{y\in S\setminus\{x\}}\Theta^x_y(\frac{\eta}{N})H_{xy}(\eta)$.

We can see that

$$H_x(\eta)=1 \text{ if } \eta_x\geq(1-3\epsilon)N, \tag{5.3}$$

$$H_x(\eta)=0 \text{ if } \eta_x\leq 2\epsilon N. \tag{5.4}$$

Since $\hat{H}$ and Θ^x_y's are Lipschitz continuous, there exist a constant C_ϵ which depends on ϵ, not N such that

$$\max_{z,w\in S}|H(\sigma^{zw}\eta)-H(\eta)|<\frac{C_\epsilon}{N} \tag{5.5}$$

for all $\eta\in E_{N,S}$.

We will define some sets in $E_{N,S}$. Let a sequence $(\tilde{\ell}_N:N\geq 1)$ be such that $\tilde{\ell}_N\leq\ell_N$, $\lim_{N\to\infty}\frac{\tilde{\ell}_N^{1+(L-2)\alpha}}{N}\to 0$ and $1<<\tilde{\ell}_N<<N$.

Define $\tilde{T}^{xy}_N:=\{\eta\in E_N:\eta_x+\eta_y\geq N-\tilde{\ell}_N\}$ and $\tilde{T}^x_N:=\cup_{y\in S\setminus\{x\}}\tilde{T}^{xy}_N$.

By multiplying H to the equation (4.1) we get

$$\int_{E_{N,S}} -\theta_N L_N F_N\, H\,d\mu=\int_{E_{N,S}} hH\,d\mu. \tag{5.6}$$

Let us consider the left hand side of this equation.

$$\begin{aligned}(LHS) \;&=\; \int_{E_N} -\theta_N L_N F_N^{a,b}(\eta) H\,d\mu\\ &=\; N^{1+\alpha}\sum_{\eta\in E_N}\sum_{z,w\in S} -\mu(\eta)g(\eta_z)r(z,w)\left(F(\sigma^{zw}\eta)-F(\eta)\right)H(\eta)\\ &=\; \frac{N^{1+\alpha}}{2}\sum_{\eta\in E_N}\sum_{z,w\in S}\mu(\eta)g(\eta_z)r(z,w)\left(F(\sigma^{zw}\eta)-F(\eta)\right)\left(H(\sigma^{zw}\eta)-H(\eta)\right)\end{aligned}$$

since the process is reversible.

For functions $F, G : E_N \to \mathbb{R}$ and a subset $\mathcal{A}$ of E_N, define

$$D_N(F,G;\mathcal{A}) = \frac{1}{2}\sum_{\eta\in\mathcal{A}}\sum_{z,w\in S}\mu(\eta)g(\eta_z)r(z,w)(F(\sigma^{zw}\eta)-F(\eta))(G(\sigma^{zw}\eta)-G(\eta)).$$

Then,

$$\begin{aligned}(LHS) &= \theta_N D_N(F,H;E_N)\\ &= \theta_N D_N(F,H;(\tilde{T}_N^x)^{\complement}) + \theta_N D_N(F,H;\tilde{T}_N^x)\\ &= \theta_N D_N(F,H;(\tilde{T}_N^x)^{\complement}) + \sum_{y\in S,\, y\neq x}\theta_N D_N(F,H;\tilde{T}^{xy}),\end{aligned} \tag{5.7}$$

for sufficiently large N because of (5.3), (5.4).

Consider the first term $\theta_N D_N(F,H;(\tilde{T}_N^x)^{\complement})$.
We use the following lemma.

Lemma 5.2. *For sufficiently large N,*

$$\frac{N^{1+\alpha}}{2}\sum_{\eta\in(\tilde{T}_N^x)^{\complement}}\sum_{z,w\in S}\mu(\eta)g(\eta_z)r(z,w)\left(H(\sigma^{zw}\eta)-H(\eta)\right)^2 \le \frac{C_\epsilon}{(\epsilon\tilde{\ell}_N)^{\alpha-1}},$$

where C_ϵ is a constant only depends on ϵ.

Proof. See the proof of Lemma 5.2 in [4]. ☐

The first term in (5.7) is
$\theta_N D_N(F,H;(\tilde{T}_N^x)^{\complement})$
$= \frac{N^{1+\alpha}}{2}\sum_{\eta\in(\tilde{T}_N^x)^{\complement}}\sum_{z,w\in S}\mu(\eta)g(\eta_z)r(z,w)\left(F(\sigma^{zw}\eta)-F(\eta)\right)\left(H(\sigma^{zw}\eta)-H(\eta)\right)$
$\le \left(\frac{N^{1+\alpha}}{2}\sum_{\eta\in(\tilde{T}_N^x)^{\complement}}\sum_{z,w\in S}\mu(\eta)g(\eta_z)r(z,w)\left(F(\sigma^{zw}\eta)-F(\eta)\right)^2\right)^{1/2}\times$
$\left(\frac{N^{1+\alpha}}{2}\sum_{\eta\in(\tilde{T}_N^x)^{\complement}}\sum_{z,w\in S}\mu(\eta)g(\eta_z)r(z,w)\left(H(\sigma^{zw}\eta)-H(\eta)\right)^2\right)^{1/2}$
$\le \bar{c}\frac{C_\epsilon}{(\epsilon\tilde{\ell}_N)^{\frac{\alpha-1}{2}}}$ by the previous lemma and Proposition 4.1 (2).
Thus $\lim_{N\to\infty}\theta_N D_N(F_N,H_N;(\tilde{T}_N^x)^{\complement}) = 0$.

Consider the second term $\sum_{y\in S,\, y\neq x}\theta_N D_N(F,H;\tilde{T}^{xy})$ in (5.7).

$$\begin{aligned}\theta_N D_N(F,H;\tilde{T}^{xy}) &= \frac{N^{1+\alpha}}{2}\sum_{z,w\in S}\sum_{\zeta+\mathfrak{d}_z\in\tilde{T}_N^{xy}}\frac{N^\alpha}{Z_N}\frac{1}{a(\zeta)}r(z,w)\left(F(\zeta+\mathfrak{d}_w)-F(\zeta+\mathfrak{d}_z)\right)\\ &\quad\times\ \left(H(\zeta+\mathfrak{d}_w)-H(\zeta+\mathfrak{d}_z)\right)\\ &= \frac{N^{1+\alpha}}{2}\sum_{\substack{\zeta\in E_{N-1}\\ \zeta_x+\zeta_y\ge N-\tilde{\ell}}}\sum_{z,w\in S}\frac{N^\alpha}{Z_N}\frac{1}{a(\zeta)}r(z,w)\left(F(\zeta+\mathfrak{d}_w)-F(\zeta+\mathfrak{d}_z)\right)\\ &\quad\times\ \left(H(\zeta+\mathfrak{d}_w)-H(\zeta+\mathfrak{d}_z)\right) + \Lambda_N^{xy}\end{aligned} \tag{5.8}$$

Lemma 5.3. $|\Lambda_N^{xy}| \le \frac{C_\epsilon}{\tilde{\ell}^{\alpha/2}}$ *where C_ϵ is a constant only depends on ϵ.*

Proof. Write $\eta = \zeta + \mathfrak{d}_z$. If $\eta_x + \eta_y > N - \tilde{\ell}$,then $\zeta_x + \zeta_y \geq N - \tilde{\ell}$. If $\eta_x + \eta_y = N - \tilde{\ell}$, then $\zeta_x + \zeta_y \geq N - \tilde{\ell}$ only if $z = x$, $w \neq y$ or $z = y$, $w \neq x$.

So

$$
\begin{aligned}
|\Lambda_N^{xy}| &\leq \frac{N^{1+\alpha}}{2} \sum_{\eta \in \tilde{T}_N^{xy}} \sum_{z,w \in S} \frac{N^\alpha}{Z_N} \frac{g(\eta_z)}{a(\eta)} r(z,w) \left| F(\sigma^{zw}\eta) - F(\eta) \right| \left| H(\sigma^{zw}\eta) - H(\eta) \right| \\
&\leq \left(\frac{N^{1+\alpha}}{2} \sum_{\eta \in \tilde{T}_N^{xy}} \sum_{z,w \in S} \frac{N^\alpha}{Z_N} \frac{g(\eta_z)}{a(\eta)} r(z,w) \left| F(\sigma^{zw}\eta) - F(\eta) \right|^2 \right)^{1/2} \\
&\quad \times \left(\frac{N^{1+\alpha}}{2} \sum_{\eta \in \tilde{T}_N^{xy}} \sum_{z,w \in S} \frac{N^\alpha}{Z_N} \frac{g(\eta_z)}{a(\eta)} r(z,w) \left| H(\sigma^{zw}\eta) - H(\eta) \right|^2 \right)^{1/2}
\end{aligned}
$$

The first term is bounded by the Proposition 4.1 (2).

Consider the second term. $\hat{\eta}$ is the restriction of η to sites $z \neq x, y$.

$$
\begin{aligned}
&\sum_{\eta \in \tilde{T}_N^{xy}} \sum_{z,w \in S} \frac{N^\alpha}{Z_N} \frac{g(\eta_z)}{a(\eta)} r(z,w) \left| H(\sigma^{zw}\eta) - H(\eta) \right|^2 \\
&= \sum_{\hat{\eta} \in E_{\tilde{\ell}}} \sum_{\substack{\lfloor 2\epsilon N \rfloor \leq \eta_x \leq N - \lfloor 3\epsilon N \rfloor \\ \eta_y = N - \tilde{\ell} - \eta_x}} \sum_{z,w \in S} \frac{N^\alpha}{Z_N} \frac{g(\eta_z)}{a(\eta)} r(z,w) \left| H(\sigma^{zw}\eta) - H(\eta) \right|^2 \text{ by 5.3, 5.4}
\end{aligned}
$$

From now C is a constant which can vary line by line and C_ϵ is a constant depending only on ϵ which can vary line by line too. We have that $g(\eta_z)$ is bounded and $|H(\zeta + \mathfrak{d}_w) - H(\zeta + \mathfrak{d}_z)| \leq \frac{C_\epsilon}{N}$ by (5.5). Also $\sum_{z,w \in S} r(z,w)$ is bounded.

$$
\begin{aligned}
\sum_{\hat{\eta} \in E_{\tilde{\ell}}} \sum_{\substack{\lfloor 2\epsilon N \rfloor \leq \eta_x \leq N - \lfloor 3\epsilon N \rfloor \\ \eta_y = N - \tilde{\ell} - \eta_x}} \frac{N^\alpha}{Z_N} \frac{1}{a(\eta)} &= \sum_{\hat{\eta} \in E_{\tilde{\ell}}} \sum_{\substack{\lfloor 2\epsilon N \rfloor \leq \eta_x \leq N - \lfloor 3\epsilon N \rfloor \\ \eta_y = N - \tilde{\ell} - \eta_x}} \frac{N^\alpha}{Z_N} \frac{1}{a(\hat{\eta}) a(\eta_x) a(\eta_y)} \\
&= \frac{N^\alpha}{Z_N} \sum_{\hat{\eta} \in E_{\tilde{\ell}}} \frac{1}{a(\hat{\eta})} \sum_{\substack{\lfloor 2\epsilon N \rfloor \leq \eta_x \leq N - \lfloor 3\epsilon N \rfloor \\ \eta_y = N - \tilde{\ell} - \eta_x}} \frac{1}{a(\eta_x) a(\eta_y)}.
\end{aligned}
$$

By the Proposition 2.1, $\sum_{\hat{\eta} \in E_{\tilde{\ell}}} \frac{1}{a(\hat{\eta})} = O(\tilde{\ell}^{-\alpha})$.

$\sum_{\substack{\lfloor 2\epsilon N \rfloor \leq \eta_x \leq N - \lfloor 3\epsilon N \rfloor \\ \eta_y = N - \tilde{\ell} - \eta_x}} \frac{1}{a(\eta_x) a(\eta_y)} = \sum_{\lfloor 2\epsilon N \rfloor \leq \eta_x \leq N - \lfloor 3\epsilon N \rfloor} \frac{1}{\eta_x^\alpha \left(N - \tilde{\ell} - \eta_x \right)^\alpha}$

Let $N' = N - \tilde{\ell}$. Since $\tilde{\ell} << N$, $\sum_{\lfloor 2\epsilon N \rfloor \leq \eta_x \leq N - \lfloor 3\epsilon N \rfloor} \frac{1}{\eta_x^\alpha \left(N - \tilde{\ell} - \eta_x \right)^\alpha}$

$= \sum_{\lfloor 2\epsilon N \rfloor \leq \eta_x \leq N - \lfloor 3\epsilon N \rfloor} \frac{1}{\left(\frac{\eta_x}{N'} \right)^\alpha \left(\frac{N' - \eta_x}{N'} \right)^\alpha} \frac{1}{N'} N'^{1-2\alpha}$

$= \int_{2\epsilon}^{1-3\epsilon} \frac{1}{u^\alpha (1-u)^\alpha} \, du \, O(N'^{1-2\alpha}) = C_\epsilon O(N^{1-2\alpha})$

Summarizing these,

$$
\left(\frac{N^{1+\alpha}}{2} \sum_{\eta \in \tilde{T}_N^{xy}} \sum_{z,w \in S} \frac{N^\alpha}{Z_N} \frac{g(\eta_z)}{a(\eta)} r(z,w) \left| H(\sigma^{zw}\eta) - H(\eta) \right|^2 \right)^{1/2} = C_\epsilon O(\tilde{\ell}^{-\alpha/2})
$$

Thus $|\Lambda_N^{xy}| \le \frac{C_\epsilon}{\ell^{\alpha/2}}$. □

Consider the first term of the equation (5.8).
Define

$$\tilde{S}_N^{xy} = \left\{\zeta \in E_{N-1} : \zeta_z + \zeta_y \ge N - \tilde{\ell}\right\}.$$

Also define

$$\tilde{S}_N^{xy}(a,b) = \left\{\zeta \in E_{N-1} : \zeta_z + \zeta_y \ge N - \tilde{\ell}, a \le \zeta_x \le b\right\}.$$

Then the first term of the equation (5.8) is

$$\frac{N^{1+\alpha}}{2} \sum_{\zeta \in \tilde{S}_N^{xy}} \sum_{z,w \in S} \frac{N^\alpha}{Z_N} \frac{r(z,w)}{a(\zeta)} \left(F(\zeta + \mathfrak{d}_w) - F(\zeta + \mathfrak{d}_z)\right)\left(H(\zeta + \mathfrak{d}_w) - H(\zeta + \mathfrak{d}_z)\right)$$

$$\begin{aligned}
&= \frac{N^{1+\alpha}}{2} \sum_{\zeta \in \tilde{S}_N^{xy}(\lfloor 4\epsilon N \rfloor,\, N - \lfloor 4\epsilon N \rfloor)} \sum_{z,w \in S} \frac{N^\alpha}{Z_N} \frac{1}{a(\zeta)} r(z,w) \left(F(\zeta + \mathfrak{d}_w) - F(\zeta + \mathfrak{d}_z)\right) \\
&\quad \times \left(H(\zeta + \mathfrak{d}_w) - H(\zeta + \mathfrak{d}_z)\right) \\
&\quad + \frac{N^{1+\alpha}}{2} \sum_{\zeta \in \tilde{S}_N^{xy}(1,\, \lfloor 4\epsilon N \rfloor - 1)} \sum_{z,w \in S} \frac{N^\alpha}{Z_N} \frac{1}{a(\zeta)} r(z,w) \left(F(\zeta + \mathfrak{d}_w) - F(\zeta + \mathfrak{d}_z)\right) \\
&\quad \times \left(H(\zeta + \mathfrak{d}_w) - H(\zeta + \mathfrak{d}_z)\right) \\
&\quad + \frac{N^{1+\alpha}}{2} \sum_{\zeta \in \tilde{S}_N^{xy}(N - \lfloor 4\epsilon N \rfloor + 1,\, N)} \sum_{z,w \in S} \frac{N^\alpha}{Z_N} \frac{1}{a(\zeta)} r(z,w) \left(F(\zeta + \mathfrak{d}_w) - F(\zeta + \mathfrak{d}_z)\right) \\
&\quad \times \left(H(\zeta + \mathfrak{d}_w) - H(\zeta + \mathfrak{d}_z)\right).
\end{aligned}$$

Let the first term, second term, and last term in the previous expression be $\Omega_1, \Omega_{21}, \Omega_{22}$.

Lemma 5.4. *If N is sufficiently large so that $\epsilon N >> \tilde{\ell}_N >> 1$, then $|\Omega_{21}| \le C\epsilon^{\frac{\alpha+1}{2}}$, $|\Omega_{22}| \le C\epsilon^{\frac{\alpha+1}{2}}$ where C is a constant independent of N, ϵ.*

Proof. In this proof, a constant C can vary line by line.

Consider Ω_{21}. Assume $\zeta \in E_{N-1}$, $\zeta_x + \zeta_y \ge N - \tilde{\ell}$, $\zeta_x \le \lfloor 4\epsilon N \rfloor - 1$.

$$H(\zeta + \mathfrak{d}_w) - H(\zeta + \mathfrak{d}_z) = \frac{1}{I_\alpha} \int_{\phi\left(\frac{J \cdot \zeta + J_z}{N}\right)}^{\phi\left(\frac{J \cdot \zeta + J_w}{N}\right)} u^\alpha (1-u)^\alpha \, du.$$

By the fundamental theorem of calculus, there exists $u_0$between $\frac{J \cdot \zeta + J_w}{N}$, $\frac{J \cdot \zeta + J_z}{N}$ such that

$$H(\zeta + \mathfrak{d}_w) - H(\zeta + \mathfrak{d}_z) = \frac{1}{I_\alpha} \left(\phi\left(\frac{J \cdot \zeta + J_w}{N}\right) - \phi\left(\frac{J \cdot \zeta + J_z}{N}\right)\right) u_0^\alpha (1-u_0)^\alpha.$$

Here $u_0 \le \frac{\zeta_x + \tilde{\ell} + 1}{N} \le 5\epsilon N$ and $|\phi'(v_0)| \le \frac{1}{1-6\epsilon}$.

So

$$|H(\zeta + \mathfrak{d}_w) - H(\zeta + \mathfrak{d}_z)| \le \frac{1}{I_\alpha} \frac{|J_w - J_z|}{N} \frac{1}{1-6\epsilon} (5\epsilon)^\alpha \le C \frac{\epsilon^\alpha}{N} \tag{5.9}$$

for some constant C. We used the condition $\epsilon < \frac{1}{12}$.

$$\begin{aligned}
\Omega_{21} &= \frac{N^{1+2\alpha}}{2Z_N} \sum_{\zeta \in \tilde{S}_N^{xy}(1,\, \lfloor 4\epsilon N \rfloor - 1)} \sum_{z,w \in S} \frac{1}{a(\zeta)} r(z,w) \left(F(\zeta + \mathfrak{d}_w) - F(\zeta + \mathfrak{d}_z)\right) \\
&\quad \times \left(H(\zeta + \mathfrak{d}_w) - H(\zeta + \mathfrak{d}_z)\right).
\end{aligned}$$

By the Cauchy-Schwartz inequity,

$$\Omega_{21}^2 \le \left(\frac{N^{1+2\alpha}}{2Z_N}\right)^2 \left(\sum_{\zeta\in\tilde{S}_N^{xy}(1,\lfloor 4\epsilon N\rfloor-1)} \sum_{z,w\in S} \frac{1}{a(\zeta)} r(z,w)\ (F(\zeta+\mathfrak{d}_w)-F(\zeta+\mathfrak{d}_z))^2\right)$$
$$\times \left(\sum_{\zeta\in\tilde{S}_N^{xy}(1,\lfloor 4\epsilon N\rfloor-1)} \sum_{z,w\in S} \frac{1}{a(\zeta)} r(z,w)\ (H(\zeta+\mathfrak{d}_w)-H(\zeta+\mathfrak{d}_z))^2\right).$$

By the Proposition 4.1 (2),

$$\sum_{\zeta\in\tilde{S}_N^{xy}(1,\lfloor 4\epsilon N\rfloor-1)} \sum_{z,w\in S} \frac{1}{a(\zeta)} r(z,w)\ (F(\zeta+\mathfrak{d}_w)-F(\zeta+\mathfrak{d}_z))^2 = O(N^{-(1+2\alpha)}),$$

and

$$\sum_{\zeta\in\tilde{S}_N^{xy}(1,\lfloor 4\epsilon N\rfloor-1)} \sum_{z,w\in S} \frac{1}{a(\zeta)} r(z,w)\ (H(\zeta+\mathfrak{d}_w)-H(\zeta+\mathfrak{d}_z))^2$$
$$= \sum_{z,w\in S} r(z,w) \sum_{\tilde{S}_N^{xy}(\lfloor 2\epsilon N\rfloor,\lfloor 4\epsilon N\rfloor-1)} \frac{1}{a(\zeta)}\ (H(\zeta+\mathfrak{d}_w)-H(\zeta+\mathfrak{d}_z))^2 \text{ by 5.4}$$
$$\le L^2C\frac{\epsilon^{2\alpha}}{N^2}\left(\sum_{\zeta\in\tilde{S}_N^{xy}(\lfloor 2\epsilon N\rfloor,\lfloor 4\epsilon N\rfloor-1)} \frac{1}{a(\zeta)}\right).$$

The term inside the parentheses is

$$\sum_{\zeta\in\tilde{S}_N^{xy}(\lfloor 2\epsilon N\rfloor,\lfloor 4\epsilon N\rfloor-1)} \frac{1}{a(\zeta)} \le \sum_{k=0}^{\tilde{\ell}} \sum_{\hat{\zeta}\in E_{k,S\setminus\{x,y\}}} \frac{1}{a(\hat{\zeta})} \sum_{\substack{\lfloor 2\epsilon N\rfloor\le\zeta_x\le\lfloor 4\epsilon N\rfloor-1\\ \zeta_y=N-k-\zeta_x}} \frac{1}{a(\zeta_x)a(\zeta_y)}$$

where $\hat{\zeta}$ is the restriction of ζ to $S\setminus\{x,y\}$

$$\le \left(\sum_{k=0}^{\infty} \sum_{\hat{\zeta}\in E_{k,S\setminus\{x,y\}}} \frac{1}{a(\hat{\zeta})}\right)(\lfloor 4\epsilon N\rfloor-\lfloor 2\epsilon N\rfloor)\frac{1}{\lfloor 2\epsilon N\rfloor^\alpha\left(\frac{N}{2}\right)^\alpha}$$
$$\le C\Gamma(\alpha)^{L-2}\epsilon^{1-\alpha}N^{1-2\alpha} \quad \text{where } C \text{ is a constant.}$$

So $\sum_{\zeta\in\tilde{S}_N^{xy}(1,\lfloor 4\epsilon N\rfloor-1)} \sum_{z,w\in S} \frac{1}{a(\zeta)} r(z,w)\ (H(\zeta+\mathfrak{d}_w)-H(\zeta+\mathfrak{d}_z))^2 \le C\epsilon^{1+\alpha}N^{-1-2\alpha}$.

Thus $|\Omega_{21}|\le C\epsilon^{\frac{1+\alpha}{2}}$ for some constant C. Similarly we can get $|\Omega_{22}|\le C\epsilon^{\frac{\alpha+1}{2}}$.

□

Consider the term Ω_1.

Assume $\lfloor 4\epsilon N\rfloor\le\zeta_x\le N-\lfloor 4\epsilon N\rfloor$, $\zeta\in E_{N-1}$, and $\zeta_x+\zeta_y\ge N-\tilde{\ell}$. Also assume N is sufficiently large so that $\epsilon N >> \tilde{\ell}_N >> 1$.

Consider

$$H(\zeta+\mathfrak{d}_w)-H(\zeta+\mathfrak{d}_z) = \frac{1}{I_\alpha}\int_{\phi\left(\frac{J\cdot\zeta+J_z}{N}\right)}^{\phi\left(\frac{J\cdot\zeta+J_w}{N}\right)} u^\alpha(1-u)^\alpha\,du.$$

Since $3\epsilon\le\frac{J\cdot\zeta+J_w}{N}$ and $\frac{J\cdot\zeta+J_z}{N}\le 1-3\epsilon N$, $\phi'(\frac{J\cdot\zeta+J_w}{N})=\phi'(\frac{J\cdot\zeta+J_z}{N})=\frac{1}{1-6\epsilon}$.

By the fundamental theorem of calculus, there exists $u_0$between $\frac{J\cdot\zeta+J_w}{N}$, $\frac{J\cdot\zeta+J_z}{N}$ such that

$$H(\zeta+\mathfrak{d}_w)-H(\zeta+\mathfrak{d}_z)=\frac{1}{I_\alpha}\left(\phi\left(\frac{J\cdot\zeta+J_w}{N}\right)-\phi\left(\frac{J\cdot\zeta+J_z}{N}\right)\right)u_0^\alpha(1-u_0)^\alpha.$$

Write $u_0=\frac{J\cdot\zeta+v_0}{N}$ where v_0 is a real number between J_w and J_z.

Then $u_0=\frac{\zeta_x+\sum_{z\neq x}J_z\zeta_z+v_0}{N}\leq\frac{\zeta_x}{N}+\frac{\tilde{\ell}+1}{N}$.

Since $\zeta_x>3\epsilon N$, $\frac{\zeta_x}{N}\leq u_0\leq\frac{\zeta_x}{N}\left(1+\frac{\tilde{\ell}+1}{N}\right)=\frac{\zeta_x}{N}\left(1+O(\frac{\tilde{\ell}}{\epsilon N})\right)$.

Thus $u_0=\frac{\zeta_x}{N}\left(1+O(\frac{\tilde{\ell}}{\epsilon N})\right)$. We get $1-u_0=\frac{\sum_z(1-J_z)\zeta_z+1-c_0}{N}$. By changing the role of $(J_z:z\in S)$ and ζ_x to $(1-J_z:z\in S)$ and ζ_y, we get $1-u_0=\frac{\zeta_y}{N}\left(1+O(\frac{\tilde{\ell}}{\epsilon N})\right)$.

So

$$\begin{aligned}
&H(\zeta+\mathfrak{d}_w)-H(\zeta+\mathfrak{d}_z)\\
&=\frac{1}{I_\alpha(1-6\epsilon)}\frac{J_w-J_z}{N}\left(\frac{\zeta_x}{N}\left(1+O(\frac{\tilde{\ell}}{\epsilon N})\right)\right)^\alpha\left(\frac{\zeta_y}{N}\left(1+O(\frac{\tilde{\ell}}{\epsilon N})\right)\right)^\alpha\\
&=\frac{1}{I_\alpha(1-6\epsilon)}\frac{J_w-J_z}{N}\left(\frac{\zeta_x}{N}\right)^\alpha\left(\frac{\zeta_y}{N}\right)^\alpha\left(1+O(\frac{\tilde{\ell}}{\epsilon N})\right)\\
&=\frac{N^{-1-2\alpha}}{I_\alpha(1-6\epsilon)}\left(J_w-J_z\right)\zeta_x^\alpha\zeta_y^\alpha\left(1+O(\frac{\tilde{\ell}}{\epsilon N})\right)\\
&=\frac{N^{-1-2\alpha}}{I_\alpha(1-6\epsilon)}\left(J_w-J_z\right)\zeta_x^\alpha\zeta_y^\alpha+\hat{R}(\zeta,w,z),
\end{aligned}$$

where

$$\left|\hat{R}(\zeta,w,z)\right|\leq C\frac{\tilde{\ell}}{\epsilon N}N^{-1-2\alpha}\zeta_x^\alpha\zeta_y^\alpha. \tag{5.10}$$

Define

$$\begin{aligned}
\tilde{C}_N^{xy}&:=\tilde{S}_N^{xy}\left(\lfloor 4\epsilon N\rfloor,\,N-\lfloor 4\epsilon N\rfloor\right)\\
&=\left\{\zeta\in E_{N-1}:\zeta_z+\zeta_y\geq N-\tilde{\ell},\lfloor 4\epsilon N\rfloor\leq\zeta_x\leq N-\lfloor 4\epsilon N\rfloor\right\}.
\end{aligned}$$

Let

$$\begin{aligned}
\Omega_{11}=\frac{N^{1+\alpha}}{2}&\sum_{\zeta\in\tilde{C}_N^{xy}}\sum_{z,w\in S}\frac{N^\alpha}{Z_N}\frac{1}{a(\zeta)}r(z,w)\left(F(\zeta+\mathfrak{d}_w)-F(\zeta+\mathfrak{d}_z)\right)\\
&\times\frac{1}{(1-6\epsilon)I_\alpha}(J_w-J_z)\zeta_x^\alpha\zeta_y^\alpha N^{-1-2\alpha}
\end{aligned}$$

and

$$\Omega_{12}=\frac{N^{1+\alpha}}{2}\sum_{\zeta\in\tilde{C}_N^{xy}}\sum_{z,w\in S}\frac{N^\alpha}{Z_N}\frac{1}{a(\zeta)}r(z,w)\left(F(\zeta+\mathfrak{d}_w)-F(\zeta+\mathfrak{d}_z)\right)\hat{R}(\zeta,w,z).$$

Then $\Omega_1=\Omega_{11}+\Omega_{12}$.

Consider Ω_{11}, which is

$$\Omega_{11} = \frac{1}{2I_\alpha Z_N(1-6\epsilon)} \sum_{\zeta \in \tilde{C}_N^{xy}} \frac{1}{a(\hat{\zeta})} \sum_{z,w\in S} r(z,w)(F(\zeta+\mathfrak{d}_w) - F(\zeta+\mathfrak{d}_z))(J_w - J_z),$$

where $\hat{\zeta}$ is the restriction of ζ to sites $z \neq a, b$.

Fix ζ. Then,

$\frac{1}{2} \sum\limits_{z,w\in S} r(z,w)\,(F(\zeta+\mathfrak{d}_w) - F(\zeta+\mathfrak{d}_z))(J_w - J_z)$

$= \frac{1}{2} \sum\limits_{z,w\in S} r(z,w)\,(F(\zeta+\mathfrak{d}_w) - F(\zeta+\mathfrak{d}_z))(J_w - J_z)\frac{1}{L}L$

Recall that $\bar{L}$ is the infinitesimal generator of the underlying random walk and $\bar{L}f(z) = \sum_{w\in S} r(z,w)\,(f(w) - f(z))$ for the function f on S.

Then the previous expression is

$-\sum_{z\in S}\sum_{w\in S} r(z,w)\,F(\zeta+\mathfrak{d}_z))(J_w - J_z)\frac{1}{L}L$ since the underlying random walk is reversible with the uniform measure

$= -\sum_{z\in S} F(\zeta+\mathfrak{d}_z)\bar{L}J(z)\frac{1}{L}L$

$= -\bar{L}J(x)F(\zeta+\mathfrak{d}_x) - \bar{L}J(y)F(\zeta+\mathfrak{d}_y)$ by the definition of J.

$= L\mathrm{cap}_{\bar{L}}(x,y)\,(F(\zeta+\mathfrak{d}_x) - F(\zeta+\mathfrak{d}_y))$.

Write $\eta = (\hat{\eta};\eta_x,\eta_y)$ where $\hat{\eta}$ is the restriction of η to sites without x, y. We have

(5.11)

$$\begin{aligned}
\Omega_{11} &= \frac{L\mathrm{cap}_{\bar{L}}(x,y)}{I_\alpha Z_N(1-6\epsilon)} \sum_{\zeta\in\tilde{C}_N^{xy}} \frac{1}{a(\hat{\zeta})}\,(F(\zeta+\mathfrak{d}_x) - F(\zeta+\mathfrak{d}_y)) \\
&= \frac{L\mathrm{cap}_{\bar{L}}(x,y)}{I_\alpha Z_N(1-6\epsilon)} \sum_{k=0}^{\tilde{\ell}} \sum_{\hat{\zeta}\in E_{k,S\setminus\{x,y\}}} \frac{1}{a(\hat{\zeta})} \sum_{\substack{\lfloor 4\epsilon N\rfloor \le \zeta_x \le N - \lfloor 4\epsilon N\rfloor \\ \zeta_y = N-k-\zeta_x}} (F(\zeta+\mathfrak{d}_x) - F(\zeta+\mathfrak{d}_y)) \\
&= \frac{L\mathrm{cap}_{\bar{L}}(x,y)}{I_\alpha Z_N(1-6\epsilon)} \sum_{k=0}^{\tilde{\ell}} \sum_{\hat{\zeta}\in E_{k,S\setminus\{x,y\}}} \frac{1}{a(\hat{\zeta})} \\
&\quad \times \Big(F(\zeta; N-\hat{\ell}+1, \hat{\ell}-k-1) - F(\hat{\zeta}; \hat{\ell}, N-k-\hat{\ell})\Big),
\end{aligned}$$

where $\hat{\ell} = \lfloor 4\epsilon N\rfloor$.

Denote by $\eta^{center,\mathcal{E}^z}$ for $z \in S$ the configuration where every particles are on the site z. Then $\eta^{center,\mathcal{E}^x} = (0; N, 0)$. Let $\eta^{1,x} = \Big(0; N-\tilde{\ell}+1, \tilde{\ell}-1\Big)$, $\eta^{2,x} = \Big(0; N-\hat{\ell}+1, \hat{\ell}-1\Big)$ and $\eta^{3,x} = \Big(\hat{\zeta}; N-\hat{\ell}+1, \hat{\ell}-k-1\Big)$.

From now, C is a constant which can vary line by line.

As in the proof of Proposition 4.1 (3), we can see for the configuration $\eta^1, \eta^2 \in E_{N,S}$,

$$\begin{aligned}
\big|F(\eta^1) - F(\eta^2)\big|^2 \le C\,&(\text{Number of jumps to go from } \eta^1 \text{ to } \eta^2) \\
&\times \Big(\max_{\zeta \text{ in the path from } \eta^1 \text{ to } \eta^2} a(\zeta)\Big)\, N^{-1-2\alpha}
\end{aligned}$$

where $\zeta = \eta - \mathfrak{d}_z$ when we move a particle at z to w in the configuration of η.

Consider a path from $\eta^{center,\mathcal{E}^x} = (0; N, 0)$ to $\eta^{1,x} = \left(0; N-\tilde{\ell}+1, \tilde{\ell}-1\right)$. We move a particle at x to y one by one. We can make $\hat{\zeta} = 0$ in this path. Number of length of the path is $O(\tilde{\ell})$ and $a(\zeta) = a(\hat{\zeta})a(\zeta_x)a(\zeta_y) \le N^\alpha \tilde{\ell}^\alpha$.

So $\left|F(\eta^{center,\mathcal{E}^x}) - F(\eta^{1,x})\right| \le C\sqrt{\frac{\tilde{\ell}^{1+\alpha}}{N^{1+\alpha}}}$.

Consider a path from $\eta^{1,x} = \left(0; N-\tilde{\ell}+1, \tilde{\ell}-1\right)$ to $\eta^{2,x} = \left(0; N-\hat{\ell}+1, \hat{\ell}-1\right)$. We move a particle at x to y one by one. We can make $\hat{\zeta} = 0$ in this path. Number of length of the path is $O(\hat{\ell})$ and $a(\zeta) = a(\hat{\zeta})a(\zeta_x)a(\zeta_y) \le N^\alpha \hat{\ell}^\alpha$. So $\left|F(\eta^{1,x}) - F(\eta^{2,x})\right| \le C\sqrt{\frac{\hat{\ell}^{1+\alpha}}{N^{1+\alpha}}}$.

Also consider a path from $\eta^{2,x} = \left(0; N-\hat{\ell}+1, \hat{\ell}-1\right)$ to $\eta^{3,x} = \left(\hat{\zeta}; N-\hat{\ell}+1, \hat{\ell}-k-1\right)$. Move a particle at y to a site in $S\setminus\{x,y\}$ one by one. Number of length of the path is $O(\tilde{\ell})$ and $a(\zeta) = a(\hat{\zeta})a(\zeta_x)a(\zeta_y) \le C\tilde{\ell}^{(L-2)\alpha}N^\alpha\hat{\ell}^\alpha$. So $\left|F(\eta^{2,x}) - F(\eta^{3,x})\right| \le C\sqrt{\frac{\tilde{\ell}^{1+(L-2)\alpha}\hat{\ell}^\alpha}{N^{1+\alpha}}}$.

Then

$$\begin{aligned}
\left|F(\eta^{center,\mathcal{E}^x}) - F(\hat{\zeta}; \hat{\ell}, N-k-\hat{\ell})\right| &= \left|F(\eta^{center,\mathcal{E}^x}) - F(\eta^{3,x})\right| \\
&\le \left|F(\eta^{center,\mathcal{E}^x}) - F(\eta^{1,x})\right| \\
&\quad + \left|F(\eta^{1,x}) - F(\eta^{2,x})\right| + \left|F(\eta^{2,x}) - F(\eta^{3,x})\right| \\
&\le C\sqrt{\frac{\tilde{\ell}^{1+\alpha}}{N^{1+\alpha}}} + C\sqrt{\frac{\hat{\ell}^{1+\alpha}}{N^{1+\alpha}}} + C\sqrt{\frac{\tilde{\ell}^{1+(L-2)\alpha}\hat{\ell}^\alpha}{N^{1+\alpha}}}.
\end{aligned}$$

Similarly consider a path from $\eta^{center,\mathcal{E}^y} = (0; 0, N)$ to $\eta^{1,y} = \left(0; \tilde{\ell}, N-\tilde{\ell}\right)$. We have $\left|F(\eta^{center,\mathcal{E}^y}) - F(\eta^{1,y})\right| \le C\sqrt{\frac{\tilde{\ell}^{1+\alpha}}{N^{1+\alpha}}}$. By considering a path from $\eta^{1,y} = \left(0; \tilde{\ell}, N-\tilde{\ell}\right)$ to $\eta^{2,y} = \left(0; \hat{\ell}, N-\hat{\ell}\right)$, we get $\left|F(\eta^{1,y}) - F(\eta^{2,y})\right| \le C\sqrt{\frac{\hat{\ell}^{1+\alpha}}{N^{1+\alpha}}}$. By considering a path from $\eta^{2,y} = \left(0; \hat{\ell}, N-\hat{\ell}\right)$ to $\eta^{3,y} = \left(\hat{\zeta}; \hat{\ell}, N-\hat{\ell}-k\right)$, we have $\left|F(\eta^{2,y}) - F(\eta^{3,y})\right| \le C\sqrt{\frac{\tilde{\ell}^{1+(L-2)\alpha}\hat{\ell}^\alpha}{N^{1+\alpha}}}$.

So

$$\left|F(\eta^{center,\mathcal{E}^y}) - F\left(\hat{\zeta}; \hat{\ell}, N-\hat{\ell}-k\right)\right| \le C\sqrt{\frac{\tilde{\ell}^{1+\alpha}}{N^{1+\alpha}}} + C\sqrt{\frac{\hat{\ell}^{1+\alpha}}{N^{1+\alpha}}} + C\sqrt{\frac{\tilde{\ell}^{1+(L-2)\alpha}\hat{\ell}^\alpha}{N^{1+\alpha}}}.$$

Thus

$$\begin{aligned}
\Omega_{11} = \frac{L\mathrm{cap}_{\bar{L}}(x,y)}{I_\alpha Z_N(1-6\epsilon)} \sum_{k=0}^{\tilde{\ell}} \sum_{\hat{\zeta}\in E_{k,S\setminus\{x,y\}}} \frac{1}{a(\hat{\zeta})} &\Big(F(\eta^{center,\mathcal{E}^x}) - F(\eta^{center,\mathcal{E}^y}) \,. \\
&+O(\sqrt{\frac{\tilde{\ell}^{1+\alpha}}{N^{1+\alpha}}}) + O(\sqrt{\frac{\hat{\ell}^{1+\alpha}}{N^{1+\alpha}}}) + O(\sqrt{\frac{\tilde{\ell}^{1+(L-2)\alpha}\hat{\ell}^\alpha}{N^{1+\alpha}}})\Big)
\end{aligned}$$

Since $\lim_{N\to\infty}\sum_{k=0}^{\tilde{\ell}}\sum_{\hat{\zeta}\in E_{k,S\setminus\{x,y\}}}\frac{1}{a(\hat{\zeta})} = \Gamma(\alpha)^{L-2}$, $\lim_{N\to\infty} Z_N = Z_S$,

$$\liminf_{N\to\infty}\Omega_{11} = \frac{L\mathrm{cap}_{\bar{L}}(x,y)}{I_\alpha Z(1-6\epsilon)}\Gamma(\alpha)^{L-2} \times \liminf_{N\to\infty}\Big(F_N(\eta^{center,\mathcal{E}_N^x}) - F_N(\eta^{center,\mathcal{E}_N^y})\Big) + O(\epsilon^{\frac{\alpha+1}{2}}),$$

$$\limsup_{N\to\infty}\Omega_{11} = \frac{L\mathrm{cap}_{\bar{L}}(x,y)}{I_\alpha Z(1-6\epsilon)}\Gamma(\alpha)^{L-2} \times \limsup_{N\to\infty}\Big(F_N(\eta^{center,\mathcal{E}_N^x}) - F_N(\eta^{center,\mathcal{E}_N^y})\Big) + O(\epsilon^{\frac{\alpha+1}{2}}).$$

Define $g_N(x) = \int_{\mathcal{E}_N^x} F^N(\eta)\, d\mu_N$ for $s \in S$.
By the Proposition 4.1 (3),

$$\lim_{\epsilon\to 0}\liminf_{N\to\infty}\Omega_{11} = \frac{L\mathrm{cap}_{\bar{L}}(x,y)}{I_\alpha Z_S}\Gamma(\alpha)^{L-2}\liminf_{N\to\infty}\left(g_N(x) - g_N(y)\right), \tag{5.12}$$

$$\lim_{\epsilon\to 0}\limsup_{N\to\infty}\Omega_{11} = \frac{L\mathrm{cap}_{\bar{L}}(x,y)}{I_\alpha Z_S}\Gamma(\alpha)^{L-2}\limsup_{N\to\infty}\left(g_N(x) - g_N(y)\right). \tag{5.13}$$

Consider Ω_{12}, which is

$$\Omega_{12} = \frac{N^{1+\alpha}}{2}\sum_{\zeta\in\tilde{C}_N^{xy}}\sum_{z,w\in S}\frac{N^\alpha}{Z_N}\frac{1}{a(\zeta)}\, r(z,w)\left(F(\zeta+\mathfrak{d}_w) - F(\zeta+\mathfrak{d}_z)\right)\hat{R}(\zeta,w,z).$$

Because of (5.10),

$$|\Omega_{12}| \le \frac{1}{2I_\alpha Z_N(1-6\epsilon)}\left(C\frac{\tilde{\ell}}{\epsilon N}\right)\sum_{\zeta\in\tilde{C}_N^{xy}}\sum_{z,w\in S} r(z,w)\,\frac{1}{a(\hat{\zeta})}\left|F(\zeta+\mathfrak{d}_w) - F(\zeta+\mathfrak{d}_z)\right|$$

By Cauchy-Schwarz inequality,

$$\sum_{\zeta\in\tilde{C}_N^{xy}}\sum_{z,w\in S} r(z,w)\,\frac{1}{a(\hat{\zeta})}\left|F(\zeta+\mathfrak{d}_w) - F(\zeta+\mathfrak{d}_z)\right|$$

$$\le \left(\sum_{\zeta\in\tilde{C}_N^{xy}}\sum_{z,w\in S} r(z,w)\,\frac{a(\zeta_x)a(\zeta_y)}{a(\hat{\zeta})}\right)^{1/2} \times \left(\sum_{\zeta\in\tilde{C}_N^{xy}}\sum_{z,w\in S} r(z,w)\,\frac{1}{a(\zeta)}\left(F(\zeta+\mathfrak{d}_w) - F(\zeta+\mathfrak{d}_z)\right)^2\right)^{1/2}.$$

By the Proposition 4.1 (2),

$$\left(\sum_{\zeta\in\tilde{C}_N^{xy}}\sum_{z,w\in S} r(z,w)\,\frac{1}{a(\zeta)}\left(F(\zeta+\mathfrak{d}_w) - F(\zeta+\mathfrak{d}_z)\right)^2\right)^{1/2} = O(N^{-\frac{1+2\alpha}{2}}).$$

Also

$$\sum_{\zeta\in\tilde{C}_N^{xy}}\sum_{z,w\in S} r(z,w)\,\frac{a(\zeta_x)a(\zeta_y)}{a(\hat{\zeta})} \le \sum_{z,w\in S} r(z,w) \sum_{\zeta\in\tilde{C}_N^{xy}} \frac{a(\zeta_x)a(\zeta_y)}{a(\hat{\zeta})}$$
$$\le L^2 \sum_{\zeta\in\tilde{C}_N^{xy}} \frac{a(\zeta_x)a(\zeta_y)}{a(\hat{\zeta})}$$
$$\le L^2 \sum_{k=0}^{\tilde{\ell}} \sum_{\hat{\zeta}\in E_{k,S\setminus\{a,b\}}} \frac{1}{a(\hat{\zeta})} \sum_{\substack{\lfloor 4\epsilon N\rfloor \le \zeta_x \le N-\lfloor 4\epsilon N\rfloor \\ \zeta_y = N-k-\zeta_x}} a(\zeta_x)a(\zeta_y)$$

The last summation in the last line of the previous equation equals

$$N^{1+2\alpha} \sum_{\lfloor 4\epsilon N\rfloor \le \zeta_x \le N-\lfloor 4\epsilon N\rfloor} \left(\frac{\zeta_x}{N}\right)^{\alpha} \left(\frac{N-k-\zeta_x}{N}\right)^{\alpha} \frac{1}{N}.$$

By sending N to the infinity,

$$\lim_{N\to\infty} \sum_{\lfloor 4\epsilon N\rfloor \le \zeta_x \le N-\lfloor 4\epsilon N\rfloor} \left(\frac{\zeta_x}{N}\right)^{\alpha} \left(\frac{N-k-\zeta_x}{N}\right)^{\alpha} \frac{1}{N} = I_\alpha.$$

So $\displaystyle\sum_{\substack{\lfloor 4\epsilon N\rfloor \le \zeta_x \le N-\lfloor 4\epsilon N\rfloor \\ \zeta_y = N-k-\zeta_x}} a(\zeta_x)a(\zeta_y) = O(N^{-\frac{1+2\alpha}{2}})$.

And $\sum_{k=0}^{\tilde{\ell}} \sum_{\hat{\zeta}\in E_{k,S\setminus\{a,b\}}} \frac{1}{a(\hat{\zeta})} \le \Gamma(\alpha)^{L-2}$.

So

$$\left(\sum_{\zeta\in\tilde{C}_N^{xy}}\sum_{z,w\in S} r(z,w)\,\frac{a(\zeta_x)a(\zeta_y)}{a(\hat{\zeta})}\right)^{1/2} = O(N^{-\frac{1+2\alpha}{2}}). \tag{5.14}$$

Thus $|\Omega_{12}| = O(\frac{\tilde{\ell}}{\epsilon N})$ and $\lim_{N\to\infty}\Omega_{12} = 0$.

putting together estimates for $\Omega_{11}, \Omega_{12}, \Omega_{21}, \Omega_{22}$, we have

$$\lim_{\epsilon\to 0}\liminf_{N\to\infty}(LHS \text{ of } 5.6) = \frac{L\mathrm{cap}_{\bar{L}}(x,y)}{I_\alpha Z_S}\Gamma(\alpha)^{L-2}\liminf_{N\to\infty}\sum_{y\in S}(g_N(x)-g_N(y))), \tag{5.15}$$

$$\lim_{\epsilon\to 0}\limsup_{N\to\infty}(LHS \text{ of } 5.6) = \frac{L\mathrm{cap}_{\bar{L}}(x,y)}{I_\alpha Z_S}\Gamma(\alpha)^{L-2}\limsup_{N\to\infty}\sum_{y\in S}(g_N(x)-g_N(y))). \tag{5.16}$$

Consider (RHS) of (5.6), which is

$$(RHS) = \int_{E_N} \left(\mathbf{1}\{\eta\in\mathcal{E}^a\} - \mathbf{1}\{\eta\in\mathcal{E}^b\}\right) H(\eta)\, d\mu_N(\eta)$$
$$= \mu_N(\mathcal{E}^a)(\mathbf{1}\{x=a\} - \mathbf{1}\{x=b\}), \text{ since } \mu_N(\mathcal{E}^a) = \mu_N(\mathcal{E}^b)$$

By sending N to infinity,

$$\lim_{N\to\infty}(RHS) = \frac{\mathbf{1}\{x=a\} - \mathbf{1}\{x=b\}}{L}. \tag{5.17}$$

By (5.15), (5.16), (5.17) we have

$$\frac{L\mathrm{cap}_{\bar{L}}(x,y)}{I_\alpha Z_S}\Gamma(\alpha)^{L-2}\lim_{N\to\infty}\sum_{y\in S}(g_N(x)-g_N(y)))=\frac{\mathbf{1}\{x=a\}-\mathbf{1}\{x=b\}}{L}.$$

Substituting $L\Gamma(\alpha)^{L-1}$ for Z_S,

$$\lim_{N\to\infty}\frac{L\mathrm{cap}_{\bar{L}}(x,y)}{I_\alpha\Gamma(\alpha)}\sum_{y\in S}(g_N(x)-g_N(y))=\mathbf{1}\{x=a\}-\mathbf{1}\{x=b\}.$$

That is

$$\lim_{N\to\infty}-\mathfrak{L}g_N(x)=\mathbf{1}\{x=a\}-\mathbf{1}\{x=b\}.$$

Also g_N satisfies $\lim_{N\to\infty}\sum_{x\in S}g_N(x)=0$ by Proposition 2.3 and 2.2.

Since S is a finite set, we can think $\mathfrak{L}$ as a matrix and g_N, $f_{a,b}$ are vectors. The function $f_{a,b}$ is defined by (5.1), (5.2). As a matrix, $\mathfrak{L}$ has a rank $L-1$. Also we know that $\sum_{x\in S}f_{a,b}(x)=0$ and $\lim_{N\to\infty}\sum_{x\in S}g_N(x)=0$. So we can think $f_{a,b}$ as a solution for a system of linear equations and g_N as an approximate solution, where the matrix for the system has full rank. This implies that $\lim_{N\to\infty}g_N(x)=f_{a,b}(x)$ for all $x\in S$.

By the Proposition 4.1 (3),

$$\lim_{N\to\infty}F_N^{a,b}(\eta^N)=\lim_{N\to\infty}g_N(x)=f_{a,b}(x).$$

This proves the proposition.

5.2. **Proof of Proposition 5.1 for The Non-reversible Case.** Definition of H is same to the reversible case except the definition of J.

Let $\bar{L}^*$ be the adjoint of the infinitesimal generator of the underlying random walk.

In the definition of H, $J_{xy}:S\to[0,1]$ solves $\begin{cases}\bar{L}^*J_{xy}(z) &=0, \quad z\neq x,y\\ J_{xy}(x) &=1\\ J_{xy}(y) &=0\end{cases}$ and $J\cdot\eta=\sum_z J_z\eta_z$, the dot product where $J_z=J(z)$ for $z\in S$.

As in the reversible case, multiply H to the equation (4.1). We get

$$\int_{E_{N,S}}-\theta_N L_N F_N\, H\, d\mu=\int_{E_{N,S}}hH\, d\mu. \tag{5.18}$$

Consider the left hand side of the previous equation. Denote by L_N^* the adjoint operator of L_N and by r^* the jump rate for the adjoint underlying random walk. Since the uniform measure is invarint measure of underlying random walk, $r^*(x,y)=r(y,x)$.

$$
\begin{aligned}
(LHS) &= -\theta_N \int_{E_N} L_N F_N^{a,b}(\eta) H \, d\mu \\
&= -\theta_N \int_{E_N} F_N^{a,b}(\eta) \, L_N^* H \, d\mu \\
&= -N^{1+\alpha} \sum_{\eta \in E_N} \sum_{z,w \in S} \mu(\eta) g(\eta_z) r^*(z,w) \, F(\eta) \, \left(H(\sigma^{zw}\eta) - H(\eta) \right) \qquad (5.19) \\
&= -N^{1+\alpha} \sum_{\zeta \in E_{N-1}} \frac{N^\alpha}{Z_N} \frac{1}{a(\zeta)} \sum_{z,w \in S} r^*(z,w) \, F(\zeta + \mathfrak{d}_z) \\
&\qquad \times \left(H(\zeta + \mathfrak{d}_w) - H(\zeta + \mathfrak{d}_z) \right)
\end{aligned}
$$

Define $\bar{F}(\zeta) = \frac{1}{L} \sum_{u \in S} F(\zeta + \mathfrak{d}_u)$.

Since the uniform measure is an invariant measure for the underlying random walk,

$$
\sum_{z,w \in S} r^*(z,w) \, \left(H(\sigma^{zw}\eta) - H(\eta) \right) = 0.
$$

So the expression of the equation (5.19) equals

$$
-N^{1+\alpha} \sum_{\zeta \in E_{N-1}} \sum_{z,w \in S} \frac{N^\alpha}{Z_N} \frac{r^*(z,w)}{a(\zeta)} \left(F(\zeta + \mathfrak{d}_z) - \bar{F}(\zeta) \right) \left(H(\zeta + \mathfrak{d}_w) - H(\zeta + \mathfrak{d}_z) \right)
$$

Define $\hat{F}_z(\eta) = \begin{cases} \bar{F}(\eta - \mathfrak{d}_z) & \text{if } \eta_z > 0 \\ 0 & \text{if } \eta_z = 0 \end{cases}$.

Then the previous expression is

$$
-N^{1+\alpha} \sum_{\eta \in E_N} \sum_{z,w \in S} \mu(\eta) g(\eta_z) r^*(z,w) \, \left(F(\eta) - \hat{F}_z(\eta) \right) \, \left(H(\sigma^{zw}\eta) - H(\eta) \right).
$$

For functions F, G on $E_{N,S}$, and a set $\mathcal{B} \subset E_{N,S}$, define

$$
A_N(F,G;\mathcal{B}) = -\sum_{\eta \in B} \sum_{z,w \in S} \mu(\eta) g(\eta_z) r^*(z,w) \, \left(F(\eta) - \hat{F}_z(\eta) \right) \, \left(H(\sigma^{zw}\eta) - H(\eta) \right)
$$

Then the equation (5.19) is

$$
\begin{aligned}
\theta_N A_N(F,H;E_{N,S}) &= \theta_N A_N(F,H;(\tilde{T}_N^x)^{\complement}) + \theta_N A_N(F,H;\tilde{T}_N^x) \\
&= \theta_N A_N(F,H;(\tilde{T}_N^x)^{\complement}) + \sum_{y \in S,\, y \neq x} \theta_N A_N(F,H;\tilde{T}^{xy}), \qquad (5.20)
\end{aligned}
$$

for sufficiently large N because of (5.3), (5.4).

We will use the following lemma.

Lemma 5.5. *For any function F on $E_{N,S}$, there is a constant C which doesn't depend on N such that*

$$
\sum_{\eta \in E_{N,S}} \sum_{z,w \in S} \mu(\eta) g(\eta_z) r^*(z,w) \, \left(F(\eta) - \hat{F}_z(\eta) \right)^2 \le C \, D_N(F).
$$

Proof. The idea of this proof is in the proof of Lemma 4.2 in [18].

$$
\begin{aligned}
&\sum_{\eta\in E_{N,S}}\sum_{z,w\in S}\mu(\eta)g(\eta_z)r^*(z,w)\,\left(F(\eta)-\hat{F}_z(\eta)\right)^2\\
&=\sum_{\zeta\in E_{N-1}}\frac{N^\alpha}{Z_N}\frac{1}{a(\zeta)}\sum_{z,w\in S}r^*(z,w)\,\left(\bar{F}(\zeta)-F(\zeta+\mathfrak{d}_z)\right)^2 \qquad (5.21)
\end{aligned}
$$

The last summation in z,w in the previous expression is

$$
\begin{aligned}
&\sum_{z,w\in S}r^*(z,w)\,\left(\sum_{u\in S}\frac{1}{L}F(\zeta+\mathfrak{d}_u)-F(\zeta+\mathfrak{d}_z)\right)^2 \qquad (5.22)\\
&=\sum_{z,w\in S}r^*(z,w)\,\left(\sum_{u\in S}\frac{F(\zeta+\mathfrak{d}_u)-F(\zeta+\mathfrak{d}_z)}{L}\right)^2
\end{aligned}
$$

Define $P=\{(z,w)\in S\times S\,:\,r^*(z,w)>0\}$. Let

$$
C_1=\min_{(z,w)\in P}r^*(z,w)\quad\text{and}\quad C_2=\max_{(z,w)\in P}r^*(z,w).
$$

For $u,v\in S$,consider a *canonical path*

$$
u=z_1(u,v),\,z_2(u,v),\,\cdots,\,z_{k(u,v)}=v,
$$

where $(z_i(u,v),\,z_{i+1}(u,v))\in P$ for $1\le i\le k(u,v)-1$ and $z_i(u,v)$'s are different. There exists a canonical path since the underlying random walk is irreducible. We can see $k(u,v)\le L$.

The equation (5.22) is bounded above by

$$
\begin{aligned}
&\sum_{z\in S}\frac{C_2(L-1)}{L^2}\,L\sum_{u\in S}\left(F(\zeta+\mathfrak{d}_u)-F(\zeta+\mathfrak{d}_z)\right)^2\ \text{ by Cauchy-Schwarz inequality}\\
&\le\frac{C_2(L-1)}{L}\sum_{u,z\in S}L\sum_{i=1}^{k(u,z)-1}\left(F(\zeta+\mathfrak{d}_{z_i})-F(\zeta+\mathfrak{d}_{z_{i+1}})\right)^2\\
&\le C_2(L-1)L^2\sum_{(z,w)\in P}\left(F(\zeta+\mathfrak{d}_w)-F(\zeta+\mathfrak{d}_z)\right)^2\\
&\le\frac{C_2(L-1)L^2}{C_1}\sum_{(z,w)\in S}r^*(z,w)\left(F(\zeta+\mathfrak{d}_w)-F(\zeta+\mathfrak{d}_z)\right)^2\\
&=C\sum_{(z,w)\in S}r^*(z,w)\left(F(\zeta+\mathfrak{d}_w)-F(\zeta+\mathfrak{d}_z)\right)^2.
\end{aligned}
$$

So the equation (5.21) is bounded above by

$$
\sum_{\zeta\in E_{N-1}}\frac{N^\alpha}{Z_N}\frac{1}{a(\zeta)}C\sum_{z,w\in S}r^*(z,w)\,\left(F(\zeta+\mathfrak{d}_w)-F(\zeta+\mathfrak{d}_z)\right)^2=C\,D_N(F).
$$

□

Consider the first term $\theta_N A_N(F,H;(\tilde{T}_N^x)^\complement)$ in the equation (5.20).

As in the reversible case,

$\theta_N A_N(F,H;(\tilde{T}_N^x)^\complement)$

$=-\theta_N\sum_{\eta\in(\tilde{T}_N^x)^\complement}\sum_{z,w\in S}\mu(\eta)g(\eta_z)r^*(z,w)\,\left(F(\eta)-\hat{F}_z(\eta)\right)\,(H(\sigma^{zw}\eta)-H(\eta))$

$$\leq \left(N^{1+\alpha} \sum_{\eta\in(\tilde{T}_N^x)^{\complement}} \sum_{z,w\in S} \mu(\eta) g(\eta_z) r(z,w) \left(F(\eta) - \hat{F}_z(\eta)\right)^2\right)^{1/2} \times$$
$$\left(N^{1+\alpha} \sum_{\eta\in(\tilde{T}_N^x)^{\complement}} \sum_{z,w\in S} \mu(\eta) g(\eta_z) r(z,w) \left(H(\sigma^{zw}\eta) - H(\eta)\right)^2\right)^{1/2}$$

The first summation is bounded above by a constant because of the Proposition 4.1 (2) and the Lemma 5.5. The second summation is bounded above by $\frac{C_\epsilon}{(\epsilon\tilde{\ell}_N)^{\frac{\alpha-1}{2}}}$ as in the reversible case.

So $\lim_{N\to\infty} \theta_N A_N(F_N, H_N; (\tilde{T}_N^x)^{\complement}) = 0$.

Consider the second term $\sum_{y\in S,\, y\neq x} \theta_N A_N(F, H; \tilde{T}^{xy})$ in the equation (5.20).

$$\begin{aligned}\theta_N A_N(F,H;\tilde{T}^{xy}) &= -N^{1+\alpha} \sum_{z,w\in S} \sum_{\zeta+\mathfrak{d}_z\in\tilde{T}_N^{xy}} \frac{N^\alpha}{Z_N}\frac{1}{a(\zeta)} r^*(z,w)\left(F(\zeta+\mathfrak{d}_z) - \bar{F}(\zeta)\right) \\ &\quad\times\ \left(H(\zeta+\mathfrak{d}_w) - H(\zeta+\mathfrak{d}_z)\right) \\ &= -N^{1+\alpha} \sum_{\substack{\zeta\in E_{N-1}\\ \zeta_x+\zeta_y\geq N-\tilde{\ell}}} \sum_{z,w\in S} \frac{N^\alpha}{Z_N}\frac{1}{a(\zeta)} r^*(z,w)\left(F(\zeta+\mathfrak{d}_z) - \bar{F}(\zeta)\right) \\ &\quad\times\ \left(H(\zeta+\mathfrak{d}_w) - H(\zeta+\mathfrak{d}_z)\right) + \Lambda_N^{xy} \end{aligned} \tag{5.23}$$

The proof for the Lemma 5.3 for the non-reversible case, which states that $|\Lambda_N^{xy}| \leq \frac{C_\epsilon}{\tilde{\ell}^{\alpha/2}}$, is similar. The proof is the following.

As in the proof for the reversible case,

$$|\Lambda_N^{xy}| \leq N^{1+\alpha} \sum_{\substack{\eta\in E_N\\ \eta_x+\eta_y=N-\tilde{\ell}}} \sum_{z,w\in S} \frac{N^\alpha}{Z_N}\frac{g(\eta_z)}{a(\eta)} r^*(z,w) \left|F(\eta) - \hat{F}_z(\eta)\right| \left|H(\sigma^{zw}\eta) - H(\eta)\right|$$
$$\leq \left(N^{1+\alpha} \sum_{\substack{\eta\in E_N\\ \eta_x+\eta_y=N-\tilde{\ell}}} \sum_{z,w\in S} \frac{N^\alpha}{Z_N}\frac{g(\eta_z)}{a(\eta)} r(z,w) \left|F(\eta) - \hat{F}_z(\eta)\right|^2\right)^{1/2}$$
$$\times \left(N^{1+\alpha} \sum_{\substack{\eta\in E_N\\ \eta_x+\eta_y=N-\tilde{\ell}}} \sum_{z,w\in S} \frac{N^\alpha}{Z_N}\frac{g(\eta_z)}{a(\eta)} r(z,w) \left|H(\sigma^{zw}\eta) - H(\eta)\right|^2\right)^{1/2}$$

The first term is bounded above by a constant by the Proposition 4.1 (2) and the Lemma 5.5. The second term is bounded above by $C_\epsilon O(\tilde{\ell}^{-\alpha/2})$ as in the reversible case. This proves the lemma 5.3 for the non-reversible case.

Consider the first term of the equation (5.23).

As the reversible case, define

$$\tilde{S}_N^{xy} = \left\{\zeta \in E_{N-1} : \zeta_z + \zeta_y \geq N - \tilde{\ell}\right\}.$$

Also define

$$\tilde{S}_N^{xy}(a,b) = \left\{\zeta \in E_{N-1} : \zeta_z + \zeta_y \geq N - \tilde{\ell}, a \leq \zeta_x \leq b\right\}.$$

Then

$$\begin{aligned}
&-N^{1+\alpha}\sum_{\zeta\in\tilde{S}_N^{xy}}\sum_{z,w\in S}\frac{N^\alpha}{Z_N}\frac{r^*(z,w)}{a(\zeta)}\left(F(\zeta+\mathfrak{d}_z)-\bar{F}(\zeta)\right)\\
&\quad\times\quad (H(\zeta+\mathfrak{d}_w)-H(\zeta+\mathfrak{d}_z))\\
&=-N^{1+\alpha}\sum_{\zeta\in\tilde{S}_N^{xy}(\lfloor 4\epsilon N\rfloor,\,N-\lfloor 4\epsilon N\rfloor)}\sum_{z,w\in S}\frac{N^\alpha}{Z_N}\frac{1}{a(\zeta)}r^*(z,w)\left(F(\zeta+\mathfrak{d}_z)-\bar{F}(\zeta)\right)\\
&\quad\times\quad (H(\zeta+\mathfrak{d}_w)-H(\zeta+\mathfrak{d}_z))\\
&=-N^{1+\alpha}\sum_{\zeta\in\tilde{S}_N^{xy}(1,\,\lfloor 4\epsilon N\rfloor-1)}\sum_{z,w\in S}\frac{N^\alpha}{Z_N}\frac{1}{a(\zeta)}r^*(z,w)\left(F(\zeta+\mathfrak{d}_z)-\bar{F}(\zeta)\right)\\
&\quad\times\quad (H(\zeta+\mathfrak{d}_w)-H(\zeta+\mathfrak{d}_z))\\
&=-N^{1+\alpha}\sum_{\zeta\in\tilde{S}_N^{xy}(N-\lfloor 4\epsilon N\rfloor+1,\,N)}\sum_{z,w\in S}\frac{N^\alpha}{Z_N}\frac{1}{a(\zeta)}r^*(z,w)\left(F(\zeta+\mathfrak{d}_z)-\bar{F}(\zeta)\right)\\
&\quad\times\quad (H(\zeta+\mathfrak{d}_w)-H(\zeta+\mathfrak{d}_z))
\end{aligned}$$

Let the first term, second term, and last term be $\Omega_1, \Omega_{21}, \Omega_{22}$ as the reversible case.

The Lemma 5.4 holds for the non-reversible case, which states that $|\Omega_{21}| \leq C\epsilon^{\frac{\alpha+1}{2}}$, $|\Omega_{22}| \leq C\epsilon^{\frac{\alpha+1}{2}}$. The proof is the following.

$$\Omega_{21}=-\frac{N^{1+2\alpha}}{Z_N}\sum_{\zeta\in\tilde{S}_N^{xy}(1,\,\lfloor 4\epsilon N\rfloor-1)}\sum_{z,w\in S}\frac{1}{a(\zeta)}\,r^*(z,w)\,\left(F(\zeta+\mathfrak{d}_z)-\bar{F}(\zeta)\right)(H(\zeta+\mathfrak{d}_w)-H(\zeta+\mathfrak{d}_z))\,.$$

By the Cauchy-Schwartz inequity,

$$\begin{aligned}
\Omega_{21}^2\leq\left(\frac{N^{1+2\alpha}}{Z_N}\right)^2&\left(\sum_{\zeta\in\tilde{S}_N^{xy}(1,\,\lfloor 4\epsilon N\rfloor-1)}\sum_{z,w\in S}\frac{1}{a(\zeta)}\,r^*(z,w)\,\left(F(\zeta+\mathfrak{d}_z)-\bar{F}(\zeta)\right)^2\right)\\
&\times\left(\sum_{\zeta\in\tilde{S}_N^{xy}(1,\,\lfloor 4\epsilon N\rfloor-1)}\sum_{z,w\in S}\frac{1}{a(\zeta)}\,r^*(z,w)\,(H(\zeta+\mathfrak{d}_w)-H(\zeta+\mathfrak{d}_z))^2\right).
\end{aligned}$$

The first summation in the previous expression is

$$\begin{aligned}
&\sum_{\zeta\in\tilde{S}_N^{xy}(1,\,\lfloor 4\epsilon N\rfloor-1)}\sum_{z,w\in S}\frac{1}{a(\zeta)}\,r^*(z,w)\,\left(F(\zeta+\mathfrak{d}_z)-\bar{F}(\zeta)\right)^2\\
&\leq\sum_{\zeta\in E_{N-1}}\sum_{z,w\in S}\frac{1}{a(\zeta)}\,r^*(z,w)\,\left(F(\zeta+\mathfrak{d}_z)-\bar{F}(\zeta)\right)^2\\
&\leq C\left(\sum_{\zeta\in E_{N-1}}\sum_{z,w\in S}\frac{1}{a(\zeta)}\,r^*(z,w)\,(F(\zeta+\mathfrak{d}_w)-F(\zeta+\mathfrak{d}_z))^2\right)\\
&=O_N(N^{-1-2\alpha}),
\end{aligned}$$

by the Proposition 4.1 (2) and the Lemma 5.5.

As we show in the reversible case,

$$\sum_{\zeta\in\tilde S_N^{xy}(1,\,\lfloor 4\epsilon N\rfloor-1)}\sum_{z,w\in S}\frac{1}{a(\zeta)}\,r^*(z,w)\,\left(H(\zeta+\mathfrak{d}_w)-H(\zeta+\mathfrak{d}_z)\right)^2=\epsilon^{1+\alpha}O_N(N^{-1-2\alpha}).$$

This proves $|\Omega_{21}|\le C\epsilon^{\frac{1+\alpha}{2}}$ for some constant C. We can get $|\Omega_{22}|\le C\epsilon^{\frac{\alpha+1}{2}}$ similarly.
Next we consider the term Ω_1.

Since the definition of H is same as the one except the function J, as in the reversible case

$$\begin{aligned}&H(\zeta+\mathfrak{d}_w)-H(\zeta+\mathfrak{d}_z)\\&=\frac{N^{-1-2\alpha}}{I_\alpha(1-6\epsilon)}\,(J_w-J_z)\,\zeta_x^\alpha\zeta_y^\alpha+\hat R(\zeta,w,z),\end{aligned}$$

where

$$\left|\hat R(\zeta,w,z)\right|\le C\frac{\tilde\ell}{\epsilon N}N^{-1-2\alpha}\zeta_x^\alpha\zeta_y^\alpha. \tag{5.24}$$

Define

$$\begin{aligned}\tilde C_N^{xy}&:=\tilde S_N^{xy}\left(\lfloor 4\epsilon N\rfloor,\,N-\lfloor 4\epsilon N\rfloor\right)\\&=\left\{\zeta\in E_{N-1}:\zeta_z+\zeta_y\ge N-\tilde\ell,\lfloor 4\epsilon N\rfloor\le\zeta_x\le N-\lfloor 4\epsilon N\rfloor\right\}.\end{aligned}$$

Let

$$\begin{aligned}\Omega_{11}=-N^{1+\alpha}&\sum_{\zeta\in\tilde C_N^{xy}}\sum_{z,w\in S}\frac{N^\alpha}{Z_N}\frac{1}{a(\zeta)}\,r^*(z,w)\,(F(\zeta+\mathfrak{d}_z)-\bar F(\zeta))\\&\times\ \frac{1}{(1-6\epsilon)I_\alpha}(J_w-J_z)\zeta_x^\alpha\zeta_y^\alpha N^{-1-2\alpha}\end{aligned}$$

and

$$\Omega_{12}=-N^{1+\alpha}\sum_{\zeta\in\tilde C_N^{xy}}\sum_{z,w\in S}\frac{N^\alpha}{Z_N}\frac{1}{a(\zeta)}\,r^*(z,w)\,(F(\zeta+\mathfrak{d}_z)-\bar F(\zeta))\hat R(\zeta,w,z).$$

Then $\Omega_1=\Omega_{11}+\Omega_{12}$.
Consider Ω_{11}. The computation is almost same as one of the reversible case.

$$\Omega_{11}=\frac{1}{I_\alpha Z_N(1-6\epsilon)}\sum_{\zeta\in\tilde C_N^{xy}}\frac{1}{a(\hat\zeta)}\left(-\sum_{z,w\in S}r^*(z,w)\,(F(\zeta+\mathfrak{d}_z)-\bar F(\zeta))(J_w-J_z)\right)$$

where $\hat\zeta$ is the restriction of ζ to sites $z\ne a,b$.

Fix $\hat{\zeta}$. Then

$$
\begin{aligned}
&-\sum_{z,w\in S} r^*(z,w)\,(F(\zeta+\mathfrak{d}_z)-\bar{F}(\zeta))(J_w-J_z)\\
&=\left(-\sum_{z,w\in S} r^*(z,w)\,(F(\zeta+\mathfrak{d}_z)-\bar{F}(\zeta))(J_w-J_z)\frac{1}{L}\right)L\\
&=\left(-\sum_{z,w\in S} r^*(z,w)\,F(\zeta+\mathfrak{d}_z)(J_w-J_z)\frac{1}{L}\right)L \qquad \text{since} \sum_{z\in S}\bar{L}^*J(z)\frac{1}{L}=0\\
&=\left(\sum_{z\in S} -F(\zeta+\mathfrak{d}_z)\bar{L}^*J(z)\frac{1}{L}\right)L\\
&=-\bar{L}^*J(x)F(\zeta+\mathfrak{d}_x)-\bar{L}^*J(y)F(\zeta+\mathfrak{d}_y) \qquad \text{by the definition of } J\\
&=L\mathrm{cap}_{\bar{L}^*}(x,y)\,(F(\zeta+\mathfrak{d}_x)-F(\zeta+\mathfrak{d}_y))\\
&=L\mathrm{cap}_{\bar{L}}(x,y)\,(F(\zeta+\mathfrak{d}_x)-F(\zeta+\mathfrak{d}_y))
\end{aligned}
$$

By writing $\eta=(\hat{\eta};\eta_x,\eta_y)$ where $\hat{\eta}$ is the restriction of η to sites without x,y,

$$\Omega_{11}=\frac{L\mathrm{cap}_{\bar{L}}(x,y)}{I_\alpha Z_N(1-6\epsilon)}\sum_{\zeta\in\tilde{C}_N^{xy}}\frac{1}{a(\hat{\zeta})}\,(F(\zeta+\mathfrak{d}_x)-F(\zeta+\mathfrak{d}_y))$$

which is same to the equation (5.11) in the reversible case. So we can get the following equations which is same as (5.12), (5.13).

$$\lim_{\epsilon\to0}\liminf_{N\to\infty}\Omega_{11}=\frac{L\mathrm{cap}_{\bar{L}}(x,y)}{I_\alpha Z_S}\Gamma(\alpha)^{L-2}\liminf_{N\to\infty}\left(g_N(x)-g_N(y)\right),$$

$$\lim_{\epsilon\to0}\limsup_{N\to\infty}\Omega_{11}=\frac{L\mathrm{cap}_{\bar{L}}(x,y)}{I_\alpha Z_S}\Gamma(\alpha)^{L-2}\limsup_{N\to\infty}\left(g_N(x)-g_N(y)\right),$$

where $g_N(x)=\int_{\mathcal{E}_N^x}F^N(\eta)\,d\mu_N$ for $s\in S$.

Consider Ω_{12},which is

$$\Omega_{12}=-N^{1+\alpha}\sum_{\zeta\in\tilde{C}_N^{xy}}\sum_{z,w\in S}\frac{N^\alpha}{Z_N}\frac{1}{a(\zeta)}\,r^*(z,w)\,(F(\zeta+\mathfrak{d}_z)-\bar{F}(\zeta))\hat{R}(\zeta,w,z).$$

By (5.24),

$$|\Omega_{12}|\le\frac{1}{I_\alpha Z_N(1-6\epsilon)}\left(C\frac{\tilde{\ell}}{\epsilon N}\right)\sum_{\zeta\in\tilde{C}_N^{xy}}\sum_{z,w\in S}r^*(z,w)\,\frac{1}{a(\hat{\zeta})}\left|F(\zeta+\mathfrak{d}_z)-\bar{F}(\zeta)\right|$$

By Cauchy-Schwarz inequality,

$$
\begin{aligned}
&\sum_{\zeta\in\tilde{C}_N^{xy}}\sum_{z,w\in S}r^*(z,w)\,\tfrac{1}{a(\hat{\zeta})}\left|F(\zeta+\mathfrak{d}_z)-\bar{F}(\zeta)\right|\\
&\le\left(\sum_{\zeta\in\tilde{C}_N^{xy}}\sum_{z,w\in S}r^*(z,w)\,\tfrac{a(\zeta_x)a(\zeta_y)}{a(\hat{\zeta})}\right)^{1/2}\\
&\quad\times\left(\sum_{\zeta\in\tilde{C}_N^{xy}}\sum_{z,w\in S}r^*(z,w)\,\tfrac{1}{a(\zeta)}\left(F(\zeta+\mathfrak{d}_z)-\bar{F}(\zeta)\right)^2\right)^{1/2}.
\end{aligned}
$$

By the Lemma 5.5 and Proposition 4.1 (2),

$$\left(\sum_{\zeta\in\tilde{C}_N^{xy}}\sum_{z,w\in S} r^*(z,w)\,\frac{1}{a(\zeta)}\left(F(\zeta+\mathfrak{d}_z)-\bar{F}(\zeta)\right)^2\right)^{1/2} = O(N^{-\frac{1+2\alpha}{2}}).$$

For the reversible case, we showed the equation (5.14). By changing $r(z,w)$ to $r^*(z,w)$ in the derivation of this equation, we can get

$$\left(\sum_{\zeta\in\tilde{C}_N^{xy}}\sum_{z,w\in S} r^*(z,w)\,\frac{a(\zeta_x)a(\zeta_y)}{a(\hat{\zeta})}\right)^{1/2} = O(N^{-\frac{1+2\alpha}{2}}).$$

Thus $|\Omega_{12}| = O(\frac{\tilde{\ell}}{\epsilon N})$ and $\lim_{N\to\infty}\Omega_{12}=0$.

So we have the same estimates for $\Omega_{11},\Omega_{12},\Omega_{21},\Omega_{22}$ as ones of the reversible case.

By applying arguments of the end of the previous subsection, we can conclude

$$\lim_{N\to\infty} F_N^{a,b}(\eta^N) = \lim_{N\to\infty} g_N(x) = f_{a,b}(x).$$

This proves the proposition for the non-reversible case.

6. Tightness and convergence of processes

In this section, we prove Proposition 3.1 and Theorem 3.2.

Recall the definitions of $\mathcal{T}_t^{\mathcal{E}_N}$, $\mathcal{S}_t^{\mathcal{E}_N}$, which are

$$\mathcal{T}_t^{\mathcal{E}_N} := \int_0^t \mathbf{1}\{\eta_s\in\mathcal{E}_N\}\,ds,\; t\geq 0$$

and $\mathcal{S}_t^{\mathcal{E}_N}$ as the generalized inverse of $\mathcal{T}_t^{\mathcal{E}_N}$;

$$\mathcal{S}_t^{\mathcal{E}_N} := \sup\{s\geq 0 : \mathcal{T}_s(\eta_\cdot)\leq t\}.$$

We use shorthands $\mathcal{T}_t$ for $\mathcal{T}_t^{\mathcal{E}_N}$ and $\mathcal{S}_t$ for $\mathcal{S}_t^{\mathcal{E}_N}$.

Then $\eta_t^{\mathcal{E}_N} = \eta_{\mathcal{S}_t}^N$. Define $\mathcal{S}_t' := \frac{\mathcal{S}_{\theta_N t}}{\theta_N}$, which satisfies $\eta_{\theta_N t}^{\mathcal{E}_N} = \eta_{\theta_N\mathcal{S}_t'}^N$. Define $\mathcal{T}_t' = \frac{\mathcal{T}_{\theta_N t}}{\theta_N}$. $\mathcal{S}_t'$ is a stopping time with respect to $\left(\eta_{\theta_N t}^N : t\geq 0\right)$. (For proof, refer to Lemma 8.1. in [15].)

Proof of Proposition 3.1. To prove tightness, we use the Aldous criterion(see Theorem 16.10 in [6]).

Let $\epsilon>0$ and $T>0$. Let $\mathfrak{T}_T$ be the set of all stopping times bounded by T.

We need to prove

$$\lim_{\delta\downarrow 0}\lim_{N\to\infty}\sup_{\gamma\leq\delta}\sup_{\tau\in\mathfrak{T}_T}\mathbb{P}_{\xi_N}^N\left[\left|X_{\theta_N(\tau+\gamma)}^N - X_{\theta_N\tau}^N\right|>\epsilon\right]=0.$$

The expression inside brackets is

$$\begin{aligned}\left|X^N_{\theta_N(\tau+\gamma)} - X^N_{\theta_N\tau}\right| > \epsilon \quad &\Rightarrow \quad X^N_{\theta_N(\tau+\gamma)} \neq X^N_{\theta_N\tau} \\ &\Rightarrow \quad \gamma \geq \inf\left\{t \geq 0 : X^N_{\theta_N(\tau+t)} \neq X^N_{\theta_N\tau}\right\} \\ &\Rightarrow \quad \gamma \geq \inf\left\{t \geq 0 : \Psi_N(\eta^{\mathcal{E}_N}_{\theta_N(\tau+t)}) \neq \Psi_N(\eta^{\mathcal{E}_N}_{\theta_N\tau})\right\} \\ &\Rightarrow \quad \gamma \geq \inf\left\{t \geq 0 : \eta^{\mathcal{E}_N}_{\theta_N(\tau+t)} \in \check{\mathcal{E}}^{\Psi_N(\eta^{\mathcal{E}_N}_{\theta_N\tau})}\right\}.\end{aligned}$$

For $\zeta \in \mathcal{E}_N$, denote the hitting time $\inf\left\{t \geq 0 : \eta^{\mathcal{E}_N}_{\theta_N t} \in \check{\mathcal{E}}^{\Psi_N(\zeta)} \text{ where } \eta^{\mathcal{E}_N}_0 = \zeta\right\}$ by σ_ζ.

If $\gamma \leq \delta$, then $\gamma \geq \sigma_{\eta^{\mathcal{E}_N}_{\theta_N\tau}}$ implies $\delta \geq \sigma_{\eta^{\mathcal{E}_N}_{\theta_N\tau}}$.

So

$$\begin{aligned}&\sup_{\gamma\leq\delta}\sup_{\tau\in\mathfrak{T}_T} \mathbb{P}^N_{\xi_N}\left[\left|X^N_{\theta_N(\tau+\gamma)} - X^N_{\theta_N\tau}\right| > \epsilon\right] \\ \leq\ &\sup_{\tau\in\mathfrak{T}_T} \mathbb{P}^N_{\xi_N}\left[\delta \geq \sigma_{\eta^{\mathcal{E}_N}_{\theta_N\tau}}\right] \\ \leq\ &\sup_{\zeta\in\mathcal{E}_N} \mathbb{P}^N_\zeta\left[\delta \geq \sigma_\zeta\right].\end{aligned}$$

We can estimate $\mathbb{P}^N_\zeta\left[\delta \geq \sigma_\zeta\right]$ as the following.

Fix $x \in S$. We can choose functions $\bar{h}$, $f : S \to \mathbb{R}$ such that $\bar{h}(x) = 1$, $f(z) = 0$ for $z \neq x$, $z \in S$, $f(x) > 0$ and $-\mathfrak{L}f = \bar{h}$ in the following way. Define $f_1 : S \to \mathbb{R}$ by $f_1(x) = 1$, $f_1(z) = 0$ for $z \neq x$, $z \in S$. Let $\tilde{f} = -\mathfrak{L}f_1(x)$. Define $f = \frac{f_1}{\tilde{f}}$ and $\bar{h} = -\mathfrak{L}f$. Then $\bar{h}$, f satisfies the conditions.

Define $h_N : E_N \to \mathbb{R}$ by $h_N = \sum_{z\in S} \bar{h}(z)\, \mathbf{1}_{\mathcal{E}^z_N}$. We can choose a sequence of functions $(F_N : E_N \to \mathbb{R},\ N \geq 1)$ such that

$$-\theta_N L_N F_N = h_N$$

and for $z \in S$ and a sequence $(\eta^N \in \mathcal{E}^z_N : N \geq 1)$,

$$\lim_{N\to\infty} F_N(\eta^N) = f(z)$$

as follows. Since $\sum_{x\in S} \bar{h}(x) = 0$, $\bar{h}$ can be written as

$$\bar{h} = \sum_{a,b\in S} c_{a,b}\left(\mathbf{1}\{z = a\} - \mathbf{1}\{z = b\}\right)$$

for some coefficients $c_{a,b} \in \mathbb{R}$. Define $G_N : E_N \to \mathbb{R}$ by $G_N = \sum_{a,b\in S} c_{a,b} F^{a,b}_N$ where $F^{a,b}_N$ is defined by (4.1) and (4.2). Define $\bar{f} = f(x)$. Define $F_N = G_N + \frac{\bar{f}}{L}$. Then F_N satisfies the conditions because of Proposition 5.1 and linearity.

Since $\left(\eta^N_{\theta_N t} : t \geq 0\right)$ is a Markov process,

$$\bar{M}^N_t = F_N(\eta^N_{\theta_N t}) - F_N(\eta^N_0) - \int_0^t \theta_N L_N F_N(\eta^N_{\theta_N s})\, ds$$

is a martingale.

Consider a sequence $\left(\zeta^N \in \mathcal{E}^x_N : N \geq 1\right)$. Let a hitting time

$$\bar{\sigma}_{\zeta^N} = \inf\left\{t \geq 0 : \eta^N_{\theta_N t} \in \check{\mathcal{E}}^x_N \text{ where } \eta^N_0 = \zeta^N\right\}.$$

We use the optional sampling theorem for 0 and the $\bar{\sigma}_{\zeta^N} \wedge \mathcal{S}_t^{'}$. We use shorthand $\mathbb{E}$ for $\mathbb{E}_{\zeta^N}$ and $\mathbb{P}$ for $\mathbb{P}_{\zeta^N}$.

Since $\bar{\sigma}_{\zeta^N} \wedge \mathcal{S}_t^{'}$ is an unbounded stopping time, we need to check the following conditions(See Theorem 3.97 in [9].)

(i) $\bar{\sigma}_{\zeta^N} \wedge \mathcal{S}_t^{'}$ is finite a.s.,

(ii) $\mathbb{E}\left[\left|\bar{M}^N_{\bar{\sigma}_{\zeta^N} \wedge \mathcal{S}_t^{'}}\right|\right] < \infty$,

(iii) $\lim_{T\to\infty} \mathbb{E}\left[\bar{M}^N_T \mathbf{1}_{\bar{\sigma}_{\zeta^N} \wedge \mathcal{S}_t^{'} > T}\right] = 0$.

The condition (i) is true, since $\bar{\sigma}_{\zeta^N}$ is a hitting time for a recurrent Markov process.

Consider the condition (ii). The term inside the brackets is

$$\left|\bar{M}^N_{\bar{\sigma}_{\zeta^N} \wedge \mathcal{S}_t^{'}}\right| \le \left|F_N(\eta^N_{\bar{\sigma}_{\zeta^N} \wedge \mathcal{S}_t^{'}})\right| + \left|F_N(\eta^N_0)\right| + \left|\int_0^{\bar{\sigma}_{\zeta^N} \wedge \mathcal{S}_t^{'}} \theta_N L_N F_N(\eta^N_{\theta_N s})\, ds\right|.$$

Before the time $\bar{\sigma}_\zeta$,

$$-\theta_N L_N F_N(\eta^N_{\theta_N s}) = \begin{cases} 1 & , \eta_{\eta^N_{\theta_N s}} \in \mathcal{E}_N^x \\ 0 & , \text{otherwise.} \end{cases} \tag{6.1}$$

So

$$\begin{aligned} \int_0^{\bar{\sigma}_{\zeta^N} \wedge \mathcal{S}_t^{'}} -\theta_N L_N F_N(\eta^N_{\theta_N s})\, ds &\le \int_0^{\mathcal{S}_t^{'}} -\theta_N L_N F_N(\eta^N_{\theta_N s})\, ds \\ &\le \int_0^{\mathcal{S}_t^{'}} \mathbf{1}_{\eta^N_{\theta_N s} \in \mathcal{E}_N}\, ds \\ &= t. \end{aligned}$$

Since $\|F_N\|_{L^\infty} < \infty$, $\left|\bar{M}^N_{\bar{\sigma}_{\zeta^N} \wedge \mathcal{S}_t^{'}}\right|$ is bounded. So the condition (ii) holds.

If $\bar{\sigma}_{\zeta^N} \wedge \mathcal{S}_t^{'} > T$, then

$$\begin{aligned} \int_0^T -\theta_N L_N F_N(\eta^N_{\theta_N s})\, ds &\le \int_0^{\bar{\sigma}_{\zeta^N} \wedge \mathcal{S}_t^{'}} -\theta_N L_N F_N(\eta^N_{\theta_N s})\, ds \\ &\le t\,, \end{aligned}$$

the first inequality is because of the equation (6.1) and the we showed the second inequality in showing condition (ii).

So $\left|\bar{M}^N_T\right| \le 2\,\|F_N\|_{L^\infty} + t$ if $\bar{\sigma}_{\zeta^N} \wedge \mathcal{S}_t^{'} > T$. Since $\|F_N\|_{L^\infty}$ is uniformly bounded in N, $\left|\bar{M}^N_T\right|$ is uniformly bounded.

The Markov process $\eta^{\mathcal{E}_N}_{\cdot}$ is recurrent. So $\lim_{T\to\infty} \mathbb{P}\left[\mathcal{S}_t^{'} > T\right] = 0$. This implies $\lim_{T\to\infty} \mathbb{P}\left[\bar{\sigma}_{\zeta^N} \wedge \mathcal{S}_t^{'} > T\right] = 0$. We get

$$\lim_{T\to\infty} \left|\mathbb{E}\left[\bar{M}^N_T \mathbf{1}_{\bar{\sigma}_{\zeta^N} \wedge \mathcal{S}_t^{'} > T}\right]\right| \le \lim_{T\to\infty} (2\,\|F_N\|_{L^\infty} + t)\mathbb{P}\left[\bar{\sigma}_{\zeta^N} \wedge \mathcal{S}_t^{'} > T\right] = 0.$$

So the condition (iii) holds.

Thus we get

$$\mathbb{E}\left[\bar{M}^N_{\bar{\sigma}_{\zeta^N} \wedge \mathcal{S}_t^{'}}\right] = \mathbb{E}\left[\bar{M}^N_0\right] = 0.$$

That is

$$\mathbb{E}\left[F_N\left(\eta^N_{\bar\sigma_{\zeta^N}\wedge\mathcal{S}_t'}\right) - F_N\left(\eta^N_0\right)\right] = \mathbb{E}\left[\int_0^{\bar\sigma_{\zeta^N}\wedge\mathcal{S}_t'} \theta_N L_N F_N(\eta^N_{\theta_N s})\, ds\right]$$
$$\le t,$$

as we did in showing the condition (ii).

The left hand side of the previous equation is

$$\begin{aligned}\mathbb{E}\left[F_N\left(\eta^N_{\bar\sigma_{\zeta^N}\wedge\mathcal{S}_t'}\right) - F_N\left(\eta^N_0\right)\right] &\ge \mathbb{E}\left[F_N\left(\eta^N_{\bar\sigma_{\zeta^N}}\right) \mid \bar\sigma_{\zeta^N} \le \mathcal{S}_t'\right] + o_N(1) \\ &\quad \text{since } F_N \ge o_N(1) \\ &= \left(\bar f + o_N(1)\right)\mathbb{P}\left[\bar\sigma_{\zeta^N} \le \mathcal{S}_t'\right] + o_N(1) \\ &= \bar f \mathbb{P}\left[\bar\sigma_{\zeta^N} \le \mathcal{S}_t'\right] + o_N(1).\end{aligned}$$

Thus $\mathbb{P}\left[\bar\sigma_{\zeta^N} \le \mathcal{S}_t'\right] \le \frac{t}{\bar f} + o_N(1)$.

Since $\bar f$ depends on $x \in S$ by the definition and S is finite, for $\zeta \in \mathcal{E}_N$

$$\mathbb{P}\left[\bar\sigma_\zeta \le \mathcal{S}_t'\right] \le Ct + o_N(1) \text{ for some constant } C.$$

Also by the definitions of $\sigma_\zeta, \bar\sigma_\zeta$, and $\mathcal{S}_t'$, $\mathbb{P}\left[\bar\sigma_\zeta \le \mathcal{S}_t'\right] = \mathbb{P}\left[\sigma_\zeta \le t\right]$.

In conclusion,

$$\begin{aligned}\lim_{\delta\downarrow 0}\lim_{N\to\infty}\sup_{\gamma\le\delta}\sup_{\tau\in\mathfrak{T}_T}\mathbb{P}^N_{\xi_N}\left[\left|X^N_{\tau+\gamma} - X^N_\tau\right| > \epsilon\right] &\le \lim_{\delta\downarrow 0}\lim_{N\to\infty}\sup_{\zeta\in\mathcal{E}_N}\mathbb{P}^N_\zeta\left[\delta \ge \sigma_\zeta\right] \\ &\le \lim_{\delta\downarrow 0}\lim_{N\to\infty}\left(C\delta + o_N(1)\right) \\ &= \lim_{\delta\downarrow 0} C\delta \\ &= 0,\end{aligned}$$

this proves tightness.

We showed the tightness of the sequence of laws, which is Proposition 3.1. We need to show the uniqueness of limit points. Let $\mathbb{Q}_N$ be the law of $(X_{\theta_N t} : t \ge 0)$ under $\mathbb{P}^N_{\xi_N}$. Without loss of generality, assume that $\mathbb{Q}_N$ converges to $\mathbb{Q}$. By the property of the martingale problem, it's enough to show the following lemma for the uniqueness of the limit points.

Lemma. *Under* $\mathbb{Q}$, $X_0 = x$,

$$M_t = f(X_t) - f(X_0) - \int_0^t \mathfrak{L} f(X_s)\, ds$$

is a martingale for every function f *from* S *to* $\mathbb{R}$.

Proof of the Lemma. It's enough to prove this lemma for f satisfying

$$-\mathfrak{L} f(x) = \mathbf{1}\{x = a\} - \mathbf{1}\{x = b\} \text{ for } a \ne b \in S$$

and

$$\sum_{x\in S} f(x) = 0.$$

This is because the following set spans the vector space of all functions from S to $\mathbb{R}$, which is

$$\{f : S \to \mathbb{R} \mid -\mathfrak{L}f(x) = \mathbf{1}\{x = a\} - \mathbf{1}\{x = b\} \text{ for some } a \neq b \in S \text{ and } \sum_{x \in S} f(x) = 0\}$$

$\cup \{f : S \to \mathbb{R} \mid f \text{ is a constant function}\}$.

Assume that f satisfies $-\mathfrak{L}f(x) = \mathbf{1}\{x = a\} - \mathbf{1}\{x = b\}$ for $a \neq b \in S$ and $\sum_{x \in S} f(x) = 0$.

We need to show that

$$\mathbb{E}^{\mathbb{Q}}\left[g((X_u : 0 \leq u \leq s))(f(X_t) - f(X_s) - \int_s^t \mathfrak{L}f(X_u)\,du)\right] = 0,$$

for all $0 \leq s < t$ and all bounded, continuous functions $g : D([0, s], S) \to \mathbb{R}$.

The left hand side of the previous equation is

$\mathbb{E}^{\mathbb{Q}}\left[g((X_u : 0 \leq u \leq s))(f(X_t) - f(X_s) - \int_s^t \mathfrak{L}f(X_u)\,du)\right]$

$= \lim_{N\to\infty} \mathbb{E}^{\mathbb{Q}_N}\left[g((X_u : 0 \leq u \leq s))(f(X_t) - f(X_s) - \int_s^t \mathfrak{L}f(X_u)\,du\,)\right]$

$= \lim_{N\to\infty} \mathbb{E}^{\mathbb{P}^N_{\xi_N}}\Big[g((\,\Psi(\eta^{\mathcal{E}_N}_{\theta_N u}) : 0 \leq u \leq s))(f(\Psi(\eta^{\mathcal{E}_N}_{\theta_N t})) - f(\Psi(\eta^{\mathcal{E}_N}_{\theta_N s}))$

$\quad - \int_s^t \mathfrak{L}f(\Psi(\eta^{\mathcal{E}_N}_{\theta_N u}))\,du\,)\Big]$

$= \lim_{N\to\infty} \mathbb{E}^{\mathbb{P}^N_{\xi_N}}\Big[g((\,\Psi(\eta^{\mathcal{E}_N}_{\theta_N u}) : 0 \leq u \leq s))(F(\eta^{\mathcal{E}_N}_{\theta_N t}) - F(\eta^{\mathcal{E}_N}_{\theta_N s})$

$\quad - \int_s^t \theta_N L_N F(\eta^{\mathcal{E}_N}_{\theta_N u})\,du\,)\Big]$, F is the function defined by the equation (4.1) and we use (3) in Proposition 4.1.

$= \lim_{N\to\infty} \mathbb{E}^{\bar{\mathbb{P}}^N_{\xi_N}}\Big[g((\,\Psi(\eta_{\theta_N \mathcal{S}'_u}) : 0 \leq u \leq s))\,(\,F(\eta_{\theta_N \mathcal{S}'_t}) - F(\eta_{\theta_N \mathcal{S}'_s})$

$\quad - \int_s^t \theta_N L_N F(\eta_{\theta_N \mathcal{S}'_u})\,du\,)\Big]$, $\bar{\mathbb{P}}^N_{\xi_N}$ is the law of $\eta^N_\cdot$ starting at ξ_N.

The last expression above is

$\int_s^t \theta_N L_N F(\eta_{\theta_N \mathcal{S}'_u})\,du = \int_{\mathcal{S}'_s}^{\mathcal{S}'_t} \theta_N L_N F(\eta_{\theta_N v})\,\frac{d\mathcal{T}'_v}{dv}dv$, since $\mathcal{T}'_{\mathcal{S}'_u} = u$.

$\quad = \int_{\mathcal{S}'_s}^{\mathcal{S}'_t} \theta_N L_N F(\eta_{\theta_N v})\,\frac{d\mathcal{T}'_v}{dv}dv$

Since $\frac{d\mathcal{T}'_v}{dv} = \begin{cases} 1 & , \eta_{\theta_N v} \in \mathcal{E}_N \\ 0 & , \eta_{\theta_N v} \notin \mathcal{E}_N \end{cases}$ and $\theta_N L_N F(\eta_{\theta_N v}) = 0$ if $\eta_{\theta_N v} \notin \mathcal{E}_N$,

$$\int_{\mathcal{S}'_s}^{\mathcal{S}'_t} \theta_N L_N F(\eta_{\theta_N v})\,\frac{d\mathcal{T}'_v}{dv}dv = \int_{\mathcal{S}'_s}^{\mathcal{S}'_t} \theta_N L_N F(\eta_{\theta_N v})\,dv.$$

We apply the optional sampling theorem to the martingale

$$\bar{M}^N_t = F_N(\eta^N_{\theta_N t}) - F_N(\eta^N_0) - \int_0^t \theta_N L_N F_N(\eta^N_{\theta_N s})\,ds$$

and stopping times $\mathcal{S}'_t \geq \mathcal{S}'_s$. Since $\mathcal{S}'_t$ is unbounded, we need to show the following conditions like we did in the proof for tightness. We use shorthands $\mathbb{E}$ for $\mathbb{E}_{\zeta^N}$ and $\mathbb{P}$ for $\mathbb{P}_{\zeta^N}$.

(i) $\mathcal{S}'_t$ is finite a.s.,

(ii) $\mathbb{E}\left[\left|\bar{M}^N_{\mathcal{S}'_t}\right|\right] < \infty$,

(iii) $\lim_{T\to\infty} \mathbb{E}\left[\bar{M}^N_T \mathbf{1}_{\mathcal{S}'_t > T}\right] = 0$.

Since the process $\left(\eta^N_{\theta_N t} : t \geq 0\right)$ is irreducible and recurrent, a stopping time $\mathcal{S}_t'$ is finite a.s. So the condition (i) is true.

Let us check the condition (ii). The term inside the brackets is

$$\left|\bar{M}^N_{\mathcal{S}_t'}\right| \leq \left|F_N(\eta^N_{\mathcal{S}_t'})\right| + \left|F_N(\eta^N_0)\right| + \left|\int_0^{\mathcal{S}_t'} \theta_N L_N F_N(\eta^N_{\theta_N s})\, ds\right|.$$

By the definition of F_N,

$$\left|\theta_N L_N F_N(\eta^N_{\theta_N s})\right| = \begin{cases} 1 & , \eta^N_{\theta_N s} \in \mathcal{E}^a \cup \mathcal{E}^b \\ 0 & , \text{otherwise.} \end{cases}$$

So

$$\begin{aligned} \left|\int_0^{\mathcal{S}_t'} \theta_N L_N F_N(\eta^N_{\theta_N s})\, ds\right| &\leq \int_0^{\mathcal{S}_t'} \left|\theta_N L_N F_N(\eta^N_{\theta_N s})\right|\, ds \\ &\leq \int_0^{\mathcal{S}_t'} \mathbf{1}_{\eta^N_{\theta_N s} \in \mathcal{E}_N}\, ds \\ &= \mathcal{T}'_{\mathcal{S}_t'} = t. \end{aligned}$$

Since $\|F_N\|_{L^\infty} < \infty$, $\left|\bar{M}^N_{\mathcal{S}_t'}\right|$ is bounded . So the condition (ii) holds.

If $\mathcal{S}_t' > T$, then

$$\begin{aligned} \left|\int_0^{T} \theta_N L_N F_N(\eta^N_{\theta_N s})\, ds\right| &\leq \int_0^{T} \left|\theta_N L_N F_N(\eta^N_{\theta_N s})\right|\, ds \leq \int_0^{\mathcal{S}_t'} \left|\theta_N L_N F_N(\eta^N_{\theta_N s})\right|\, ds \\ &\leq \int_0^{\mathcal{S}_t'} \mathbf{1}_{\eta^N_{\theta_N s} \in \mathcal{E}_N}\, ds = \mathcal{T}'_{\mathcal{S}_t'} = t. \end{aligned}$$

So $\left|\bar{M}^N_T\right| \leq 2\|F_N\|_{L^\infty} + t$ if $\mathcal{S}_t' > T$. Since $\|F_N\|_{L^\infty}$ is uniformly bounded in N, $\left|\bar{M}^N_T\right|$ is uniformly bounded.

Since the Markov process $\eta^{\mathcal{E}_N}_\cdot$ is irreducible and recurrent, $\lim_{T\to\infty} \mathbb{P}\left[\mathcal{S}_t' > T\right] = 0$.

So $\lim_{T\to\infty}\left|\mathbb{E}\left[\bar{M}^N_T \mathbf{1}_{\mathcal{S}_t' > T}\right]\right| \leq \lim_{T\to\infty}(2\|F_N\|_{L^\infty} + t)\mathbb{P}\left[\mathcal{S}_t' > T\right] = 0$.

Thus the condition (iii) holds.

Let's get back to the original equation,

$$\lim_{N\to\infty} \mathbb{E}^{\bar{\mathbb{P}}^N_{\xi_N}}\Big[g((\Psi(\eta^N_{\theta_N \mathcal{S}_u'}) : 0 \leq u \leq s))(F(\eta^N_{\theta_N \mathcal{S}_t'}) - F(\eta^N_{\theta_N \mathcal{S}_s'}) - \textstyle\int_s^t \theta_N L_N F(\eta^N_{\theta_N \mathcal{S}_u'})\, du)\Big]$$

$$= \lim_{N\to\infty} \mathbb{E}^{\bar{\mathbb{P}}^N_{\xi_N}}\Big[g((\Psi(\eta^N_{\theta_N \mathcal{S}_u'}) : 0 \leq u \leq s))(F(\eta^N_{\theta_N \mathcal{S}_t'}) - F(\eta^N_{\theta_N \mathcal{S}_s'}) - \textstyle\int_{\mathcal{S}_s'}^{\mathcal{S}_t'} \theta_N L_N F(\eta^N_{\theta_N v})\, dv)\Big]$$

$= 0$ by the optional sampling theorem. Here the function $g((\Psi(\eta_{\theta_N \mathcal{S}_u'}) : 0 \leq u \leq s))$ is measurable by $\mathscr{F}_{\theta_N \mathcal{S}_s'}$,the filtration at time $\theta_N \mathcal{S}_s'$ for $\eta^N_\cdot$.

So we proved the lemma. □

This proves the Theorem 3.2. □

Next we prove Theorem 3.3.

Proof of Theorem 3.3. Denote the sample space for $\mathbb{P}^N_{\nu_N}$ as Ω_N. Then,

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}^N_{\nu_N}}\left[\int_0^T 1\left\{\eta^N\left(N^{1+\alpha}s\right)\in\Delta_N\right\}ds\right]\\
&=\int_{\Omega_N}\int_0^T 1\left\{\eta^N\left(N^{1+\alpha}s\right)\in\Delta_N\right\}ds\,d\mathbb{P}^N_{\nu_N}\\
&=\int_0^T\int_{\Omega_N} 1\left\{\eta^N\left(N^{1+\alpha}s\right)\in\Delta_N\right\}d\mathbb{P}^N_{\nu_N}\,ds \qquad \text{by Fubini's theorem}\\
&=\int_0^T\sum_{\eta\in E_N} 1\left\{\eta\in\Delta_N\right\}\,\nu_N(\eta,N^{1+\alpha}s)\,ds\\
&\qquad\text{, where } \nu_N(\eta,N^{1+\alpha}s) \text{ is the distribution of } \eta^N(\cdot) \text{ at time } N^{1+\alpha}s\\
&=\int_0^T\sum_{\eta\in E_N} 1\left\{\eta\in\Delta_N\right\}\,f_N(\eta,N^{1+\alpha}s)\,\mu_N(\eta)\,ds\\
&\qquad\text{, where } f_N(\eta,N^{1+\alpha}s)=\frac{\nu_N(\eta,N^{1+\alpha}s)}{\mu_N(\eta)}.
\end{aligned}
$$

The square of the summation in the last equation is equal or less than

$$
\begin{aligned}
&\left(\sum_{\eta\in E_N}\left(1\left\{\eta\in\Delta_N\right\}\right)^2\,\mu_N(\eta)\right)\left(\sum_{\eta\in E_N} f_N^2(\eta,N^{1+\alpha}s)\,\mu_N(\eta)\right)\\
&=\mu_N(\Delta_N)\left(\sum_{\eta\in E_N} f_N^2(\eta,N^{1+\alpha}s)\,\mu_N(\eta)\right).
\end{aligned}
$$

By differentiating the summation in the previous equation in s,

$$
\begin{aligned}
&\frac{d}{ds}\left(\sum_{\eta\in E_N} f_N^2(\eta,N^{1+\alpha}s)\,\mu_N(\eta)\right)\\
&=N^{1+\alpha}\sum_{\eta\in E_N} 2f_N(\eta,N^{1+\alpha}s)\,L_Nf_N(\eta,N^{1+\alpha}s)\,\mu_N(\eta)\\
&=-2N^{1+\alpha}D_N(f_N)\\
&\le 0.
\end{aligned}
$$

So

$$
\sum_{\eta\in E_N} f_N^2(\eta,N^{1+\alpha}s)\,\mu_N(\eta)\le\sum_{\eta\in E_N} f_N^2(\eta,0)\,\mu_N(\eta)\le M
$$

for some M, since $\sum_{\eta\in E_N} f_N^2(\eta,0)\,\mu_N(\eta)$ is uniformly bounded in N by the assumption of the theorem.

Thus

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}^N_{\nu_N}}\left[\int_0^T 1\left\{\eta^N\left(N^{1+\alpha}s\right)\in\Delta_N\right\}ds\right]\\
&\le \int_0^T \sqrt{\mu_N(\Delta_N)}\sqrt{M}\,ds\\
&= T\sqrt{\mu_N(\Delta_N)}\sqrt{M}
\end{aligned}
$$

By the Theorem 2.2, which is $\lim_{N\to\infty}\mu_N(\Delta_N)=0$, we get

$$
\lim_{N\to\infty}\mathbb{E}^{\mathbb{P}^N_{\nu_N}}\left[\int_0^T 1\left\{\eta^N\left(N^{1+\alpha}s\right)\in\Delta_N\right\}ds\right]=0.
$$

□

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