Growth Models and Hamilton-Jacobi PDEs

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1 Introduction

Hamilton–Jacobi equation (HJE) is one of the most popular and studied PDE which enjoys vast applications in numerous areas of science. Originally HJEs were formulated in connection with the completely integrable Hamiltonian ODEs of celestial mechanics. They have also been used to study the evolution of the value functions in control and differential game theory. HJE associated with space-time stationary Hamiltonian functions are used to study turbulence in hydrodynamics. Several growth models in physics and biology are described by such HJEs and their viscous variants. In these models, a random interface separates regions associated with different phases and the interface can be locally approximated by the graph of a solution to a HJE. Naturally we would like to understand how the randomness affects the solutions and how the statistics of solutions are propagated with time. Lagrangian techniques in Aubry-Mather theory for action-minimizing trajectories, PDE techniques of weak KAM theory, and probabilistic methods related to first/last passage percolation problems have been employed to study long-time behavior of solutions. Most notably, a unique invariant measure has been constructed for any prescribed average velocity for some important examples of Hamiltonian functions. In these lectures I will give an overview of some of the existing results for the statistics of random solutions to HJEs. In particular, I will discuss a systematic approach for constructing Gibbsian solutions to Hamilton- Jacobi PDEs by exploring the Eularian description of the shock dynamics. Such Gibbsian solutions depend on kernels satisfying kinetic-like equations reminiscent of the Smoluchowski model for coagulating and fragmenting particles.

Given a C^2 Hamiltonian function $H : \mathbb{R} \times [0, \infty) \times \mathbb{R} \to \mathbb{R}$, we consider the HJE

$$u_t = H(x, t, u_x), \quad t \geq s, \tag{1.1}$$

or the corresponding scalar conservation law

$$\rho_t = H(x,t,\rho)_x, \qquad t \geq s. \tag{1.2}$$

We assume that the Hamiltonian function $H(x,t,\rho)$ is convex in the *momentum* variable ρ.

Definition 1.1 Given $z=(y,s)\in\mathbb{R}^{d+1}$, by a *fundamental solution* $W(\cdot;z):\mathbb{R}\times(s,\infty)\to\mathbb{R}$ associated with z we mean

$$W(x,t;z) = \sup\left\{\int_s^t L\big(\xi(\theta),\theta,\dot\xi(\theta)\big)\,d\theta:\ \xi\in C^1\big([s,t];\mathbb{R}^d\big),\ \xi(s)=y,\ \xi(t)=x\right\}, \tag{1.3}$$

where L is the Legendre transform of H in the p-variable:

$$L(x,t,v)=\inf_p\big(p\cdot v+H(x,t,p)\big), \qquad H(x,t,p)=\sup_v\big(L(x,t,v)-p\cdot v\big).$$

We also set $M(x,t;z)=W_x(x,t;z)$ for the x-derivative of W. □

Under our conditions on H, the function W is a Lipschitz function of (x,t) for $t>s$, and $M(x,t)$ is well-defined a.e.. A representation of M is given as follows. For each (x,t), we may find a maximizing C^1 path $\xi(\theta)=\xi(\theta;x,t;z)$. This maximizing path satisfies a Newton's like equation

$$\big(L_v(\xi(th),\theta,\dot\xi(\theta))\big)_\theta = L_x(\xi(th),\theta,\dot\xi(\theta)).$$

If we set $p(\theta)=L_v(\xi(\theta),\theta,\dot\xi(\theta))$, then the pair $(\xi(\theta),p(\theta))$ satisfies the Hamiltonian ODE

$$\dot\xi(\theta)=-H_p(\xi(\theta),\theta,p(\theta)), \qquad \dot p(\theta)=H_x(\xi(th),\theta,p(\theta)).$$

The function M is continuous at (x,t) if and only if the maximizing path is unique. When this is the case, we simply have

$$M(x,t)=L_v\big(\xi(t),t,\dot\xi(t)\big)=L_v\big(x,t,\dot\xi(t)\big). \tag{1.4}$$

In general $M(x,t)$ could be multi-valued; for each maximizing path, the right-hand side of (1.4) offers a possible value for $M(x,t)$. Under our assumptions on H, the set of discontinuity points of $M(\cdot,t)$ is countable for each t.

The Cauchy problem associated with (1.1) has a representation of the form

$$u(x,t)=\sup_y\big(u^0(y)+W(x,t;y,s)\big). \tag{1.5}$$

In other words, u given by (1.5), satisfies (1.1) in viscosity sense for $t>s$, and $u(x,s)=u^0(x)$. We also use the notation

$$\mathcal{T}_s^t(u^0)(x)=u(x,t),$$

so that the family $\{\mathcal{T}_s^t:\ s \le t\}$ is a semigroup. Let us observe that the nonlinear operator $\mathcal{T}_s^t$ has the following strong monotonicity property:

$$\mathcal{T}_s^t\big(\sup_{\alpha\in I} g^\alpha\big) = \sup_{\alpha\in I} \mathcal{T}_s^t\big(g^\alpha\big), \tag{1.6}$$

where $\{g^\alpha : \alpha \in I\}$ is a family of initial data.

Example 1.1(i) When H does not depend on (x,t), then

$$W(x,t;y,s) = (t-s)L\left(\frac{x-y}{t-s}\right), \qquad M(x,t;y,s) = \nabla L\left(\frac{x-y}{t-s}\right).$$

This leads to the formula

$$u(x,t) = \sup_y \left(u^0(y) + (t-s)L\left(\frac{x-y}{t-s}\right)\right). \tag{1.7}$$

Moreover, if we set $s=0$, and write $y(x,t)$ for a maximizing y in (1.7), then

$$\rho(x,t) = \nabla L\left(\frac{x-y(x,t)}{t}\right).$$

Also observe that for $w^p(x,t) = p\cdot x + (t-s)H(p) - c$ is also solution for $p\in\mathbb{R}^d,\ c\in\mathbb{R}$. From this and (1.6) we learn that for every function g^*, the function

$$u(x,t) = \sup_p \big(p\cdot x + (t-s)H(p) - g^*(p)\big), \tag{1.8}$$

is a solution that is convex in (x,t), and has the initial data

$$u(x,s) = \sup_p \big(p\cdot x - g^*(p)\big).$$

□

Example 1.2 Several examples of stochastic growth models and random fluids can be formulated as HJE with $H(x,t,\rho) = H_0(\rho) - V(x,t)$, with H_0 convex, and a potential V which is stationary process in (x,t). Note that L takes the form $L(x,t,v) = L_0(v) + V(x,t)$, with L_0 a concave function given by

$$L_0(v) = \inf_p \big(p\cdot v + H_0(p)\big).$$

(i) As our first example of a stationary potential, consider

$$V(x,t) = \sum_{i=1}^\infty V_i(x)\dot B_i(t),$$

where $(V_i:\ i\in\mathbb{N})$ is a collection of 1-periodic functions, and $(B_i:\ i\in\mathbb{N})$ is a collection of i.i.d standard Brownian motions. This model was studied in [EKMS].

(ii) As an example of a percolation-like model, we assume that the stationary potential V is formally given by

$$V(x,t)=\sum_{i\in I}\delta_{s_i}(t)1\!\!1(x=a_i), \tag{1.9}$$

where $\omega=\big\{(a_i,s_i):\ i\in I\big\}$, is a realization of a *Poisson Point Process* of intensity 1 in $\mathbb{R}^2$. In practice, we may approximate V by

$$V_\varepsilon(x,t)=\sum_{i\in I}\varepsilon^{-1}\zeta\left(\frac{t-s_i}{\varepsilon}\right)\eta\left(\frac{x-a_i}{\delta(\varepsilon)}\right),$$

where $\delta(\varepsilon)\to0$, in small ε-limit, and η and ζ are two smooth functions of compact support such that $\int\zeta(t)\ dt=1$, and $\eta(x)=1$ in a neighborhood of the origin. Replacing V with V_ε yields a Hamiltonian function H^ε for which the equation (1.1) is well-defined and its solution u^ε has a limit u as $\varepsilon\to0$. A variational representation as in (1.5) for u^ε would yield a variational representation for u as well. It is not hard to show that the minimizing path ξ of the variational problem (1.3) is a concatenation of line segments between Poisson points of ω. In other words,

$$W(x,t;y,s)=W(x,t;y,s;\omega)=\sup\left(N(\mathbf{z})+\sum_{i=1}^{N(\mathbf{z})}(s_{i+1}-s_i)L_0\left(\frac{a_{i+1}-a_i}{s_{i+1}-s_i}\right)\right), \tag{1.10}$$

where the supremum is over sequences $\mathbf{z}=\big((a_0,s_0),(a_1,s_1),\dots,(a_n,s_n),(a_{n+1},s_{n+1})\big)$, such that $N(\mathbf{z})=n$, and

$$\begin{aligned}&s_0<s_1<\dots<s_{n+1}, &&(a_1,s_1),\dots,(a_n,s_n)\in\omega,\\ &(a_0,s_0)=(y,s), &&(a_{n+1},s_{n+1})=(x,t).\end{aligned} \tag{1.11}$$

This model was defined and studied in Bakhtin [B] and Bakhtin et al. [BCK] when $H_0(p)=p^2/2$ (which leads to $L_0(v)=-v^2/2$).

(iii) If $H_0(p)=|p|$ in part **(ii)**, then $L_0(v)=-\infty\ 1\!\!1(|v|>1)$. In this case,

$$W(x,t;y,s)=W(x,t;y,s;\omega)=\sup N(\mathbf{z}),$$

where the supremum is over sequences $\mathbf{z}$ as in (1.1), with the additional requirement

$$|a_{i+1}-a_i|\le s_{i+1}-s_i.$$

Note that the fundamental solutions take value in $\mathbb{N}$. Moreover, if the height function takes value in $\mathbb{Z}$ initially, then the same is true at later times. The corresponding $u(x,t)$ is a stochastic growth model that is known as *Polynuclear Growth* (in short PNG). We refer to [PS] for more details. □

1.1 Discrete models

The HJE offers a growth model in the continuum i.e., $x \in \mathbb{R}^d$, $u \in \mathbb{R}$, and $t \in \mathbb{R}$. There are many interesting and well-studied growth models such that some of the parameters are discrete. We already discussed PNG model in Example 1.2**(iii)** where the height function takes value in $\mathbb{Z}$. PNG model is an example of a completely integrable model because of a *determinantal* description of its correlation function. We now discuss another family of stochastic growth models where the time is discrete. This family includes an integrable model that is known as *semi-discrete polymer* (SDP in short).

Ginzburg-Landau (GZ) model is a diffusion $\mathbf{h}(t) = (h_i(t) : i \in \mathbb{Z})$ which satisfies the SDE

$$\begin{aligned} \frac{dh_i}{dt} &= \left(\sigma + \frac{\gamma\beta^2}{2}\right) V'(h_{i+1} - h_i) + \left(\sigma - \frac{\gamma\beta^2}{2}\right) V'(h_i - h_{i-1}) + \beta\frac{dB_i}{dt} \\ &= \sigma(V'(h_{i+1} - h_i)) + V'(h_i - h_{i-1})) \\ &\quad + \frac{\gamma\beta^2}{2}(V'(h_{i+1} - h_i) - V'(h_i - h_{i-1})) + \beta\frac{dB_i}{dt}. \end{aligned} \tag{1.12}$$

where $\beta, \gamma > 0$ and B_i's are independent Brownian motions. If we interpret h_i as the height at site i, and write $r_i = h_i - h_{i-1}$, for the height difference, then

$$\begin{aligned} \frac{dr_i}{dt} &= \left(\sigma + \frac{\gamma\beta^2}{2}\right) V'(r_{i+1}) + \left(\sigma - \frac{\gamma\beta^2}{2}\right) V'(r_i) - \left(\sigma + \frac{\gamma\beta^2}{2}\right) V'(r_i) \\ &\quad - \left(\sigma - \frac{\gamma\beta^2}{2}\right) V'(r_{i-1}) + \beta\left(\frac{dB_i}{dt} - \frac{dB_{i-1}}{dt}\right). \end{aligned}$$

Writing D_i and $D_{i,i+1}$ for $\frac{\partial}{\partial r_i} - \frac{\partial}{\partial r_{i+1}}$ and $\frac{\partial}{\partial r_i}$ respectively, the generator of $\mathbf{h}$ can be written as $\mathcal{L} = \sigma\mathcal{A} + \beta^2\mathcal{S}$, where

$$\begin{aligned} \mathcal{A} &= \sum_i (V'(r_{i+1}) + V'(r_i))D_{i,i+1} = \sum_i (V'(r_{i+1}) - V'(r_{i-1}))D_i, \\ \mathcal{S} &= \frac{1}{2}\sum_i D_{i,i+1}^2 - \gamma(V'(r_i) - V'(r_{i+1}))D_{i,i+1}. \end{aligned} \tag{1.13}$$

We now argue that $\mathcal{A}$ is invariant with respect to

$$\nu_\alpha(d\mathbf{r}) = \prod_i \frac{1}{Z(\alpha)} e^{\alpha r_i - \gamma V(r_i)},$$

with $Z(\alpha) = \int e^{\alpha r - V(r)} dr$.

$$\int g\ \mathcal{S}f\ d\nu_\alpha = \int f\ \mathcal{S}g\ d\nu_\alpha = -\frac{1}{2}\int (D_{i,i+1}f)(D_{i,i+1}g) d\nu_\alpha,$$

and $\int \mathcal{A}f \; d\nu_\alpha = 0$, for every nice (local) function f and g. To see this, observe

$$\int \sum_i (V'(r_{i+1}) - V'(r_{i-1}))\frac{\partial f}{\partial r_i} d\nu_\alpha = \int \sum_i (V'(r_{i+1}) - V'(r_{i-1}))(-\alpha + \gamma V'(r_i)) d\nu_\alpha = 0.$$

As an example, choose $\sigma = -1/2$, and $\gamma = \beta = 1$ so that (1.12) simplifies to

$$dh_i = -V'(r_i) \; dt + dB_i.$$

In particular, when $V(r) = e^{-r} + r/2$, we arrive at

$$dh_i = \left(e^{h_{i-1} - h_i} - \frac{1}{2}\right) \; dt + dB_i. \tag{1.14}$$

This is closely related to the stochastic discrete heat equation of the form

$$dZ_i = Z_{i-1} \; dt + Z_i dB_i. \tag{1.15}$$

One can show that $Z_i(t) > 0$ for all $i \in \mathbb{Z}$, and $t > 0$, if this is so initially i.e., $Z_i(0) > 0$, for all $i \in \mathbb{Z}$. If we set $h_i = \log Z_i$, then $h = (h_i : \; i \in \mathbb{Z})$ satisfies

$$dh_i = \frac{dZ_i}{Z_i} - \frac{Z_i^2}{2Z_i^2} \; dt = \frac{Z_{i-1}}{Z_i} \; dt - \frac{1}{2} \; dt + dB_i,$$

which is (1.14).

2 Homogeneous Hamiltonian

Even when H is deterministic and independent of (x, t), some interesting mathematics emerges as we start from a random initial data. As a warm up, let us study the solution ρ of (1.2) for a white noise initial data. Equivalently, the initial data for (1.1) is a two-sided Brownian motion. In the case of Burgers Turbulence, we have $d = 1, s = 0$, $H(p) = p^2/2$, and the initial data is white noise. From (1.7) we learn

$$u(x, t) = \sup_y \left(\sigma B(y) - \frac{(x-y)^2}{2t}\right), \tag{2.1}$$

where B is a standard two-sided Brownian motion. If we set

$$\hat{B}(y) = \lambda^{-1} B(\lambda^2 y),$$

then $\hat{B}$ has the same law as B, and $\hat{u}$, given by

$$\begin{aligned}\hat{u}(x,t) &= \sup_y \left(\sigma \hat{B}(y) - \frac{(x-y)^2}{2t}\right) = \sup_y \left(\sigma\lambda^{-1}B\big(\lambda^2 y\big) - \frac{(x-y)^2}{2t}\right)\\ &= \lambda^{-1}\sup_y \left(\sigma B(y) - \lambda\frac{\big(x-\lambda^{-2}y\big)^2}{2t}\right) = \lambda^{-1}\sup_y\left(\sigma B(y) - \frac{\big(\lambda^2 x - y\big)^2}{2t\lambda^3}\right)\\ &= \lambda^{-1}u\big(\lambda^2 x, \lambda^3 t\big),\end{aligned}$$

has the same law as u. By choosing $\lambda = t^{-1/3}$, we learn that the process $x \mapsto u(x,t)$, and $x \mapsto t^{1/3}u\big(t^{-2/3}x, 1\big) := t^{1/3}u\big(t^{-2/3}x\big)$ have the same law. Groenboom [G] discovered that the process $x \mapsto u_x(x,1) =: \rho(x)$ is linear motion that interrupted by random jumps. More specifically, it is a Markov process with a drift equals to -1, and a jump rate density of the form

$$f(\rho_-, \rho_+) = \frac{J(\rho_-)}{J(\rho_+)}K\big(\rho_+ - \rho_-\big), \qquad \rho_- < \rho_+, \tag{2.2}$$

where J and K are explicitly known; when $\sigma = 2^{-1/2}$, then their Laplace transforms

$$j(z) = \int_{-\infty}^{\infty} e^{-z\rho}J(\rho)\ d\rho, \qquad k(z) = \int_0^{\infty} e^{-z\rho}K(\rho)\ d\rho,$$

are given by

$$j(z) = Ai(z)^{-1}, \qquad k(z) = -2\frac{d^2}{dz^2}\log Ai(z),$$

with Ai denoting the Airy function defined by

$$Ai(z) = \frac{1}{\pi}\int_0^{\infty}\cos\left(\frac{\theta^3}{3} + x\theta\right)\ d\theta.$$

Moreover the process $u(x)$ is a stationary Markov process with an invariant measure that is explicitly given by

$$\pi(d\rho) = J(\rho)J(-\rho)\ d\rho. \tag{2.3}$$

One way to interpret the work of [G] is that if we regard the initial white noise data as a (singular) Markov process, then this Markov property persists under the evolution of Burgers' equation. In 2010, Menon and Srinivasan formulated a conjecture about the evolution of Markovian solutions of (1.2) when the Hamiltonian is convex and independent of (x,t). This conjecture was established in [KR1] and [KR2]. As was demonstrated in [OR], we may use the work of [KR2] to give a new proof of (2.2). This is carried out in two steps.

(1) The jump rate $f(t,\rho_-,\rho_+) = t^{-1/3} f\big(t^{1/3}\rho_-, t^{1/3}\rho_+\big)$ is compatible with the Burgers' equation.

(2) The law of the process $x \mapsto u(x,t)$ (or equivalently the process $x \mapsto t^{1/3}u\big(t^{-2/3}x\big)$) converges to the law of the Brownian motion $2^{-1/2}B$, in low t-limit.

In fact **(2)** is a central limit theorem (CLT) for the Markov process $u(x)$: For $\varepsilon = t^{-1/3}$,

$$\lim_{\varepsilon\to 0} u^\varepsilon(x) := \lim_{\varepsilon\to 0} \varepsilon u\big(\varepsilon^{-2}x\big) = \lim_{\varepsilon\to 0} \varepsilon \int_0^{\varepsilon^{-2}x} \rho(y)\ dy = 2^{-1/2}B(x). \tag{2.4}$$

As we will see, if the generator of the Markov process $\rho(x)$ is denoted by $\mathcal{L}$, then we can explicitly determine $\mathcal{L}^{-1}$. On the other hand, if $V(\rho) = \rho$, then V is of 0 average with respect to the invariant measure, and we can explicitly calculate the diffusion coefficient as we establish a central limit theorem for u^ε. Let us describe a general strategy for establishing (2.4):

(i) First we find a function w so that $-\mathcal{L}w = V$. In principle such w exists because $\int V\ d\pi = 0$. In terms of w, we can write

$$\varepsilon \int_0^{\varepsilon^{-2}x} V\big(\rho(y)\big)\ dy = M_\varepsilon(x) + \varepsilon w(\rho(0)) - \varepsilon w\big(\rho(\varepsilon^{-2}x)\big), \tag{2.5}$$

where $M_\varepsilon = M(\varepsilon^{-2}x)$, with $M(x)$ a martingale given by

$$M(x) = w(x) - w(0) - \int_0^x \mathcal{L}w\big(\rho(y)\big)\ dy.$$

For a reasonable solution w, we expect $\varepsilon w\big(\rho(\varepsilon^{-2}x)\big) \to 0$ in small ε limit. Hence the desired CLT for the left-hand side of (2.5) would follow if we establish a CLT for M^ε.

(ii) To establish a CLT for $\varepsilon M\big(\varepsilon^{-2}x\big)$, we need to control its variance. Using the martingale

$$N(x) := M(x)^2 - \int_0^x \big(\mathcal{L}w^2 - 2w\mathcal{L}w\big)(\rho(y))\ dy,$$

we can assert

$$\begin{aligned}\mathbb{E}M_\varepsilon(x)^2 = \mathbb{E}\varepsilon^2 M(\varepsilon^{-2}x)^2 &= \mathbb{E}\int_0^{\varepsilon^{-2}x} \big(\mathcal{L}w^2 - 2w\mathcal{L}w\big)(\rho(y))\ dy \\ &= x\int \big(\mathcal{L}w^2 - 2w\mathcal{L}w\big)\ d\pi = -x\int 2w\mathcal{L}w\ d\pi =: 2x\|w\|^2_{\mathcal{H}^1}.\end{aligned}$$

This allows us to evaluate the variance of the limit in (2.4). □

Going back to the Markov process $\rho(x)$, and the function $V(\rho) = \rho$, we have $w = (-\mathcal{L})^{-1}V = -J'/(2J)$, and

$$\begin{aligned} -2\int w\mathcal{L}w\ d\pi &= -Z^{-1}\int J'(\rho)J(-\rho)\rho\ d\rho \\ &= -Z^{-1}\int J(\rho)J'(-\rho)\rho\ d\rho + Z^{-1}\int J(\rho)J(-\rho)\ d\rho \\ &= Z^{-1}\int J(\rho)J'(-\rho)\rho\ d\rho + 1, \end{aligned}$$

where we performed an integration by parts to deduce the second equality. This implies

$$Z^{-1}\int J'(\rho)J(-\rho)\rho\ d\rho = 1/2.$$

Motivated with the Markovian description of Groeneboom in the case of quadratic Hamiltonian, we wish to find a general scheme for finding Markovian solutions of (1.2). Let us first observe that in the case of Burgers' equation, if we write $I(t)$ for the smallest set for which

$$u(x,t) = \max_{y\in I(t)}\left(\sigma B(y) - \frac{(x-y)^2}{2t}\right), \tag{2.6}$$

holds, the $I(t)$ is discrete set for every $t > 0$, and $I(t) \subseteq I(s)$ whenever $s \le t$. This means that as soon as t becomes positive, all but a discrete set of y are redundant in the maximization problem (2.1). Moreover, as we increase time, more points become redundant in (2.1). From this interpretation it is not hard to guess that the same phenomenon occurs even when H is a general Hamiltonian function. In other words there would be a monotonically nonincreasing family of discrete sets $(I(t):\ t > 0)$ such that

$$u(x,t) = \max_{y\in I(t)}\left(\sigma B(y) + tL\left(\frac{x-y}{t}\right)\right) = \max_{i\in\mathbb{Z}}\left(g_i(t) + tL\left(\frac{x-y_i(t)}{t}\right)\right), \tag{2.7}$$

where $g_i(t) = \sigma B(y_i(t))$, and $I(t) = \big\{y_i(t):\ i\in\mathbb{Z}\big\}$, with $y_i(t) < y_{i+1}(t)$ for every $i\in\mathbb{Z}$. As a consequence, there exists a discrete set $\big\{x_i(t):\ i\in\mathbb{Z}\big\}$ such that $x_i \le x_{i+1}$ for each i, and

$$u(x,t) = \sum_i \left(\sigma B(y_i(t)) + tL\left(\frac{x-y_i(t)}{t}\right)\right), \mathbb{1}\big(x\in[x_i(t),x_{i+1}(t))\big).$$

As for our Markov process ρ, we have

$$\rho(x,t) = M(x,t;y(x,t)) = \sum_i M(x,t;y_i(t))\mathbb{1}\big(x\in[x_i(t),x_{i+1}(t))\big), \tag{2.8}$$

where $M(x,t;y) = L'\left(\frac{x-y}{t}\right)$. The process $x \mapsto \rho(x,t)$, or equivalently the process $x \mapsto y(x,t)$ is a Markov process for every t. Indeed the latter is simply a non-decreasing jump process with jump rate density $g(x,t,y_-,y_+)$. In this context we can formulate the following general theorem that was established in [R2]:

Theorem 2.1 *Assume that the kernel $g(x,t,y_-,y_+)$ satisfies the following (kinetic) equation:*

$$g_t - (\hat{v}g)_x = Q(g) = Q^+(g) - Q^-(g) = Q^+(g) - gL(g), \tag{2.9}$$

where

$$\begin{aligned}
v(x,t,y_-,y_+) &= \frac{H\big(M(x,t;y_+)\big) - H\big(M(x,t;y_-)\big)}{M(x,t;y_+) - M(x,t;y_-)}, \\
Q^+(g)(y_-,y_+) &= \int \big(v(y_*,y_+) - v(y_-,y_*)\big) g(y_-,y_*) g(y_*,y_+)\ dy_*, \\
L(g)(y_-,y_+) &= \big(A(vg)(y_+) - A(vg)(y_-)\big) - v(y_-,y_+)\big(A(g)(y_+) - A(g)(y_-)\big).
\end{aligned}$$

Here we have not displayed the dependence of our functions on (x,t) for a compact notation, and

$$A(h)(y) = \int_y^\infty h(y,y_*)\ dy_*.$$

If $\rho(x,s) = M(x,t;y^0(x)$, for some $s > 0$, ans for y^0 a Markov jump process associated with $g(x,s,y_-,y_+)$, then foe $t > s$, we have $\rho(x,t) = M(x,t;y(x,t))$, where $y(\cdot,t)$ is a Markov jump process associated with $g(x,t,y_-,y_+)$.

We end this section with some open questions

Open Questions(i) Ouaki [O] in 2022 has found an explicit formula for the law of the solution $\rho(x,t)$ when $\rho(x,0)$ is white noise, and $H(p)$ is an arbitrary C^2 convex function. It remains to be seen how Ouaki's formula is compatible with the kinetic equation of [MS] and [KR2].

(ii) Let $B(x),\ x = (x_1,\dots,x_d)$ be a Brownian sheet. Consider the HJE $w_t = 2^{-1}w_{x_1}^2$ (or more generally $w_t = H(w_{x_1})$), with the initial condition $w(x,0) = B(x)$. Determine the law of the random field $x \mapsto w(x,t)$, for $t > 0$.

(iii) Let $B(x),\ x = (x_1,\dots,x_d)$ be as in **(ii)**. Consider the HJE $u_t = 2^{-1}|u_x|^2$ (or more generally $u_t = H(u_x)$), with the initial condition $u(x,0) = B(x)$. Determine the law of the random field $x \mapsto u(x,t)$, for $t > 0$. □

Note the questions **(ii)** and **(iii)** are closely related. For example, when $d = 2$, then

$$u(x_1,x_2,t) = \sup_{y_1,y_2}\left(B(y_1,y_2) - \frac{(x_1-y_1)^2}{2t} - \frac{(x_1-y_1)^2}{2t}\right) = \sup_{y_2}\left(w(x_1,y_2,t) - \frac{(x_1-y_1)^2}{2t}\right).$$

3 Inhomogeneous Stationary Hamiltonian

We now assume that the Hamiltonian function is a random stationary process in (x,t). Regarding u as a height function in a stochastic growth model, we wish to find the invariant measures. By invariant measures we mean a family of solutions $\{u^P : P \in \mathbb{R}^d\}$, such that the following conditions are met for $\rho^P = u_x^P$:

- The process $x \mapsto \rho(x,t)$ is stationary and ergodic with respect to the spatial translation, and $\mathbb{E}\rho(x,t) = P$.
- The law of the process $\rho(\cdot, t)$ is independent of t.

Example 3.1 In the case of PNG model of Example 1.2**(iii)**, let us assume that Poisson point process ω is of intensity 2. For $P = m - m^{-1}$, with $m > 0$, we have the following candidate for ρ^P: Take two independent Poisson point processes $\{a_i : i \in \mathbb{Z}\}$ and $\{b_i : i \in \mathbb{Z}\}$ of intensities m and m^{-1} respectively. Then an initial data

$$\rho_m(x,s) = \sum_{i \in \mathbb{Z}} \left(\delta_{a_i} - \delta_{b_i}\right),$$

would yield a solution $\rho_m(x,t),\ t > s$, which is our candidate for ρ^P. □

Example 3.1 offers explicit invariant measures when H is as in Example 1.2, with $H_0(p) = |p|$. Bakhtin et al. [BCK] have constructed (non explicit) invariant measures when $H_0(p) = p^2/2$. Even though their recipe for invariant measures is expected to work for general Hamiltonian functions, their arguments use $H_0(p) = p^2/2$ in an essential way. The existence/construction of invariant measures for general Hamiltonian remains open. We now briefly describe [BCK] recipe for ρ^P.

Definition 3.1(i) By a geodesic from (y,s) to (x,t), we mean a sequence $\mathbf{z}$ as in (**??**) which maximizes in (**??**). Given such a sequence, we can also construct a piecewise function $x : [s,t] \to \mathbb{R}$ that passes through the points of the sequence $\mathbf{z}$.

(ii) By a semi-infinite geodesic, we mean a path $x : (-\infty, t] \to \mathbb{R}$ such that the restriction of x to any interval $[s,t],\ s < t$ is a geodesic. □

The following theorem was established in [BCK] when $H_0(p) = p^2/2$.

Theorem 3.1 **(i)** *If $x(\cdot)$ is a semi-infinite geodesic, then the asymptotic velocity*

$$v = \lim_{t \to -\infty} t^{-1} x(t) \in [-\infty, \infty],$$

exists.

(ii) *Given v, almost surely, for every (x,t), there exists at least one semi-infinite geodesic $\xi : (-\infty, t] \to \mathbb{R}$, such that $\xi(t) = x$.*

(ii) *Given (v,x), there exists a unique semi-infinite geodesic $\xi : (-\infty, 0] \to \mathbb{R}$, such that $\xi(0) = x$.*

With the aid of semi-infinite geodesics, we can define a random function $\rho(x;v) = L_0'(\xi(0;x,v))$, where $\xi(\cdot;x,v) : (-\infty,0] \to \mathbb{R}$ is a semi-infinite geodesic with asymptotic velocity v such that $\xi(0;x,v) = x$. Except for countably many points, this semi-infinite geodesic is unique i.e., $\rho(x;v)$ is single-valued except for a discrete set of x's. In the case of $H_0(p) = p^2/2$, we have $L_0'(a) = -a$, and $\rho^P(x) = \rho(x;P)$. In the general case, the relationship between P and P is more complicated. There would exist a *homogenized Hamiltonian* $\bar{H}$ such that $v = \bar{H}'(P)$. What plays a role in [BCK] is that $\bar{H}$ is C^2 and uniformly convex. This is not known in general. However, when $H_0(p) = p^2/2$, then $\bar{H}(p) = p^2/2 + c$ for a suitable constant c.

We can also construct Markovian solutions analogous to (3.1) when the Hamiltonian H is as in Example 1.2**(ii)**. To explain this, let us write $M(x,t;y,s), t \geq s$ for $W_x(x,t;y,s)$, where W is defined by (1.10). We are interested in Markov processes ρ of the form

$$\rho(x,t) = M(x,t;y(x,t),t_0) = \sum_i M(x,t;y_i(t),t_0)\mathbb{1}\big(x \in [x_i(t), x_{i+1}(t))\big). \tag{3.1}$$

The process $x \mapsto y(x,t)$ a non-decreasing jump process with jump rate density $g(x,t,y_-,y_+)$ as before. We have the following general theorem that appears in [R2]:

Theorem 3.2 *Assume that the kernel $g(x,t,y_-,y_+)$ satisfies the kinetic equation* (2.9) *with*

$$v(x,t,y_-,y_+) = \frac{H\big(x,t,M(x,t;y_+)\big) - H\big(x,t,M(x,t;y_-)\big)}{M(x,t;y_+) - M(x,t;y_-)}.$$

If $\rho(x,s) = M(x,t;y^0(x)$, for some $s > t_0$, and for y^0 a Markov jump process associated with $g(x,s,y_-,y_+)$, then foe $t > s$, we have $\rho(x,t) = M(x,t;y(x,t))$, where $y(\cdot,t)$ is a Markov jump process associated with $g(x,t,y_-,y_+)$.

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