Kinetic statistics of scalar conservation laws with piecewise-deterministic Markov process data

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Abstract

In 2010 Menon and Srinivasan published a conjecture for the statistical structure of solutions ρ to scalar conservation laws with certain Markov initial conditions, proposing a kinetic equation that should suffice to describe $\rho(x,t)$ as a stochastic process in x with t fixed, or as a stochastic process in t with x fixed. In this article we largely resolve this conjecture.

1 Introduction

In this article we show the statistics of $\rho(x,t)$ solving the scalar conservation law

$$\begin{cases} \rho_t = H(\rho)_x & \text{in } \mathbb{R}\times(0,\infty), \\ \rho = \rho^0 & \text{in } \mathbb{R}\times\{0\}, \end{cases} \tag{1.1}$$

admits an exact kinetic description when the initial data $\rho^0 = \rho^0(x)$ is a piecewise-deterministic Markov process (PDMP), determined by a generator $\mathcal{A}^0$ acting on test functions $\psi(p)$ according to

$$(\mathcal{A}^0\psi)(p) = b^0(p)\psi'(p) + \int_p^\infty \big(\psi(p_+) - \psi(p)\big)\, f^0(p,p_+)\, dp_+. \tag{1.2}$$

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The random path $\rho^0(x)$ may be constructed by solving (deterministically) the ODE $d\rho^0/dx = b^0(\rho^0)$, interrupted by jumps which occur stochastically: the probability that ρ^0 makes a jump in the short interval $(x, x+dx)$ is

$$(1.3) \qquad \left(\int_p^\infty f^0(\rho^0(x), p_+)\, dp_+ \right) dx + O((dx)^2),$$

and the new value of ρ^0 following the jump is selected with probability density proportional to $p_+ \mapsto f^0(\rho^0(x), p_+)$.

We largely resolve a conjecture of Menon and Srinivasan [MS], and extend our own results [KR] in the case without drift ($b = 0$), verifying that the process $x \mapsto \rho(x,t)$ (for fixed $t > 0$) is again a PDMP, with generator

$$(1.4) \qquad \big(\mathcal{A}^t \psi\big)(p) = b(p,t)\psi'(p) + \int_p^\infty \big(\psi(p_+) - \psi(p)\big)\, f(p, p_+, t)\, dp_+.$$

Here $b(p,t)$ and $f(p_-, p_+, t)$ are obtained from their initial ($t = 0$) conditions

$$(1.5) \qquad b(p,0) = b^0(p), \qquad f(p_-, p_+, 0) = f^0(p_-, p_+),$$

by solving an ODE with parameter,

$$(1.6) \qquad b_t(p,t) = H''(p) b(p,t)^2,$$

and a kinetic (integro-)PDE

$$(1.7) \qquad f_t = Q(f,f) + C(f),$$

where $Q(f,f) = Q^+(f,f) - Q^-(f,f)$ is a coagulation-like collision operator and C is a linear first order differential operator. More precisely,

(i) The quadratic operator Q^+ is defined as

$$(1.8) \qquad Q^+(f,f)(\rho_-, \rho_+) := \int_{\rho_-}^{\rho_+} \big(H(\rho_*, \rho_+) - H(\rho_-, \rho_*)\big) f(\rho_-, \rho_*) f(\rho_*, \rho_+)\, d\rho_*,$$

where

$$H(p_1, p_2) = \frac{H(p_2) - H(p_1)}{p_2 - p_1}.$$

(ii) The quadratic operator Q^- is of the form $Q^-(f,f) = fLf$, for a linear operator L. This linear operator is defined as

$$(1.9) \qquad (Lf)(\rho_-, \rho_+) = L(\rho_-, \rho_+) := A(\rho_+) - A(\rho_-) - H(\rho_-, \rho_+)\big(\lambda(\rho_+) - \lambda(\rho_-)\big),$$

where

$$\lambda(\rho_-) = \lambda(f)(\rho_-) = \int_{\rho_-}^{\infty} f(\rho_-, \rho_+)\, d\rho_+,$$

$$A(\rho_-) = A(f)(\rho_-) = \int_{\rho_-}^{\infty} H(\rho_-, \rho_+) f(\rho_-, \rho_+)\, d\rho_+.$$

(iii) Given a C^1 kernel f, we define the linear operator C by

$$\begin{aligned}(Cf)(\rho_-, \rho_+) = {} & b(\rho_-, t) H_{\rho_-}(\rho_-, \rho_+) f(\rho_-, \rho_+) \\ & + \left[H(\rho_-, \rho_+) - H'(\rho_-)\right] b(\rho_-, t) f_{\rho_-}(\rho_-, \rho_+) \\ & + \left[\big(H(\rho_-, \rho_+) - H'(\rho_+)\big) b(\rho_+, t) f(\rho_-, \rho_+)\right]_{\rho_+}.\end{aligned} \tag{1.10}$$

Here and below, by the expression X_a we mean the partial derivative of X with respect to the variable a. For example the last term on the right-hand side of (1.10) represents the partial derivative of the expression in brackets with respect to ρ_+.

Remark 1.1 As in [MS], we may write the operator C in a more symmetric way:

$$\begin{aligned}(Cf)(\rho_-, \rho_+) = {} & \left[b(\rho_-, t) H''(\rho_-) - \big(H(\rho_-, \rho_+) - H'(\rho_-)\big) b_{\rho_-}(\rho_-, t)\right] f(\rho_-, \rho_+) \\ & + \left[\big(H(\rho_-, \rho_+) - H'(\rho_-)\big) b(\rho_-, t) f(\rho_-, \rho_+)\right]_{\rho_-} \\ & + \left[\big(H(\rho_-, \rho_+) - H'(\rho_+)\big) b(\rho_+, t) f(\rho_-, \rho_+)\right]_{\rho_+}.\end{aligned}$$

Though it is the expression (1.10) that will appear more naturally in our calculations as we derive the equation (1.7). □

1.1 Motivation: Burgers turbulence

In the particular case $H(p) = -p^2/2$, (1.1) reduces to Burgers equation [Bu]. The field of study concerned with Burgers equation and random initial data or stochastic forcing is known as *Burgers turbulence*. Among the motivations for continued investigation in this area is the desire to confront, in a simpler setting, the delicate interplay between nonlinear dynamics and statistical structure that arise in genuine turbulence [VF]. Significant recent advances [DS], [I] on the PDE side underscore the need for continued effort on the statistical side.

Among those works in the Burgers context, those most closely related to our own are the following:

- Groeneboom [Gr] determined the statistics of solutions to Burgers equation with white noise initial data. Burgers equation is not explicitly mentioned—the paper is rather concerned with asymptotic behavior of nonparametric estimators, and discusses convex minorants of Brownian motion with parabolic drift—but these problems are connected by the Hopf-Lax-Oleinik solution formula and the Legendre transform.

- Sinai [S] and Aurell, Frisch, She [AFS] considered Burgers equation with Brownian motion initial data, relating the statistics of solutions to convex hulls and addressing pathwise properties, such as the almost-sure Hausdorff dimension of the set where the derivative of the convex hull grows. In the same setting, Avellaneda and E [AE] showed the Markov property (in space) of the initial data is preserved forward in time.
- Carraro and Duchon [CD1-2] considered *statistical* solutions, which need not coincide with genuine (entropy) solutions, but realized in this context that Lévy process initial data (of which Brownian motion is an example) should interact nicely with Burgers equation. Bertoin [Be] showed this intuition was correct on the level of entropy solutions, arguing in a Lagrangian style and using Getoor's [Ge] notion of splitting times.

Developing an alternative treatment to that given by Bertoin, which relies less on particulars of Burgers equation and happens to be more Eulerian, was among the goals of [MS], [KR], and the present work.

1.2 Motivation: A solvable model in kinetic theory

The operator $Q(f, f)$ in the kinetic equation (1.7) is, as we will see, closely related to the Smoluchowski coagulation equation, a model for mean-field binary coalesce such as one observes in aerosols. Indeed, in the case of Burgers equation with Lévy initial data, it is exactly the Smoluchowski coagulation equation with additive rate which determines the jump statistics [Be], [MP].

The typical situation, for Smoluchowski [HR] and other kinetic equations [RV], is that we have some (stochastic or deterministic) dynamics defined on a *finite* system, and these kinetic equations emerge upon passage to a scaling limit. The dynamics might not be definable for the infinite system, and the kinetic equation should describe statistics only approximately for a large but finite system. In the setting of [Be], [MS], [KR], and the present work, the kinetic equations give statistics *exactly* without passage to a rescaled limit. We view this unusual circumstance as demanding an explanation. Further, our treatment (tracking shocks as inelastically colliding particles) seems quite at home in the kinetic context.

1.3 Motivation: Integrability

The evolution of the (spatial) generator $\mathcal{A}^t$ implied by (1.6) and (1.7), can be expressed as a Lax pair

$$\frac{d}{dt}\mathcal{A}^t = [\mathcal{A}^t, \mathcal{B}^t] = \mathcal{A}^t\mathcal{B}^t - \mathcal{B}^t\mathcal{A}^t \tag{1.11}$$

where

$$(1.12)\qquad (\mathcal{B}^t\psi)(p) = H'(p)b(p,t)\psi'(p) + \int_p^\infty \big(\psi(p_+) - \psi(p)\big)H(p,p_+)f(p,p_+,t)\,dp_+,$$

where $H(p,p_+) = (H(p) - H(p_+))/(p - p_+)$. In the pure-jump case (drift $b = 0$), and when the initial data $\rho^0(x)$ is allowed to assume values only in a fixed, finite set of states, the operators $\mathcal{A}^t$ and $\mathcal{B}^t$ in (1.11) can be represented by triangular matrices. The integrability of this matrix evolution has been investigated by Menon [M2] and Li [Li]. For generic matrices—where the genericity assumptions unfortunately exclude the triangular case—this evolution is completely integrable in the Liouville sense. Though the triangular case technically fails to be Liouville integrable, much can still be said; the evolution is conjugate to straight-line motion through an appropriate change of variables.

1.4 Main Result

In this section we provide a statistical description of solutions to the scalar conservation law when the initial condition is a piecewise-deterministic Markov process (PDMP) with drift b^0 and jump rate kernel f^0. For this we require some assumptions on the initial rate kernel f^0 and the Hamiltonian H.

Hypothesis 1.1(i) The initial condition $\rho^0 = \rho^0(x)$ is 0 for $x < 0$, and is a Markov process for $x \geq 0$ that starts at $\rho^0(0) = 0$. This Markov process has an infinitesimal generator in the form (1.2) for a drift b^0.

(ii) The rate kernel $f^0(p_-, p_+)$ is C^1 and is supported on

$$\{(p_-, p_+) : P_- \leq p_- \leq p_+ \leq P_+\},$$

for some constants $P_\pm$.

(iii) The Hamiltonian function $H : [P_-, P_+] \to \mathbb{R}$ is C^2, convex, has positive right-derivative at $p = P_-$ and finite left-derivative at $p = P_+$.

(iii) The initial drift b^0 is C^1 and satisfies $b^0 \leq 0$ with $b^0(\rho) = 0$ whenever $\rho \notin [P_-, P_+]$. □

Our statistical description consists of a one-dimensional marginal, a drift, and a rate kernel generating the rest of the path. The evolution of the drift and the rate kernel are given by (1.6) and the kinetic equation (1.7). Evolution of the marginal will be described in terms of the solutions to these equations. We continue with some definitions.

Definition 1.1(i) We write $\mathcal{B}^{t*}$ for the adjoint of the operator $\mathcal{B}^t$ that acts on measures. More precisely, for a probability measure ν, we have

$$\big(\mathcal{B}^{t*}\nu\big)(d\rho) = \left[\int_{-\infty}^{\rho} H(\rho_*, \rho)f(\rho_*, \rho)\ \nu(d\rho_*)\right]\ d\rho - A(f)(\rho)\ \nu(d\rho) - \frac{d}{d\rho}\big(H'(\rho)b(\rho,t)\nu(d\rho)\big),$$

with the last term is interpreted in weak sense. When the measure ν is absolutely continuous with respect to the Lebesgue measure with a C^1 Radon-Nykodym derivative, then $\mathcal{B}^{t*}\nu$ is also absolutely continuous with respect to the Lebesgue measure. The action of the operator $\mathcal{B}^{t*}$ on ν can be described in terms of its action on the corresponding Radon-Nykodym derivative. By slight abuse of notation, we write $\mathcal{B}^{t*}$ for the corresponding operator that now acts on C^1 functions. In other words, when $\nu(d\rho) = \bar{\nu}(\rho)\ d\rho$, then $\mathcal{B}^{t*}\nu = (\mathcal{B}^{t*}\bar{\nu})\ d\rho$, with

$$\big(\mathcal{B}^{t*}\bar{\nu}\big)(\rho) = \int_{-\infty}^{\rho} H(\rho_*,\rho)f(\rho_*,\rho)\bar{\nu}(\rho_*)\ d\rho_* - A(f)(\rho)\bar{\nu}(\rho) - \frac{d}{d\rho}\big(H'(\rho)b(\rho,t)\bar{\nu}(\rho)\big).$$

(ii) We write $\mathcal{M}$ for the set of measures and $\mathcal{M}_1$ for the set of probability measures. □

Theorem 1.1 *Under Hypothesis 1.1, the kinetic equation* (1.7) *has a unique* C^1 *solution subject to the initial condition* $f(p_-,p_+,0) = f^0(p_-,p_+)$. *Moreover, given a* C^1 *rate* f, *there exists a unique* $\ell^c : [0,\infty) \to \mathcal{M}_1$ *such that* $\ell^c(d\rho,0) = \delta_c(d\rho)$, *and*

$$\frac{d\ell^c}{dt}(d\rho,t) = \big(\mathcal{B}^{t*}\ell^c(\cdot,t)\big)(d\rho,t). \tag{1.13}$$

The kernels described by Theorem 1.1 are precisely what we need to describe the statistics of the solution ρ, which brings us to our main result:

Theorem 1.2 *When Hypothesis 1.1 holds, the entropy solution* ρ *to* (1.1) *for each fixed* $t > 0$ *has* $x = 0$ *marginal given by* $\ell^0(d\rho_0,t)$ *and for* $0 < x < \infty$ *evolves according to a Markov process with the generator* $\mathcal{A}^t$. *Moreover, the process* $t \mapsto \rho(a,t)$ *is an inhomogeneous Markov process with generator* $\mathcal{B}^t$, *for every* $a \geq 0$.

Remark 1.2(i) According to Hypothesis 1.1**(ii)**, the function H is increasing. This restriction on H can be relaxed almost completely. The main role of the condition $H' > 0$ is that all shock discontinuities of ρ travel with negative velocity so that they cross any fixed location, say $x = a$ eventually. This allows us to assert that if $\rho(a,t)$ is known, then the law of $\rho(x,t)$ can be determined uniquely for all $x > a$. We are doing this for all $t > 0$. In general, we may try to determine $\rho(x,t)$ for $x > a(t)$, provided that $\rho(a(t),t)$ is specified. The condition $H' > 0$, allows us to choose $a(t)$ constant. If instead we can find a negative constant c such that $H'(\rho) > c$, then $\hat{\rho}(x,t) := \rho(x-ct,t)$ satisfies

$$\hat{\rho}_t = \hat{H}(\hat{\rho})_x,$$

for $\hat{H}(\rho) = H(\rho) - c\rho$, which is increasing. Hence, the process $t \mapsto \hat{\rho}(x,t) = \rho(x-ct,t)$ is now Markovian with a generator $\hat{\mathcal{B}}^t$ that we obtain from $\mathcal{B}^t$ by replacing H with $\hat{H}$. Even an upper bound on H' can lead to a result similar to Theorem 1.2. For example if $H' < 0$, then $x \mapsto \rho(x,t)$ is a Markov process but now as we decrease x.

(ii) The condition $\rho \in [P_-, P_+]$ is used only in Theorem 1.1, which guarantees the existence of a unique classical solution to (1.7).

(iii) If we drop the assumption $b \le 0$, then Theorem 1.2 is still valid so long as b stays finite.

(iv) We refer to [R] for more heuristics and discussions about Theorem 1.2. Most notably, it is shown in [R] that one may arrive at the equation (1.7) by taking an initial condition with only two jump discontinuities! The reader may take this derivation of (1.7) as a heuristic explanation for the very form of the operators Q and C in (1.8)-(1.10). □

We continue with an outline of the paper:

(i) In Section 2, we show that the evolution of the PDE (1.1) for piecewise smooth solutions is equivalent to a particle system in $\mathbb{R} \times [P_-, P_+]$. We restrict this particle system to a large finite interval $[0, L]$ and introduce a stochastic boundary condition at L. This restriction allows us to reduce our main result to a finite system; the precise statement can be found in Theorem 2.1 of Section 2.

(ii) The strategy of the proof of Theorem 2.1 will be described in Section 3. Our strategy is similar to the one that was utilized in our previous work [KR]: Since we have a candidate for the generator of the process $x \mapsto \rho(x,t)$, we have a candidate measure, say $\mu(\cdot, t)$ for the law of $\rho(\cdot, t)$. We establish Theorem 2.1 by showing that this candidate measure satisfies the *forward equation* associated with Markovian dynamics of the underlying particle system (see the equation (3.4) in Section 3). The particle system has a deterministic evolution inside the interval and a stochastic (Markovian) dynamics at the right end boundary point.

(iii) The rigorous derivation of the forward equation will be carried out in Section 4.

(iv) Section 5 is devoted to the proof of Theorem 2.1.

(v) In Section 6, we show that the equation (1.7) has a unique classical solution. □

We are now in a position to compare the proof that was carried out in [KR] when $b = 0$, with the proof we provide in the present paper:

- When $b = 0$, the velocity $\mathbf{v}$ of the particle configuration is constant and the dynamics inside the interval $[0, L]$ can be recast as a billiard. The rigorous verification of the forward equation (3.4) was achieved in [KR] by comparing the billiard domain with its translation in the direction of the velocity $\mathbf{v}$. Theorem 4.1 in Section 4 offers a more robust approach for rigorous verification of the forward equation that would work even when the velocity $\mathbf{v}$ in a billiard-type model changes with time, space and density. This non-constant feature of the velocity is responsible of the emergence of the first order operator C in (1.7).

- Theorem 4.1 of Section 4, reduces the proof of the main theorem to an identity, namely the forward equation (3.4) of Section 3. The verification of this equation when $b = 0$ is rather straightforward. When b is nonzero, the verification of (3.4) is significantly more involved and requires various identities related to the integro-differential equation (1.7) and the flow of the vector field b. These identities are collected in Lemma 2.1. We also use Proposition 5.1 to organize the left-hand side of the forward equation as a sum of 9 terms.

2 Particle System

Let us assume that the initial condition ρ^0, in the PDE (1.1) is of the following form

- $\rho^0(x) = 0$ for $x \le 0$.
- There exists a discrete set $I^0 = \{x_i : i \in \mathbb{N}\}$, with $0 < x_1 < \cdots < x_i < \dots$ such that for every $x > 0$ with $x \notin I^0$, we have $\rho^0_x(x) = b^0(\rho^0(x))$. Here by ρ^0_x denotes the derivative of ρ^0 with respect to its argument x.
- If $\rho_i^\pm = \rho^0(x_i\pm)$ denote the right and left values of ρ^0 at x_i, then $\rho_i^- < \rho_i^+$.

Now if ρ is an entropic solution of (1.1) with initial ρ^0, then we may apply the *method of characteristics* to show that for each $t \ge 0$, the function $\rho(\cdot, t)$ has a similar form. More precisely, there are pairs $\mathbf{q}(t) = \big((x_i(t), \rho_i(t)) : \ i = 0, 1, \dots\big)$, with

$$0 = x_0(t) < x_1(t) < \cdots < x_i(t) < \dots,$$

such that $\rho(x_i(t)+, t) = \rho_i(t)$ and that for $x > 0$ and $x \ne x_i(t)$ for $i \in \mathbb{N}$, we have

$$\rho_x(x, t) = b(\rho(x, t), t), \tag{2.1}$$

where b is the solution to (1.6), subject to the initial condition $b(x, 0) = b^0(x)$. Because of (2.1), the data $\mathbf{q}(t)$ determines $\rho(\cdot, t)$ completely. To explain this, let us write $\phi_z(m; t)$ for the flow of the ODE (2.1). More precisely, if $\rho(x) = \phi_x(m; t)$, then $\rho_x(x) = b(\rho(x), t)$, and $\rho(0) = m$. Then

$$\rho\big(x, t\big) = \sum_{i=0}^{\infty} \phi_{x - x_i(t)}\big(\rho_i(t); t\big) \mathbb{1}\big(x_i(t) \le x < x_{i+1}(t)\big), \tag{2.2}$$

for $x \ge 0$. Because of this, we can fully describe the evolution of $\rho(\cdot, t)$ by describing an evolution of the particle system $\mathbf{q}(t)$. Indeed from the PDE (1.1) and celebrated Rankine-Hugoniot Formula, we have

$$\dot{x}_i = -H(\hat{\rho}_{i-1}, \rho_i), \qquad \dot{\rho}_0 = H'(\rho_0) b(\rho_0, t), \qquad \dot{\rho}_i = \big(H'(\rho_i) - H(\hat{\rho}_{i-1}, \rho_i)\big) b(\rho_i, t), \tag{2.3}$$

for $i \in \mathbb{N}$, where $\hat{\rho}_{i-1}(t) = \phi_{x_i - x_{i-1}}(\rho_{i-1}(t), t)$. Here by $\dot{f}$ we mean the time derivative of the function f with respect to t, and we regard (2.3) as a system of ODEs. We note that (2.3) gives a complete description of $\mathbf{q}$ in an inductive fashion; once (x_{i-1}, ρ_{i-1}) is determined, then we use (2.3) to write a system of two equations for the pair (x_i, ρ_i). Moreover (2.3) holds so long as x_i's do not collide. When there is a collision between x_i and x_{i+1}, for some $i = 0, 1, \dots$, we remove x_{i+1} from the system, replace ρ_i with ρ_{i+1}, and relabel (x_j, ρ_j) as (x_{j-1}, ρ_{j-1}) for $j > i + 1$.

Proposition 2.1 **(i)** *The function $\rho(x,t)$, defined by equation (2.2), with $\mathbf{q}(t)$ evolving as above, is the unique entropy solution of (1.1) for $x, t \geq 0$.*

We do not prove Proposition 2.1 because a variant of it will be proved below as Proposition 2.2.

According to Theorem 1.2 if $\rho(\cdot, 0)$ is a PDMP with drift b^0 and jump rate f^0, then $\rho(\cdot, t)$ is also a PDMP with drift $b(\cdot, t)$ and $f(\cdot, \cdot, t)$. We may translate this as a statement about the law of our particle system $\mathbf{q}(t)$. However, since the dynamics of $\mathbf{q}$ are infinite dimensional (involves infinite number of particles to the right of the origin), we may take advantage of the finiteness of propagation speed in (1.1) and reduce Theorem 1.2 to an analogous claim for a finite interval $[0, L]$.

Since $H' > 0$ by Hypothesis 1.1**(iii)**, all particles travel to left. Because of this, we need to choose appropriate boundary dynamics at the right boundary L only; the shocks and characteristics only flow outward across $x = 0$, and any boundary condition we would assign at $x = 0$, would thus be irrelevant. The involved analysis will all pertain to the following result.

Theorem 2.1 *Assume Hypothesis 1.1. For any fixed $L > 0$, consider the scalar conservation law*

$$(2.4) \qquad \begin{cases} \rho_t = H(\rho)_x & (x,t) \in (0, L) \times (0, \infty) \\ \rho = \rho^0 & x \in [0, L] \times \{t = 0\} \\ \rho = \zeta & (x,t) \in \{x = L\} \times (0, \infty) \end{cases}$$

with initial condition ρ^0 (restricted to $[0, L]$), open boundary at $x = 0$, and random boundary ζ at $x = L$. Suppose the process ζ has $\zeta(0) = \rho^0(L)$ and evolves according to the time-dependent rate kernel $H(\rho_,\rho_+) f(\rho_,\rho_+, t)$ and drift $b(\rho, t) H'(\rho)$, independently of ρ^0 (given $\rho^0(L)$). Then for all $t > 0$ and $a \in [0, L)$, the law of $\big(\rho(x,t) : \ x \in [a, L]\big)$ is as follows:

- **(i)** *The $x = a$ marginal is $\ell^c(d\rho_0, t)$, for $c = \rho^0(a)$.*
- **(ii)** *The rest of the path is a PDMP with generator $\mathcal{A}^t$ (rate kernel $f(\rho_-, \rho_+, t)$ and drift $b(\rho, t)$).*

To prove our main result Theorem 1.2, we can send $L \to \infty$, applying Theorem 2.1 on each $[0, L]$, and use bounded speed of propagation to limit the respective influences of far away particles (unbounded system) or truncation with random boundary (bounded system). The argument is quite short and can be found in [KR].

We prove Theorem 2.1 by showing that the particle system $\mathbf{q}(t)$ restricted to the interval $[0, L]$ has the correct law predicted by this theorem. For this we have two tasks at hand:

- **(i)** Give a precise description for the evolution of $\mathbf{q}$ restricted to $[0, L]$.
- **(ii)** Give a precise description of the law of $\mathbf{q}(t)$, when the corresponding $x \mapsto \rho(x,t)$ is a Markov process with generator $\mathcal{A}^t$.

To carry out our first task, let us make some definitions.

Definition 2.1(i) The configuration space for our particle system $\mathbf{q}$, is the set

$$\Delta_L = \cup_{n=0}^{\infty} \bar{\Delta}_n^L,$$

where $\bar{\Delta}_n^L$ is the topological closure of Δ_n^L, with Δ_n^L denoting the set

$$\big\{\mathbf{q} = \big((x_i, \rho_i) : i = 0, 1, \dots, n\big) : \; x_0 = 0 < x_1 < \cdots < x_n < x_{n+1} = L, \quad \rho_0, \dots, \rho_n \in \mathbb{R}\big\}.$$

We write $\mathbf{n}(\mathbf{q})$ for the number of particles i.e., $\mathbf{n}(\mathbf{q}) = n$ means that $\mathbf{q} \in \Delta_n^L$. What we have in mind is that $\rho_i(t) = \rho(x_i(t)+, t)$ with $x_1, \dots, x_n$ denoting the locations of all shocks in $(0, L)$.

(ii) Given a realization $\mathbf{q} = \big(0, \rho_0, x_1, \rho_1, \dots, x_n, \rho_n\big) \in \bar{\Delta}_n^L$, we define

$$\rho\big(x, t; \mathbf{q}\big) = R_t(\mathbf{q})(x) = \sum_{i=0}^{n} \phi_{x - x_i}\big(\rho_i; t\big) 1\!\!1\big(x_i \le x < x_{i+1}\big),$$
$$\hat{\rho}_{i-1}(t) = \rho(x_i(t)-, t; \mathbf{q}(t)) = \phi_{x_i(t) - x_{i-1}(t)}(\rho_{i-1}(t); t).$$

(iii) The process $\mathbf{q}(t)$ evolves according to the following rules:

- **(1)** So long as x_i remains in (x_{i-1}, x_{i+1}), it satisfies

$$\dot{x}_i = -H(\hat{\rho}_{i-1}, \rho_i).$$

- **(2)** We have $\dot{\rho}_0 = H'(\rho_0) b(\rho_0, t)$, and for $i > 0$,

$$\dot{\rho}_i = \big(H'(\rho_i) - H(\hat{\rho}_{i-1}, \rho_i)\big) b(\rho_i, t).$$

(3) With rate

$$H(\hat{\rho}_n, \rho_{n+1}) f\big(\hat{\rho}_n, \rho_{n+1}, t\big),$$

the configuration $\mathbf{q}$ gains a new particle (x_{n+1}, ρ_{n+1}), with $x_{n+1} = L$. This new configuration is denoted by $\mathbf{q}(\rho_{n+1})$.

(4) When x_1 reaches the origin, we relabel particles $(x_i, \rho_i),\ i \geq 1$, as (x_{i-1}, ρ_{i-1}).

(5) When $x_{i+1} - x_i$ becomes 0, then $\mathbf{q}(t)$ becomes $\mathbf{q}^i(t)$, that is obtained from $\mathbf{q}(t)$ by omitting (ρ_i, x_i) and relabeling particles to the right of the i-th particle.

□

Remark 2.1(i) Recall that we expect the process $t \mapsto \rho(L, t)$ to be an inhomogeneous Markov process with generator $\mathcal{B}^t$. From the way the boundary dynamics is described in **(3)**, the process $t \mapsto \rho(L, t; \mathbf{q}(t)) =: m(t)$ may appear not exactly what we expected because of its dependence on the particle system to the right of L. Once an explicit construction of a process $t \mapsto \bar{m}(t)$, with generator $\mathcal{B}^t$ will be given below, it will be clear that indeed $m(t)$ is a realization of $\bar{m}(t)$; it may be regarded as as an inhomogeneous Markov process with infinitesimal generator $\mathcal{B}^t$ and initial condition $m(0) = \rho^0(L)$.

The process $\bar{m}$ with generator $\mathcal{B}^t$ may be realized with the aid of a sequence of independent standard exponential random variables $\big(\tau_i :\ i \in \mathbb{N}\big)$. Let us write $\beta_s^t(a)$ for the flow of the ODE associated with speed $\hat{b}(m, t) := H'(m) b(m, t)$. In other words, if $\bar{m}(t) = \beta_s^t(a)$, then

$$\frac{d}{dt}\bar{m}(t) = \hat{b}\big(\bar{m}(t), t\big), \qquad \bar{m}(s) = a.$$

We also set $g(\rho_-, \rho_+, t) = H(\rho_-, \rho_+) f(\rho_-, \rho_+, t)$, and

$$\eta(m, t) = \int_m^\infty g(m, \rho_+, t)\ d\rho_+.$$

Now construct a sequence $\mathbf{z} = \big((\sigma_i, m_i) :\ i = 0, 1, \dots\big)$ inductively by the following recipe:

- $m_0 = \rho^0(L)$, and $\sigma_0 = 0$.
- Given (σ_i, m_i), we set

$$\sigma_{i+1} = \min\left\{ s > \sigma_i :\ \int_{\sigma_i}^s \eta\big(\beta_{\sigma_i}^\theta(m_i), \theta\big)\ d\theta \geq \tau_{i+1} \right\},$$
$$\hat{m}_i = \beta_{\sigma_i}^{\sigma_{i+1}}(m_i).$$

- We select m_{i+1} randomly according to the probability measure

$$\eta\big(\hat{m}_i, \sigma_{i+1}\big)^{-1}\ g\big(\hat{m}_i, m_{i+1}, \sigma_{i+1}\big)\ dm_{i+1}.$$

Using our sequence $\mathbf{z}$, we construct $m(t)$ by

$$\bar{m}(t) = \sum_{i=0}^{\infty} \beta^t_{\sigma_i}(m_i) 1\!\!1\big(t \in [\sigma_i, \sigma_{i+1})\big).$$

By induction on i, we can readily show that if at time σ_i, there are exactly n particle to the left of L, then $\hat{\rho}_n = \hat{m}_{i-1}$. This is an immediate consequence of Proposition 2.1, namely, in between the jumps at $x = L$, the function $\zeta(t) = \phi_{L-x_n(t)}(\rho_n(t); t)$ satisfies

$$\dot{\zeta}(t) = H'(\zeta(t)) b(\zeta(t), t).$$

(See the proof of Proposition 2.2**(i)** below.) Hence the recipe we gave in **(3)** above is compatible with our expectation: the process $\rho(L, t)$ is a Markov process with generator $\mathcal{B}^t$.

(ii) A similar recipe may be used to construct a realization of a process generated by $\mathcal{A}^t$. Such a construction allows us to write down an explicit formula for the law of the corresponding process as we will see in Definition 2.2**(ii)** below. □

The following variant of Proposition 2.1 provides us with a stability of solutions to (2.4).

Proposition 2.2 **(i)** *The function $\rho(x, t) = \rho(x, t; \mathbf{q}(t))$, with $\mathbf{q}(t)$ evolving as above, is the unique entropy solution of* (2.4)*, with the boundary condition $\rho(L, t) = m(t)$.*

(ii) *If ρ and ρ' are entropy solutions of* (1.1) *in the interval $[0, L]$, and $s < t$, then*

$$\int_0^L |\rho'(x,t) - \rho(x,t)|\, dx \le \int_0^L |\rho'(x,s) - \rho(x,s)|\, dx + \int_s^t \big|H(\rho'(L,\theta)) - H(\rho(L,\theta))\big|\, d\theta. \tag{2.5}$$

The proof of Proposition 2.2 will be given at the end of this section. We now turn to our second task, namely a precise description for the PDMP $\rho(\cdot, t)$ in terms of $\mathbf{q}(t)$.

Definition 2.2(i) We set

$$\Gamma(\rho, x, t) = \int_0^x \lambda(\phi_y(\rho; t), t) dy$$

$$\Gamma(\mathbf{q}, t) = \int_0^L \lambda\big(\rho(y, t; \mathbf{q})\big)\, dy = \sum_{i=0}^{n} \Gamma(\rho_i, x_{i+1} - x_i, t),$$

(ii) We define a measure $\mu(d\mathbf{q},t)$ on the set Δ_L that is our candidate for the law of $\mathbf{q}(t)$. The restriction of μ to Δ_n^L is denoted by $\mu^n(d\mathbf{q},t)$. This measure is explicitly given by

$$\ell(d\rho_0,t)\exp\left\{-\Gamma(\mathbf{q},t)\right\}\prod_{i=1}^{n} f\left(\phi_{x_i-x_{i-1}}(\rho_{i-1};t),\rho_i,t\right)\,dx_i d\rho_i,$$

where f solves (1.7) and ℓ solves (1.13).

(iii) When the kernel f depends on t, we write $\lambda(\rho_-,t)$, $A(\rho_-,t)$, $L(\rho_-,\rho_+,t)$, $Q(f,f)(\rho_-,\rho_+,t)$, $Q^{\pm}(f,f)(\rho_-,\rho_+,t)$,, and $(Cf)(\rho_-,\rho_+,t)$, for the resulting λ, A, L, Q, $Q^{\pm}$, and C. □

There are several identities that we will need for the proof of Proposition 2.2 and Theorem 2.1. We prove them in the following Lemma.

Lemma 2.1 *Let us write $T_xh(\rho)=h(\phi_x(\rho;t))$ and $(\mathcal{D}h)(\rho)=b(\rho,t)h'(\rho)$, then*

$$\frac{dT_x}{dx}=\mathcal{D}T_x=T_x\mathcal{D}. \tag{2.6}$$

Moreover

$$b(\rho,t)\Gamma_\rho(\rho,x,t)=\lambda\big(\phi_x(\rho;t),t\big)-\lambda(\rho,t), \tag{2.7}$$
$$\big[\phi_x(\rho;t)\big]_t=\big[H'(\phi_x(\rho;t))-H'(\rho)\big]b(\phi_x(\rho;t),t), \tag{2.8}$$
$$\lambda_t(\rho,t)+H'(\rho)b(\rho,t)\lambda_\rho(\rho,t)=b(\rho,t)A_\rho(\rho,t), \tag{2.9}$$
$$\Gamma_t(\rho,x,t)=A\big(\phi_x(\rho;t),t\big)-A(\rho,t)-H'(\rho)\left(\lambda\big(\phi_x(\rho;t),t\big)-\lambda(\rho,t)\right), \tag{2.10}$$
$$\big[\phi_x(\rho;t)\big]_\rho b(\rho,t)=b(\phi_x(\rho;t),t). \tag{2.11}$$

Proof The family of operators $\{T_x:\ x\in\mathbb{R}\}$, is a group in x. The equation (2.6) is an immediate consequence of

$$\begin{aligned}T_{x+z}h&=T_x\big(h\circ\phi_z(\cdot;t)\big)=T_x\big(h+z\mathcal{D}h+o(z)\big)=T_xh+zT_x\mathcal{D}h+o(z),\\ T_{x+z}h&=T_z\big(T_xh\big)=\big(T_xh\big)\circ\phi_z(\cdot;t)=T_xh+z\mathcal{D}\big(T_xh\big)+o(z).\end{aligned}$$

For the proof of (2.7) use the definition of Γ to write,

$$\begin{aligned}b(\rho,t)\Gamma_\rho(\rho,x,t)&=\int_0^x b(\rho,t)\left[\lambda\big(\phi_y(\rho;t),t\big)\right]_\rho\,dy=\int_0^x\left[\lambda\big(\phi_y(\rho;t),t\big)\right]_y\,dy\\ &=\lambda\big(\phi_x(\rho;t),t\big)-\lambda(\rho,t),\end{aligned}$$

where we used (2.6) for the second equality. This completes the proof of (2.7).

Set

$$X(\rho, x, t) := \big[\phi_x(\rho;t)\big]_t - \big[H'(\phi_x(\rho;t)) - H'(\rho)\big] b(\phi_x(\rho;t), t).$$

We wish to show that $X(\rho, x, t) = 0$ for all (ρ, x, t). This is true for $x = 0$. Differentiating with respect to x yields

$$\begin{aligned}
X_x(\rho, x, t) &= \big[b(\phi_x(\rho;t), t)\big]_t - \big[H'(\phi_x(\rho;t))\big]_x b(\phi_x(\rho;t), t) \\
&\qquad - \big[H'(\phi_x(\rho;t)) - H'(\rho)\big]\big[b(\phi_x(\rho;t), t)\big]_x \\
&= b_t(\phi_x(\rho;t), t) + b_\rho(\phi_x(\rho;t), t)\big[\phi_x(\rho;t)\big]_t - H''(\phi_x(\rho;t)) b^2(\phi_x(\rho;t), t) \\
&\qquad - \big[H'(\phi_x(\rho;t)) - H'(\rho)\big](bb_\rho)(\phi_x(\rho;t), t) \\
&= b_\rho(\phi_x(\rho;t), t)\big[\phi_x(\rho;t)\big]_t - \big[H'(\phi_x(\rho;t)) - H'(\rho)\big](bb_\rho)(\phi_x(\rho;t), t) \\
&= b_\rho(\phi_x(\rho;t), t) X(\rho, x, t),
\end{aligned}$$

where we used (1.6) for the third equality. As a result.

$$X(\rho, x, t) = X(\rho, 0, t) \exp\left[\int_0^x b_\rho(\phi_z(\rho;t), t)\ dz\right] = 0.$$

This completes the proof of (2.8).

For (2.9), we integrate both sides of (1.7) with respect to ρ_+ to assert

$$\begin{aligned}
\lambda_t(\rho, t) &= \int \big[Q(f, f)(\rho, \rho_+, t) + (Cf)(\rho, \rho_+, t)\big]\ d\rho_+ \\
&= \iint \mathbb{1}\big(\rho \le \rho_* \le \rho_+\big)\big(H(\rho_*, \rho_+) - H(\rho, \rho_*)\big) f(\rho, \rho_*, t) f(\rho_*, \rho_+, t)\ d\rho_* d\rho_+ \\
&\quad - \int_\rho^\infty \big[A(\rho_+, t) - A(\rho, t) - H(\rho, \rho_+)\big(\lambda(\rho_+, t) - \lambda(\rho, t)\big)\big]\ f(\rho, \rho_+, t)\ d\rho_+ \\
&\quad + b(\rho, t)\int_\rho^\infty \big\{H_\rho(\rho, \rho_+) f(\rho, \rho_+, t) + \big[H(\rho, \rho_+) - H'(\rho)\big] f_\rho(\rho, \rho_+, t)\big\}\ d\rho_+ \\
&\quad + \int_\rho^\infty \big[\big(H(\rho, \rho_+) - H'(\rho_+)\big) b(\rho_+, t) f(\rho, \rho_+, t)\big]_{\rho_+}\ d\rho_+ \\
&= \int_\rho^\infty \big(A(\rho_*, t) - H(\rho, \rho_*)\lambda(\rho_*, t)\big) f(\rho, \rho_*, t)\ d\rho_* \\
&\quad - \int_\rho^\infty \big[A(\rho_+, t) - A(\rho, t) - H(\rho, \rho_+)\big(\lambda(\rho_+, t) - \lambda(\rho, t)\big)\big]\ f(\rho, \rho_+, t)\ d\rho_+ \\
&\quad + b(\rho, t) A_\rho(\rho, t) - H'(\rho) b(\rho, t)\lambda_\rho(\rho, t) \\
&= b(\rho, t) A_\rho(\rho, t) - H'(\rho) b(\rho, t)\lambda_\rho(\rho, t),
\end{aligned}$$

as desired. Here we have used the fact that $f(\rho,\rho_+,t)=0$ for $\rho_+>P_+$, and

$$\lim_{\rho_+\to\rho} H(\rho,\rho_+)=H'(\rho),$$

for replacing the integral on the fifth line with 0.

We now turn to the proof of (2.10). With the aid of (2.7), we may rewrite (2.10) as

$$\int_0^x \big[\lambda(\phi_y(\rho;t),t)\big]_t\,dy+H'(\rho)b(\rho,t)\int_0^x \big[\lambda(\phi_y(\rho;t),t)\big]_\rho\,dy=\int_0^x \big[A\big(\phi_y(\rho;t),t\big)\big]_y\,dy.$$

For this, it suffices to check

$$\big[\lambda(\phi_x(\rho;t),t)\big]_t+H'(\rho)b(\rho,t)\ \big[\lambda(\phi_x(\rho;t),t)\big]_\rho=\big[A\big(\phi_x(\rho;t),t\big)\big]_x, \tag{2.12}$$

for every x. Note that by (2.6)

$$\begin{aligned}&b(\rho,t)\ \big[\lambda(\phi_x(\rho;t),t)\big]_\rho=b(\phi_x(\rho;t),t)\lambda_\rho(\phi_x(\rho;t),t),\\&\big[A\big(\phi_x(\rho;t),t\big)\big]_x=b(\phi_x(\rho;t),t)A_\rho\big(\phi_x(\rho;t),t\big).\end{aligned}$$

Hence (2.12) is equivalent to

$$\big[\lambda(\phi_x(\rho;t),t)\big]_t+H'(\rho)b(\phi_x(\rho;t),t)\lambda_\rho(\phi_x(\rho;t),t)=b(\phi_x(\rho;t),t)A_\rho\big(\phi_x(\rho;t),t\big). \tag{2.13}$$

We carry out the time differentiation of the first term and use (2.8) to rewrite (2.13) as

$$\lambda_t(\phi_x(\rho;t),t)+H'(\phi_x(\rho;t))b(\phi_x(\rho;t),t)\lambda_\rho(\phi_x(\rho;t),t)=b(\phi_x(\rho;t),t)A_\rho\big(\phi_x(\rho;t),t\big). \tag{2.14}$$

But (2.14) is an immediate consequence of (2.9). This completes the proof of (2.10).

We finally turn to the proof of (2.11). Set

$$Y(x)=Y(x,\rho,t)=b(\phi_x(\rho;t),t)-\big[\phi_x(\rho;t)\big]_\rho b(\rho,t).$$

Evidently, $Y(0)=0$. On the other hand

$$\begin{aligned}Y'(x)&=b_\rho\big(\phi_x(\rho;t),t\big)b\big(\phi_x(\rho;t),t\big)-\big[b\big(\phi_x(\rho;t),t\big)\big]_\rho b(\rho,t)\\&=b_\rho\big(\phi_x(\rho;t),t\big)b\big(\phi_x(\rho;t),t\big)-b_\rho\big(\phi_x(\rho;t),t\big)\big[\phi_x(\rho;t)\big]_\rho b(\rho,t)\\&=b_\rho\big(\phi_x(\rho;t),t\big)Y(x).\end{aligned}$$

As a result,

$$Y(x)=Y(0)\exp\left(\int_0^x b_\rho\big(\phi_y(\rho;t),t\big)dy\right)=0,$$

as desired. ☐

Remark 2.2 As an immediate consequence of (2.8) and (2.11), we have

$$\frac{d\hat{\rho}_i}{dt} = \big[H'(\hat{\rho}_i) - H(\hat{\rho}_i, \rho_{i+1})\big] b(\hat{\rho}_i, t),$$

because

$$\hat{\rho}_i(t) = \phi_{x_{i+1}(t)-x_i(t)}\big(\rho_i(t); t\big).$$

However, if we do not vary (x_i, x_{i+1}, ρ_i) with time, and set

$$\hat{\rho}_i'(t) = \phi_{x_{i+1}-x_i}\big(\rho_i; t\big),$$

then instead we have the following formula that will be used in the proof of Theorem 2.1,

$$\frac{d\hat{\rho}_i'}{dt} = \big[H'(\hat{\rho}_i') - H(\rho_i)\big] b(\hat{\rho}_i', t), \tag{2.15}$$

by (2.8). ☐

We are now ready to establish Proposition 2.2.

Proof of Proposition 2.2(i) We first show that ρ solves (1.1) classically away from the shock curves. For this, take a point (x, t) such that $x \in \big(x_i(t), x_{i+1}(t)\big)$, for some nonnegative integer i. Let us write $\hat{\phi}_x(\rho; t)$ for $\big[\phi_x(\rho; t)\big]_\rho$. Then

$$\begin{aligned}
\rho_t(x,t) =& \big(\phi_{x-x_i(t)}(\rho_i(t); t)\big)_t \\
=& \big[H'(\rho(x,t)) - H'(\rho_i(t))\big] b(\rho(x,t),t) + b(\rho(x,t),t) H\big(\hat{\rho}_{i-1}(t), \rho_i(t)\big) \\
&+ \hat{\phi}_{x-x_i(t)}(\rho_i(t); t)\big[H'(\rho_i(t)) - H(\hat{\rho}_{i-1}(t), \rho_i(t))\big] b(\rho_i(t), t) \\
=& \big[H'(\rho(x,t)) - H'(\rho_i(t))\big] b(\rho(x,t),t) + b(\rho(x,t),t) H\big(\hat{\rho}_{i-1}(t), \rho_i(t)\big) \\
&+ \big[H'(\rho_i(t)) - H(\hat{\rho}_{i-1}(t), \rho_i(t))\big] b(\rho(x,t), t) \\
=& H'(\rho(x,t)) b(\rho(x,t),t) = H'(\rho(x,t))\rho_x(x,t),
\end{aligned}$$

as desired. Here we used (2.8) and (2.11) for the second and third equalities respectively. Since Rankine-Hugoniot Formula is valid at shock curves and (1.1) holds classically away from the shock curves, we deduce that ρ is a weak solution of (1.1). On the other hand, since initially $\rho(x_i(0)-, 0) < \rho(x_i(0)+, 0)$, and this inequality persists at later times by the way the dynamics of $\mathbf{q}(t)$ is defined, we deduce that ρ is an entropy solution. We are done if we can show that there is at most one entropy solution for given initial data and boundary condition. This is an immediate consequence of the second part.

(ii) The proof of (2.5) with no boundary condition can be found in Lax [La]. We only sketch the proof of (2.5) because it is straightforward adaptation of the proof of Theorem 3.4 in [La].

Take a sequence

$$0 = y_0(t) < y_1(t) < \cdots < y_n(t) < y_{n+1}(t) = L,$$

such that on each interval $(y_i(t), y_{i+1}(t))$, either $\rho'(\cdot,t)-\rho(\cdot,t)$ is positive or negative. Without loss of generality, we may assume that $\rho'(x,t)-\rho(x,t) > 0$ for $x \in (0, y_1(t))$. Then we can write

$$\int_0^L |\rho'(x,t)-\rho(x,t)|\, dx = \sum_{i=0}^n (-1)^i \int_{y_i(t)}^{y_{i+1}(t)} \big(\rho'(x,t)-\rho(x,t)\big)\, dx$$

As in [La], we can readily show that in between the jumps of ρ or ρ', the expression

$$\frac{d}{dt}\int_0^L |\rho'(x,t)-\rho(x,t)|\, dx$$

equals to

$$\sum_{i=0}^n (-1)^i \left[\int_{y_i(t)}^{y_{i+1}(t)} (\rho'_t-\rho_t)(x,t)\, dx + \big(\rho'-\rho\big)(y_{i+1}(t)-,t)\frac{dy_{i+1}}{dt}(t) - \big(\rho'-\rho\big)(y_i(t)+,t)\frac{dy_i}{dt}(t)\right]$$

$$= \sum_{i=0}^n (-1)^i \left[\big(H(\rho')-H(\rho)\big)(y_{i+1}(t)-,t) - \big(H(\rho')-H(\rho)\big)(y_i(t)+,t)\right]$$

$$+ \sum_{i=0}^n (-1)^i \left[\big(\rho'-\rho\big)(y_{i+1}(t),t)\frac{dy_{i+1}}{dt}(t) - \big(\rho'-\rho\big)(y_i(t),t)\frac{dy_i}{dt}(t)\right].$$

From the convexity of H and the entropy condition it follows that each summand associated with y_i, for $i = 1,\ldots,n$ contributes non-positively. The proof of this is exactly as in [La] and is omitted. On the other hand, the contribution of the y_0 term is exactly $H(\rho(0,t)) - H(\rho'(0,t)$ which is nonpositive because $\rho(0,t) \le \rho'(0,t)$, and H is increasing. As a result,

$$\frac{d}{dt}\int_0^L |\rho'(x,t)-\rho(x,t)|\, dx \le \big|H(\rho'(L,t)) - H(\rho(L,t))\big|, \tag{2.16}$$

provided that no jump occurs for either ρ or ρ' at (L,t). If the jumps of ρ or ρ' at L occur at $s_1,\ldots,s_k$ with $s = s_0 < s_1 < \cdots < s_k < s_{k+1} = t$, then we integrate (2.16) over intervals $(s_i, s_{i+1}), i = 0,\ldots,k$ and add up over i to obtain (2.5). □

3 Main Strategy and Some Preliminaries

We first explain our strategy for establishing Theorem 2.1. Without loss of generality, we may assume that $a = 0$. Let us write Γ for the set of piecewise C^1 functions $\rho : [0, L] \to \mathbb{R}$, and regard Γ as Banach space with total variation norm. We also write $S^t\rho^0(x) = \rho(x,t)$ for the solution in (2.4). Because of the stochastic boundary condition, the operator S^t is random, and we write $\mathbb{E}$ for the corresponding expected value. According to Theorem 2.1, we have a candidate for the law of the solution $\rho(\cdot, t) \in \Gamma$ whenever the assumptions of Theorem 2.1 are met. Let us write $\nu\big(d\rho(\cdot), t\big)$ for this candidate, which is a probability measure on Γ. (The measure ν is the measure μ of Definition 2.2, expressed in terms of ρ instead of $\mathbf{q}$.) Theorem 2.1 is equivalent to the claim

$$\int F\big(\rho(\cdot)\big)\ \nu\big(d\rho(\cdot), t\big) = \mathbb{E}\int F\big(S^t\rho^0(\cdot)\big)\ \nu\big(d\rho^0(\cdot), 0\big), \tag{3.1}$$

for every bounded continuous function $F : \Gamma \to \mathbb{R}$. For this, it suffices to establish (3.1) for F of the form

$$F(\rho) = \exp\left[\int_0^L J(x)\rho(x)\ dx\right], \tag{3.2}$$

where J is a continuous function.

As we have seen in Proposition 2.2, there is a simple recipe for building a density $\rho \in \Gamma$ from a configuration $\mathbf{q} \in \Delta_n^L$, namely the function $R_t : \Delta^L \to \Gamma$ defined by

$$R_t(\mathbf{q})(x) = \sum_{i=0}^{\mathbf{n}(\mathbf{q})} \phi_{x-x_i}(\rho_i; t)\mathbb{1}\big(x_i \le x < x_{i+1}\big).$$

If we set $\hat{F}(\mathbf{q}, t) = F\big(R_t(\mathbf{q})\big)$, then (3.1) reads as

$$\int \hat{F}\big(\mathbf{q}, t\big)\ \mu\big(d\mathbf{q}, t\big) = \mathbb{E}\int \hat{F}\big(\Psi_0^t\mathbf{q}, t\big)\ \mu\big(d\mathbf{q}, 0\big),$$

where $\Psi_0^t\mathbf{q}$ denotes $\mathbf{q}(t)$ with the initial condition $\mathbf{q}(0) = \mathbf{q}$. To ease the notation, we set $G(\mathbf{q}) = \hat{F}(\mathbf{q}, t)$. Observe that the function $G : \Delta^L \to \mathbb{R}$ satisfies the following conditions: For every $\mathbf{q} = \big((x_0, \rho_0), \dots, (x_n, \rho_n)\big) \in \Delta^L$,

(i) $G(\mathbf{q}) = G(\mathbf{q}^i)$, whenever $x_{i+1} = x_i$;

(ii) $G(\mathbf{q}) = G\big(\mathbf{q}(\rho_{n+1})\big)$.

This is an immediate consequence of the fact that G is a function of the expression $\int_0^L JR_t(\mathbf{q})\,dx$. We wish to show

$$\int G(\mathbf{q})\ \mu(d\mathbf{q},t) = \mathbb{E}\int G(\Psi_0^t\mathbf{q})\ \mu(d\mathbf{q},0). \tag{3.3}$$

In fact formally $\mathbf{q}(t)$ has a generator $\mathcal{L} = \mathcal{L}^*$ that is a sum of first order operators (coming from the deterministic motion of particles inside the interval $(0,L)$), and a pure jump part (coming from the stochastic dynamics at the boundary $x = L$). We establish (3.3) by verifying that the time derivatives of both sides of (3.3) match: a variant of the equality

$$\dot{\mu}^n = \big(\mathcal{L}^*\mu\big)^n, \tag{3.4}$$

for all $n \geq 0$, where $\mathcal{L}^*$ is the adjoint of the operator $\mathcal{L}$. Here and below, we write ν^n for the restriction of a measure ν to Δ_n^L. Also, given $G : \Delta_L \to \mathbb{R}$, we write G^n for the restriction of the function G to the set Δ_n^L. To verify (3.3) or (3.4), we show

$$\int G^n\ d\dot{\mu}^n = \int \big(\mathcal{L}G\big)^n\ d\mu^n, \tag{3.5}$$

for every C^1 admissible function G. This is achieved in three steps.

- **(i)** We differentiate μ^n with respect to time and derive an explicit formula for this derivative in the form $\dot{\mu}^n = X^n\mu^n$. We regard X^n as the Radon-Nykodym derivative of $\dot{\mu}^n$ with respect to μ^n.
- **(ii)** We differentiate the expected value of $G(\mathbf{q}(t))$, that can be expressed as the expected value of $\mathcal{L}G(\mathbf{q}(t))$. This step is more challenging to carry out because the deterministic part of the dynamics is discontinuous at collision times.
- **(iii)** We use (**i**) and (**ii**) to match both sides of (3.5).

To prepare for Step **(ii)**, we introduce some notation for the particle dynamics

Definition 3.1(i) For $0 \leq s \leq t$ and $\mathbf{q} \in \Delta^L$, we write $\psi_s^t\mathbf{q}$ for the deterministic evolution from time s to t of the configuration $\mathbf{q}$ according to the annihilating particle dynamics for the PDE, *without* random entry dynamics at $x = L$.

(ii) Given a configuration $\mathbf{q} = \big((x_0,\rho_0),\dots,(x_n,\rho_n)\big)$ and $\rho_+ \in \mathbb{R}$, write $\epsilon_{\rho_+}\mathbf{q}$ for the configuration $\big((x_0,\rho_0),\dots,(x_n,\rho_n),(L,\rho_+)\big)$.

(iii) Write $\Psi_s^t\mathbf{q}$ for the *random* evolution of the configuration according to deterministic particle dynamics interrupted with random entries at $x = L$ according to the boundary process as in **(3)** in Section 2, where the latter has been started at time s with value

$\phi_{L-x_n}(\rho_n; s)$. In particular, if the jumps between times s and t occur at times $\tau_1 < \cdots < \tau_k$ with values $m_1, \cdots, m_k$, then

$$\Psi_s^t \mathbf{q} = \psi_{\tau_k}^t \epsilon_{m_k} \psi_{\tau_{k-1}}^{\tau_k} \epsilon_{m_{k-1}} \cdots \psi_{\tau_1}^{\tau_2} \epsilon_{m_1} \psi_s^{\tau_1} \mathbf{q}. \tag{3.6}$$

(iv) For $n \geq 1$, and $i \in \{0, \dots, n-1\}$, we write $\partial_i \Delta_n^L$ for the portion of the boundary Δ_n^L such that $x_i = x_{i+1}$. Note that $\mathbf{q}(t)$ reaches the boundary set $\partial_0 \Delta_n^L$ at time τ if at this time $x_1(\tau) = 0$. For time t immediately after τ, the configuration $\mathbf{q}(t)$ belongs to Δ_{n-1}^L with ρ_0 taking new value. Similarly $\mathbf{q}(t)$ reaches the boundary set $\partial_i D_n^L$ for some $i > 0$ at time τ if at this time x_{i+1} collides with x_i. For time t immediately after τ, the configuration $\mathbf{q}(t)$ belongs to Δ_{n-1}^L.

(v) We write $\partial_{n+1} \Delta_{n+1}^L$ for the set of points $\mathbf{q} \in \Delta_{n+1}^L$ with $x_{n+1} = L$. When $\mathbf{q} \in \Delta_n^L$, and a new particle is created at L at time τ by the stochastic boundary dynamics, the configuration $\mathbf{q}(\tau+)$ is regarded as a boundary point in $\partial_{n+1} \Delta_{n+1}^L$.

(vi) We write $\mathcal{L}$ for the generator of the process $\mathbf{q}(t)$. This generator can be expressed as $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_b$, where $\mathcal{L}_0$ is the generator of the deterministic part of dynamics, and $\mathcal{L}_b$ represents the Markovian boundary dynamics. The deterministic dynamics restricted to Δ_n^L has a generator that is denoted by $\mathcal{L}_{0n}$. While $\mathbf{q}(t)$ remains in Δ_n^L, its evolution is governed by an ODE of the form

$$\frac{d\mathbf{q}}{dt}(t) = \mathbf{b}\big(\mathbf{q}(t), t\big),$$

with $\mathbf{b} = \mathbf{b}_n : \Delta_n^L \to \mathbb{R}^{2n+1}$, that can be easily described with the aid of rules **(1)** and **(2)** of Definition 2.1**(iii)**. Given this vector field, the generator $\mathcal{L}_{0n}$ is given by

$$\mathcal{L}_{0n} F = \mathbf{b} \cdot \nabla F,$$

where ∇F is the full gradient of F with respect to variables $\big(\rho_0, x_1, \rho_1, \dots, x_n, \rho_n\big)$. We also write $\mathcal{L}_{0n}^*$ for the adjoint of $\mathcal{L}_{0n}$ with respect to the Lebesgue measure:

$$\mathcal{L}_{0n}^* \mu = \nabla \cdot (\mu \mathbf{b}).$$

□

Proposition 3.1 *For any $s, \mathbf{q}$, the process $\Psi_s^t \mathbf{q}$ is strong Markov.*

This assertion follows after recognizing $\Psi_s^t q$ as a piecewise-deterministic Markov process described in some generality by Davis [Da].

Clearly (3.3) would follow if we can show

$$\frac{d}{ds} \int \mathbb{E}\ G(\Psi_s^t \mathbf{q}) \mu(d\mathbf{q}, s) = 0 \tag{3.7}$$

for $0 < s < t$. The differentiation of $\mu(d\mathbf{q}, s)$ can be carried out directly and poses no difficulty. As for the contribution of $G(\Psi_s^t \mathbf{q})$ to the s-derivative, we need to show

$$\frac{d}{ds}\mathbb{E}\ G(\Psi_s^t \mathbf{q}) = -\mathbb{E}\ \mathcal{L}G(\Psi_s^t \mathbf{q}), \tag{3.8}$$

where $\mathcal{L}$ is the infinitesimal generator of $\mathbf{q}(\cdot)$. Since the deterministic part of the evolution is discontinuous in time, the justification of (3.8) requires some work and will be carried out in Section 4. We end this section with a lemma that will be used for the proof of (3.8). Note that for the differentiation in (3.8) we will need to compare $\mathbb{E}\ G(\Psi_s^t \mathbf{q})$ and $\mathbb{E}\ G(\Psi_{s'}^t \mathbf{q})$ for $0 < s' < s \leq t$. As a warm-up we verify the Lipschitzness of the function $s \mapsto \mathbb{E}\ G(\Psi_s^t \mathbf{q})$.

Lemma 3.1 *Fix $t > 0$. There exists a constant $C_0 = C_0(P_-, P_+, J, f^0)$ such that the function $G(\mathbf{q}, s) = \mathbb{E}\ G(\Psi_s^t \mathbf{q})$ satisfies*

$$|G(\mathbf{q}, s') - G(\mathbf{q}, s)| \leq C_0(n+1)|s' - s|, \tag{3.9}$$

for all $\mathbf{q} \in \Delta_n^L$ and $s, s' \in [0, t]$.

Proof *(Step 1.)* The proof follows from the L^1-stability (2.5) and a coupling argument for the stochastic boundary dynamics. Let us write

$$\rho(x,t) = R_t\big(\Psi_s^t \mathbf{q}\big)(x), \qquad \rho'(x,t) = R_t\big(\Psi_{s'}^t \mathbf{q}\big)(x).$$

Since ρ takes value in a bounded interval $[P_-, P_+]$, and $G = F \circ R_t$ for F given by (3.2), (3.9) would follow if we can find a constant $c_1 = c_1(P_-, P_+, f^0)$, such that

$$\mathbb{E} \int_0^L \big|\rho'(x,t) - \rho(x,t)\big|\ dx \leq c_1(n+1)|s' - s|, \tag{3.10}$$

for all $\mathbf{q} \in \Delta_n^L$ and $s, s' \in [0, t]$. On account of (2.5), it suffices to find constants $c_2 = c_2(P_-, P_+, f^0)$ and $c_3 = c_3(P_-, P_+, f^0)$ such that

$$\mathbb{E} \int_0^L \big|\rho'(x,s) - \rho(x,s)\big|\ dx \leq c_2(n+1)|s' - s|, \tag{3.11}$$

$$\mathbb{E} \int_s^t \big|\rho'(L,\theta) - \rho(L,\theta)\big|\ d\theta \leq c_3|s' - s|, \tag{3.12}$$

for all $\mathbf{q} \in \Delta_n^L$ and $s, s' \in [0, t]$.

(Step 2.) By definition, (3.11) means

$$\mathbb{E} \int_0^L \big|R_s\big(\Psi_{s'}^s \mathbf{q}\big)(x) - R_s(\mathbf{q})(x)\big|\ dx \leq c_2(n+1)|s' - s|. \tag{3.13}$$

Let us write E_0 for the event that no jump occurs at $x = L$ in (s', s), and E_1 for the complement of E_0. Since the jump rate at $x = L$ is $H(\rho_-, \rho_+)f(\rho_-, \rho_+, t)$ with f bounded by a constant that depends on $P_\pm$ and f^0, we can assert

$$\mathbb{P}\big(E_1\big) \le c_4|s' - s|, \tag{3.14}$$

for a constant $c_4 = c_4(P_-, P_+, f^0)$. Hence, (3.13) would follow if we can find a constant $c_5 = c_5(P_-, P_+)$ such that

$$\int_0^L \big|R_s\big(\psi_{s'}^s \mathbf{q}\big)(x) - R_s(\mathbf{q})(x)\big|\ dx \le c_5(n+1)|s' - s|, \tag{3.15}$$

for all $\mathbf{q} \in \Delta_n^L$ and $s, s' \in [0, t]$. Since $\rho'(x, s) = R_s\big(\psi_{s'}^s \mathbf{q}\big)(x), s > s'$ solves the first equation in (2.4) in the interval $[0, L]$, we may use the method of characteristic to express

$$\rho'(x, s) = \rho\big(y(x, s - s'), s'\big),$$

where $y(x, s - s')$ is the location of a backward characteristic at time s' that emanates from x at time s. For $c_0 = H'(P_+)$, we have $|y(x, s - s') - x| \le c_0|s - s'| =: \delta$. Note that if $\mathbf{q} \in \Delta_n^L$, and $\psi_{s'}^s(\mathbf{q}) \in \Delta_{n'}^L$, then $n' \le n$. Let $x_1, \dots, x_{n'}$ be the locations of the particles (shock discontinuities) at time s, and for each i with $x_i + \delta < x_{i+1} - \delta$, set

$$I_i := \big(x_i + \delta, x_{i+1} - \delta\big).$$

We write I for the union of such intervals. Note that $|I| \le 2\delta n' \le 2\delta n$. If $x \notin I$, then there is no jump discontinuity between x and $y(x, s - s')$ at time s'. Hence

$$\begin{aligned}|\rho'(x, s) - \rho'(x, s')| &= \big|\rho'\big(y(x, s - s'), s'\big) - \rho'(x, s')\big| \le \max|b(\cdot, s')|\ |y(x, s - s') - x| \\ &\le \max|b(\cdot, s')|\ c_0|s - s'|.\end{aligned}$$

From this and $|I| \le 2c_0 n|s - s'|$, we deduce

$$\int_0^L |\rho'(x, s) - \rho'(x, s')|\ dx \le c_5(n+1)|s - s'|.$$

Hence (3.15) would follow if we can show

$$\int_0^L |\rho(x, s) - \rho'(x, s')|\ dx = \int_0^L \big|R_s\big(\mathbf{q}\big)(x) - R_{s'}(\mathbf{q})(x)\big|\ dx \le c_6(n+1)|s' - s|,$$

for a constant c_6. The existence of finite c_6 is an immediate consequence of the Lipschitz regularity of $b(x, s)$ with respect to s. This completes the proof of (3.13).

(Step 3.) For (3.12), recall that by Remark 2.1**(i)**, the processes $\theta \mapsto m(\theta) := \rho(L,\theta)$ and $\theta \mapsto m'(\theta) := \rho'(L,\theta)$ are Markov processes with generator $\mathscr{B}^\theta$ in the interval $[s,t]$. Observe $m(s) = R_s(\mathbf{q})(L)$ and $m'(s) = R_s\big(\Psi^s_{s'}\mathbf{q}\big)(L)$. We first claim that there exists a constant c_0 such that

$$\mathbb{E}|m'(s) - m(s)| \le c_7|s - s'|. \tag{3.16}$$

The bound (3.16) is an immediate consequence of (3.14) and the elementary fact that in E_0,

$$|m(s') - m(s)| \le c_8|s - s'|,$$

for

$$c_8 = \sup_{[P-,P_+]\times[0,t]} |\hat{b}|,$$

with $\hat{b}(\rho,t) = H'(\rho)b(\rho,t)$. Next we define a coupling for the pair (m,m'). Recall

$$g(\rho_-,\rho_+,\theta) = H(\rho_-,\rho_+)f(\rho_-,\rho_+,\theta), \qquad \eta(\rho_-,\theta) = \int_{\rho_-}^{\infty} g(\rho_-,\rho_+,\theta)\ d\rho_+.$$

The generator of the coupled process (m,m') is given by

$$\begin{aligned}
\tilde{\mathscr{B}}^\theta F(m,m') =& \hat{b}(m,\theta)F_m(m,m') + \hat{b}(m',\theta)F_{m'}(m,m') \\
&+ 1\!\!1(m < m') \int_m^{m'} g(m,m_+,\theta)\big(F(m_+,m') - F(m,m')\big)\ dm_+ \\
&+ 1\!\!1(m' < m) \int_{m'}^{m} g(m,m_+,\theta)\big(F(m,m_+) - F(m,m')\big)\ dm_+ \\
&+ \int_{m\vee m'}^{\infty} \hat{g}(m,m',m_+,\theta)\big(F(m_+,m_+) - F(m,m')\big)\ dm_+ \\
&+ \int_{m\vee m'}^{\infty} \big[g(m',m_+,\theta) - \hat{g}(m,m',m_+,\theta)\big]\big(F(m,m_+) - F(m,m')\big)\ dm_+ \\
&+ \int_{m\vee m'}^{\infty} \big[g(m,m_+,\theta) - \hat{g}(m,m',m_+,\theta)\big]\big(F(m_+,m') - F(m,m')\big)\ dm_+
\end{aligned}$$

where $\hat{g}(m,m',m_+,\theta) = g(m,m_+,\theta) \wedge g(m',m_+,\theta)$. Since both $f(\rho_-,\rho_+,t)$ and $H(\rho_-,\rho_+)$ are bounded Lipschitz functions in $\rho_\pm \in [P_-,P_+]$, we can find a constant c_9 such that

$$\big|g(m,m_+,\theta) - \hat{g}(m,m',m_+,\theta)\big| \le c_8|m' - m|. \tag{3.17}$$

We then use the identity

$$\frac{d}{d\theta}\ \mathbb{E}\ F\big(m(\theta),m'(\theta)\big) = \mathbb{E}\ \big(\tilde{\mathscr{B}}^\theta F\big)\big(m(\theta),m'(\theta)\big),$$

for $F(m,m') = |m - m'|$. From (3.17) and the Lipschitzness of $\hat{b}$ we deduce

$$\frac{d}{d\theta}\ \mathbb{E}\ |m'(\theta) - m(\theta)| \le c_{10}\mathbb{E}\ |m'(\theta) - m(\theta)|,$$

for a constant c_{10}. This and (3.16) imply (3.12). □

4 Forward Equation

This section is devoted to the rigorous verification of a variant of the forward equation (3.4).

Theorem 4.1 *For $G(\mathbf{q}, s) = \mathbb{E}\ G\big(\Psi_s^t(\mathbf{q})\big)$, we have*

$$\lim_{s'\uparrow s}\ (s - s')^{-1} \int \big(G(\mathbf{q}, s) - G(\mathbf{q}, s')\big)\ \mu(d\mathbf{q}, s) = -\int (\mathcal{L}G)(\mathbf{q}, s)\ \mu(d\mathbf{q}, s). \tag{4.1}$$

Proof *(Step 1.)* Let $0 < s' < s \le t$. To facilitate the calculation of the derivative, we show that we can separate the deterministic and stochastic portions of the dynamics over the time interval $[s', s]$, when the $s - s'$ is small. Write $\tau = \tau(\mathbf{q}, s')$ for the first time a jump occurs at $x = L$ after the time s', and let E_1 denote the event that $\tau \in (s', s)$. We claim that there exists a constant $C_1 = C_1(P_-, P_+, J, f^0)$ so that for $\mathbf{q} \in \Delta_n^L$,

$$\begin{aligned} G(\mathbf{q}, s') =&(s - s') \int \big(\mathbb{E}\left[G\big(\epsilon_{\rho_+}\psi_{s'}^{\tau}\mathbf{q}\big)\ \big|\ E_1\right] - G(\mathbf{q}, s)\big)\, H(\rho_n, \rho_+) f(\rho_n, \rho_+, s)\ d\rho_+ \\ &+ G\big(\psi_{s'}^s\mathbf{q}, s\big) + (s - s')^2 R(\mathbf{q}, s', s), \end{aligned} \tag{4.2}$$

with $\big|R(\mathbf{q}, s', s)\big| \le C_1(n+1)$, and $\hat{\rho}_n = R_s(\mathbf{q})(L)$. Note that by the Markov property of the random flow Ψ,

$$G(\mathbf{q}, s') = \mathbb{E}\ G\big(\Psi_s^t\Psi_{s'}^s\mathbf{q}\big) = \mathbb{E}\ G\big(\Psi_{s'}^s\mathbf{q}, s\big).$$

Let $\mathbf{q} = \big((0, \rho_0), (x_1, \rho_1), \ldots, (x_n, \rho_n)\big)$ be fixed, and write $\hat{\rho}_n' := \big(R_{s'}\mathbf{q}\big)(L)$. Let E_0 be the event that there is no jump at $x = L$ in (s', s), and recall that E_1 is the complement of E_0. Observe that on E_0 we see only the deterministic flow ψ over the time interval (s', s):

$$\mathbb{E}\ G\big(\Psi_{s'}^s\mathbf{q}, s\big)\mathbb{1}_{E_0} = G\big(\psi_{s'}^s\mathbf{q}, s\big)\ \mathbb{P}\big(E_0\big) = G\big(\psi_{s'}^s\mathbf{q}, s\big) - G\big(\psi_{s'}^s\mathbf{q}, s\big)\ \mathbb{P}\big(E_1\big). \tag{4.3}$$

On the other hand, using the Lipschitz regularity of η (which is the consequence of the Lipschitz regularity of f),

$$\mathbb{P}\big(E_1\big) = \int_{s'}^{s} \eta\big(\beta_{s'}^{\theta}(\hat{\rho}_n'), \theta\big)\ d\theta + O((s - s')^2) = (s - s')\eta\big(\hat{\rho}_n, s\big) + O((s - s')^2), \tag{4.4}$$

with both errors bounded uniformly over $\mathbf{q}$. From this and (4.3) we learn

$$
\begin{aligned}
\mathbb{E}\; G\big(\Psi^s_{s'}\mathbf{q}, s\big)1\!\!1_{E_0} &= G\big(\psi^s_{s'}\mathbf{q}, s\big) - (s-s')G\big(\psi^s_{s'}\mathbf{q}, s\big)\eta\big(\hat\rho_n, s\big) + O((s-s')^2) \\
&= G\big(\psi^s_{s'}\mathbf{q}, s\big) - (s-s')G\big(\psi^s_{s'}\mathbf{q}, s\big)\int g(\hat\rho_n, \rho_+, s)\; d\rho_+ + O((s-s')^2). \qquad (4.5)
\end{aligned}
$$

In E_1, recall that $\tau \in (s', s)$ is the first time a random entry occurs for $\Psi^s_{s'}$, and ρ_+ for the new boundary value. We have

$$G\big(\Psi^s_{s'}\mathbf{q}, s\big)1\!\!1_{E_1} = G\big(\Psi^s_\tau\epsilon_{\rho_+}\psi^\tau_{s'}\mathbf{q}, s\big)1\!\!1_{E_1}.$$

Using the strong Markov property for the random boundary at the stopping time τ,

$$\mathbb{E}\; G\big(\Psi^s_{s'}\mathbf{q}, s\big)1\!\!1_{E_1} = \mathbb{E}\; G\big(\Psi^s_\tau\epsilon_{\rho_+}\psi^\tau_{s'}\mathbf{q}, s\big)1\!\!1_{E_1} = \mathbb{E}\; G\big(\epsilon_{\rho_+}\psi^\tau_{s'}\mathbf{q}, \tau\big)1\!\!1_{E_1}. \qquad (4.6)$$

Since $\mathbb{P}(E_1) = O(s-s')$, we can afford to make $o(1)$ modifications to this by (3.9):

$$\big|\mathbb{E}\; G\big(\epsilon_{\rho_+}\psi^\tau_{s'}\mathbf{q}, \tau\big)1\!\!1_{E_1} - \mathbb{E}\; G\big(\epsilon_{\rho_+}\psi^\tau_{s'}\mathbf{q}, s\big)1\!\!1_{E_1}\big| \le C_0(n+1)\mathbb{P}(E_1)(s-s'). \qquad (4.7)$$

Next we modify the distribution from which ρ_+ is selected; at present, ρ_+ is selected according to a random measure with density

$$\hat g\big(\tilde\rho_n, \rho_+, \tau\big) := \eta\big(\tilde\rho_n, \tau\big)^{-1} g\big(\tilde\rho_n, \rho_+, \tau\big),$$

where $\tilde\rho_n := \beta^\tau_{s'}(\hat\rho'_n)$. From the Lipshitzness of $b(x,s)$ in s, it is not hard to show that there exists a constant c_1 such that

$$\big|\hat\rho'_n - \hat\rho_n\big| \le c_1|s'-s|, \qquad \big|\hat\rho'_n - \tilde\rho_n\big| \le c_1|s'-s|. \qquad (4.8)$$

Let us write $\hat\rho_+$ for an independent random variable distributed as $\hat g(\hat\rho_n, \rho_+, s)\; d\rho_+$. Observe

$$\eta(m,\theta) = \int_m^\infty H(m,\rho_+)f(m,\rho_+,\theta)\; d\rho_+ \ge H'(P_-)\int_m^\infty f(m,\rho_+,\theta)\; d\rho_+ = H'(P_-)\lambda(m,\theta).$$

According to Theorem 1.1, $\rho_+ \mapsto f(m,\rho_+,\theta)$ is a non-zero continuous kernel. As a result, $\lambda(m,\theta)$ is uniformly positive in $[P_-, P_+]\times[0,T]$. From this, (4.8), and the Lipschitzness of f we can readily show

$$\big|\hat g(\hat\rho_n, \rho_+, s) - \hat g\big(\tilde\rho_n, \rho_+, \tau\big)\big| \le c_2|s'-s|, \qquad (4.9)$$

for a constant c_2 that depends on $P_\pm$ only. We then use (4.8) and (4.9) to assert that there exists a constant c_3 such that that the expression

$$\big|\mathbb{E}\;\big[G\big(\epsilon_{\rho_+}\psi^\tau_{s'}\mathbf{q}, s\big) - G\big(\epsilon_{\hat\rho_+}\psi^\tau_{s'}\mathbf{q}, s\big)\big]\,1\!\!1_{E_1}\big|,$$

is bounded above by

$$\left|\mathbb{E}\ \mathbb{1}_{E_1}\int_{\tilde\rho_n\vee\hat\rho_n}^{\infty} G\big(\epsilon_{\rho_+}\psi_{s'}^{\tau}\mathbf{q},s\big)\big(\hat g\big(\tilde\rho_n,\rho_+,\tau\big)-\hat g(\hat\rho_n,\rho_+,s)\big)\ d\rho_+\right| \\ +\left|\mathbb{E}\ \mathbb{1}_{E_1}\int_{\hat\rho_n}^{\tilde\rho_n} G\big(\epsilon_{\rho_+}\psi_{s'}^{\tau}\mathbf{q},s\big)\ \hat g\big(\tilde\rho_n,\rho_+,\tau\big)\ d\rho_+\right| \le c_3(s'-s)\mathbb{P}(E_1).$$

Here we used $\tilde\rho_n \ge \hat\rho_n$, which follows from $b\le 0$ and $\tau\in(s',s)$. From this, (4.6), (4.7), and (4.4) we learn

$$\begin{aligned}\mathbb{E}\ G\big(\Psi_{s'}^{s}\mathbf{q},s\big)\mathbb{1}_{E_1} =&\mathbb{E}\left[G\big(\epsilon_{\hat\rho_+}\psi_{s'}^{\tau}\mathbf{q},s\big)\ \big|\ E_1\right]\mathbb{P}(E_1)+(s-s')^2R_1\\ =&\mathbb{E}\left[\int G\big(\epsilon_{\rho_+}\psi_{s'}^{\tau}\mathbf{q},s\big)\ g(\hat\rho_n,\rho_+,s)\ d\rho_+\ \big|\ E_1\right]\eta(\hat\rho_n,s)^{-1}\ \mathbb{P}(E_1)+(s-s')^2R_1\\ =&(s-s')\int\mathbb{E}\left[G\big(\epsilon_{\rho_+}\psi_{s'}^{\tau}\mathbf{q},s\big)\ g(\hat\rho_n,\rho_+,s)\ \big|\ E_1\right]\ d\rho_+ +(s-s')^2R_2.\end{aligned}$$

where R_1 and R_2 are bounded by a constant multiple of $n+1$. This and (4.5) complete the proof of (4.2).

(Step 2.) We wish to calculate the left-hand side of (4.1). (4.2) allows us to separate the deterministic dynamics from the stochastic boundary dynamics. For the deterministic part, we wish to evaluate

$$\lim_{s'\uparrow s}\ (s-s')^{-1}\int\Big(G\big(\psi_s^t(\mathbf{q})\big)-G\big(\psi_{s'}^t(\mathbf{q})\big)\Big)\ \mu(d\mathbf{q},s). \tag{4.10}$$

Let us define $H(\mathbf{q},s)=G\big(\psi_s^t(\mathbf{q})\big)$. By the group property of the flow ψ, we may write

$$G\big(\psi_{s'}^t(\mathbf{q})\big)=G\big(\psi_s^t\psi_{s'}^s(\mathbf{q})\big)=H\big(\psi_{s'}^s(\mathbf{q}),s\big).$$

In terms of H, (4.10) can be written as

$$-\lim_{s'\uparrow s}\ (s-s')^{-1}\int\Big(H\big(\psi_{s'}^s(\mathbf{q}),s\big)-H(\mathbf{q},s)\Big)\ \mu(d\mathbf{q},s). \tag{4.11}$$

If we simply write $K(\mathbf{q})$ for $H(\mathbf{q},s)$, then (4.11) reads as

$$-\lim_{s'\uparrow s}\int(s-s')^{-1}\Big(K\big(\psi_{s'}^s(\mathbf{q})\big)-K(\mathbf{q})\Big)\ \mu(d\mathbf{q},s). \tag{4.12}$$

Recall that the restriction of a function K to Δ_n^L is denoted by K^n, and that the measure $\mu(d\mathbf{q},s)$ restricted to Δ_n^L has a density $\mu^n(\mathbf{q},s)$ with respect to the Lebesgue measure. For (4.12), we need to evaluate

$$-\lim_{s'\uparrow s}\int(s-s')^{-1}\Big(K^n\big(\psi_{s'}^s(\mathbf{q})\big)-K^n(\mathbf{q})\Big)\ \mu^n(\mathbf{q},s)\ d\mathbf{q}. \tag{4.13}$$

Here we are using the fact that if $\mathbf{q} \in \Delta_n^L$ and $s - s'$ is sufficiently small, then $\psi_{s'}^s(\mathbf{q}) \in \Delta_n^L$. Note that if $K : \Delta_n^L \to \mathbb{R}$ were differentiable, then we would have had a simple candidate for the limit in (4.13), namely

$$-\int \mathcal{L}_{0n} K(\mathbf{q})\ \mu^n(\mathbf{q}, s)\ d\mathbf{q}. \tag{4.14}$$

To show that this is indded the limit, we need to examine the set in which K is C^1. For this, let us take any $\mathbf{q} \in \Delta_n^L$, and define $\sigma_1(\mathbf{q}) < \sigma_2(\mathbf{q}) < \cdots < \sigma_{n'}(\mathbf{q}), n' = n'(\mathbf{q})$ to be the times after s at which a collision between two particles occur. Evidently $n' \le n$. Set

$$\Lambda_n = \Lambda_n(s,t) = \big\{\mathbf{q} \in \Delta_n^L :\ \sigma_i(\mathbf{q}) = t \quad \text{for some } i \in \{1, \dots, n'(\mathbf{q})\}\big\}.$$

It is not hard to show that the set $\Lambda_n(s)$ is a C^1-subset of Δ_n^L of codimension one, and K is C^1 in $\Delta_n^L \setminus \Lambda_n$. By Lemma 3.1**(i)**, we know that the function

$$(s - s')^{-1}\big(K^n\big(\psi_{s'}^s(\mathbf{q})\big) - K^n(\mathbf{q})\big),$$

is uniformly bounded in the set Δ_n^L. As a result, we may use Bounded Convergence Theorem to assert

$$\lim_{s' \uparrow s}\ (s - s')^{-1} \int \big(K^n\big(\psi_{s'}^s(\mathbf{q})\big) - K^n(\mathbf{q})\big)\ \mu^n(\mathbf{q}, s)\ d\mathbf{q} = \int \mathcal{L}_{0n} K(\mathbf{q})\ \mu^n(\mathbf{q}, s)\ d\mathbf{q}. \tag{4.15}$$

(Step 3.) We now turn our attention to the first term on the right-hand side of (4.2). On account of (4.15), our claim (4.1) would follow if we can show

$$\lim_{s' \uparrow s} \left| \int \mathbb{E}\left[G\big(\varepsilon_{\rho_+} \psi_{s'}^\tau \mathbf{q}\big) - G\big(\varepsilon_{\rho_+} \mathbf{q}\big) \big|\ E_1\right]\ g(\hat{\rho}_n, \rho_+, s)\ d\rho_+\ \mu(d\mathbf{q}, s) \right| = 0, \tag{4.16}$$

simply because

$$\mathbb{E}\ \left[G\big(\varepsilon_{\rho_+} \mathbf{q}\big) \big|\ E_1\right] = G\big(\varepsilon_{\rho_+} \mathbf{q}\big).$$

Here we are using the fact that the event E_1 depends only on the stochastic boundary that is independent from the law of ρ_+.

It remains to verify (4.16). Let us we write $\sigma(\mathbf{q}, s')$ for the first time $\sigma > s'$ at which $\psi_{s'}^\sigma \mathbf{q}$ experiences a collision between particles of $\mathbf{q}$. Recall that $\mathbf{n}(\mathbf{q})$ denotes the number of particles of $\mathbf{q}$. We can readily show

$$\int \mathbb{1}\big(\sigma(\mathbf{q}, s') \le s\big)\ \mu(d\mathbf{q}, s) \le c_0(s - s') \int \mathbf{n}(\mathbf{q})\ \mu(d\mathbf{q}, s) \le c_1(s - s'), \tag{4.17}$$

with $c_0 = H'(P_+)$ which is an upper bound on the speed of particles. The bound (4.17) is an immediate consequence of the following two facts:

- If $\sigma(\mathbf{q}, s') \le s$, then for some i, we have $|x_i - x_{i+1}| \le c_0|s - s'|$, where $x_1 < \cdots < x_n$ denote the locations of the particles in $\mathbf{q}$.
- If we choose δ_0 so that $\lambda(\rho_-, s) \ge \delta_0$ for all ρ_-, then there exists a Poisson random variable N_{δ_0} of intensity $\delta_0 L$ such that $\mathbf{n}(\mathbf{q}) \le N_{\delta_0}$ almost surely.

Because of (4.17), the claim (4.16) is equivalent to

$$\lim_{s' \uparrow s} \big|X(s')\big| = 0, \tag{4.18}$$

where $X(s')$ is the expression

$$\sum_{n=0}^{\infty} \int_{\Delta_n^L} \int \mathbb{E}\left[G\big(\varepsilon_{\rho_+} \psi_{s'}^{\tau} \mathbf{q}\big) - G\big(\varepsilon_{\rho_+} \mathbf{q}\big)\big|\ E_1\right] 1\!\!1\big(\sigma(\mathbf{q}, s') > s\big)\ g(\hat{\rho}_n, \rho_+, s)\ d\rho_+\ \mu^n(\mathbf{q}, s)\ d\mathbf{q}.$$

On account of (4.4), the claim (4.18) would follow if we can show

$$\lim_{s' \uparrow s} (s - s')^{-1} \big|Y(s')\big| = 0, \tag{4.19}$$

where $Y(s') = Y_+(s') - Y_-(s')$, with

$$Y_+(s') = \sum_{n=0}^{\infty} \int_{\Delta_n^L} \int \mathbb{E}\ G\big(\varepsilon_{\rho_+} \psi_{s'}^{\tau} \mathbf{q}\big) 1\!\!1\big(\sigma(\mathbf{q}, s') > s > \tau(\mathbf{q}, s')\big)\ g(\hat{\rho}_n, \rho_+, s)\ d\rho_+\ \mu^n(\mathbf{q}, s)\ d\mathbf{q},$$

$$Y_-(s') = \sum_{n=0}^{\infty} \int_{\Delta_n^L} \int \mathbb{E}\ G\big(\varepsilon_{\rho_+} \mathbf{q}\big) 1\!\!1\big(\sigma(\mathbf{q}, s') > s > \tau(\mathbf{q}, s')\big)\ g(\hat{\rho}_n, \rho_+, s)\ d\rho_+\ \mu^n(\mathbf{q}, s)\ d\mathbf{q}.$$

(Final Step.) The expected value in the definition of $Y_\pm$ is for the random variable $\tau = \tau(\mathbf{q}, s')$. As was explained in Remark 2.1**(i)**, the variable τ can be expressed in terms of $\hat{\rho}_n$ and a standard exponential random variable. More precisely,

$$\tau = \tau(\mathbf{q}, s') = \ell(r, \hat{\rho}_n, s'),$$

with $r > 0$ a random variable with distribution $e^{-r}\ dr$, and $\ell(r, \hat{\rho}_n, s')$ denoting the inverse of the map

$$\tau \mapsto r = \int_{s'}^{\tau} \eta\big(\beta_{s'}^{\theta}(\hat{\rho}_n), \theta\big)\ d\theta, \qquad \tau \in (s', \infty).$$

Note we may replace the expected values in (4.19) with an integration with respect to $e^{-r}\ dr$. On the other hand,

$$1\!\!1(r > 0)\ e^{-r}\ dr = 1\!\!1(\tau > s')\ e^{-r} \eta\big(\beta_{s'}^{\tau}(\hat{\rho}_n), \tau\big)\ d\tau = 1\!\!1(\tau > s')\ \big(\eta\big(\hat{\rho}_n, s'\big) + O(\tau - s')\big)\ d\tau$$

Because of this, for (4.19), it suffices to show

$$\lim_{s'\uparrow s}(s-s')^{-1}\big|Z(s')\big|=0, \tag{4.20}$$

where $Z(s')=Z_+(s')-Z_-(s')$, with

$$Z_+(s')=\sum_{n=0}^{\infty}\int_{\Delta_n^L}\int\int_{s'}^{s}G\big(\varepsilon_{\rho_+}\psi_{s'}^{\tau}\mathbf{q}\big)\mathbb{1}\big(\sigma(\mathbf{q},s')>s\big)\eta\big(\hat\rho_n,s'\big)g(\hat\rho_n,\rho_+,s)\mu^n(\mathbf{q},s)\ d\tau d\rho_+ d\mathbf{q},$$

$$Z_-(s')=\sum_{n=0}^{\infty}\int_{\Delta_n^L}\int\int_{s'}^{s}G\big(\varepsilon_{\rho_+}\mathbf{q}\big)\mathbb{1}\big(\sigma(\mathbf{q},s')>s\big)\eta\big(\hat\rho_n,s'\big)g(\hat\rho_n,\rho_+,s)\mu^n(\mathbf{q},s)\ d\tau d\rho_+ d\mathbf{q}.$$

To prove (4.20), we carry out $d\mathbf{q}$ integration. Fix $\tau>0$ and ρ_+, and make a change of variables $\mathbf{q}'=\psi_{s'}^{\tau}\mathbf{q}$ for this integration. Recall the vector field $\mathbf{b}$, that was defined in Definition 3.1**(vi)**. Since the map $\mathbf{q}\mapsto\psi_{s'}^{\tau}\mathbf{q}$ is the flow of the ODE associated with vector field $\mathbf{b}$, its Jacobian has the expansion

$$1+(\tau-s')div(\mathbf{b})+\mathbf{n}(\mathbf{q})\ o(\tau-s').$$

Since $div(\mathbf{b})=O(\mathbf{n}(\mathbf{q}))$, a change of variable $\mathbf{q}'=\psi_{s'}^{\tau}\mathbf{q}$ causes a Jacobian factor of the form

$$1+\mathbf{n}(\mathbf{q})O(\tau-s')=1+\mathbf{n}(\mathbf{q})O(s-s').$$

Moreover, we can readily show,

$$\eta\big(\hat\rho_n,s'\big)g(\hat\rho_n,\rho_+,s)\mu^n(\mathbf{q},s)=\eta\big(\hat\rho_n',s'\big)g(\hat\rho_n',\rho_+,s)\mu^n(\mathbf{q}',s)\big(1+\mathbf{n}(\mathbf{q})O(s-s')\big).$$

From all this we deduce That $Z_+(s')=\hat Z_+(s')+Err$, where

$$\hat Z_+(s')=\sum_{n=0}^{\infty}\int_{\Delta_n^L}\int\int_{s'}^{s}G\big(\varepsilon_{\rho_+}\mathbf{q}'\big)\mathbb{1}\big(\sigma(\psi_{\tau}^{s'}\mathbf{q}',s')>s\big)\eta\big(\hat\rho_n',s'\big)g(\hat\rho_n,\rho_+',s)\mu^n(\mathbf{q}',s)\ d\tau d\rho_+ d\mathbf{q}'.$$

and the Err is an error term that satisfies

$$|Err|\le c_2(s-s')^2\int\mathbf{n}(\mathbf{q})^2\ \mu(d\mathbf{q},s)=c_3(s-s')^2.$$

By $\psi_{\tau}^{s'}$ we mean the inverse of $\psi_{s'}^{\tau}$. Renaming $\mathbf{q}'$ as $\mathbf{q}$ and comparing $\hat Z_+(s')$ with $Z_-(s')$ leads to

$$\hat Z_+(s')-Z_-(s')=\sum_{n=0}^{\infty}\int_{\Delta_n^L}\int\int_{s'}^{s}G\big(\varepsilon_{\rho_+}\mathbf{q}\big)\chi(\mathbf{q};s',\tau,s)\eta\big(\hat\rho_n,s'\big)g(\hat\rho_n,\rho_+,s)\mu^n(\mathbf{q},s)\ d\tau d\rho_+ d\mathbf{q},$$

where $\chi(\mathbf{q}; s', \tau, s) = 1\!\!1\big(\sigma(\psi_\tau^{s'}\mathbf{q}, s') > s\big) - 1\!\!1\big(\sigma(\mathbf{q}, s') > s\big)$. After replacing G with an upper bound, and carrying out the ρ_+ integration, we obtain

$$\big|\hat{Z}_+(s') - Z_-(s')\big| \le c_4 \int_{s'}^{s} \int \chi(\mathbf{q}; s', \tau, s)\ \mu(d\mathbf{q}, s)d\tau.$$

Finally, since $\chi(\mathbf{q}; s, s) = 0$, we can show

$$\lim_{s' \uparrow s} (s - s')^{-1} \big(\hat{Z}_+(s') - Z_-(s')\big) = 0,$$

completing the proof of (4.20), that in turn completes the proof of Theorem. □

5 Proof of Theorem 2.1

Without loss of generality, we may assume that $a = 0$. The proof of Theorem 2.1 is carried out in three steps that were described right after (3.5). For the first step, we calculate the time derivative our candidate measure $\mu^n = \mu^n(s)$ that was defined in Definition 2.2**(ii)**. To simplify our presentation, we assume that $\ell = \ell^0$ has a density with respect to the Lebesgue measure. With a slight abuse of notation, we write $\ell(\rho, s)$ for this density: $\ell(d\rho, s) = \ell(\rho, s)\ d\rho$.

Proposition 5.1 *We have that* $\dot{\mu}^n = X^n \mu^n$, *for* $X^n = \sum_{i=1}^{9} X_i^n$, *where*

$$\begin{aligned}
X_1^n &= X_1 = \frac{\int H(\rho_*, \rho_0) f(\rho_*, \rho_0, s)\ \ell(d\rho_*, s)}{\ell(\rho_0, s)} \\
X_2^n &= X_2 = -\frac{\big(H'(\rho_0) b(\rho_0, s) \ell(\rho_0, s)\big)_{\rho_0}}{\ell(\rho_0, s)} \\
X_3^n &= \sum_{i=1}^{n} \frac{Q^+(f,f)(\hat{\rho}_{i-1}, \rho_i, s)}{f(\hat{\rho}_{i-1}, \rho_i, s)} \\
X_4^n &= H'(\rho_0)\big(\lambda(\hat{\rho}_0, s) - \lambda(\rho_0, s)\big) - A(\hat{\rho}_n, s) \\
X_5^n &= \sum_{i=1}^{n} \big(H'(\rho_i) - H(\hat{\rho}_{i-1}, \rho_i)\big)\big(\lambda(\hat{\rho}_i, s) - \lambda(\rho_i, s)\big) \\
X_6^n &= \sum_{i=1}^{n} H(\hat{\rho}_{i-1}, \rho_i)\big(\lambda(\hat{\rho}_i, s) - \lambda(\hat{\rho}_{i-1}, s)\big) \\
X_7^n &= \sum_{i=1}^{n} \frac{\Big[\big(H(\hat{\rho}_{i-1}, \rho_i) - H'(\rho_i)\big) b(\rho_i, s) f(\hat{\rho}_{i-1}, \rho_i, s)\Big]_{\rho_i}}{f(\hat{\rho}_{i-1}, \rho_i, s)} \\
X_8^n &= \sum_{i=1}^{n} b(\hat{\rho}_{i-1}, s) H_{\rho_-}(\hat{\rho}_{i-1}, \rho_i) \\
X_9^n &= \sum_{i=1}^{n} \big[H(\hat{\rho}_{i-1}, \rho_i) - H'(\rho_{i-1})\big] b(\hat{\rho}_{i-1}, s) \frac{f_{\rho_-}(\hat{\rho}_{i-1}, \rho_i, s)}{f(\hat{\rho}_{i-1}, \rho_i, s)}.
\end{aligned}$$

Proof Direct differentiation yields

$$X^n = -\Gamma_s(\mathbf{q}, s) + \frac{\ell_s(\rho_0, s)}{\ell(\rho_0, s)} + \sum_{i=1}^{n} \frac{[f(\hat{\rho}_{i-1}, \rho_i, s)]_s}{f(\hat{\rho}_{i-1}, \rho_i, s)}. \tag{5.1}$$

Moreover, from (1.13), we can readily show

$$\frac{\ell_s}{\ell} = X_1 + X_2 - A(\rho_0, s). \tag{5.2}$$

On the other hand,

$$\Gamma_s(\mathbf{q}, s) = \sum_{i=0}^{n} \Big\{ \big(A(\hat{\rho}_i, s) - A(\rho_i, s)\big) - H'(\rho_i)\big(\lambda(\hat{\rho}_i, s) - \lambda(\rho_i, s)\big) \Big\},$$

by (2.10). From this, (5.1), (5.2), and (2.15) we deduce

$$X^n = \sum_{i=0}^{n} \big\{ \big(A(\rho_i, s) - A(\hat\rho_i, s)\big) + H'(\rho_i)\big(\lambda(\hat\rho_i, s) - \lambda(\rho_i, s)\big) \big\}$$
$$+ X_1 + X_2 - A(\rho_0, s) + \sum_{i=1}^{n} \frac{f_s(\hat\rho_{i-1}, \rho_i, s)}{f(\hat\rho_{i-1}, \rho_i, s)} + Z^n,$$

where

$$Z^n = \sum_{i=1}^{n} \big[H'(\hat\rho_{i-1}) - H'(\rho_{i-1})\big] b(\hat\rho_{i-1}, s) \frac{f_{\rho_-}(\hat\rho_{i-1}, \rho_i, s)}{f(\hat\rho_{i-1}, \rho_i, s)}.$$

Let us set

$$W^n = \sum_{i=1}^{n} \frac{(Cf)(\hat\rho_{i-1}, \rho_i, s)}{f(\hat\rho_{i-1}, \rho_i, s)}.$$

We now use the kinetic equation and the form of Q^- to obtain

$$X^n = X_1^n + X_2^n + X_3^n + X_4^n + Z^n + W^n$$
$$+ \sum_{i=1}^{n} \big\{ H'(\rho_i)\big(\lambda(\hat\rho_i, s) - \lambda(\rho_i, s)\big) + H(\hat\rho_{i-1}, \rho_i)\big(\lambda(\rho_i, s) - \lambda(\hat\rho_{i-1}, s)\big) \big\}$$
$$= X_1^n + X_2^n + X_3^n + X_4^n + X_5^n + X_6^n + Z^n + W^n.$$

To simplify this further, we use the definition of the operator C to write

$$W^n = X_7^n + X_8^n + \sum_{i=1}^{n} \big[H(\hat\rho_{i-1}, \rho_i) - H'(\hat\rho_{i-1})\big] b(\hat\rho_{i-1}, s) \frac{f_{\rho_-}(\hat\rho_{i-1}, \rho_i, s)}{f(\hat\rho_{i-1}, \rho_i, s)}.$$

We are done because the last term plus Z^n is exactly X_9^n. □

Armed with Theorem 4.1 and Proposition 5.1, we are now ready to present the proof of our main result:

Proof of Theorem 2.1 As we demonstrated in Section 3, we only need to establish (3.7). Recall that we write $G(\mathbf{q}, s)$ for $\mathbb{E}\ G\big(\Psi_s^t \mathbf{q}\big)$. Evidently,

$$(5.3) \qquad (s - s')^{-1} \left[\int G(\mathbf{q}, s)\ \mu(d\mathbf{q}, s) - \int G(\mathbf{q}, s')\ \mu(d\mathbf{q}, s') \right] = \Omega_1(s') + \Omega_2(s') - \Omega_3(s'),$$

where

$$\Omega_1(s') = (s-s')^{-1}\int \big(G(\mathbf{q},s)-G(\mathbf{q},s')\big)\mu(d\mathbf{q},s)$$
$$\Omega_2(s') = (s-s')^{-1}\int G(\mathbf{q},s)\ \big(\mu(d\mathbf{q},s)-\mu(d\mathbf{q},s')\big)$$
$$\Omega_3(s') = (s-s')^{-1}\int \big(G(\mathbf{q},s)-G(\mathbf{q},s')\big)\ \big(\mu(d\mathbf{q},s)-\mu(d\mathbf{q},s')\big).$$

With the aid of Lemma 3.1 and Proposition 5.1, we can show

$$\limsup_{s'\uparrow s}\ (s-s')^{-1}\big|\Omega_3(s')\big| \le C_0\int \big(\mathbf{n}(\mathbf{q})+1\big)\ (s-s')^{-1}\left|\big(\mu(d\mathbf{q},s)-\mu(d\mathbf{q},s')\big)\right|$$
$$\le c_1C_0(s-s')\int \big(\mathbf{n}(\mathbf{q})+1\big)\mathbf{n}(\mathbf{q})\ \mu(d\mathbf{q},s)\le c_2,$$

for constants c_1 and c_2. (For example, c_1 is a uniform bound on X^n of Proposition 5.1.) As a result,

$$\lim_{s'\uparrow s}\big|\Omega_3(s')\big| = 0. \tag{5.4}$$

By Proposition 5.1, we also know

$$\lim_{s'\downarrow s}\Omega_2(s') = \sum_{i=1}^{9}\int G(\mathbf{q},s)X_i^n(\mathbf{q})\ \mu^n(d\mathbf{q},s). \tag{5.5}$$

On the other hand, by Theorem 4.1,

$$\lim_{s'\downarrow s}\Omega_1(s') = -\sum_{n=0}^{\infty}\int \big(\mathcal{L}_0G+\mathcal{L}_bG\big)(\mathbf{q},s)\ \mu(d\mathbf{q},s) =: -(Y_0+Y_b). \tag{5.6}$$

Observe

$$Y_b = \sum_{n=0}^{\infty}\int\int H(\hat\rho_n,\rho_+)f(\hat\rho_n,\rho_+,s)\big(G^{n+1}\big(\varepsilon_{\rho_+}\mathbf{q},s\big)-G^n(\mathbf{q},s)\big)\mu^n(\mathbf{q},s)\ d\mathbf{q}\ d\rho_+$$
$$= \sum_{n=1}^{\infty}Y_{b,+}^n - \sum_{n=0}^{\infty}Y_{b,-}^n, \tag{5.7}$$

where

$$Y_{b,+}^n = \int\int H(\hat\rho_{n-1},\rho_+)f(\hat\rho_{n-1},\rho_+,s)G^n\big(\varepsilon_{\rho_+}\mathbf{q},s\big)\mu^{n-1}(\mathbf{q},s)\ d\mathbf{q}\ d\rho_+$$
$$Y_{b,-}^n = \int A(\hat\rho_n,s)G^n(\mathbf{q},s)\mu^n(\mathbf{q},s)\ d\mathbf{q}.$$

We now concentrate on Y_0. We wish to integrate by parts and apply $\mathcal{L}_0^*$ on μ. Recall the function $G(\mathbf{q}, s)$ is only piecewise C^1; it is continuously differentiable in the complement of a finite union of C^1 manifolds of codimension 1. This union, denoted by $\Lambda = \Lambda(s,t)$, is exactly the set of points $\mathbf{q}$ for which a collision occurs at time t. However, since $G(\mathbf{q}, s)$ is continuous, there will be no boundary contribution coming from the non-differentiability set Λ. Because of this, the only boundary contributions come from the boundary of the set Δ_n^L. In other words, if we write

$$Y_0 = \sum_{n=0}^{\infty} Y_0^n := \sum_{n=0}^{\infty} \int_{\Delta_n^L} \big(\mathcal{L}_{0,n} G\big)^n(\mathbf{q}, s)\ \mu^n(\mathbf{q}, s)\ d\mathbf{q}, \tag{5.8}$$

then

$$Y_0^n = Y_{01}^n + Y_{02}^n := \int_{\Delta_n^L} G^n(\mathbf{q}, s)\ \mathcal{L}_{0,n}^* \mu^n(\mathbf{q}, s)\ d\mathbf{q} + \sum_{i=0}^{n} Y_{02i}^n + \hat{Y}_{02}^n, \tag{5.9}$$

where Y_{0i}^n is the boundary contribution coming from the condition $x_i = x_{i+1}$, and $\hat{Y}_{02}^n$ is the boundary contribution coming from the condition $x_n = L$. Note carefully that Y_{0i}^n, for $i = 0, \dots, n$, comes from boundary terms as integrate by parts with respect to an integration over Δ_{n+1}^L with configurations of $n+1$ particles $\big(x_0, \rho_0), \dots, (x_{n+1}, \rho_{n+1}\big)$, so that when $x_i = x_{i+1}$ for some $i \in \{0, 1, \dots, n\}$, then we obtain a configuration in Δ_n^L. However, for $\hat{Y}_{02}^n$ we integrate by parts with respect to an integration over Δ_n^L with configurations of n particles $\big(x_0, \rho_0), \dots, (x_n, \rho_n\big)$; when $x_n = L$ we still regard this configuration as a member of $\bar{\Delta}_n^L$.

Indeed $\mathcal{L}_{0,n}^* \mu^n = Z^n \mu^n$, with

$$Z^n = Z_{11} + Z_{12} + Z_{13} + Z_{21}^n + Z_{22}^n + Z_{23}^n + Z_{31}^n + Z_{32}^n, \tag{5.10}$$

where

$$
\begin{aligned}
Z_{11} &= H'(\rho_0)b(\rho_0,t)\Gamma_\rho(\rho_0,x_1,t) = H'(\rho_0)\big(\lambda(\hat\rho_0,t)-\lambda(\rho_0,t)\big)\\
Z_{12} &= -H'(\rho_0)b(\rho_0,t)\frac{\big[f\big(\hat\rho_0,\rho_1,t\big)\big]_{\rho_0}}{f\big(\hat\rho_0,\rho_1,t\big)}\\
Z_{13} &= -\frac{\big(H'(\rho_0)b(\rho_0,t)\ell(\rho_0,t)\big)_{\rho_0}}{\ell(\rho_0,t)}\\
Z_{21}^n &= \sum_{i=1}^n \big(H'(\rho_i)-H(\hat\rho_{i-1},\rho_i)\big)b(\rho_i,t)\Gamma_\rho(\rho_i,x_{i+1}-x_i,t)\\
&= \sum_{i=1}^n \big(H'(\rho_i)-H(\hat\rho_{i-1},\rho_i)\big)\big(\lambda(\hat\rho_i,t)-\lambda(\rho_i,t)\big),\\
Z_{22}^n &= \sum_{i=1}^{n-1}\frac{\big[\big(H(\hat\rho_{i-1},\rho_i)-H'(\rho_i)\big)b(\rho_i,t)f(\hat\rho_{i-1},\rho_i,t)f(\hat\rho_i,\rho_{i+1},t)\big]_{\rho_i}}{f\big(\hat\rho_{i-1},\rho_i,t\big)f\big(\hat\rho_i,\rho_{i+1},t\big)}\\
Z_{23}^n &= \frac{\big[\big(H(\hat\rho_{n-1},\rho_n)-H'(\rho_n)\big)b(\rho_n,t)f(\hat\rho_{n-1},\rho_n,t)\big]_{\rho_n}}{f\big(\hat\rho_{n-1},\rho_n,t\big)},\\
Z_{31}^n &= \sum_{i=1}^{n-1}\frac{\big[H\big(\hat\rho_{i-1},\rho_i\big)f\big(\hat\rho_{i-1},\rho_i,t\big)f\big(\hat\rho_i,\rho_{i+1},t\big)\big]_{x_i}}{f\big(\hat\rho_{i-1},\rho_i,t\big)f\big(\hat\rho_i,\rho_{i+1},t\big)} + \frac{\big[H\big(\hat\rho_{n-1},\rho_n\big)f\big(\hat\rho_{n-1},\rho_n,t\big)\big]_{x_n}}{f\big(\hat\rho_{n-1},\rho_n,t\big)},\\
Z_{32}^n &= \sum_{i=1}^n H\big(\hat\rho_{i-1},\rho_i\big)\,\big(\lambda\big(\hat\rho_i,t\big)-\lambda\big(\hat\rho_{i-1},t\big)\big)\,.
\end{aligned}
$$

Here,

- The sum $Z_{11}+Z_{12}+Z_{13}$ comes from an integration by parts with respect to the variable ρ_0. The dynamics of ρ_0 as in rule **(2)** of Definition 2.1**(iii)** is responsible for this contribution. We have used (2.7) for the second equality on the first line.
- The i-terms in Z_{21}^n, Z_{22}^n and Z_{23}^n come from an integration by parts with respect to the variable ρ_i. The dynamics of ρ_i as in rule **(2)** of Definition 2.1**(iii)** is responsible for these three contributions. The equality on line 5 is a consequence of (2.7).
- The i-th terms in Z_{31}^n and Z_{32}^n come from an integration by parts with respect to the variable x_i. The dynamics of x_i as in rule **(1)** of Definition 2.1**(iii)** is responsible for this contribution.

On the other hand, for $i=0,\dots,n$

$$
Y_{02i}^n = \int_{\Delta_n^L} G^n(\mathbf{q},s)W_i^n(\mathbf{q},s)\mu^n(\mathbf{q},s)\ d\mathbf{q}, \tag{5.11}
$$

where

$$W_0^n = W_0 = \frac{\int H(\rho_*, \rho_0) f(\rho_*, \rho_0, s)\ \ell(d\rho_*, s)}{\ell(\rho_0, s)}, \qquad W_i^n = \frac{Q^+(f,f)\big(\hat\rho_{i-1}, \rho_i, t\big)}{f\big(\hat\rho_{i-1}, \rho_i, t\big)},$$

for $i = 1, \dots, n$. Here,

- The term W_0^n comes from the boundary term $x_1 = 0$ in the integration by parts with respect to the variable x_1. This boundary condition represents the event that x_1 has reached the origin after which ρ_0 becomes ρ_1, and (x_i, ρ_i) is relabeled as (x_{i-1}, ρ_{i-1}) for $i \geq 2$.
- The term W_i^n comes from the boundary term $x_i = x_{i+1}$. The relative distance $x_{i+1} - x_i$ travels with speed
$$-\big[H(\hat\rho_i, \rho_{i+1}) - H(\hat\rho_{i-1}, \rho_i)\big],$$
As x_{i+1} catches up with x_i, the particle x_i disappears and its density $\rho_i = \hat\rho_i$ is renamed ρ_*, and is integrated out. (The resulting integral is $Q^+(f,f)\big(\hat\rho_{i-1}, \rho_i, t\big)$.) We then relabel (x_j, ρ_j), $j > i$, as (x_{j-1}, ρ_{j-1}).

As for $\hat Y_{02}^n$, we simply have

$$\hat Y_{02}^n = -Y_{b,+}^n, \tag{5.12}$$

where $Y_{b,+}^n$ was defined by (5.7).

Recall that we wish to establish (3.7). From (5.3)-(5.12), we learn that for (3.7) it suffices to verify the equality

$$\sum_{i=1}^{9} X_i^n = Z_{11} + Z_{12} + Z_{13} + Z_{21}^n + Z_{22}^n + Z_{23}^n + Z_{31}^n + Z_{32}^n - A(\hat\rho_n, s) + W_0 + W^n, \tag{5.13}$$

where

$$W^n = \sum_{i=1}^{n} W_i^n.$$

Since

$$X_1 = W_0, \quad X_2 = Z_{13}, \quad X_3^n = W^n, \quad X_4^n = Z_{11} - A(\hat\rho_n, s), \quad X_5^n = Z_{21}^n, \quad X_6^n = Z_{32}^n,$$

the equality (5.13) is equivalent to the equality

$$X_7^n + X_8^n + X_9^n = Z_{12} + Z_{22}^n + Z_{23}^n + Z_{31}^n. \tag{5.14}$$

Observe that $Z_{22}^n + Z_{23}^n = \widehat{Z}_{22}^n + \widehat{Z}_{23}^n$, and $Z_{31}^n = Z_{311}^n + Z_{312}^n + Z_{313}^n$, where

$$\begin{aligned}
\widehat{Z}_{22}^n &= \sum_{i=1}^{n} \frac{\big[\big(H(\hat\rho_{i-1},\rho_i) - H'(\rho_i)\big)b(\rho_i,t)f(\hat\rho_{i-1},\rho_i,t)\big]_{\rho_i}}{f(\hat\rho_{i-1},\rho_i,t)},\\
\widehat{Z}_{23}^n &= \sum_{i=1}^{n-1} \big(H(\hat\rho_{i-1},\rho_i) - H'(\rho_i)\big)b(\rho_i,t)\frac{\big[f(\hat\rho_i,\rho_{i+1},t)\big]_{\rho_i}}{f(\hat\rho_i,\rho_{i+1},t)},\\
Z_{311}^n &= \sum_{i=1}^{n} \big[H\big(\hat\rho_{i-1},\rho_i\big)\big]_{x_i} = \sum_{i=1}^{n} b(\hat\rho_{i-1},t)H_{\rho_-}\big(\hat\rho_{i-1},\rho_i\big),\\
Z_{312}^n &= \sum_{i=1}^{n} H\big(\hat\rho_{i-1},\rho_i\big)\frac{\big[f\big(\hat\rho_{i-1},\rho_i,t\big)\big]_{x_i}}{f\big(\hat\rho_{i-1},\rho_i,t\big)},\\
Z_{313}^n &= \sum_{i=1}^{n-1} H\big(\hat\rho_{i-1},\rho_i\big)\frac{\big[f\big(\hat\rho_i,\rho_{i+1},t\big)\big]_{x_i}}{f\big(\hat\rho_i,\rho_{i+1},t\big)}.
\end{aligned}$$

Since $X_7^n = \hat{Z}_{22}^n$ and $X_8^n = Z_{311}^n$, the equality (5.14) is equivalent to $X_9^n = \widehat{Z}^n$, for

$$\widehat{Z}^n := Z_{12} + \widehat{Z}_{23}^n + Z_{312}^n + Z_{313}^n.$$

By the group property (2.6),

$$\big(h(\hat\rho_{i-1})\big)_{x_i} = b(\rho_{i-1},t)\big(h(\hat\rho_{i-1})\big)_{\rho_{i-1}}, \qquad \big(h(\hat\rho_i)\big)_{x_i} = -b(\rho_i,t)\big(h(\hat\rho_i)\big)_{\rho_i}.$$

This allows us to write

$$\begin{aligned}
Z_{312}^n + Z_{313}^n = \sum_{i=1}^{n-1} H\big(\hat\rho_{i-1},\rho_i\big)&\left\{b(\rho_{i-1},t)\frac{\big[f\big(\hat\rho_{i-1},\rho_i,t\big)\big]_{\rho_{i-1}}}{f\big(\hat\rho_{i-1},\rho_i,t\big)} - b(\rho_i,t)\frac{\big[f\big(\hat\rho_i,\rho_{i+1},t\big)\big]_{\rho_i}}{f\big(\hat\rho_i,\rho_{i+1},t\big)}\right\}\\
&+ H\big(\hat\rho_{n-1},\rho_n\big)b(\rho_{n-1},t)\frac{\big[f\big(\hat\rho_{n-1},\rho_i,t\big)\big]_{\rho_{n-1}}}{f\big(\hat\rho_{n-1},\rho_n,t\big)}.
\end{aligned}$$

Hence

$$\begin{aligned}
\widehat{Z}^n &= -\sum_{i=0}^{n-1} H'(\rho_i)b(\rho_i,t)\frac{\big[f(\hat\rho_i,\rho_{i+1},t)\big]_{\rho_i}}{f(\hat\rho_i,\rho_{i+1},t)} + \sum_{i=1}^{n} H\big(\hat\rho_{i-1},\rho_i\big)b(\rho_{i-1},t)\frac{\big[f\big(\hat\rho_{i-1},\rho_i,t\big)\big]_{\rho_{i-1}}}{f\big(\hat\rho_{i-1},\rho_i,t\big)}\\
&= \sum_{i=1}^{n} \big[H\big(\hat\rho_{i-1},\rho_i\big) - H'(\rho_{i-1})\big]\, b(\rho_{i-1},t)\frac{\big[f\big(\hat\rho_{i-1},\rho_i,t\big)\big]_{\rho_{i-1}}}{f\big(\hat\rho_{i-1},\rho_i,t\big)}\\
&= \sum_{i=1}^{n} \big[H\big(\hat\rho_{i-1},\rho_i\big) - H'(\rho_{i-1})\big]\, b(\hat\rho_{i-1},t)\frac{f_{\rho_-}\big(\hat\rho_{i-1},\rho_i,t\big)}{f\big(\hat\rho_{i-1},\rho_i,t\big)} = X_9^n,
\end{aligned}$$

where we have used (2.6) for the third equality. This completes the proof. □

6 The Kinetic Equation

In this section we develop a well-posedness result for the kinetic equation (1.7) which is adequate for our present purposes. Consider the following integro-PDE for $f(p_-, p_+, t)$:

$$f_t + V \cdot \nabla f = Q(f, f) + M(p_-, p_+, t)f \tag{6.1}$$

where Q is as in Definition 1.1**(iii)**, $V = \big(V^-, V^+\big)$, and

$$M(p_-, p_+, t) = H_{p_-}(p_-, p_+)b(p_-, t) - V^+_{p_+},$$

with

$$V^\pm(p_-, p_+, t) = \big(H'(p_\pm) - H(p_-, p_+)\big)b(p_\pm, t). \tag{6.2}$$

In (6.1) we have chosen to express (6.2) with only transport terms on the left-hand side; the two are formally equivalent.

6.1 Estimates

For functions (or kernels) f and g we write

$$Q(f, g) = Q^+(f, g) - L(g)f + N(g)f$$

where

$$\begin{aligned}
Q^+(f, g)(p_-, p_+) &= \int (H(p_*, p_+) - H(p_-, p_*))f(p_-, p_*)g(p_*, p_+)\, dp_* \\
L(g)(p_-, p_+) &= \int (H(p_+, p_*) - H(p_-, p_+))g(p_+, p_*)\, dp_* \\
N(g)(p_-, p_+) &= \int (H(p_-, p_*) - H(p_-, p_+))g(p_-, p_*)\, dp_*.
\end{aligned}$$

We write also

$$\begin{aligned}
Q_{p_\pm}(f, g) &= Q^+_{p_\pm}(f, g) - L_{p_\pm}(g)f + N_{p_\pm}(g)f \\
Q^+_{p_\pm}(f, g) &= \pm \int H_{p_\pm}(p_\pm, p_*)f(p_-, p_*)g(p_*, p_+)\, dp_* \\
L_{p_-}(g) &= \int -H_{p_-}(p_-, p_+)g(p_+, p_*)\, dp_* \\
L_{p_+}(g) &= \int \big(H_{p_+}(p_+, p_*) - H_{p_+}(p_-, p_+)\big)g(p_+, p_*)\, dp_* \\
N_{p_-}(g) &= \int \big(H_{p_-}(p_-, p_*) - H_{p_-}(p_-, p_+)\big)g(p_-, p_*)\, dp_* \\
N_{p_+}(g) &= -\int H_{p_+}(p_-, p_+)g(p_-, p_*)\, dp_*.
\end{aligned}$$

Lemma 6.1 *Suppose that $H'(p)$ and $H''(p)$ are bounded and we have absolutely continuous kernels determined by functions $f(p_-, p_+)$ and $g(p_-, p_+)$. Writing*

$$\|f\|_{L^\infty(L^1)} = ess \sup_{p_-} \int |f(p_-, p_+)| \, dp_+,$$

we have the following estimates.

(i) *Components Q^+, L, N:*

$$\begin{aligned}
\|Q^+(f,g)\|_{L^\infty} &\le \|H'\|_{L^\infty(L^1)} \|f\|_{L^\infty(L^1)} \|g\|_{L^\infty(L^1)} \\
\|Q^+(f,g)\|_{L^\infty} &\le \|H'\|_{L^\infty} \|f\|_{L^\infty(L^1)} \|g\|_{L^\infty} \\
\|L(g)\|_{L^\infty} &\le \|H'\|_{L^\infty} \|g\|_{L^\infty(L^1)} \\
\|N(g)\|_{L^\infty} &\le \|H'\|_{L^\infty} \|g\|_{L^\infty(L^1)}
\end{aligned}$$

(ii) *Q as a whole:*

$$\begin{aligned}
\|Q(f,g)\|_{L^\infty(L^1)} &\le 3\|H'\|_{L^\infty} \|f\|_{L^\infty(L^1)} \|g\|_{L^\infty(L^1)} \\
\|Q(f,g)\|_{L^\infty} &\le \|H'\|_{L^\infty} \big(\|f\|_{L^\infty(L^1)} \|g\|_{L^\infty} + 2\|f\|_{L^\infty} \|g\|_{L^1(L^\infty)} \big)
\end{aligned}$$

(iii) *Derivative with respect to p_-:*

$$\begin{aligned}
\|Q(f,g)_{p_-}\|_{L^\infty(L^1)} &\le \frac{3}{2} \|H''\|_{L^\infty} \|f\|_{L^\infty(L^1)} \|g\|_{L^\infty(L^1)} \\
&\quad + \|H'\|_{L^\infty} \big(3\|f_{p_-}\|_{L^\infty(L^1)} \|g\|_{L^\infty(L^1)} + \|f\|_{L^\infty(L^1)} \|g_{p_-}\|_{L^\infty(L^1)} \big) \\
\|Q(f,g)_{p_-}\|_{L^\infty} &\le \frac{1}{2} \|H''\|_{L^\infty} \big(\|f\|_{L^\infty(L^1)} \|g\|_{L^\infty} + 2\|f\|_{L^\infty} \|g\|_{L^1(L^\infty)} \big) \\
&\quad + \|H'\|_{L^\infty} \big(\|f_{p_-}\|_{L^\infty(L^1)} \|g\|_{L^\infty} + 2\|f_{p_-}\|_{L^\infty} \|g\|_{L^1(L^\infty)} \big) \\
&\quad + \|H'\|_{L^\infty} \|f\|_{L^\infty} \|g_{p_-}\|_{L^1(L^\infty)}.
\end{aligned}$$

(iv) *Derivative with respect to p_+:*

$$\begin{aligned}
\|Q(f,g)_{p_+}\|_{L^\infty(L^1)} &\le \frac{3}{2} \|H''\|_{L^\infty} \|f\|_{L^\infty(L^1)} \|g\|_{L^\infty(L^1)} \\
&\quad + 2\|H'\|_{L^\infty} \big(\|f\|_{L^\infty(L^1)} \|g_{p_+}\|_{L^\infty(L^1)} + \|f_{p_+}\|_{L^\infty(L^1)} \|g\|_{L^\infty(L^1)} \big) \\
\|Q(f,g)_{p_+}\|_{L^\infty} &\le \frac{1}{2} \|H''\|_{L^\infty} \big(\|f\|_{L^\infty(L^1)} \|g\|_{L^\infty} + 2\|f\|_{L^\infty} \|g\|_{L^1(L^\infty)} \big) \\
&\quad + \|H'\|_{L^\infty} \big(\|f\|_{L^\infty(L^1)} \|g_{p_+}\|_{L^\infty} + \|f\|_{L^\infty} \|g_{p_+}\|_{L^\infty(L^1)} \big) \\
&\quad + 2\|H'\|_{L^\infty} \|f_{p_+}\|_{L^\infty} \|g\|_{L^\infty(L^1)}
\end{aligned}$$

Proof The proof in **(i)** is routine and omitted. The proof of **(ii)** follows from adding the component estimates in **(i)**. For **(iii)** and **(iv)** we observe (formally for now, but see below the proof of Theorem 6.1) that

$$Q(f,g)_{p_-} = Q_{p_-}(f,g) + Q^+(f_{p_-},g) - L(g)f_{p_-} + N(g)f_{p_-} + N(g_{p_-})f$$
$$Q(f,g)_{p_+} = Q_{p_+}(f,g) + Q^+(f,g_{p_+}) - L(g_{p_+})f - L(g)f_{p_+} + N(g)f_{p_+}.$$

Now $Q_{p_\pm}$, $L_\pm$, and $N_\pm$, admit estimates similar to those of Q, L, and N respectively, with $\frac{1}{2}H''$ replacing H', and in the remaining terms we simply use whichever estimates from **(i)** are appropriate. □

6.2 Classical solutions

Now consider the space of functions $f(p_-,p_+)$

$$X = \{f \in C_0^1(\mathbb{R}^2,\mathbb{R}) : f, f_{p_-}, f_{p_+} \in L^\infty(L^1)\cap L^\infty\}$$

equipped with norm

$$\|f\|_X = \|f\|_{L^\infty(L^1)} + \|f\|_{L^\infty} + \|f_{p_-}\|_{L^\infty(L^1)} + \|f_{p_-}\|_{L^\infty} + \|f_{p_+}\|_{L^\infty(L^1)} + \|f_{p_+}\|_{L^\infty}.$$

Lemma 6.2 *Concerning the space X, the quadratic operator Q, and the multiplication operator associated with M, we have the following:*

(i) $(X, \|\cdot\|_X)$ *is a Banach space.*

(ii) $\|Q(f,f)\|_X \le C_1\|f\|_X^2$ *for a constant* $C_1 = C_1(\|H'\|_{L^\infty}, \|H''\|_{L^\infty})$.

(iii) $\|M(\cdot,\cdot,t)f\|_X \le C_2\|f\|_X$ *for a constant*

$$C_2 = C_2(t; \|H^{(k)}\|_\infty, k=0,1,2,3; \|b(\cdot,t)\|_\infty; \|b_p(\cdot,t)\|_\infty),$$

which admits a uniform bound over bounded interval of time prior to blow-up of b *(if any).*

(iv) *For given* $f_0 \in X$ *there exists* $T>0$ *(depending only on* $\|f_0\|_X$ *and* C_1, C_2 *from* **(ii)** *and* **(iii)***) such that there is a unique solution* $f \in C^1([0,T],X)$ *to* $f_t = Q(f,f) + Mf$ *with* $f = f_0$ *at* $t=0$.

Proof (i) It is straightforward to show that $Q(f,f) \in C_0^1(\mathbb{R}^2,\mathbb{R})$. The only other issue is completeness. If a sequence f_n is Cauchy in X, it is also Cauchy in the Banach space $C_0^1(\mathbb{R}^2,\mathbb{R})$ (which corresponds to the L^∞ part of the X-norm), and we thereby obtain a limit $f \in C_0^1(\mathbb{R}^2,\mathbb{R})$, including consistency of the limits of f_n and the derivatives of f_n. We then rely on completeness of $L^\infty(L^1)$ *separately* for each of f_n, $(f_n)_{p_-}$, and $(f_n)_{p_+}$, to verify the convergence holds in X.

(ii) This is immediate from the estimates in Lemma 6.1.

(iii) This is obvious.

(iv) The bounds in **(ii)** and **(iii)** imply $f \mapsto Q(f,f) + Mf$ is locally Lipschitz, and the standard Picard theorem (in the Banach-valued setting) applies. $\square$

Lemma 6.3 *Concerning the local solution $f \in C^1([0,T], X)$ constructed in Lemma 6.2:*

(i) *If the initial kernel $f_0 \geq 0$, then there exists $T_+ \in (0,T]$ such that $f \geq 0$ for $t \in [0, T_+)$.*

(ii) *For each p_-, we have*

$$\partial_t \int f(p_-, p_+, t)\, dp_+ \leq \|M(\cdot,\cdot,t)\|_{L^\infty} \int |f(p_-,p_+,t)|\, dp_+. \tag{6.3}$$

Proof(i) Set $h = L(f) - N(f) + M$ and $c = \sup\left\{|h(p_-,p_+,t)| : \ (p_-,p_+,t) \in \mathbb{R}^2 \times [0,T]\right\}$ which is a finite constant. Define

$$k(p_-,p_+,t) = \exp\left(\int_0^t h(p_-,p_+,s)\, ds\right),$$

and observe that the function $\tilde{f} = kf$. is the unique solution to the related equation

$$\tilde{f}_t = kQ^+\big(k^{-1}\tilde{f}, k^{-1}\tilde{f}\big).$$

Using Lemma 6.1 this admits estimates similar to $f_t = Q(f,f)$ (adjusted by a constant factor e^{3cT}) and so we can obtain a local solution (possibly over a shorter time interval $[0,T_+)$) using Picard iteration. Noting that Q^+ preserves positivity, we are done.

(ii) Swapping the labels p_* and p_+ in $L(f)f$ we find $\int \big(Q^+(f,f) - L(f)f\big)\, dp_+ = 0$, and $\int N(f)f\, dp_+ = 0$ by symmetry. So the only nonzero contribution to the integral is from $\int Mf\, dp_+$, which we bound as in (6.3). $\square$

Theorem 6.1 *For any nonnegative initial kernel $f^0 \in X$ the problem*

$$f_t = Q(f,f) + Mf, \qquad f|_{t=0} = f^0,$$

has a unique nonnegative classical solution $f(p_-,p_+,t)$ defined for all $t \geq 0$ prior to blow-up of b (if any), and for each p_- we have

$$\lambda(p_-,t) = \int f(p_-,p_+,t)\, dp_+$$

growing at most linearly in time.

Proof Lemma 6.2 gives a classical solution defined locally in time, and using Lemma 6.3 we find (possibly on a shorter time interval) that solution is nonnegative and $\lambda(t, p_-)$ grows linearly in time. Using *both* these facts,

$$\|f(\cdot,\cdot,t)\|_{L^\infty(L^1)} = ess \sup_{p_-} \lambda(p_-, t)$$

grows linearly in time. Once this is known, the quadratic estimates for $Q(f,f)$, $Q(f,f)_{p_-}$, and $Q(f,f)_{p_+}$ in Lemma 6.1 are effectively linear, which prevents blow-up (as long as b itself has not done so). □

Theorem 6.2 *Assuming $f^0 \in X$, the full kinetic equation*

$$f_t + V \cdot \nabla f = Q(f,f) + Mf \tag{6.4}$$

has a unique C^1 solution defined for all times where $b(p,t)$ is finite.

Proof Let $(P_-(p_-,p_+,t), P_+(p_-,p_+,t))$ be the flow corresponding to the vector field $V = V(p_-,p_+,t)$. If $f(p_-,p_+,t)$ is a classical solution of (6.4), we see that

$$\tilde{f}(p_-,p_+,t) = f(P_-,P_+,t) \tag{6.5}$$

solves

$$\tilde{f}_t = f_t + V \cdot \nabla f = Q(f,f) + Mf = Q(\tilde{f},\tilde{f}) + M\tilde{f}. \tag{6.6}$$

By the invertibility of the flow, uniqueness for $\tilde{f}$ translates to uniqueness for f. Conversely, if we (classically) solve $\tilde{f}_t = Q(\tilde{f},\tilde{f}) + M\tilde{f}$ and then define f by (6.5), we see f is a classical solution of (6.4). □

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