Stochastic Solutions to Hamilton-Jacobi Equations

Fraydoun Rezakhanlou*
Departmet of Mathematics, UC Berkeley

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Abstract

1 Introduction

The primary goal of these notes is to give an overview of the statistical properties of solutions to the Cauchy problem for the Hamilton-Jacobi Equation

$$\begin{aligned} u_t &= H(x,t,u_x) &&\text{in} \quad \mathbb{R}^d\times(0,\infty) \\ u &= u^0 &&\text{on} \quad \mathbb{R}^d\times\{t=0\}, \end{aligned} \tag{1.1}$$

or, the scalar conservation law

$$\begin{aligned} \rho_t &= H(x,t,\rho)_x &&\text{in} \quad \mathbb{R}\times(0,\infty) \\ \rho &= \rho^0 &&\text{on} \quad \mathbb{R}\times\{t=0\}, \end{aligned} \tag{1.2}$$

where either H or $\rho^0=\rho^0(x)$ is random. Note that if u satisfies (1.1) and $d=1$, then $\rho=u_x$ satisfies (1.2). As is well-known, the PDE (1.1) or (1.2) does not possess classical solutions even when the initial data is smooth. In the case of equation (1.1), we may consider *viscosity solutions* to guarantee the uniqueness under some standard assumptions on the initial data and H. In the case of (1.2) with $d=1$, we consider the so-called *entropy solutions*.

We will be mostly concerned with the following two scenarios:

(1) $d=1$, $H(x,t,p)=H(p)$ is convex in p and independent of (x,t), with initial data ρ^0 that is either a white noise, or a Markov process.

(2) $d\geq 1$, and $H(x,t,p)$ is a stationary ergodic process in (x,t).

Our aim is to give an overview of various classical and recent results and formulate a number of open problems.

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2 Scalar Conservation Law with Random Initial data

We first recall the following important features of the solutions to (1.2) when $d = 1$, $H(x,t,p) = H(p)$ is convex in p, and independent of (x,t):

(i) If a discontinuity of ρ occurs at $x = x(t)$, and $\rho_\pm = \rho(x(t)\pm, t)$, then for a *weak solution* of (1.2) we must have the Rankin-Hugoniot Equation:

$$\frac{dx}{dt} = -H[\rho_-, \rho_+] =: -\frac{H(\rho_+) - H(\rho_-)}{\rho_+ - \rho_-}.$$

(ii) By an entropy solution, we mean a week solution for which the entropy condition is satisfied. In the case of convex H, the entropy condition is equivalent to the requirement

$$\rho_- < \rho_+.$$

(iii) If ρ^0 has a discontinuity with $\rho_- > \rho_+$, then such a discontinuity disappears instantaneously by inserting a rarefaction wave between between ρ_- and ρ_+. That is a solution of the form

$$G\left(\frac{x-c}{t}\right),$$

where $G = (H')^{-1}$.

We next state three results.

(i) *(Burgers Equation with Lévy Initial Data)*

When $H(p) = \frac{1}{2}p^2$, (1.2) is the well-known inviscid Burgers' equation, which has often been considered with random initial data. Burgers studied (1.2) in his investigation of turbulence [Bu]. Carraro and Duchon [CD] defined a notion of *statistical solution* to Burgers' equation and realized that it was natural to consider Lévy process initial data. In fact any (random) entropy solution is also a statistical solution, but the converse is not true in general. In 1998, Bertoin [Be] proved a closure theorem for Lévy initial data.

Theorem 2.1 *Consider Burgers' equation with initial data $\rho^0(x)$ which is a Lévy process without negative jumps for $x \geq 0$, and $\rho^0(x) = 0$ for $x < 0$. Assume that the expected value of $\rho^0(1)$ is non-positive, $\mathbb{E}\rho^0(1) \leq 0$. Then, for each fixed $t > 0$, the process $x \mapsto \rho(x,t) - \rho(0,t)$ is also a Lévy process with*

$$\mathbb{E}\exp\big(-s(\rho(x,t) - \rho(0,t))\big) = \exp(x\psi(s,t)),$$

where the exponent ψ solves the following equation:

$$\psi_t + \psi\psi_s = 0. \tag{2.1}$$

Remark 2.1 The requirement $\mathbb{E}\rho^0(1) \le 0$ can be relaxed with minor modifications to the theorem, in light of the following elementary fact. Suppose that $\rho^0(x)$ and $\hat{\rho}^0(x)$ are two different initial conditions for Burgers' equation, which are related by $\hat{\rho}^0(x) = \rho^0(x) + cx$. It is easy to check that the corresponding solutions $\rho(x,t)$ and $\hat{\rho}(x,t)$ are related for $t > 0$ by

$$\hat{\rho}(x,t) = \frac{1}{1+ct}\left[\rho\left(\frac{x}{1+ct}, \frac{t}{1+ct}\right) + cx\right].$$

Using this we can adjust a statistical description for a case where $\mathbb{E}0$ to cover the case of a Lévy process with general mean drift. □

(ii) *(Burgers Equation with white noise initial data)*

Groeneboom [Gr] considers the white noise initial data. In other words, take two independent Brownian motions $B^{\pm}$, and take a two sided Brownian motion for the initial data

$$u^0(x) = \begin{cases} B^+(x) & \text{if} \quad x \ge 0 \\ B^-(x) & \text{if} \quad x \le 0, \end{cases} \tag{2.2}$$

Theorem 2.2 *Let $\rho = u_x$, where u is a viscosity solution of the PDE $u_t = \frac{1}{2}u_x^2$, subject to the initial condition $u(x,0) = u^0(x)$, with u^0 given as in (2.2). Then the process $x \mapsto \rho(x,t)$ is a Markov jump process with drift $-t^{-1}$ and a suitable jump measure $\nu(t, \rho_-, \rho_+)\ d\rho_+$.*

(iii) A different particular case,

$$-H(p) = \begin{cases} 0 & \text{if} \quad |p| \le 1, \\ \infty & \text{otherwise.} \end{cases}$$

corresponds to the problem of determining Lipschitz minorants, and has been investigated by Abramson and Evans [AE].

3 Menon-Srinivasan Conjecture

In 2007 Menon and Pego [MP] used the Lévy-Khintchine representation for the Laplace exponent and observed that the evolution according to Burgers' equation in (2.1) corresponds to a Smoluchowski coagulation equation [A], with additive collision kernel, for the jump measure of the Lévy process $\nu(\cdot, t)$. The jumps of $\nu(\cdot, t)$ correspond to shocks in the solution $\rho(\cdot, t)$. Regarding the sizes of the jumps as the usual masses in the Smoluchowski equation, it is plausible that Smoluchowski equation with additive kernel should be relevant.

It is natural to wonder whether this evolution through Markov processes with simple statistical descriptions is specific to the Burgers-Lévy case, or an instance of a more general phenomenon. The biggest step toward understanding the problem for a wide class of H is found in a 2010 paper of Menon and Srinivasan [MS]. Here it is shown that when the initial condition ρ^0 is a strong Markov process with positive jumps only, the solution $\rho(\cdot, t)$ remains Markov for fixed $t > 0$. The argument is adapted from that of [Be] and both [MS] and [Be] use the notion of splitting times (due to Getoor [Ge]) to verify the Markov property according to its bare definition. In the Burgers-Lévy case, the independence and homogeneity of the increments can be shown to survive, from which additional regularity is immediate using standard results about Lévy processes. As [MS] points out, without these properties it is not clear whether a Feller process initial condition leads to a Feller process in x at later times. Nonetheless, [MS] presents a very interesting conjecture for the evolution of the generator of $\rho(\cdot, t)$, which has a remarkably nice form.

To prepare for the statement of Menon-Srinivasan Conjecture, we first examine the following simple scenario for the solutions of the PDE

$$\rho_t = H(\rho)_x = H'(\rho)\rho_x. \tag{3.1}$$

Imagine that the initial data ρ^0 satisfies an ODE of the form

$$\frac{d\rho^0}{dx}(x) = b^0\big(\rho^0(x)\big), \tag{3.2}$$

for some C^1 function $b^0 : \mathbb{R} \to \mathbb{R}$. We may wonder whether or not this feature of ρ^0 survives at later times. That is, for some function $b(\rho, t)$, we also have

$$\rho_x(x, t) = b\big(\rho(x, t), t\big), \tag{3.3}$$

for $t > 0$. For (3.3) to be consistent with (3.1), observe

$$\rho_t = H'(\rho)\rho_x = H'(\rho)b(\rho, t),$$

and as we calculate mixed derivatives, we arrive at

$$\begin{aligned}
\rho_{xt} &= b_\rho(\rho, t)\rho_t + b_t(\rho, t) = b_\rho(\rho, t)H'(\rho)b(\rho, t) + b_t(\rho, t),\\
\rho_{tx} &= H''(\rho)b(\rho, t)\rho_x + H'(\rho)b_\rho(\rho, t)\rho_x = H''(\rho)b^2(\rho, t) + H'(\rho)b_\rho(\rho, t)b(\rho, t).
\end{aligned}$$

As a result b must satisfy

$$b_t(\rho,t) = H''(\rho)b^2(\rho,t). \tag{3.4}$$

For a classical solution, all we need to do is solving the ODE (3.3) for the initial data $b(\rho,0) = b^0(\rho)$ for each ρ. When H is convex, the solution may blow up in finite time. More precisely,

- If $b^0(\rho) \le 0$, then $b^0(\rho) \le b(\rho,t) \le 0$ for all t and there would be no blow-up.
- If $b^0(\rho) > 0$, then there exists some finite $T(\rho) > 0$ such that $b(\rho,t)$ is finite in the interval $[0,T(\rho))$, and $b(\rho,T(\rho)) = \infty$.

In fact the equation (3.3) is really "the method of characteristics" in disguise, and the blow-up of solutions is equivalent to the occurrence of shock discontinuity.

To go beyond what (3.4) offers, we now take a jump kernel $f^0(\rho, d\rho_*)$ and assume that $\rho^0(x)$ is a realization of a Markov process with infinitesimal generator

$$\mathcal{L}^0 h(\rho) = b^0(\rho)h'(\rho) + \int_\rho^\infty \big(h(\rho_*) - h(\rho)\big)\ f^0(\rho, d\rho_*).$$

In words, ρ^0 solves the ODE (3.3), with some occasional random jumps with rate f^0. We are assuming that the jumps are all positive to avoid rarefaction waves. We may wonder whether the same picture is valid at later times. That is, for fixed $t > 0$, the solution $\rho(x,t)$, as a function of x is a Markov process with the generator

$$\mathcal{L}^t h(\rho) = b(\rho,t)h'(\rho) + \int_\rho^\infty \big(h(\rho) - h(\rho)\big)\ f(\rho, d\rho_*, t). \tag{3.5}$$

Menon-Srinivasan Conjecture roughly suggests that if H is convex, and we start with a Markov process with generator $\mathcal{L}^0$, then we have a Markov process at a later time with a generator of the form $\mathcal{L}^t$. Moreover, the drift of the generator satisfies (3.4), and the jump kernel $f(\rho, d\rho_*, t)$ solves an integral equation. Before we derive an equation for the evolution of f, observe that when we assert that $\rho(x,t)$ is a Markov process in x, we are specifying a direction for x. More precisely, we are asserting that if $\rho(a,t)$ is known, then the law of $\rho(x,t)$ can be determined uniquely for all $x > a$. We are doing this for all $t > 0$. In practice, we may try to determine $\rho(x,t)$ for $x > a(t)$, provided that $\rho(a(t),t)$ is specified. For example, we may wonder whether or not we can determine the law of $\rho(x,t)$ with the aid of the following procedure:

- The process $t \mapsto \rho(a(t),t)$ is a Markov process and its generator can be determined. Using this Markov process, we take a realization of $\rho(a(t),t)$, with some initial choice for $\rho(a(0),0)$.

- Once $\rho(a(t),t)$ is selected, we use the generator $\mathcal{L}^t$, to produce a realization of $\rho(x,t)$ for $x \geq a(t)$.

To materialize the above procedure, we need to make sure that for some choice of $a(t)$, the process $\rho(a(t),t)$ is Markovian with a generator that can be described. For a start, we may wonder whether or not we can even choose $a(t) = a$ a constant function. Put it differently, not only $x \mapsto \rho(x,t)$ is a Markov process for fixed $t \geq 0$, the process $t \mapsto \rho(x,t)$ is a Markov process for fixed x. As it turns out, this is the case if H is also increasing. In general, if we can find a negative constant c such that $H'(\rho) > c$, then $\hat{\rho}(x,t) := \rho(x - ct, t)$ satisfies

$$\hat{\rho}_t = \hat{H}(\hat{\rho})_x,$$

for $\hat{H}(\rho) = H(\rho) - c\rho$, which is increasing. Hence, the process $t \mapsto \hat{\rho}(x,t) = \rho(x - ct, t)$ is expected to be Markovian. In summary

- If H is increasing in the range of ρ, then ρ is also Markovian on vertical lines $x = constant$.
- If H' is bounded below by a negative constant c, then ρ is Markovian on straight lines that are titled to the right with the slope $-c$.

To simplify the matter, from now on, we make two assumptions on H:

$$H' > 0, \qquad H'' \geq 0. \tag{3.6}$$

The main consequences of these two assumptions are

- All the jump discontinuities are positive i.e. $\rho_- < \rho_+$.
- The speed of shocks are always negative.

We now argue that in fact the process $t \mapsto \rho(x,t)$ is a Markov process with a generator $\mathcal{M}$ that is independent of x because the PDE (3.1) is homogeneous (i.e. H is independent of x). Indeed

$$\mathcal{M}h(\rho) = H'(\rho)b(\rho,t)h'(\rho) + \int_\rho^\infty \big(h(\rho_*) - h(\rho)\big)H[\rho,\rho_*]f(\rho, d\rho_*, t). \tag{3.7}$$

To explain the form of $\mathcal{M}$ heuristically, observe that the ODE $\frac{d\rho}{dx} = b(\rho,t)$ leads to the ODE

$$\frac{d\rho}{dt} = H'(\rho)b(\rho,t).$$

On the other hand, if we fix x, then $\rho(x,t)$ experiences a jump discontinuity when a shock on the right of x crosses x. Given any $t > 0$, a shock would occur at some $s > t$ because all shock speeds are negative; it is just a matter of time for a shock on the right of x to cross x. We can also calculate the rate at which this happens because we have the law of the first shock on the right of x, and its speed. Observe

- The process $x \mapsto \rho(x,t)$ is a homogeneous Markov process with a generator that changes with time.
- The process $t \mapsto \rho(x,t)$ is an inhomogeneous Markov process with a generator that does not depend on x. It is only the initial data $\rho(x,0)$ that is responsible for the changes of the statistics of $\rho(x,t)$, as x varies.

We are now in a position to derive formally an evolution equation for the generator $\mathcal{L}^t$, under the assumption (3.6). Indeed if we define

$$w(x,t;\rho) = \mathbb{E}^{\rho(0,0)=\rho} h(\rho(x,t)),$$

then we expect

$$w_t = \mathcal{M}w, \qquad w_x = \mathcal{L}^t w.$$

differentiating these equations yields

$$w_{tx} = \mathcal{M}w_x = \mathcal{M}\mathcal{L}^t w, \qquad w_{xt} = \frac{d\mathcal{L}^t}{dt}w + \mathcal{L}^t w_t = \frac{d\mathcal{L}^t}{dt}w + \mathcal{L}^t\mathcal{M}w.$$

As a result

$$\frac{d\mathcal{L}^t}{dt} = \mathcal{M}\mathcal{L}^t - \mathcal{L}^t\mathcal{M}. \tag{3.8}$$

As we match the drift parts of both sides of (3.8), we simply get (3.4). Matching the jump parts yields a kinetic-type equation of the form

$$f_t = Q(f,f) + Cf, \tag{3.9}$$

for a quadratic operator Q and a linear operator C. The operator Q is independent of b and is given by

$$\begin{aligned} Q(f,f)(\rho_-, d\rho_+) = &\int_{\rho_-}^{\rho^+} \big(H[\rho_*,\rho_+] - H[\rho_-,\rho_*]\big) f(\rho_-, d\rho_*) f(\rho_*, d\rho_+) \\ &+ \int_{\rho_+}^{\infty} \big(H[\rho_+,\rho_*] - H[\rho_-,\rho_+]\big) f(\rho_+, d\rho_*) f(\rho_-, d\rho_+) \\ &+ \int_{\rho_-}^{\infty} \big(H[\rho_-,\rho_+] - H[\rho_-,\rho_*]\big) f(\rho_-, d\rho_*) f(\rho_-, d\rho_+). \end{aligned}$$

If we set

$$\begin{aligned} \lambda(\rho_-) = \lambda(f)(\rho_-) &= \int_{\rho_-}^{\infty} f(\rho_-, d\rho_+)\ d\rho_+, \\ A(\rho_-) = A(f)(\rho_-) &= \int_{\rho_-}^{\infty} H[\rho_-,\rho_+] f(\rho_-, d\rho_+), \end{aligned}$$

then $Q = Q^+ - Q^-$, with

$$
\begin{aligned}
Q^+(f,f)(\rho_-, d\rho_+) &= \int_{\rho_-}^{\rho^+} \big(H[\rho_*, \rho_+] - H[\rho_-, \rho_*]\big) f(\rho_-, d\rho_*) f(\rho_*, d\rho_+, t) \\
Q^-(f,f)(\rho_-, d\rho_+) &= \big\{A(\rho_+) - A(\rho_-) - H[\rho_-, \rho_+]\big(\lambda(\rho_+) - \lambda(\rho_-)\big)\big\}\, f(\rho_-, d\rho_+).
\end{aligned} \tag{3.10}
$$

To define the operator C we need to assume that $f(\rho_-, d\rho_+) = f(\rho_-, \rho_+) d\rho_+$ has a C^1 density. With a slight abuse of notion, we write $f(\rho_-, \rho_+)$ for the density of the measure $f(\rho_-, d\rho_+)$, and write C again for the action of the operator C on the density f:

$$
\begin{aligned}
(Cf)(\rho_-, \rho_+) = b(\rho_-, t) f(\rho_-, \rho_+) &\big(H[\rho_-, \rho_+]\big)_{\rho_-} \\
&+ \big[H[\rho_-, \rho_+] - H'(\rho_-)\big] b(\rho_-, t) f_{\rho_-}(\rho_-, \rho_+, t) \\
&+ \big[\big(H[\rho_-, \rho_+] - H'(\rho_+)\big) b(\rho_-, t) f(\rho_-, \rho_+, t)\big]_{\rho_+}.
\end{aligned}
$$

Menon-Srinivasan Conjecture has been established in [KR1] and [KR2]:

Theorem 3.1 *Assume H is a C^2 function that satisfies* (3.6) *and ρ is an entropic solution of* (3.1) *in the first quadrant. Assume that b and f satisfy* (3.4) *and* (3.9) *respectively. If $x \mapsto \rho(x, 0)$ and $t \mapsto \rho(0, t)$ are Markov processes with generators $\mathcal{L}^0$ and $\mathcal{M}$ with initial condition $\rho(0, 0) = \rho^0$, then $x \mapsto \rho(x, t)$ is a Markov process with generator $\mathcal{L}^t$ for $t > 0$, and $t \mapsto \rho(x, t)$ is a Markov process with generator $\mathcal{M}$ for $x > 0$.*

4 Heuristics for the Proof of Theorem 3.1

Let us write $x_i(t)$ for the location of the i-th shock and $\rho_i(t) = \rho(x_i(t)+, t)$. We also write $\phi_x(m_0; t)$ for the flow associated with the velocity b; the function $m(x) = \phi_x(m_0; t)$ satisfies

$$m'(x) = b(m(x), t), \qquad m(0) = m_0.$$

We can readily find the evolution $\mathbf{q} = (x_i, \rho_i : i \in \mathbb{Z})$, and $\hat{\mathbf{q}} = (z_i, \rho_i : i \in \mathbb{Z})$, with $z_i = x_{i+1} - x_i$:

- $$\dot{x}_i = -v^i := -H[\hat{\rho}_{i-1}, \rho_i], \qquad \dot{z}_i = -(v^{i+1} - v^i),$$
  where $\hat{\rho}_{i-1}(t) = \phi_{z_{i-1}}(\rho_{i-1}(t), t)$.
- $$\dot{\rho}_i = w^i := \big(H'(\rho_i) - H[\hat{\rho}_{i-1}, \rho_i]\big) b(\rho_i, t).$$
- When z_i becomes 0, the pair (ρ_i, z_i) is omitted from $\hat{\mathbf{q}}(t)$. The outcome after a relabeling is denoted by $\hat{\mathbf{q}}^i(t)$.

Write

$$\Delta = \big\{(z_i, \rho_i : i \in \mathbb{Z}) : \ z_i \geq 0, \ \rho_i \in \mathbb{R} \quad \text{for all} \quad i \in \mathbb{Z}\big\}.$$

We think of $\hat{\mathbf{q}}(t)$ as a deterministic process that has an infinitesimal generator

$$\mathcal{A}G = \sum_{i \in \mathbb{Z}} \big(w^i G_{\rho_i} - (v^{i+1} - v^i) G_{z_i}\big),$$

in the interior of Δ. We only take those G such that on the boundary face of Δ with $z_i = 0$, we have $G\big(\hat{\mathbf{q}}\big) = G\big(\hat{\mathbf{q}}^i\big)$. This stems from the fact that we are interested in the function $\rho(x) = \rho(x; \hat{\mathbf{q}})$ associated with $\hat{\mathbf{q}}$ (or $\mathbf{q}$) that is defined by

$$\sum_i \phi_{z_i}(x_i; x - x_i) 1\!\!1\big(x \in [x_i, x_{i+1})\big).$$

Note that $\rho(x; \hat{\mathbf{q}}) = \rho(x; \hat{\mathbf{q}}^i)$ whenever $z_i = 0$.

We make an ansatz that the law of $\hat{\mathbf{q}}(t)$ is of the form:

$$\mu\big(t, d\hat{\mathbf{q}}\big) = \prod_{i=-\infty}^{\infty} e^{-\int_0^{z_i} \lambda(\phi_y(\rho_i; t), t) dy} \ f\big(\phi_{z_i}(\rho_i; t), \rho_{i+1}, t\big) \ dz_i d\rho_{i+1}.$$

For this to be the correct candidate we need

$$\dot{\mu} = \mathcal{A}^* \mu. \tag{4.1}$$

This equation should determine f and λ if our ansatz is correct. To determine $\mathcal{A}^*$, we take a test function G and carry out the following calculation: After some integration by parts,we formally have

$$\int G\ d\mathcal{A}^*\mu = \int \mathcal{A}G\ d\mu = \int G \sum_i \left[w^i \Omega_i^1 - w^i_{\rho_i} + (v^{i+1} - v^i)\Omega_i^2 + v^{i+1}_{z_i} - \Omega_i^3\right]\ d\mu,$$

where

$$\begin{aligned}
\Omega_i^1 &= \int_0^{z_i} \left[\lambda\big(\phi_y(\rho_i;t),t\big)\right]_{\rho_i} dy - \frac{\left[f\big(\hat\rho_i,\rho_{i+1},t\big)\right]_{\rho_i}}{f\big(\hat\rho_i,\rho_{i+1},t\big)} - \frac{f_{\rho+}\big(\hat\rho_{i-1},\rho_i,t\big)}{f\big(\hat\rho_{i-1},\rho_i,t\big)},\\
\Omega_i^2 &= -\lambda\big(\hat\rho_i,t\big) + \frac{\left[f\big(\hat\rho_i,\rho_{i+1},t\big)\right]_{z_i}}{f\big(\hat\rho_{i-1},\rho_{i+1},t\big)},\\
\Omega_i^3 &= \frac{\int_{\hat\rho_{i-1}}^{\rho_i} H\big(\hat\rho_{i-1},\rho_*,\rho_i\big) f\big(\hat\rho_{i-1},\rho_*,t\big) f\big(\phi_{z_{i-1}}(\rho_*;t),\rho_i,t\big)\ d\rho_*}{f\big(\hat\rho_{i-1},\rho_i,t)\big)},
\end{aligned}$$

where Ω_i^3 represents the boundary contribution associated with $z_i = 0$, and

$$H(a,b,c) := H[b,c] - H[a,b].$$

On the other hand

$$\dot\mu = \sum_i \left[\Gamma_i^1 + \Gamma_i^2\right]\ \mu = \sum_i \left\{ \frac{\left[f\big(\phi_{z_i}(\rho_i;t),\rho_{i+1},t\big)\right]_t}{f\big(\phi_{z_i}(\rho_i;t),\rho_{i+1},t\big)} - \int_0^{z_i} \left[\lambda\big(\phi_y(\rho_i;t),t\big)\right]_t dy \right\}\ \mu.$$

To make the above formal calculation rigorous, we would like to switch from the infinite sum to a finite sum. For this, we restrict the dynamics to an interval, say $[0,L]$. The configuration now belongs to

$$\Delta_L = \cup_{n=0}^{\infty} \Delta_n^L,$$

with Δ_n^L denoting the set

$$\big\{\mathbf{q} = \big((x_i,\rho_i) : i = 0,1,\dots,n\big) :\ x_0 = 0 < x_1 < \dots < x_n < x_{n+1} = L, \quad \rho_0,\dots,\rho_n \in \mathbb{R}\big\}.$$

Again, what we have in mind is that $\rho_i(t) = \rho(x_i(t)+,t)$ with $x_1,\dots,x_n$ denoting the location of all shocks in $(0,L)$. We wish to define a measure $\mu(t,d\mathbf{q})$ on Δ_L. The restriction of μ to Δ_L^n is denoted by μ^n and is given by

$$\ell(t,d\rho_0) \exp\left\{-\sum_{i=0}^n \int_0^{x_{i+1}-x_i} \lambda(\phi_y(\rho_i;t),t)dy\right\} \prod_{i=0}^{n-1} f\big(\phi_{x_{i+1}-x_i}(\rho_i;t),\rho_{i+1},t\big)\ dx_{i+1}d\rho_{i+1},$$

where f solves (3.9) and ℓ is the law of $\rho(0,t)$, which is a Markov process with generator $\mathcal{M}$:

$$\dot{\ell} = \mathcal{M}^*\ell. \tag{4.2}$$

To simplify the presentation, we assume

$$\ell(t, d\rho_0) = \ell(t, \rho_0)\ d\rho_0.$$

As for the dynamics of $\mathbf{q}$, we have the following rules:

(i) So long as x_i remains in (x_{i-1}, x_{i+1}), it satisfies

$$\dot{x}_i = -v^i := -H[\hat{\rho}_{i-1}, \rho_i],$$

where $\hat{\rho}_{i-1}(t) = \phi_{z_{i-1}}(\rho_{i-1}(t), t)$.

(ii) We have $\dot{\rho}_0 = w^0 := H'(\rho_0)b(\rho_0, t)$ and for $i > 0$,

$$\dot{\rho}_i = w^i := \big(H'(\rho_i) - H[\hat{\rho}_{i-1}, \rho_i]\big)b(\rho_i, t).$$

(iii) When $z_i = x_{i+1} - x_i$ becomes 0, then $\mathbf{q}(t)$ becomes $\mathbf{q}^i(t)$, that is obtained from $\mathbf{q}(t)$ by omitting (ρ_i, x_i).

(iv) With rate

$$H[\hat{\rho}_n, \rho_{n+1}] f\big(\hat{\rho}_n, \rho_{n+1}, t\big)\ d\rho_{n+1},$$

the configuration $\mathbf{q}$ gains a new particle (x_{n+1}, ρ_{n+1}), with $x_{n+1} = L$. This new configuration is denoted by $\mathbf{q}(\rho_{n+1})$.

We note that since H is increasing, all velocities are negative. Moreover, when the first particle of location x_1 crosses the origin, a particle is lost.

We wish to establish (4.1). We write G^n for the restriction of a smooth function $G : \Delta^L \to \mathbb{R}$ to Δ^L_n. Recall that we only consider those test functions G that cannot differentiate between $\mathbf{q}$ and $\mathbf{q}^i$ (respectively $\mathbf{q}(\rho_{n+1})$) , when $x_i = x_{i+1}$ (respectively $x_{n+1} = L$). We need to verify

$$\dot{\mu}^n = \big(\mathcal{A}^*\mu\big)^n, \tag{4.3}$$

for all $n \geq 0$. Recall we write ν^n for the restriction of the measure ν to Δ^L_n. Also, given $H : \Delta_L \to \mathbb{R}$, we write H^n for the restriction of the function H to the set Δ^L_n. To verify (4.3), we verify

$$\int G^n\ d\dot{\mu}^n = \int \big(\mathcal{A}G\big)^n\ d\mu^n, \tag{4.4}$$

for every C^1 function G. It is instructive to see why (4.3) (or its integrated version (4.4)) is true when $n = 0$ and 1 before treating the general case. As we will see below, the cases $n = 0, 1$ are already equivalent to the equation (3.9). As a warm-up, we first assume that $n = 0$ and $b = 0$. In this case the equation (4.3) is equivalent to the fact that the law ℓ of $\rho(0, \cdot)$ is governed by a Markov process with generator $\mathcal{M}$. The case $n = 0$ and general b leads to the general form of $\mathcal{M}$ for the evolution of $\rho(0, \cdot)$, and an equation for λ that is a consequence of (3.9). The full equation (3.9) shows up when we consider the case $n = 1$.

□

The case $n = 0$ **and** $b = 0$**.** As it turns out, the function $\lambda(\rho, t) = \lambda(\rho)$ is independent of time when $b = 0$. We simply have

$$\mu^0(t, d\rho_0) = e^{-L\lambda(\rho_0)}\ell(t, d\rho_0), \qquad \mu^0_t(t, d\rho_0) = e^{-\lambda(\rho_0)L}\ell_t(t, d\rho_0). \tag{4.5}$$

On the other hand, the right-hand side of (4.4) is of the form $\Omega^1_0 + \Omega^2_0$, where Ω^1_0 comes from rule (**i**), and Ω^2_0 comes from the stochastic boundary dynamics. Indeed

$$\begin{aligned}\Omega^1_0 = &\int H[\rho_0, \rho_1]G^0(0, \rho_1)e^{-\lambda(\rho_1)L}f(\rho_0, \rho_1, t)\ d\rho_1\ \ell(t, d\rho_0)\\ &- \int H[\rho_0, \rho_1]G^1(0, \rho_0, L, \rho_1)e^{-\lambda(\rho_0)L}f(\rho_0, \rho_1, t)\ d\rho_1\ \ell(t, d\rho_0),\end{aligned} \tag{4.6}$$

which we get it from the boundary terms when we apply an integration by parts to the integral

$$-\int H[\rho_0, \rho_1]G^1_{x_1}(0, \rho_0, x_1, \rho_1)\ e^{-\lambda(\rho_0)x_1 - \lambda(\rho_1)(L - x_1)}f(\rho_0, \rho_1, t)\ d\rho_1\ \ell(t, d\rho_0).$$

Moreover,

$$\Omega^2_0 = \int H[\rho_0, \rho_1]f(\rho_0, \rho_1, t)\big(G^1(0, \rho_0, L, \rho_1) - G^0(0, \rho_0)\big)e^{-\lambda(\rho_0)L}\ d\rho_1\ \ell(t, d\rho_0).$$

From this and (4.6) we learn

$$\begin{aligned}\Omega^1_0 + \Omega^2_0 &= \int H[\rho_0, \rho_1]G^0(0, \rho_1)e^{-\lambda(\rho_1)L}f(\rho_0, \rho_1, t)\ d\rho_1\ \ell(t, d\rho_0)\\ &\qquad - \int H[\rho_0, \rho_1]f(\rho_0, \rho_1, t)G^0(0, \rho_0)e^{-\lambda(\rho_0)L}\ d\rho_1\ \ell(t, d\rho_0)\\ &= \int H[\rho_1, \rho_0]G^0(0, \rho_0)e^{-\lambda(\rho_0)L}f(\rho_1, \rho_0, t)\ d\rho_0\ \ell(t, d\rho_1)\\ &\qquad - \int H[\rho_0, \rho_1]f(\rho_0, \rho_1, t)G^0(0, \rho_0)e^{-\lambda(\rho_0)L}\ d\rho_1\ \ell(t, d\rho_0)\\ &= \int G^0(0, \rho_0)e^{-\lambda(\rho_0)L}\ \big(\mathcal{M}^*\ell\big)(t, d\rho_0) = \int G^0(0, \rho_0)e^{-\lambda(\rho_0)L}\ \ell_t(t, d\rho_0),\end{aligned}$$

as desired. $\square$

The case $n = 0$ and general b. To ease the notation, we write

$$\Gamma(\rho, x, t) = \int_0^x \lambda(\phi_y(\rho; t), t)\ dy.$$

When $n = 0$, the right-hand side of (4.4) equals

$$\begin{aligned}
&\int G^0(0,\rho_0)\left[H'(\rho_0)b(\rho_0,t)\ell(t,\rho_0)\Gamma_\rho(\rho_0,L,t) - \big(H'(\rho_0)b(\rho_0,t)\ell(t,\rho_0)\big)_{\rho_0}\right]\ e^{-\Gamma(\rho_0,L,t)}\ d\rho_0\\
&\quad + \int H[\rho_0,\rho_1]G^0(0,\rho_1)e^{-\Gamma(\rho_1,L,t)}f(\rho_0,\rho_1,t)\ d\rho_1\ \ell(t,d\rho_0)\\
&\quad - \int H\big[\phi_L(\rho_0;t),\rho_1\big]f\big(\phi_L(\rho_0;t),\rho_1,t\big)G^1(0,\rho_0,L,\rho_1)e^{-\Gamma(\rho_0,L,t)}\ d\rho_1\ \ell(t,d\rho_0)\\
&\quad + \int H\big[\phi_L(\rho_0;t),\rho_1\big]f\big(\phi_L(\rho_0;t),\rho_1,t\big)\big(G^1(0,\rho_0,L,\rho_1) - G^0(0,\rho_0)\big)e^{-\Gamma(\rho_0,L,t)}\ d\rho_1\ \ell(t,d\rho_0)
\end{aligned}$$

This simplifies to

$$\begin{aligned}
&\int G^0(0,\rho_0)\left[H'(\rho_0)b(\rho_0,t)\ell(t,\rho_0)\Gamma_\rho(\rho_0,L,t) - \big(H'(\rho_0)b(\rho_0,t)\ell(t,\rho_0)\big)_{\rho_0}\right]\ e^{-\Gamma(\rho_0,L,t)}\ d\rho_0\\
&\quad + \int H[\rho_*,\rho_0]G^0(0,\rho_0)e^{-\Gamma(\rho_0,L,t)}f(\rho_*,\rho_0,t)\ d\rho_0\ \ell(t,d\rho_*)\\
&\quad - \int H\big[\phi_L(\rho_0;t),\rho_1\big]f\big(\phi_L(\rho_0;t),\rho_1,t\big)G^0(0,\rho_0)e^{-\Gamma(\rho_0,L,t)}\ d\rho_1\ \ell(t,d\rho_0)\\
&\qquad = \int G^0(0,\rho_0)\Lambda(\rho_0,t)\ e^{-\Gamma(\rho_0,L,t)}\ d\rho_0,
\end{aligned}$$

where $\Lambda(\rho_0, t)$ equals

$$\begin{aligned}
&H'(\rho_0)b(\rho_0,t)\ell(t,\rho_0)\Gamma_\rho(\rho_0,L,t) - \big(H'(\rho_0)b(\rho_0,t)\ell(t,\rho_0)\big)_{\rho_0}\\
&\quad + \int H[\rho_*,\rho_0]f(\rho_*,\rho_0,t)\ \ell(t,d\rho_*) - \int H\big[\phi_L(\rho_0;t),\rho_1\big]f\big(\phi_L(\rho_0;t),\rho_1,t\big)\ d\rho_1\ \ell(t,\rho_0).
\end{aligned}$$

We need to match $\Lambda(\rho_0, t)$ with the corresponding term on left-hand side of (4.4), which, by (4.2) takes the form

$$\begin{aligned}
&-\Gamma_t(\rho_0,L,t)\ \ell(t,\rho_0) - \big(H'(\rho_0)b(\rho_0,t)\ell(t,\rho_0)\big)_{\rho_0}\\
&\quad + \int H[\rho_*,\rho_0]f(\rho_*,\rho_0,t)\ \ell(t,d\rho_*) - A(\rho_0,t)\ \ell(t,\rho_0),
\end{aligned}$$

where

$$A(\rho_0, t) = \int H[\rho_0, \rho_*] f(\rho_0, \rho_*, t)\ d\rho_*.$$

We are done if we can verify

$$\Gamma_t(\rho_0, L, t) + H'(\rho_0) b(\rho_0, t) \Gamma_\rho(\rho_0, L, t) = A\big(\phi_L(\rho_0; t), t\big) - A(\rho_0, t). \tag{4.7}$$

Equivalently

$$\int_0^L \big[\lambda(\phi_y(\rho_0; t), t)\big]_t\ dy + H'(\rho_0) b(\rho_0, t) \int_0^L \big[\lambda(\phi_y(\rho_0; t), t)\big]_{\rho_0}\ dy = \int_0^L \big[A\big(\phi_y(\rho_0; t), t\big)\big]_y\ dy.$$

For this, it suffices to check

$$\big[\lambda(\phi_y(\rho_0; t), t)\big]_t + H'(\rho_0) b(\rho_0, t)\ \big[\lambda(\phi_y(\rho_0; t), t)\big]_{\rho_0} = \big[A\big(\phi_y(\rho_0; t), t\big)\big]_y.$$

Note that if $u(y, \rho) = A\big(\phi_y(\rho; t), t\big)$, then

$$u_y(y, \rho_0) = b(\rho_0, t) u_\rho(y, \rho_0).$$

Hence for (4.7), it suffices to show

$$\big[\lambda(\phi_y(\rho_0; t), t)\big]_t + H'(\rho_0) b(\rho_0, t)\big[\lambda(\phi_y(\rho_0; t), t)\big]_{\rho_0} = b(\rho_0, t)\big[A\big(\phi_y(\rho_0; t), t\big)\big]_{\rho_0}. \tag{4.8}$$

This for $y = 0$ takes the form

$$\lambda_t(t, \rho_0) + H'(\rho_0) b(\rho_0, t) \lambda_\rho(t, \rho_0) = b(\rho_0, t) A_\rho(\rho_0, t). \tag{4.9}$$

It turns out that if we choose

$$\lambda(t, \rho-) = \int_{\rho-}^\infty f(\rho_-, \rho_+, t)\ d\rho_+, \tag{4.10}$$

then (4.9) follows from (3.9) after integrating both sides of (3.9) with respect to ρ_-. The verification of (4.8) requires some additional work.

Let us write $T_y h(m) = h(\phi_y(m; t))$. The family of operators $\{T_y :\ y \in \mathbb{R}\}$, is a group in y. Moreover, if $(\mathcal{B}h)(m) = b(m, t) h'(m)$, then

$$\frac{dT_y}{dy} = \mathcal{B} T_y = T_y \mathcal{B}. \tag{4.11}$$

Using this, we may rewrite (4.8) as

$$\big[\lambda(\phi_y(\rho_0; t), t)\big]_t + H'(\rho_0) b(\phi_y(\rho_0; t), t) \lambda_\rho(\phi_y(\rho_0; t), t) = b(\phi_y(\rho_0; t), t) A_\rho\big(\phi_y(\rho_0; t), t\big). \tag{4.12}$$

On account of (4.9), the claim (4.12) would follow if we can show

$$X(\rho_0, y, t) := \big[\phi_y(\rho_0;t)\big]_t - \big[H'(\phi_y(\rho_0;t)) - H'(\rho_0)\big] b(\phi_y(\rho_0;t), t) = 0. \tag{4.13}$$

This is true for $y = 0$. Differentiating with respect to y yields

$$\begin{aligned} X_y(\rho_0, y, t) &= \big[b(\phi_y(\rho_0;t),t)\big]_t - \big[H'(\phi_y(\rho_0;t))\big]_y b(\phi_y(\rho_0;t),t) \\ &\qquad - \big[H'(\phi_y(\rho_0;t)) - H'(\rho_0)\big]\big[b(\phi_y(\rho_0;t),t)\big]_y \\ &= b_t(\phi_y(\rho_0;t),t) + b_\rho(\phi_y(\rho_0;t),t)\big[\phi_y(\rho_0;t)\big]_t - H''(\phi_y(\rho_0;t)) b^2(\phi_y(\rho_0;t),t) \\ &\qquad - \big[H'(\phi_y(\rho_0;t)) - H'(\rho_0)\big](bb_\rho)(\phi_y(\rho_0;t),t) \\ &= b_\rho(\phi_y(\rho_0;t),t)\big[\phi_y(\rho_0;t)\big]_t - \big[H'(\phi_y(\rho_0;t)) - H'(\rho_0)\big](bb_\rho)(\phi_y(\rho_0;t),t) \\ &= b_\rho(\phi_y(\rho_0;t),t) X(\rho_0, y, t), \end{aligned}$$

where we used (3.4) for the third equality. As a result.

$$X(\rho_0, y, t) = X(\rho_0, 0, t) \exp\left[\int_0^y b_\rho(\phi_z(\rho_0;t),t)\ dz\right] = 0.$$

This completes the proof of (4.2), when $n = 0$. □

As we have seen so far, the case $n = 0$ is valid if an equation for λ is true and this would follow from the kinetic equation. On the other hand the case $n = 1$ is equivalent to the kinetic equation. Before embarking on the verification of (4.3) for $n = 1$, let us make some compact notions for some of the expressions that come into the proof. Given a realization $\mathbf{q} = \big(0, \rho_0, x_1, \rho_1, \dots, x_n, \rho_n\big) \in \Delta_n^L$, we define

$$\begin{aligned} \rho\big(x, t; \mathbf{q}\big) &= \sum_{i=0}^n \phi_{x - x_i}\big(\rho_i; t\big) 1\!\!1\big(x_i \le x < x_{i+1}\big), \\ \Gamma(\mathbf{q}, t) &= \int_0^L \lambda\big(\rho\big(y, t; \mathbf{q}\big)\big)\ dy = \sum_{i=0}^n \Gamma(\rho_{i-1}, x_i - x_{i-1}, t), \\ \hat\rho_{i-1} &= \rho(x_i-, t; \mathbf{q}) = \phi_{x_i - x_{i-1}}(\rho_{i-1}; t). \end{aligned}$$

Note that by (4.13),

$$\frac{d\hat\rho_i}{dt} = \big[H'(\hat\rho_i) - H'(\rho_i)\big] b(\hat\rho_i, t). \tag{4.14}$$

The case $n = 1$**.** We have $\dot\mu^1 = X_1 \mu^1$, where

$$X_1(\mathbf{q}, t) = -\Gamma_t(\mathbf{q}, t) + \frac{\ell_t(t, \rho_0)}{\ell(t, \rho_0)} + \frac{\big[f\big(\hat\rho_0, \rho_1, t\big)\big]_t}{f\big(\hat\rho_0, \rho_1, t\big)}.$$

On the other hand $(\mathcal{A}^*\mu)^1 = Y_1\mu^1$, with

$$Y_1(\mathbf{q},t) = \sum_{j=1}^{7} Y_{1j}(\mathbf{q},t) = \sum_{j=1}^{7} Y_{1j},$$

where

$$\begin{aligned}
Y_{11} &= H'(\rho_0)b(\rho_0,t)\left[\Gamma_\rho(\rho_0,x_1,t) - \frac{\big[f\big(\hat\rho_0,\rho_1,t\big)\big]_{\rho_0}}{f\big(\hat\rho_0,\rho_1,t\big)}\right] - \frac{\big(H'(\rho_0)b(\rho_0,t)\ell(t,\rho_0)\big)_{\rho_0}}{\ell(t,\rho_0)}\\
Y_{12} &= \big(H'(\rho_1) - H[\hat\rho_0,\rho_1]\big)b(\rho_1,t)\Gamma_\rho(\rho_1,L-x_1,t)\\
Y_{13} &= \frac{\big[\big(H[\hat\rho_0,\rho_1] - H'(\rho_1)\big)b(\rho_1,t)f\big(\hat\rho_0,\rho_1,t\big)\big]_{\rho_1}}{f\big(\hat\rho_0,\rho_1,t\big)}\\
Y_{14} &= \frac{\big(H\big[\hat\rho_0,\rho_1\big]f\big(\hat\rho_0,\rho_1,t\big)\big)_{x_1}}{f\big(\hat\rho_0,\rho_1,t\big)} + H\big[\hat\rho_0,\rho_1\big]\left[\lambda\big(\phi_{L-x_1}(\rho_1;t),t\big) - \lambda\big(\hat\rho_0,t\big)\right]\\
Y_{15} &= \frac{\int H(\rho_*,\rho_0)f(\rho_*,\rho_0,t)\ \ell(t,d\rho_*)}{\ell(t,\rho_0)}\\
Y_{16} &= -\int H\big[\phi_{L-x_1}(\rho_1;t),\rho_*\big]f\big(\phi_{L-x_1}(\rho_1;t),\rho_*,t\big)\ d\rho_* = -A\big(\phi_{L-x_1}(\rho_1;t),t\big)\\
Y_{17} &= \frac{\int\big(H[\rho_*,\rho_1] - H\big[\hat\rho_0,\rho_*\big]\big)f\big(\hat\rho_0,\rho_*,t\big)f(\rho_*,\rho_1,t)\ d\rho_*}{f\big(\hat\rho_0,\rho_1,t\big)}.
\end{aligned}$$

Here,

- The term Y_{11} comes from an integration by parts with respect to the variable ρ_0. The dynamics of ρ_0 as in rule **(ii)** is responsible for this contribution.

- The terms Y_{12} and Y_{13} come from an integration by parts with respect to the variable ρ_1. The dynamics of ρ_1 as in rule **(ii)** is responsible for these two contributions.

- The term Y_{14} comes from an integration by parts with respect to the variable x_1. The dynamics of x_1 as in rule **(i)** is responsible for this contribution.

- The term Y_{15} comes from the boundary term $x_1 = 0$ in the integration by parts with respect to the variable x_1.

- The term Y_{16} comes from the boundary term $x_1 = L$ in the integration by parts with respect to the variable x_1, and the stochastic boundary dynamics as in the rule **(iv)**. The boundary term $x_1 = L$ cancels part of the contribution of the boundary dynamics as we have already seen in our calculation in the case $n = 0$.

- The rule **(iii)** is responsible for the term Y_{17}. When $n=2$, the particles at x_1 and x_2 travel towards each other with speed $H[\rho_2,\hat\rho_1]-H[\rho_1,\hat\rho_0]$. As x_2 catches up with x_1, the particle x_2 disappears and its density $\rho_1=\hat\rho_2$ is renamed ρ_*, and is integrated out.

We wish to show that $X_1=Y_1$. After some cancellation, this simplifies to

$$X_1'=Y_1':=Y_{11}'+Y_{12}+Y_{13}+Y_{14}+Y_{16}+Y_{17},$$

where

$$\begin{aligned} X_1'&=-\Gamma_t(\mathbf{q},t)-A(\rho_0,t)+\frac{\big[f(\hat\rho_0,\rho_1,t)\big]_t}{f(\hat\rho_0,\rho_1,t)},\\ Y_{11}'&=H'(\rho_0)b(\rho_0,t)\left[\Gamma_\rho(\rho_0,x_1,t)-\frac{\big[f(\hat\rho_0,\rho_1,t)\big]_{\rho_0}}{f(\hat\rho_0,\rho_1,t)}\right]. \end{aligned}$$

(The same cancellation led to the equation (4.7).) Observe that $\Gamma(\mathbf{q},t)=\Gamma(\rho_0,x_1,t)+\Gamma(\rho_1,L-x_1,t)$. Moreover, by (4.7),

$$\begin{aligned} &\Gamma_t(\rho_0,x_1,t)+H'(\rho_0)b(\rho_0,t)\Gamma_\rho(\rho_0,x_1,t)=A(\hat\rho_0,t)-A(\rho_0,t)\\ &\Gamma_t(\rho_1,L-x_1,t)+H'(\rho_1)b(\rho_1,t)\Gamma_\rho(\rho_1,L-x_1,t)=A\big(\phi_{L-x_1}(\rho_1;t),t\big)-A(\rho_1,t). \end{aligned}$$

As a result,

$$\begin{aligned} -\Gamma_t(\mathbf{q},t)-A(\rho_0,t)&=H'(\rho_0)b(\rho_0,t)\Gamma_\rho(\rho_0,x_1,t)+H'(\rho_1)b(\rho_1,t)\Gamma_\rho(\rho_1,L-x_1,t)\\ &\quad-A\big(\phi_{L-x_1}(\rho_1;t),t\big)+A(\rho_1,t)-A\big(\hat\rho_0,t\big). \end{aligned}$$

Using this, we learn that the equality $X_1'=Y_1'$ is equivalent to the identity

$$\begin{aligned} \big[f\big(\hat\rho_0,\rho_1,t\big)\big]_t&=H\big[\hat\rho_0,\rho_1\big]\left[\lambda\big(\phi_{L-x_1}(\rho_1;t),t\big)-\lambda\big(\hat\rho_0,t\big)\right]f\big(\hat\rho_0,\rho_1,t\big)\\ &\quad+\left[A\big(\hat\rho_0,t\big)-A(\rho_1,t)\right]f\big(\hat\rho_0,\rho_1,t\big)\\ &\quad+\int\big(H[\rho_*,\rho_1]-H\big[\hat\rho_0,\rho_*\big]\big)f\big(\hat\rho_0,\rho_*,t\big)f(\rho_*,\rho_1,t)\,d\rho_*\\ &\quad+\left[\big(H[\hat\rho_0,\rho_1]-H'(\rho_1)\big)b(\rho_1,t)f(\hat\rho_0,\rho_1,t)\right]_{\rho_1}\\ &\quad-H'(\rho_0)b(\rho_0,t)\big[f\big(\hat\rho_0,\rho_1,t\big)\big]_{\rho_0}-H[\hat\rho_0,\rho_1]b(\rho_1,t)\Gamma_\rho(\rho_1,L-x_1,t)f\big(\hat\rho_0,\rho_1,t\big)\\ &\quad+\big(H\big[\hat\rho_0,\rho_1\big]f\big(\hat\rho_0,\rho_1,t\big)\big)_{x_1}. \end{aligned}$$

By the semigroup property (4.11), we can assert that for any C^1 function h,

$$\big[h(\hat\rho_0)\big]_{x_1}=b(\hat\rho_0,t)h'(\hat\rho_0)=b(\rho_0)\big[h(\hat\rho_0)\big]_{\rho_0}.$$

We use this and the definition of the quadratic operator Q in (3.10) to deduce that $X_1' = Y_1'$ is equivalent to the identity

$$
\begin{aligned}
\big[f(\hat\rho_0,\rho_1,t)\big]_t = &\, Q(f,f)(\hat\rho_0,\rho_1,t) + H\big[\hat\rho_0,\rho_1\big]\,\big[\lambda\big(\phi_{L-x_1}(\rho_1;t),t\big) - \lambda\big(\rho_1,t\big)\big]\, f\big(\hat\rho_0,\rho_1,t\big)\\
&+ \big[\big(H[\hat\rho_0,\rho_1] - H'(\rho_1)\big)b(\rho_1,t)f(\hat\rho_0,\rho_1,t)\big]_{\rho_1}\\
&- H'(\rho_0)b(\hat\rho_0,t)f_{\rho_-}\big(\hat\rho_0,\rho_1,t\big) - H[\hat\rho_0,\rho_1]b(\rho_1,t)\Gamma_\rho(\rho_1,L-x_1,t)f\big(\hat\rho_0,\rho_1,t\big)\\
&+ b(\hat\rho_0,t)H[\hat\rho_0,\rho_1]f_{\rho_-}(\hat\rho_0,\rho_1,t) + b(\hat\rho_0,t)H_{\rho_-}[\hat\rho_0,\rho_1]f(\hat\rho_0,\rho_1,t).
\end{aligned}
$$

Here we are acting the quadratic operator Q on functions because we are assuming that $f(\rho_, d\rho_+,t) = f(\rho_,\rho_+,t)\ d\rho_+$, is absolutely continuous with respect to the Lebesgue measure. We now use (4.13) to rewrite $X_1' = Y_1'$ as

$$
\begin{aligned}
f_t(\hat\rho_0,\rho_1,t) = &\, Q(f,f)(\hat\rho_0,\rho_1,t) + b(\hat\rho_0,t)H_{\rho_-}[\hat\rho_0,\rho_1]f(\hat\rho_0,\rho_1,t)\\
&+ H\big[\hat\rho_0,\rho_1\big]\,\big[\lambda\big(\phi_{L-x_1}(\rho_1;t),t\big) - \lambda\big(\rho_1,t\big)\big]\, f\big(\hat\rho_0,\rho_1,t\big)\\
&+ \big[H[\hat\rho_0,\rho_1] - H'(\hat\rho_0)\big]b(\hat\rho_0,t)f_{\rho_-}\big(\hat\rho_0,\rho_1,t\big)\\
&+ \big[\big(H[\hat\rho_0,\rho_1] - H'(\rho_1)\big)b(\rho_1,t)f(\hat\rho_0,\rho_1,t)\big]_{\rho_1}\\
&- H[\hat\rho_0,\rho_1]b(\rho_1,t)\Gamma_\rho(\rho_1,L-x_1,t)f\big(\hat\rho_0,\rho_1,t\big).
\end{aligned}
$$

On the other hand, by the definition of Γ,

$$
\begin{aligned}
b(\rho_1,t)\Gamma_\rho(\rho_1,L-x_1,t) &= \int_0^{L-x_1} b(\rho_1,t)\,\big[\lambda\big(\phi_y(\rho_1;t),t\big)\big]_{\rho_1}\ dy\\
&= \int_0^{L-x_1} \big[\lambda\big(\phi_y(\rho_1;t),t\big)\big]_y\ dy \qquad (4.15)\\
&= \lambda\big(\phi_{L-x_1}(\rho_1;t),t\big) - \lambda(\rho_1,t),
\end{aligned}
$$

where we used the semigroup property (4.11) for the second equality. This leads to

$$
\begin{aligned}
f_t(\hat\rho_0,\rho_1,t) = &Q(f,f)(\hat\rho_0,\rho_1,t) + b(\hat\rho_0,t)f(\hat\rho_0,\rho_1,t)H_{\rho_-}[\hat\rho_0,\rho_1]\\
&+ \big[H[\hat\rho_0,\rho_1] - H'(\hat\rho_0)\big]b(\hat\rho_0,t)f_{\rho_-}(\hat\rho_0,\rho_1,t)\\
&+ \big[\big(H[\hat\rho_0,\rho_1] - H'(\rho_1)\big)b(\rho_1,t)f(\hat\rho_0,\rho_1,t)\big]_{\rho_1}.
\end{aligned}
$$

This is exactly our kinetic equation! □

The calculation for the general case n is similar but more tedious than the case $n = 1$, but does not pose any additional challenge. We refer to [KR2] for details

5 Homogenizations for Hamiltonian ODEs

The Hamilton-Jacobi PDE may be used to model the growth of an interface that is described as a graph of a height function. More precisely, the graph of a solution

$$u : \mathbb{R}^d \times [0, \infty) \to \mathbb{R},$$

of the Hamilton-Jacobi equation

$$u_t + H(x, u_x) = 0, \tag{5.1}$$

describes an interface at time t in microscopic coordinates. If the ratio of micro to macro scale is a large number n, then

$$u^n(x,t) = \frac{1}{n} u(nx, nt),$$

is the corresponding macroscopic height function. In practice n is large and we may obtain a simpler description of our model if the large n limit of u^n exists and satisfies a simple equation. Indeed u^n satisfies

$$u^n_t + H\big(nx, u^n_x\big) = 0,$$

and this equation must be solved for an initial condition of the form $u^n(x,0) = g(x)$, where g represents the initial macroscopic height function. Let us define

$$(\Gamma_n g)(x) = ng\left(\frac{x}{n}\right);$$

the job of the operator Γ_n is to turn a macroscopic height function to its associated microscopic height function. We also write $T_t = T^H_t$ for the semigroup associated with the PDE (5.1). More precisely, $T_t u^0(x) = u(x,t)$ means

$$\begin{cases} u_t + H(x, u_x) = 0, & t > 0, \\ u(x,t) = u^0(x), \end{cases} \tag{5.2}$$

In terms of the operators T_t and Γ_n, we simply have $u^n = \big(\Gamma_n^{-1} \circ T_{nt} \circ \Gamma_n\big)(g)$. Put it differently,

$$T^{H\circ\gamma_n}_t = \Gamma_n^{-1} \circ T^H_{nt} \circ \Gamma_n, \tag{5.3}$$

where $\gamma_n(x,p) = (nx, p)$. If we write $T(H)$ for T^H_1, then in particular we have

$$T(H \circ \gamma_n) = \Gamma_n^{-1} \circ T(H)^n \circ \Gamma_n.$$

The hope is that under some assumptions on H, the large n-limit of u^n exists and the limit $\bar{u}$ provides a reduced and simpler description of the growth model under the study. For example, when H is 1-periodic in x-variable, the high oscillations of $H \circ \gamma_n$, may result in the convergence of u^n to a function $\bar{u}$, that solves the homogenized equation

$$\bar{u}_t + \bar{H}(\bar{u}_x) = 0. \tag{5.4}$$

When this happens, we write $\mathcal{A}(H) = \bar{H}$.

More generally, write $\mathcal{H}$ for the space of all C^1 Hamiltonian functions and define the natural translation operator

$$\tau_a H(x,p) = H(x+a,p),$$

for every $a \in \mathbb{R}^d$. We then take a probability measure $\mathbb{P}$ on $\mathcal{H}$ that is translation invariant and ergodic. We wish to take advantage of the ergodicity to assert that $T_t^{H\circ\gamma_n} \to T_t^{\bar{H}}$, $\mathbb{P}$-almost surely, as $n \to \infty$. If this happens for a deterministic function $\bar{H}$, then we write $\mathcal{A}(\mathbb{P}) = \bar{H}$. We note

- If $\mathbb{P}$ is supported on the set

$$A := \left\{\tau_a H^0 : \ a \in \mathbb{R}^d\right\},$$

  for some 1-periodic Hamiltonian function H^0, then A is isomorphic to the d-dimensional torus and we are back to the periodic scenario.

- If $\mathbb{P}$ is supported on the topological closure (with respect to the uniform norm), of the set

$$A := \left\{\tau_a H^0 : \ a \in \mathbb{R}^d\right\},$$

  for some Hamiltonian function H^0, and this closure is a compact set, then H^0 is almost periodic and the homogenization would allow us to find the large n-limit of $T_t^{H\circ\gamma_n} \to T_t^{\bar{H}}$, for almost all choices of H in the compact support of $\mathbb{P}$. In this case $\bar{A}$ has the structure of a Lie group and $\mathbb{P}$ is the corresponding Haar measure.

To explore the homogenization question further, we discuss the connection between Hamiltonian ODE and Hamilton-Jacobi PDE. For a classical solution, the method of characteristics suggests that at least for short times, we can solve (5.2) in terms of the flow of the Hamiltonian ODE

$$\begin{aligned} \dot{x} &= H_p(x,p), \\ \dot{p} &= -H_x(x,p). \end{aligned} \tag{5.5}$$

Equivalently we write $\dot{z} = J\nabla H(z)$, where $z = (x,p)$, and

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},$$

with I denoting the $d \times d$ identity matrix. Writing $\phi_t = \phi_t^H$ for the flow of (5.5), we have

$$\phi_t^H \left\{ \left(x, \nabla u^0(x) \right) :\ x \in \mathbb{R}^d \right\} = \left\{ \left(x, u_x(x,t) \right) :\ x \in \mathbb{R}^d \right\}, \tag{5.6}$$

provided that the left-hand side remains a graph of a function. As we mentioned earlier, the equation (5.2) does not possess C^1 solutions in general. This has to do with the fact that if ϕ_t folds the graph of ∇u^0, then the left-hand side of (5.6) is no longer a graph of a function and (5.6) has no chance to be true. One possibility is that we *trim* the left-hand side (5.6) and hope for

$$\phi_t^H \left\{ \left(x, \nabla u^0(x) \right) :\ x \in \mathbb{R}^d \right\} \supseteq \left\{ \left(x, u_x(x,t) \right) :\ x \in \mathbb{R}^d \right\}, \tag{5.7}$$

For this to work, we have to give-up the differentiability of u. This geometric and rather naive idea does not suggest how the trimming should be carried out.

Alternatively, we may add a small viscosity term of the form $\varepsilon \Delta u$ to the right-hand side of (5.1) to guarantee the existence of a unique classical solution, and pass to the limit $\varepsilon \to 0$. The outcome is known as a *viscosity solution*. As it turns out, under some coercivity assumption on H, we can guarantee the existence of a solution that is differentiable almost every where. We can now modify the right-hand side of (5.7) accordingly and wonder whether or not

$$\phi_t^H \left\{ \left(x, \nabla u^0(x) \right) :\ x \in \mathbb{R}^d \right\} \supseteq \left\{ \left(x, u_x(x,t) \right) :\ x \in \mathbb{R}^d,\ \ u_x(x,t) \text{ exists} \right\}, \tag{5.8}$$

is true. The answer is affirmative if H is convex in p. However (5.8) may fail if we drop the convexity assumption. To explain this in the case of piecewise smooth solutions, we recall that if H is convex in p, the only discontinuity we can have is a shock discontinuity. In this case, at every point (a,t), with $t > 0$, we can find a solution $(x(s), p(s) : s \in [0,t])$ (the so-called backward characteristic) such that $x(t) = a$. If $\rho = u_x$ is continuous at a, this backward characteristic is unique and $p(t) = \rho(a,t)$. If ρ is discontinuous at (a,t), then $\rho(a,t)$ is multi-valued and for each possible value p of $\rho(a,t)$, there will a solution to the Hamiltonian ODE with $(x(t), p(t)) = (a, p)$. In both cases, we still have (5.8).

The situation is far more complex when H is not convex. What may cause the violation of (5.8) is the occurrence of a rare-faction type solutions. To explain this, let us assume that $d = 1$, and H depends on p only. There are three momenta (or densities) $a_1 < a_2 < a_3$ such that

- The graph of H is convex and below its cord in $[a_1, a_2]$.
- The graph of H is concave and above its cord in $[a_2, a_3]$.
- The graph of H is below its cord in the interval $[a_1, a_3]$.

Now imagine that we have two discontinuities at $x(t)$ and $y(t)$ with $x(t) < y(t)$, and both are shock discontinuities. Assume

- The left and right values of ρ at $x(t)$ are $a_2'(t) < a_3'(t)$.
- The left and right values of ρ at $y(t)$ are $a_3'(t) > a_1'(t)$.
- These two shock discontinuities meet at some instant t_0 with $a_i'(t_0) = a_i$.

As a result, at the moment t_0 the two shock discontinuities are replaced with a rarefaction wave. Now if we take a point (x,t) inside the fan of this rarefaction wave (for which necessarily $t > t_0$), then at such (x,t) the connection with the initial data is lost and $(x, u_x(x,t))$ does not belong to the left-hand side of (5.8).

Motivated by the failure of (5.8) for viscosity solutions, we formulate a question.

Question 5.1: Is there a notion of generalized solution for (5.1) for which (5.8) is always true?

Using some ideas from topology and symplectic geometry the notion of *geometric solution* has been developed by Chaperon, Sikarov and Viterbo. The main features of this solution is as follows:

(i) The geometric solution satisfies (5.8) always.

(ii) The geometric solution satisfies (5.2) at every differentiability point of u.

(iii) The geometric solution coincides with the viscosity solution when H is convex in p.

(iv) Writing $\hat{T}_t u^0$ for the geometric solution of (5.2) with the initial condition u^0, we do not have $\hat{T}_t \circ \hat{T}_s = \hat{T}_{t+s}$.

Needless to say the last feature of the geometric solution is a serious flaw and does not provide a satisfactory answer for Question 5.1. Nonetheless the geometric solution provides a useful notion that helps us to connect the equation (5.2) to the Hamiltonian ODEs.

Because of the intimate relation between the Hamilton-Jacobi Equation and the Hamiltonian ODE, we may wonder whether a homogenization phenomenon occurs for the latter. More precisely, does the high-n limit of

$$\phi_t^{H\circ\gamma_n} = \gamma_n^{-1} \circ \phi_{nt}^H \circ \gamma_n,$$

exist in a suitable sense? Note that $H \circ \gamma_n$ has no pointwise limit and the existence of pointwise limit of $\phi_t^{H\circ\gamma_n}$ is not expected either. Writing ϕ_H for ϕ_1^H, we may wonder in what sense, if any, the sequence $\phi_{H\circ\gamma_n}$ has a limit. We note

$$\phi_{H\circ\gamma_n} = \gamma_n^{-1} \circ \phi_H^n \circ \gamma_n =: S_n(\phi_H).$$

We now discuss the existence of some interesting metric on the space $\mathcal{H}$ that is weaker than uniform norm and is closely related to the flow properties of the Hamiltonian ODEs. More importantly, there is a chance that $H \circ \gamma_n$ converges with respect to such metrics.

There are two metrics on $\mathcal{H}$ that are well-suited for our purposes. These metrics were defined by Hofer and Viterbo; the proofs of non-triviality of these metrics are highly non-trivial. Let us write down a wish-list for what our metric should satisfy.

Let us write $\mathcal{D}$ for the space of maps φ such that $\varphi = \phi_H$ for some smooth Hamiltonian function $H : \mathbb{R}^{2d} \times [0,1] \to \mathbb{R}$. (Any such map is *symplectic* as we will see later.) Assume that there exists a function $E : \mathcal{D} \to [0,\infty)$ with the following properties: For $\varphi, \psi, \tau \in \mathcal{D}$,

- **(i)** $E(\varphi) = E(\varphi^{-1})$.
- **(ii)** $E(\varphi) = E\big(\tau^{-1}\varphi\tau\big)$.
- **(iii)** $E(\varphi\psi) \leq E(\varphi) + E(\psi)$.
- **(iv)** $E(\varphi) = 0$ if and only if $\varphi = id$.
- **(v)** $E\big(\rho_\ell^{-1}\varphi\rho_\ell\big) = \ell^{-1}E(\varphi)$, where $\rho_\ell(x,p) = (\ell x, p)$ and $\ell \in (0,\infty)$.

Here and below we simply write $\varphi\psi$ for $\varphi \circ \psi$ and think of $\mathcal{D}$ as a group with multiplication given by the map composition.

From E, we build a metric D on $\mathcal{D}$ by $D(\varphi,\psi) = E\big(\varphi\psi^{-1}\big)$. This metric has the following properties:

Proposition 5.1 **(i)** *$D(\varphi\tau, \psi\tau) = D(\tau\varphi, \tau\varphi) = D(\varphi,\psi)$ for $\varphi, \psi, \tau \in \mathcal{D}$.*

(ii) *For $\varphi_1, \psi_1 \dots, \varphi_k, \psi_k$, we have*

$$D(\varphi_1 \dots \varphi_k, \psi_1 \dots \psi_k) \leq \sum_{i=1}^{k} D(\varphi_i, \psi_i).$$

(iii) *For $S_n(\varphi) = \rho_n^{-1} \circ \varphi^n \circ \rho_n$, we have*

$$D\big(S_n(\varphi), S_n(\psi)\big) \leq D(\varphi,\psi).$$

In the case of a homogenization, we expect $S_n(\varphi) \to \bar{\varphi}$, where $\bar{\varphi} = \phi_{\bar{H}}$, for a Hamiltonian function $\bar{H}$ that is independent of x. Write $\mathcal{D}_0$ for the space of such $\bar{\varphi}$. We note that $S_n(\bar{\varphi}) = \bar{\varphi}$. As a result, for any $\bar{\varphi} \in \mathcal{D}_0$,

$$D\big(S_n(\varphi), \bar{\varphi}\big) = D\big(S_n(\varphi), S_n(\bar{\varphi})\big) \leq D\big(\varphi, \bar{\varphi}\big), \tag{5.9}$$

by Proposition 5.1**(iii)**. As was noted by Viterbo [V], (5.9) implies that the set of limit points of the sequence $(S_n(\varphi):\ n \in \mathbb{N})$ is unique: If $\bar\varphi$ and $\bar\psi$ are two limit points, then given $\delta > 0$, we find $n, m \in \mathbb{N}$ such that

$$D(S_n(\varphi), \bar\varphi) \le \delta, \qquad D(S_m(\varphi), \bar\psi) \le \delta.$$

From this and (5.9) we learn,

$$D(S_{nm}(\varphi), \bar\varphi) \le \delta, \qquad D(S_{nm}(\varphi), \bar\psi) \le \delta,$$

because $S_{nm} = S_n \circ S_m$. Hence $D(\bar\varphi, \bar\psi) \le 2\delta$. By sending $\delta \to 0$ we deduce that $\bar\varphi = \bar\psi$.

A natural question is whether we have homogenization with respect to such a metric.

Question 5.2: Given $\varphi \in \mathcal{D}$, does the large n limit of the sequence $\{S_n(\varphi)\}$ exist with respect to a metric D as above? □

6 Lagrangian Manifolds and Viterbo's Metric

The Question 5.2 has been answered affirmatively by Viterbo [V] when the Hamiltonian H is periodic in x and the metric D is the *Viterbo's metric.* We continue with a brief discussion of Viterbo's metric.

To simplify our presentation, let us assume that H is 1-periodic in x. We may also regard $u(\cdot, t)$ as a function on the d-dimensional torus $\mathbb{T}^d$.

To examine the left-hand side of (5.8), assume that the initially the solution of the ODE (5.5) satisfies the relationship $p = \nabla u^0(x)$, for some smooth function u^0. Whenever (5.6) is true, then at time t we have a similar relationship between the components of $\phi_t(x, p)$. Let us write $M^t := \phi_t(M^0)$, where

$$M^0 = \big\{(x, \nabla u^0(x)) : \ x \in \mathbb{T}^d\big\}.$$

To get a feel for $M^t = \phi_t^H(M^0)$, observe that M^0 is a graph of a an exact derivative. Let us refer to such manifolds as an *exact Lagrangian.* In general if

$$M = \big\{(x, X(x)) : \ x \in \mathbb{T}^d\big\},$$

then vectors of the form

$$\hat{a} := \begin{bmatrix} a \\ (DX)(x)a \end{bmatrix},$$

are tangents to M at x. What makes M exact is that if $X = \nabla u$, then the matrix $A = DX = D^2u$ is symmetric. To state this directly in terms of the tangent vectors, observe

$$Aa \cdot b - a \cdot Ab = \begin{bmatrix} Aa \\ -a \end{bmatrix} \cdot \begin{bmatrix} b \\ Ab \end{bmatrix} = J\hat{a} \cdot \hat{b} =: \bar{\omega}(\hat{a}, \hat{b}).$$

Hence the symmetry of A is equivalent to $\bar{\omega} \restriction_M = 0$ identically. (Here $\bar{\omega}$ is the standard 2-form of $\mathbb{R}^{2d}$.) Motivated by this we call a manifold M *Lagrangian* if the restriction of $\bar{\omega}$ to M is identically 0. The point of this definition is that if M^0 is the graph of an exact derivative, then $\varphi(M^0)$ may not be a graph of a function. However, when φ preserves the form $\bar{\omega}$, then $\varphi(M^0)$ is always a (possibly nonexact) Lagrangian. We say a map φ is *symplectic* if it preserves $\bar{\omega}$ in the following sense:

$$\bar{\omega}\big((D\varphi)(x)a, (D\varphi)(x)b\big) = \bar{\omega}(a, b),$$

for every $x \in \mathbb{T}^d$ and every pair of vectors $a, b \in \mathbb{R}^{2d}$.

It is well-known that the correct topology for the viscosity solution comes from the uniform norm; this has to do with the fact the viscous approximation of Hamilton-Jacobi Equation satisfies a *maximum principle* that survives as we send the viscous term to 0. Since we are now interested in Hamiltonian ODE, we may try to define some kind of metrics on Lagrangian manifolds of the form $\phi_t^H\big(M^0\big)$, where M^0 is an exact Lagrangian. Let us write $\mathcal{L}_0$ for the set of exact Lagrangian, and define

$$\mathcal{H}_0 = \big\{H : \mathbb{T}^d \times \mathbb{R}^d \times [0,1] \to \mathbb{R} : \ H \ \text{ is } C^1 \quad \text{and 1-periodic in } \ x\big\}$$
$$\mathcal{L} = \big\{\phi_H(M) : \quad H \in \mathcal{H}_0, \quad M \in \mathcal{L}_0\big\}$$

When M is the graph of ∇u, for some C^1 function $u : \mathbb{T}^d \to \mathbb{R}$, we refer to u as the *generating function* of M. When this is the case, we write $\mathcal{G}(M) = u$. We also write

$$L(u) := \big\{(x, \nabla u(x)) : \ x \in \mathbb{T}^d\big\}.$$

Viterbo defines a metric on $\mathcal{L}$ that is a generalization of the L^∞-metric on its generating function. In other words, the metric D is defined in such way that if M^0 and M^1 are two exact Lagrangians, then

$$D\big(M, M'\big) = \|\mathcal{G}(M) - \mathcal{G}(M')\|_\infty,$$

where by $\|\cdot\|_\infty$ we really mean the total oscillation:

$$\|u\|_\infty = \max u - \min u.$$

This is quiet natural because $\mathcal{L}(u) = \mathcal{L}(u + c)$, for any constant c.

To guess how to extend the definition of this metric to non-exact Lagrangian, we need to develop a better understanding of the Hamiltonian ODEs. First, we claim that there exists a functional $\mathcal{I} = \mathcal{I}^H$ on the space of the paths $z(\cdot) = (x,p)(\cdot)$, such that $\dot z = J\nabla H(z,t)$ if and only if $z(\cdot)$ is a critical point of $\mathcal{I}$. Writing the Hamiltonian ODE as $J\dot z + \nabla H(z,t) = 0$, it is not hard to come up an example for $\mathcal{I}$; we use a quadratic term to produce the linear part $J\dot z$, and H to produce ∇H. The following function $\mathcal{I} : C^1([0,1]; \mathbb{T}^d \times \mathbb{R}^d) \to \mathbb{R}$, is the integral of the celebrated *Cartan-Poincaré form*:

$$\mathcal{I}(z) = \int_0^1 [p(t) \cdot \dot x(t) - H(z(t), t)] \ dt.$$

Formally, $\partial \mathcal{I}(z) = -J\dot z - \nabla H(z,t)$. More precisely, if $\eta : [0,1] \to \mathbb{T}^d \times \mathbb{R}^d$, satisfies $\eta(0) = \eta(1) = 0$, then $\psi(\delta) = \mathcal{I}(z + \delta\eta)$ satisfies

$$\dot\psi(0) = -\int_0^1 \big(J\dot z(t) + \nabla H(z(t), t)\big) \cdot \eta(t) \ dt.$$

We now use this to come up with a generating-like function for $M^1 = \phi_H M^0$, where $M^0 = \mathcal{G}(u^0)$. To this end, let us define

$$\Gamma := \big\{z : [0,1] \to \mathbb{T}^d \times \mathbb{R}^d : \ z \in C^1\big\}, \qquad \Gamma(a) = \big\{z = (x,p) \in \Gamma \ : \ x(1) = a\big\}.$$

In words, $\Gamma(a)$ consists of position/momentum paths with the position component reaching a at time 1. We note that if $z \in \Gamma(a)$ and $\eta \in \Gamma(0)$, then $z + \delta\eta \in \Gamma(a)$ for all $\delta \in \mathbb{R}$. We then define $\hat{\mathcal{I}} : \Gamma(a) \to \mathbb{R}$ by

$$\hat{\mathcal{I}}(z) = u^0(x(0)) + \mathcal{I}(z) = u^0(x(0)) + \int_0^1 [p(t) \cdot \dot{x}(t) - H(z(t), t)] \ dt.$$

Since we want to use $\hat{\mathcal{I}}$ to build a generating function for M^1, observe that $\Gamma(0)$ is an infinite dimensional vector space and any $z \in \Gamma(a)$ can be written as

$$z(t) = (a, 0) + \xi(t),$$

with $\xi \in \Gamma(0)$. If M^1 is still a graph of function and has a generating function u^1, then what is happening is that we have a solution z satisfying $\dot{z} = J\nabla H(z, t)$ with

$$z(0) = \big(x(0), \nabla u^0(x(0))\big), \qquad z(1) = \big(x(1), \nabla u^1(x(1))\big).$$

Moreover, if u solves (1.1), then $u^1(x) = u(x, 1)$. Note that if $w(t) = u(x(t), t)$, then $\dot{w} = p \cdot \dot{x} - H(z, t)$, or

$$u^1(x(1)) = u^0(x(0)) + \int_0^1 [p(t) \cdot \dot{q}(t) - H(z(t), t)] \ dt.$$

To separate $x(1)$ from the rest of information in the path $z(\cdot)$, we define $\mathcal{J} : \mathbb{T}^d \times \Gamma(0) \to \mathbb{R}$, by

$$\mathcal{J}(a; \xi) = \mathcal{I}\big((a, 0) + \xi\big).$$

In other words, if $z = (a, 0) + \xi = (x, p)$, and $\xi = (x', p)$, then $x'(t) = x(t) - x(1) = x(t) - a$. Now, if we set

$$\hat{\psi}(\delta) = \hat{\mathcal{I}}(z + \delta\eta) = \mathcal{J}(a; \xi + \delta\eta),$$

for $z \in \Gamma(a)$, and $\eta = (\hat{x}, \hat{p}) \in \Gamma(0)$, then

$$\frac{d\hat{\psi}}{d\delta}(0) = \big(\nabla u^0(x(0)) - p(0)\big) \cdot \hat{x}(0) - \int_0^1 \big(J\dot{z}(t) + \nabla H(z(t), t)\big) \cdot \eta(t) \ dt.$$

We can now assert

$$\partial_\xi \mathcal{J}(a; \xi) = 0 \qquad \Longleftrightarrow \qquad p(0) = \nabla u^0(x(0)), \quad \text{and} \quad z = (a, 0) + \xi \quad \text{satisfies } \dot{z} = J\nabla H(z, t).$$

On the other hand, if we set $\bar{\psi}(\delta) = \hat{\mathcal{I}}\big(z + (\delta b, 0)\big) = \mathcal{J}(a + \delta b; \xi)$, then

$$\partial_a \mathcal{J}(a; \xi) \cdot b = \frac{d\bar{\psi}}{d\delta}(0) = \nabla u^0(x(0)) \cdot b - \int_0^1 H_x(z(t), t) \cdot b \ dt.$$

As a result, if $\partial_\xi \mathcal{J}(a;\xi)=0$, then

$$\partial_a \mathcal{J}(a;\xi)=\nabla u^0(x(0))-\int_0^1 H_x(z(t),t)\ dt=\nabla u^0(x(0))+\int_0^1 \dot{p}(t)\ dt=p(1).$$

From this we deduce

$$\phi^H M^0=\left\{\big(a,\partial_a\mathcal{J}(a,\xi)\big):\ a\in\mathbb{T}^d,\ \ \partial_\xi\mathcal{J}(a,\xi)=0\right\},$$

where $a=x(1)$ represents the position at time 1. We think of $\mathcal{J}(a;\xi)$ as a *generalized generating function* (or in short GG function) of $M=M^1$. The Lagrangian M^1 is exact if for every $(a,p)\in\mathbb{T}^d\times\mathbb{R}^d$, there is at most one solution z to the Hamiltonian ODE with $x(1)=a, p(1)=p$. Our aim is to associate a nonnegative number $E(M)$ to $M\in\mathcal{L}$ that in the case of an exact Lagrangian $M=\mathcal{G}(u)$,

$$E(M)=E^+(M)-E^-(M),$$

where $E^\pm(M)$ are two critical values of u, namely the maximum and minimum of u. In the case of a non-exact M, we may use the functional $\mathcal{J}=\mathcal{J}_M$ to select two critical points $z^\pm=(a^\pm,\xi^\pm)$ of the functional $\mathcal{J}_M$ to define

$$E^\pm(M)=\mathcal{J}_M(a^\pm,\xi^\pm)=\hat{\mathcal{I}}(z^\pm).$$

The main question now is how to select the critical paths $z^\pm$. The classical theories of Morse and Lusternik-Schnirelman would provide us with systematic ways of selecting critical values of a scalar-valued functions on a manifold. These theories are applicable if the underlying manifold is finite-dimensional and their generalizations to infinite dimensional setting are highly nontrivial. (Floer Theory is a prime example of such generalization.) However in our setting it is possible to approximate the functional $\mathcal{I}$ or $\mathcal{J}$ with a function that is defined on $\mathbb{T}^d\times\mathbb{R}^N$ for a suitable N that depends on H and u^0 and could be large. More precisely, we may try to find a generalized generating (GG) function $S:\mathbb{T}^d\times\mathbb{R}^N\to\mathbb{R}$ such that

$$M=\left\{(x,S_x(x,\xi)):\ x\in\mathbb{T}^d,\xi\in\mathbb{R}^N,S_\xi(x,\xi)=0\right\},$$

In fact any manifold of this form is automatically a Lagrangian manifold, simply because the tangent vectors at a point of the form $(x,S_x(x,\xi))$ are still of the form $\big(v,A(x,\xi)v\big);v\in\mathbb{R}^d$, where $A=S_{xx}$ is a symmetric matrix.

To explain the existence of such finite dimensional GG functions, we need to make another observation about the flows of Hamiltonian ODEs.

We may regard the symplectic property of $\varphi=\phi_1^H$, as saying that its graph

$$\left\{(x,\varphi(x)):\ x\in\mathbb{T}^d\times\mathbb{R}^d\right\},$$

is Lagrangian with respect to the 2-form $\omega \oplus (-\omega)$ in $\mathbb{R}^{4d}$. This Lagrangian manifold is exact when this graph can be expressed as a graph of the gradient of a scalar-valued function, but now because of the form of the symplectic form $\omega \oplus (-\omega)$, must be done in a twisted way. More precisely, if $\varphi(x,p) = (X,P)$, then the generating function would depend for example on (X,p). In the case of an exact symplectic map, we may find a scalar-valued function $S(X,p)$ such that

$$\varphi\big(S_p(X,p),p\big) = \big(X, S_X(X,p)\big).$$

The identity map has the generating function $p \cdot X$. This suggest writing $S(X,p) = X \cdot p - w(X,p)$ with w periodic in X. In terms of w,

$$\varphi\big(X - w_p(X,p),p\big) = \big(X, p - w_X(X,p)\big).$$

Now imagine that $M = \varphi(M^0)$, where both M^0 and φ are exact with generating functions u^0 and $S(X,p) = X \cdot p - w(X,p)$. Then

$$\hat{S}(X;x,p) = u^0(x) + p \cdot (X - x) - w(X,p) =: p \cdot (X - x) - \hat{w}(X;x,p),$$

is a GG function for M^1: If $\xi = (x,p)$, then

$$\hat{S}_\xi(X;\xi) = 0 \quad \iff \quad p = \nabla u^0(x), \quad x = X - w_p(X,p).$$

As a result

$$\hat{S}_\xi(X;\xi) = 0 \quad \implies \quad \varphi(x,p) = \big(X, \hat{S}_X(X;\xi)\big),$$

because $\hat{S}_X = p - w_X(X,p) = P$.

As we mentioned earlier, the identity map has a generating function. Using Implicit Function Theorem, it is not hard to show that any symplectic map that is C^1-close to the identity also possesses a generating function. Now if $\varphi = \phi_H$ is the time-one map associated with a smooth Hamiltonian, then we can find $\delta > 0$ sufficiently small, such that the map $\varphi = \phi^H_\delta$ is sufficiently close to the identity map and possesses a generating function. In general, each ϕ^H can be expressed as $\varphi^1 \circ \cdots \circ \varphi^N$ with each φ^i exact symplectic. If each φ^i has a generating function of the form $X \cdot p - w^i(X,p)$, then $M = \varphi(M^0)$ has a generating function of the form

$$\hat{S}\big(x_N;\xi\big) = \hat{S}\big(x_N;x_0,p_0,\dots,x_{N-1},p_{N-1}\big) := u^0(x_0) + \sum_{i=0}^{N-1}\big[p_i \cdot (x_{i+1} - x_i) - w^i(x_{i+1},p_i)\big].$$

So far we know that our Lagrangian manifolds possess finite-dimensional generating functions. The next question to address is that how we can select appropriate critical values $E^\pm(M)$ for $\hat{S}(X;\xi)$.

For the rest of this section, we assume that M is a Lagrangian manifold with a generalized generating (GG) function $S(q, \xi)$. More precisely,

$$M = \big\{(x, S_x(x,\xi)) : \ x \in \mathbb{T}^d, \xi \in \mathbb{R}^N, S_\xi(x,\xi) = 0\big\},$$

and $S(x, \xi)$ is a nice perturbation of a quadratic function in ξ. By this we mean that there exists a quadratic function $B(\xi) = A\xi \cdot \xi$ such that A is an invertible symmetric matrix, and

$$\sup_{x,\xi} |S(x,\xi) - B(\xi)|, \qquad \sup_{x,\xi} |S_\xi(x,\xi) - \nabla B(\xi)| < \infty.$$

We wish to put a metric on the space $\mathcal{L}$ of such Lagrangians. For this, we first wish to define the *size* $E(M)$ of a Lagrangian manifold M. If M is an exact Lagrangian with generating function u, we simply set

$$E(M) = \max u - \min u.$$

If M has a GG function, $E(M)$ is defined by

$$E(M) = E^+(M) - E^-(M),$$

where $E^-(M)$ and $E^+(M)$ are two critical values of the GG function that are the analog of $\min u$ and $\max u$. To explain our strategy for selecting $E^\pm(M)$, first imagine that $S(x, \xi) = u(x) + B(\xi)$. Then we still have $E^-(M) = \min u = u(x_-)$ and $E^+(M) = \max u = u(x_+)$, because both $(x_\pm, 0)$ are critical points of S. After all 0 is a critical value for B. In fact since B is a non-degenerate quadratic function, 0 could be a saddle point. We may apply *Lusternik-Schnirleman (LS) Theory*, to assert that there are two critical points of S that are very much the analogs of $(x_\pm, 0)$. We now ready to define a metric on the space of Lagrangians. If M and M' are two Lagrangian with generating functions S and S' respectively, then we define a new generating function

$$\big(S \ominus S'\big)(x, \xi_1, \xi_2) = S(x, \xi_1) - S'(x, \xi_2).$$

This new generating function produces a new Lagrangian manifold

$$\begin{aligned} M \ominus M' &= \big\{(x, S_x(x,\xi_1) - S'_x(x,\xi_2)) : \ x \in \mathbb{T}^d, \xi \in \mathbb{R}^N, \xi \in \mathbb{R}^{N'}, S_\xi(x,\xi) = 0, S'_\xi(x,\xi_2) = 0\big\} \\ &= \big\{(x, p - p') : \ (x,p) \in M, \ (x,p') \in M'\big\}. \end{aligned}$$

This generating function is a bounded perturbation of $\big(B \ominus B'\big)(\xi_1, \xi_2) = B(\xi_1) - B(\xi_2)$. We set

$$D(M, M') = E\big(S \ominus S'\big).$$

We now want to use the above metric to define a metric for Hamiltonian functions or their corresponding flows that was defined by Viterbo:

$$\mathcal{D}(H, H') = \sup\big\{D\big(\phi_H(M), \phi_{H'}(M)\big) : \ M \in \mathcal{L}\big\}.$$

Theorem 6.1 *(Viterbo [V]) The large n-limit of $H \circ \gamma_n$ exists with respect to the Viterbo Metric D. Moreover, if the limit is denoted by $\mathcal{B}(H)$, then $\mathcal{B}$ satisfies the following properties*

(i) *For every symplectic $\varphi \in \mathcal{D}$, we have $\mathcal{B}(H \circ \varphi) = \mathcal{B}(H)$.*

(ii) *If $\{H, K\} := J\nabla H \cdot K = 0$, then $\mathcal{B}(H + K) = \mathcal{B}(H) + \mathcal{B}(K)$.*

This should be compared with the Lions-Papanicolaou-Varadhan [LPV] homogenization result.

Theorem 6.2 *Assume that $H(x,p)$ is a C^1, x-periodic Hamiltonian function with*

$$\lim_{|p|\to\infty} \inf_x H(x,p) = \infty.$$

Then the large n limit of $T^{H\circ\gamma_n}$ exists. The limit is of the form $T^{\bar{H}}$, for a Hamiltonian function $\mathcal{A}(H) := \bar{H}$ that is independent of x.

In fact $\mathcal{A}(H) = \mathcal{B}(H)$ when H is convex in p; otherwise they could be different. Moreover, Theorem 6.2 has been extended to the random ergodic setting when H is convex in p in Rezakhanlou-Tarver [RT] and Souganidis [S]. A natural question is whether or not Theorem 6.1 can be extended to random

Question 6.1 Can we extend Viterbo's metric (or Hofer's metric) to the random setting and does the large n limit of $H \circ \gamma_n$ exist for stationary ergodic Hamiltonian H? □

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