Stochastically Symplectic Maps and Their Applications
to Navier-Stokes Equation

Fraydoun Rezakhanlou*
UC Berkeley
Department of Mathematics
Berkeley, CA 94720-3840

January 11, 2014

Abstract

Poincaré’s invariance principle for Hamiltonian flows implies Kelvin’s principle for solution to Incompressible Euler Equation. Iyer-Constantin Circulation Theorem offers a stochastic analog of Kelvin’s principle for Navier-Stokes Equation. Weakly symplectic diffusions are defined to produce stochastically symplectic flows in a systematic way. With the aid of symplectic diffusions, we produce a family of martingales associated with solutions to Navier-Stokes Equation that in turn can be used to prove Iyer-Constantin Circulation Theorem. We also review some basic facts in symplectic and contact geometry and their applications to Euler Equation.

1 Introduction

Hamiltonian systems appear in conservative problems of mechanics governing the motion of particles in fluid. Such a mechanical system is modeled by a Hamiltonian function $H(x,t)$ where $x = (q,p) \in \mathbb{R}^d \times \mathbb{R}^d$, $q = (q_1,\ldots,q_d)$, $p = (p_1,\ldots,p_d)$ denote the positions and the momenta of particles. The Hamiltonian’s equations of motion are

\begin{equation}
\dot{q} = H_p(q,p,t), \quad \dot{p} = -H_q(q,p,t)
\end{equation}

which is of the form

\begin{equation}
\dot{x} = J \nabla_x H(x,t), \quad J = \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix}
\end{equation}

*This work is supported in part by NSF Grant DMS-1106526.
where \( I_d \) denotes the \( d \times d \) identity matrix. It was known to Poincaré that if \( \phi_t \) is the flow of the ODE (1.2) and \( \gamma \) is a closed curve, then

\[
\frac{d}{dt} \int_{\phi_t(\gamma)} \bar{\lambda} = 0,
\]

where \( \bar{\lambda} := p \cdot dq \). We may use Stokes’ theorem to rewrite (1.3) as

\[
\frac{d}{dt} \int_{\phi_t(\Gamma)} d\bar{\lambda} = 0
\]

for every two-dimensional surface \( \Gamma \). In words, the 2-form

\[\bar{\omega} := \sum_{i=1}^{d} dp_i \wedge dq_i,\]

is invariant under the Hamiltonian flow \( \phi_t \). Equivalently,

\[
\phi_t^* \bar{\omega} = \bar{\omega}.
\]

A Hamiltonian system (1.2) simplifies if we can find a function \( u(q,t) \) such that \( p(t) = u(q(t),t) \). If such a function \( u \) exists, then \( q(t) \) solves

\[
\frac{dq}{dt} = H_p(q,u(q,t),t).
\]

The equation for the time evolution of \( p \) gives us an equation for the evolution of the velocity function \( u \); since

\[\dot{p} = (Du)\dot{q} + u_t = (Du)H_p(q,u,t) + u_t,\]

\[\dot{p} = -H_q(q,u,t),\]

the function \( u(q,t) \) must solve,

\[
u_t + (Du)H_p(q,u,t) + H_q(q,u,t) = 0.
\]

For example, if \( H(q,p,t) = \frac{1}{2}|p|^2 + P(q,t) \), then (1.7) becomes

\[
u_t + (Du)u + \nabla P(q,t) = 0,
\]

and the equation (1.6) simplifies to

\[
\frac{dq}{dt} = u(q,t).
\]
Here and below we write $Du$ and $\nabla P$ for the $q$-derivatives of the vector field $u$ and the scalar-valued function $P$ respectively. If the flow of $(1.9)$ is denoted by $Q_t$, then $\phi_t(q, u(q, 0)) = (Q_t(q), u(Q_t(q), t))$. Now $(1.3)$ means that for any closed $q$-curve $\eta$,

\[(1.10) \quad \frac{d}{dt} \int_{Q_t(\eta)} u(q, t) \cdot dq = \frac{d}{dt} \int_{\eta} (DQ_t)^* u \circ Q_t(q, t) \cdot dq = 0,\]

or equivalently

\[(1.11) \quad d(Q_t^* \alpha_t) = d\alpha_0,\]

where $\alpha_t = u(q, t) \cdot dq$. This is the celebrated Kelvin's circulation theorem. In summary Poincaré’s invariance principle $(1.3)$ implies Kelvin’s principle for Euler Equation. (Note that the incompressibility condition $\nabla \cdot u = 0$ is not needed for $(1.10)$.)

We may rewrite $(1.11)$ as

\[(1.12) \quad Q_t^*(d\alpha_t) = d\alpha_0,\]

and this is equivalent to Euler equation (the equation $(1.8)$ with the incompressibility condition $\nabla \cdot u = 0$). Moreover, when $d = 3$, $(1.12)$ can be written as

\[(1.13) \quad \xi_t \circ Q_t = (DQ_t)\xi^0, \quad \text{or} \quad \xi_t = ((DQ_t)\xi^0) \circ Q_t^{-1},\]

where $\xi^t(\cdot) = \nabla \times u(\cdot, t)$. The equation $(1.13)$ is known as Cauchy vorticity formulation of Euler Equation and is equivalent to the vorticity equation by differentiating both sides with respect to $t$:

\[(1.14) \quad \xi_t + (D\xi)u = (Du)\xi.\]

Constantin and Iyer [CI] discovered a circulation invariance principle for Navier-Stokes equation that is formulated in terms of a diffusion associated with the velocity field. Given a solution $u$ to the Navier-Stokes equation

\[(1.15) \quad u_t + (Du)u + \nabla P(q, t) = \nu \Delta u, \quad \nabla \cdot u = 0,\]

let us write $Q_t$ for the (stochastic) flow of the SDE

\[(1.16) \quad dq = u(q, t) \, dt + \sqrt{2\nu} \, dW,\]

with $W$ denoting the standard Brownian motion. If we write $A = Q^{-1}$ and $\xi^t = \nabla \times u(\cdot, t)$, and assume that $d = 3$, then Constantin and Iyer’s circulation formula reads as

\[(1.17) \quad \xi^t = E((DQ_t)\xi^0) \circ A_t,\]

where $E$ denotes the expected value.

We are now ready to state the main result of this article.

3
Theorem 1.1 Write $\alpha_t = u(q,t) \cdot dq$ with $u$ a classical solution of (1.15) and given $T > 0$, set $B_t = Q_T^{-1} \circ Q_T^{-1}$.

- (i) Then the process $\beta_t = B_t^* d\alpha_t$, $t \in [0,T]$ is a 2-form-valued martingale. When $d = 3$, this is equivalent to saying that the process 

$$M_t = ((DB_t^{-1}) \xi^{T-t}) \circ B_t, \quad t \in [0,T],$$

is a martingale.

- (ii) The process 

$$\|\beta_t\|^2 - 2\nu \int_0^t \sum_{i=1}^d \|B_s^* \xi_i\|^2 ds,$$

is a martingale, where 

$$\xi_i^\theta = \sum_{j,k=1}^d u^k_{qj} (\cdot, \theta) \, dq_j \wedge dq_k,$$

or equivalently, $\xi_i^\theta (v_1, v_2) = C(u_{qj} (\cdot, \theta)) v_1 \cdot v_2$, with $C(w) = Dw - (Dw)^*$. (The norm of a 2-form $\eta(v_1, v_2) = Cv_1 \cdot v_2$, $C^* = -C$, $C = [c_{ij}]$, is defined by $\|\eta\|^2 = \sum_{i,j} c_{ij}^2$.)

- (iii) We have the equality 

$$\|d\alpha_T\|^2 + 2\nu \mathbb{E} \int_0^T \sum_{i=1}^d \|B_s^* \xi_i\|^2 ds = \mathbb{E} \|A_T^* d\alpha_0\|^2.$$

Remark 1.1

- (i) In a subsequent paper, we will show how Theorem 1.1 can be extended to certain weak solutions. To make sense of martingales $\beta_t$ and $M_t$, we need to make sure that $DQ_t$ exists weakly and belongs to suitable $L^r$ spaces. As it turns out, a natural condition to guarantee $DQ_t \in L^r$ for all $r \in [1, \infty)$ is 

$$\int_0^T \left[ \int_{\mathbb{R}^d} |u(q,t)|^r \, dx \right]^{r'/r} dt < \infty,$$

for some $r, r' \geq 1$ such that $d/r + 2/r' \leq 1$.

- (ii) Our result takes a simpler form for if $u$ is a solution to the backward Navier-Stokes Equation. For such $u$, we simply have that $\beta_t = Q_t^* d\alpha_t$ is a martingale. When $d = 3$, we deduce that $M_t = ((DA_t)\xi^t) \circ Q_t$ is a martingale.
(iii) In some sense, Theorem 1.1(i) is compatible with a conjecture that the circulation is preserved only in some statistical sense for a singular solution of Euler Equation. We refer to [Ey1] and [Ey2] for some heuristic justification of this conjecture. We note that if for a surface $\Gamma$, the martingale $m^\nu_t = \int_{\Gamma} \beta_t$ has a limit $m^0_t$ in low $\nu$ limit, then $m^0_t$ remains a martingale. In other words, if the circulation is not conserved for a singular solution of the Euler equation, it is still a martingale and conserved in a suitable averaged sense.

The organization of the paper is as follows:

- In Section 2 we discuss Weber’s formulation of Euler Equation and show how (1.12) implies (1.13). We also discuss two fundamental results in Symplectic Geometry that are related to the so-called Clebsch variables.
- In Section 3 we address some geometric questions for stochastic flows of general diffusions and study symplectic.
- In Section 4 we use symplectic diffusions to establish Theorem 1.1.
- In Section 5 we discuss contact diffusions.

2 Euler Equation

In this section we review some basic facts in differential geometry and their applications to Euler Equation. Even though most of the discussion of this section is either well-known or part of folklore, a reader may find our discussion useful as we use similar ideas to prove Theorem 1.1. We also use this section as an excuse to demonstrate/advertise the potential use of symplectic/contact geometric ideas in fluid mechanics.

We start with giving the elementary proof of (1.4): By Cartan’s formula

\[
\frac{d}{dt} \phi_t^* \lambda = \phi_t^* L_Z \lambda = \phi_t^* dK = d(K \circ \phi_t),
\]

where $L_Z$ denotes the Lie derivative with respect to the vector field $Z$, $Z_H = J \nabla_x H$ for $H(q,p,t) = |p|^2/2 + P(q,t)$, and

\[
K(q,p,t) = p \cdot H_p(q,p,t) - H(q,p,t) = \frac{1}{2} |p|^2 - P(q,t).
\]

If we integrate both sides of (2.1) over an arbitrary (non-closed) curve of the form $(\eta, u(\eta, t))$, or equivalently restrict the form $\lambda$ to the graph of the function $u$, then we obtain

\[
\frac{d}{dt} [(DQ_t)^* u \circ Q_t] = \nabla (L \circ Q_t),
\]
where \( L(q, t) = K(q, u(q, t), t) = |u(q, t)|^2/2 - P(q, t) \). Here by \( A^* \) we mean the transpose of the matrix \( A \). Recall \( A_t = Q_t^{-1} \), so that
\[
(DQ_t)^{-1} = DA_t \circ Q_t.
\]
As a consequence of (2.3),
\[
u(\cdot, t) = (DA_t)^* u^0 \circ A_t + \nabla (R \circ A_t),
\]
for \( R = \int_0^t L \circ Q_s \, ds \). As a result,
\[(2.4) \quad u(\cdot, t) = \mathcal{P} [(DA_t)^* u^0 \circ A_t],\]
where \( u^0 \) is the initial data and \( \mathcal{P} \) denotes the Leray-Hodge projection onto the space of divergence-free vector fields. The formula (2.4) is Weber’s formulation and is equivalent to Euler’s equation.

So far we have shown that the Kelvin’s principle (1.3) is equivalent to the Weber’s formulation of Euler equation. If we use (1.5) instead, we obtain a new equivalent formulation of Euler equation, namely the vorticity equation (1.12) or (1.13). Recall \( \bar{\omega} (v_1, v_2) = Jv_1 \cdot v_2 \).

If we choose \( v_1 \) and \( v_2 \) to be tangent to the graph of \( u \), i.e. \( v_i = (w_i, Du(q, t)w_i) \) for \( i = 1, 2 \), then
\[
\bar{\omega} (v_1, v_2) = C(u) w_1 \cdot w_2,
\]
where \( C(u) = Du - (Du)^* \). Hence (1.12) really means
\[(2.5) \quad [C(u(\cdot, t)) \circ Q_t] (DQ_t) w_1 \cdot (DQ_t) w_2 = C(u(\cdot, 0)) w_1 \cdot w_2,
\quad C(u(\cdot, t)) = (DA_t)^* [C(u(\cdot, 0)) \circ A_t] (DQ_t).
\]

Let us assume now that \( d = 3 \) so that, \( C(u) w = \xi \times w \), where \( \xi = \nabla \times u \) denotes the vorticity. Hence
\[
\bar{\omega} (v_1, v_2) = (\xi \times w_1) \cdot w_2 =: [\xi, w_1, w_2].
\]
We note that the right-hand side is the volume form evaluated at the triple \( (\xi, w_1, w_2) \). Now the invariance (2.5) becomes
\[(2.6) \quad [\xi^t \circ Q_t, (DQ_t) w_1, (DQ_t) w_2] = [\xi^0, w_1, w_2],
\]
where we have written \( \xi^t \) for \( \xi(\cdot, t) \). Since \( u \) is divergence-free, the flow \( Q_t \) is volume preserving. As a result,
\[
[\xi^0, w_1, w_2] = [(DQ_t) \xi^0, (DQ_t) w_1, (DQ_t) w_2].
\]
From this and (2.6) we deduce
\[
[\xi^t \circ Q_t, (DQ_t)w_1, (DQ_t)w_2] = [(DQ_t)\xi^0, (DQ_t)w_1, (DQ_t)w_2].
\]
Since \(w_1\) and \(w_2\) are arbitrary, we conclude that (1.13) is true.

**Example 2.1** When \(d = 3\), we may use cylindrical coordinates \(q_1 = r \cos \theta, q_2 = r \sin \theta, q_3 = z\) to write \(u = ae(\theta) + cf(\theta) + be_3\), where
\[
e(\theta) = (\cos \theta, \sin \theta, 0), \quad f(\theta) = (-\sin \theta, \cos \theta, 0), \quad e_3 = (0, 0, 1).
\]
A solution is called \textit{axisymmetric} if \(a, b, c\) are functions of \((r, z)\) only and do not depend on \(\theta\). Let us write
\[
\eta = a_z - b_r, \quad \dot{\eta} = r^{-1} \eta, \quad \bar{c} = rc, \quad \dot{c} = r^{-1}c.
\]
If we write \(\dot{Q}_t(r, z, \theta)\) for the flow \(Q_t(q)\) in the cylindrical coordinates, then
\[
\dot{Q}_t(r, z, \theta) = \left(\psi_t(r, z), \theta + \int_0^t \dot{c}(\psi_s(r, z), s)\, ds\right),
\]
where \(\psi_t\) is the flow of the ODE
\[
\dot{r} = a(r, z, t), \quad \dot{z} = b(r, z, t).
\]
One can easily check that the divergence free condition \(\nabla \cdot u = 0\) means that \((ra)_r + (rb)_z = 0\).
This condition is equivalent to \(\psi^*_t \gamma = \gamma\) for the area form \(\gamma = r\, dr \wedge dz\), simply because
\[
\mathcal{L}_v \gamma = d\iota_v \gamma = d(ra\, dz - rb\, dr) = [(ra)_r + (rb)_z] \, dr \wedge dz.
\]
where \(v = (a, b)\). We also have
\[
\alpha_t = a(\cdot, t)dr + b(\cdot, t)dz + \bar{c}(\cdot, t)d\theta =: \dot{\alpha}_t + \bar{c}(\cdot, t)d\theta.
\]
To understand the meaning of (1.12) in the axisymmetric case, observe
\[
Q^*_t \alpha_t = \psi^*_t \dot{\alpha}_t + (\bar{c}_t \circ \psi_t) d\theta + (\bar{c}_t \circ \psi_t) \int_0^t d(\dot{c}_s \circ \psi_s)\, ds,
\]
where we have are written \(\bar{c}_t\) and \(\dot{c}_t\) for \(\bar{c}(\cdot, t)\) and \(\dot{c}(\cdot, t)\) respectively. As a result, the identity \(d(Q^*_t \alpha_t - \alpha_0) = 0\) is equivalent to \(\bar{c}(\psi_t(r, z), t) = \bar{c}(r, z, 0)\), and
\[
(2.7) \quad d \left[ \psi^*_t \dot{\alpha}_t + (\bar{c}_t \circ \psi_t) \int_0^t d(\dot{c}_s \circ \psi_s)\, ds \right] = d\dot{\alpha}_0.
\]
To simplify (2.7), recall that \( \bar{c}_t \circ \psi_t = c_0 \), and
\[
d(\psi_t^* \hat{\alpha}_t) = \psi_t^* (d\hat{\alpha}_t) = \psi_t^* (\hat{\eta}_t \gamma) = (\hat{\eta}_t \circ \psi_t) \gamma,
\]
where \( \hat{\eta}_t = \hat{\eta}(\cdot, t) \) and we have used \( \psi_t^* \gamma = \gamma \). On the other hand
\[
d \left[ (\bar{c}_t \circ \psi_t) \int_0^t d(\bar{c}_s \circ \psi_s) \ ds \right] = d \left[ \int_0^t \bar{c}_0 d(\bar{c} \circ \psi_s) \ ds \right] = d \left[ \int_0^t (\bar{c}_s \circ \psi_s) d(\bar{c} \circ \psi_s) \ ds \right]
\]
\[
= \int_0^t \psi_s^* d(\bar{c}_s \circ \psi_s) \ ds = \int_0^t \psi_s^* [d\bar{c}_s \wedge d\bar{c}_s] \ ds = \int_0^t \psi_s^* \{\{\bar{c}_s, \bar{c}_s\} \wedge dz \wedge dr\} \ ds
\]
\[
= -2 \int_0^t \psi_s^* [r^{-1}bb_z \wedge dz \wedge dr] \ ds = 2 \int_0^t [(r^{-2}bb_z) \circ \psi_s \gamma] \ ds,
\]
because \( \psi_s^* \gamma = \gamma \). Here we are writing \( \{\cdot, \cdot\} \) for the Poisson bracket. From this and (2.7) we deduce that (1.12) is equivalent to the equality
\[
\eta_t \circ \psi_t + 2 \int_0^t (r^{-2}bb_z) \circ \psi_s \ ds = \eta_0.
\]

\[\square\]

**Definition 2.1**

- (i) A closed 2-form \( \omega \) is symplectic if it is nondegenerate. We say that symplectic forms \( \omega^1 \) and \( \omega^2 \) are **isomorphic** if there exists a diffeomorphism \( \Psi \) such that \( \Psi^* \omega^1 = \omega^2 \).

- (ii) A 1-form \( \alpha \) is contact if \( l_x = \{ v : d\alpha(x; v, w) = 0 \text{ for every } w \} \) is a line and for every \( v \in l_x \), we have that \( \alpha(x; v) \neq 0 \). We say that contact forms \( \alpha^1 \) and \( \alpha^2 \) are **isomorphic** if there exists a diffeomorphism \( \Psi \) such that \( \Psi^* \alpha^1 = \alpha^2 \). We say that contact forms \( \alpha^1 \) and \( \alpha^2 \) are **conformally isomorphic** if there exist a diffeomorphism \( \Psi \) and a scalar-valued continuous function \( f > 0 \) such that \( \Psi^* \alpha^1 = f \alpha^2 \).

- (iii) A solution \( u \) of Euler equation is called **symplectic** if \( \omega_0 = d\alpha_0 \) is symplectic.

- (iv) A solution \( u \) of Euler equation is **contact** if there exists a scalar-valued \( C^1 \) function \( f_0 \) such that \( \alpha_0 + df_0 \) is contact. (Recall \( \alpha_t = u(\cdot, t) \cdot dx. \))

\[\square\]

**Remark 2.1**

- (i) As it is well-known, the non-degeneracy of a 2-form can only happen when the dimension \( d \) is even. Recall \( \alpha_t = u(\cdot, t) \cdot dx. \) If \( u \) is a symplectic solution, then \( \omega_t = d\alpha_t \) is symplectic for all \( t \) because by (1.12), the form \( \omega_t \) is isomorphic to \( \omega_0 \).
Let \( u \) be a symplectic solution of Euler equation, then \( \tilde{\alpha}_t = Q_t^* \alpha_0 + df_t \) is contact for all \( t \) where \( f_t = f_0 \circ Q_t \). In general \( \tilde{\alpha}_t \neq \alpha_t \). However, by equation (1.12), we have \( d\alpha_t = d\tilde{\alpha}_t \). Hence there exists a scalar-valued function \( g_t \) such that \( \alpha_t + dg_t = \tilde{\alpha}_t \) is contact.

We continue with some general properties of symplectic and contact solutions of Euler Equation.

As for symplectic solutions, assume that the dimension \( d = 2k \) is even and write \((q_1, \ldots, q_d) = (x_1, y_1, \ldots, x_k, y_k)\). A classical theorem of Darboux asserts that all symplectic forms are isomorphic to the standard form \( \bar{\omega} = d\bar{\lambda} = \sum_{i=1}^{k} dy_i \wedge dx_i \). A natural question is whether such an isomorphism exists globally.

**Definition 2.2** Let \( u \) be a symplectic solution of Euler Equation. We say that **Clebsch** variables exist for \( u \) in the interval \([0, T]\), if we can find \( C \) functions \( X_1, \ldots, X_k, Y_1, \ldots, Y_k : \mathbb{R}^d \times [0, T] \to \mathbb{R} \), \( F : \mathbb{R}^d \times [0, T] \to \mathbb{R} \) such that \( \Psi_t = (X_1, Y_1, \ldots, X_k, Y_k)(\cdot, t) \) is a diffeomorphism, and

\[
\alpha_t(x, t) = \sum_{i=1}^{k} Y_i \nabla X_i \left( x, t \right) + \nabla F(x, t),
\]

for every \( t \in [0,T] \). Alternatively, we may write \( \alpha_t = \Psi_t^* \bar{\lambda} + dF \), which implies that \( d\alpha_t = \Psi_t^* \bar{\omega} \). \( \square \)

**Proposition 2.1** Let \( u \) be a symplectic solution to Euler Equation in the interval \([0, T] \).

- (i) If Clebsch variables exist for \( t = 0 \), then they exist in the interval \([0, T] \).
- (ii) If \( d = 4 \) and Clebsch variables exist for \( t = 0 \) outside some ball \( B_r = \{ x : |x| \leq r \} \), then they exist globally in the interval \([0, T] \).

**Proof.** (i) This is an immediate consequence of (1.12): If \( \Psi_0^* \bar{\omega} = \omega_0 = d\alpha_0 \), then

\[
(Q_t \circ \Psi_0^{-1})^* d\alpha_t = \Psi_0^{-1} Q_t^* d\alpha_t = \Psi_0^{-1} d\alpha_0 = \bar{\omega},
\]

which means that we can choose \( \Psi_t = \Psi_0 \circ A_t \) for the Clebsch change of variables.

(ii) This is a consequence of a deep theorem of Gromov [Gr]: When \( d = 4 \), a symplectic form is isomorphic to standard form \( \bar{\omega} \), if this is the case outside a ball \( B_r \). \( \square \)
Observe that Euler Equation can be rewritten as

\[
\frac{d}{dt} \alpha + i_u(d\alpha) = -dH, \tag{2.8}
\]

where \( H(q,t) = P(q,t) + |u(q,t)|^2/2 \) is the Hamiltonian function. For a steady solution, \( \alpha_t \) is independent of \( t \) and we simply get

\[
i_u(d\alpha) = -dH.
\]

If \( u \) is a symplectic steady solution of Euler Equation, then \( i_u(d\alpha) = -dH \) means that \( u \) is a Hamiltonian vector field with respect to the symplectic form \( d\alpha \). Of course the associated the Hamiltonian function is \( H \). Alternatively, we may write

\[
u = -C(u)^{-1} \nabla H. \tag{2.9}
\]

**Proposition 2.2** Let \( u \) be a steady symplectic solution to Euler Equation, and let \( c \) be a regular level set of \( H(q,t) = P(q,t) + |u(q,t)|^2 \) i.e. \( \nabla H(q) \neq 0 \) whenever \( H(q) = c \). Then the restriction of the form \( \alpha \) to the submanifold \( H = c \) is contact. In words, regular level sets of \( H \) are contact submanifolds.

**Proof.** By a standard fact in Symplectic Geometry (see for example [R] or [HZ]), the level set \( H = c \) is contact if and only if we can find a Liouville vector field \( X \) that is transversal to \( M_c = \{ H = c \} \). More precisely,

\[
\mathcal{L}_X d\alpha = d\alpha, \quad X(q) \notin T_q M_c,
\]

for every \( q \in M_c \). Here \( T_q M_c \) denotes the tangent fiber to \( M_c \) at \( q \). The first condition means that \( d\iota_X d\alpha = d\alpha \). This is satisfied if \( \iota_X d\alpha = \alpha \). This really means that \( C(u)X = u \) and as a result, we need to choose \( X = C(u)^{-1}u \). It remains to show that \( X \) is never tangent to \( M_c \). For this, it suffices to check that \( X \cdot \nabla H \neq 0 \). Indeed, when \( H = c \),

\[
X \cdot \nabla H = C(u)^{-1}u \cdot \nabla H = -u \cdot C(u)^{-1} \nabla H = |C(u)^{-1} \nabla H|^2 \neq 0,
\]

by (2.9) because by assumption \( \nabla H \neq 0 \). We are done.

**Example 2.2** In this example we describe some simple solutions when the dimension is even. We use polar coordinates to write \( x_i = r_i \cos \theta_i, \ y_i = r_i \sin \theta_i \), and let \( e_i \) (respectively \( e'_i \)) denote the vector for which the \( x_i \)-th coordinate (respectively \( y_i \)-th coordinate) is 1 and any other coordinate is 0. Set

\[
e_i(\theta_i) = (\cos \theta_i)e_i + (\sin \theta_i)e'_i, \quad f_i(\theta_i) = -(\sin \theta_i)e_i + (\cos \theta_i)e'_i.
\]
We may write
\[ u = \sum_{i=1}^{k} \left( a^i e_i(\theta_i) + b^i f_i(\theta_i) \right). \]

The form \( \alpha = u \cdot dx \) can be written as
\[ \alpha = \sum_{i=1}^{k} \left( a^i dr_i + r_i b^i d\theta_i \right) =: \sum_{i=1}^{k} \left( a^i dr_i + B^i d\theta_i \right). \]

For a simple solution, let us assume that all \( a^i \)'s and \( b^i \)'s depend on \( r = (r_1, \ldots, r_k) \) only. We then have
\[ (2.10) \quad d\alpha = \sum_{i<j} \left( a_{ij}^i - a_{ij}^j \right) dr_i \wedge dr_j + \sum_{i,j} r_i^{-1} B_{ij}^i dr_j \wedge (r_id\theta_i). \]

Since
\[ \nabla = \sum_{i=1}^{d} e_i(\theta) \frac{\partial}{\partial r_i} + r_i^{-1} f_i(\theta) \frac{\partial}{\partial \theta_i}, \quad u \cdot \nabla = \sum_{i=1}^{d} a^i \frac{\partial}{\partial r_i} + r_i^{-1} b^i \frac{\partial}{\partial \theta_i}, \]

From this we learn that if \( u \) solves Euler Equation, then the vector fields \( a = (a^1, \ldots, a^k) \) and \( b = (b^1, \ldots, b^k) \) satisfy
\[ (2.11) \quad a_t + C(a) a - \left[ r_i^{-2} (b^i)^2 \right]_i + \nabla_r K = 0, \]
\[ b_t + (Db)a + \left[ r_i^{-2} (a^i b^j) \right]_i = 0, \]
\[ \sum_{i=1}^{d} (r_i a^i)_{r_i}/r_i = 0, \]

for some scalar function \( K(r) = |a|^2/2 + p(r) \). Alternatively, if we introduce the matrix \( E(b) = [r_i^{-1} B_{r_j}^i] \), the first and second equations in (2.11) may be written as
\[ (2.12) \quad a_t + C(a) a - E(b)^* b + \nabla_r H = 0, \]
\[ b_t + E(b) a = 0. \]

where \( H(r) = |b|^2/2 + K(r) = |a|^2/2 + p(r) \). (This can be derived directly from (2.8) and (2.10).) In view of (2.10), \( u \) is a symplectic solution if and only if the matrices \( E(b) \) and \( C(a) \) are invertible. When \( u \) is a steady solution, (2.12) simplifies to
\[ (2.13) \quad C(a) a - E(b)^* b + \nabla_r H = 0, \quad E(b) a = 0. \]

Moreover, by taking the dot product of both sides of the second (respectively first) equation by \( b \) (respectively \( a \), we learn
\[ (2.14) \quad a \cdot \nabla_r H = 0. \]
Also, the second equation in (2.13) really means
\[(2.15) \quad a \cdot \nabla_r B^i = 0 \quad \text{for} \quad i = 1, \ldots, k.\]

When \( d = 4 \) and \( u \) is independent of time, it is straightforward to solve (2.11): From the last equation in (2.11) we learn that there exists a function \( \psi(r_1, r_2) \) such that
\[ a^1 = \frac{\psi_{r_2}}{r_1 r_2}, \quad a^2 = -\frac{\psi_{r_1}}{r_1 r_2}. \]

From this, (2.14) and (2.15) we learn that \( \nabla_r H, \nabla_r B^1, \nabla_r B^2 \) and \( \nabla \psi \) are all parallel. So we may write
\[ H = \mu(\psi), \quad B^1 = \mu_1(\psi), \quad B^2 = \mu_2(\psi), \]
for some \( C^1 \) functions \( \mu, \mu_1, \mu_2 : \mathbb{R} \to \mathbb{R} \). Finally we go back to the first equation in (2.13) to write
\[ a^2 (a^1 r_2 - a^2 r_1) - \frac{B^1_1 B^1_{r_1}}{r_1^2} - \frac{B^2_2 B^2_{r_1}}{r_2^2} + H_{r_1} = 0. \]

Expressing this equation in terms of \( \psi \) yields the elliptic PDE
\[ (r_1 r_2)^{-1} \left[ \left( \frac{\psi_{r_1}}{r_1 r_2} \right) r_1 + \left( \frac{\psi_{r_2}}{r_1 r_2} \right) r_2 \right] = \left( \frac{\mu_1' \mu_1}{r_1^2} + \frac{\mu_2' \mu_2}{r_2^2} - \mu' \right)(\psi). \]

This equation may be compared to the Bragg-Hawthorne Equation that is solved to obtain axisymmetric steady solutions in dimension three. □

We now turn to the odd dimensions. Assume that \( d = 2k + 1 \) for \( k \in \mathbb{N} \). We write \((q_1, \ldots, q_n) = (x_1, y_1, \ldots, x_k, y_k, z)\) and when \( k = 1 \) we simply write \((q_1, q_2, q_3) = (x, y, z)\). In this case, the standard contact form is \( \lambda = \sum_{i=1}^k y_i dx_i + dz \). Again, locally all contact forms are isomorphic to \( \bar{\lambda} \).

**Definition 2.3** Let \( u \) be a solution of Euler Equation. We say that Clebsch variables exist for \( u \) in the interval \([0, T]\), if we can find \( C^1 \) functions
\[ X_1, \ldots, X_k, Y_1, \ldots, Y_k : \mathbb{R}^d \times [0, T] \to \mathbb{R}, \quad f, Z : \mathbb{R}^d \times [0, T] \to \mathbb{R} \]
such that \( \Psi_t = (X_1, Y_1, \ldots, X_k, Y_k, Z)(\cdot, t) \) is a diffeomorphism, \( f > 0 \), and
\[ (fu)(x, t) = \left( \sum_{i=1}^k Y_i \nabla X_i \right)(x, t) + \nabla Z(x, t), \]
for every \( t \in [0, T] \). Alternatively, we may write \( f \alpha_t = \Psi_t^* \bar{\lambda} \). □

As we recalled in the proof of Proposition 2.1(ii), if \( d = 4 \) and a symplectic form is isomorphic to the standard form at infinity, then the isomorphism can be extended to the whole \( \mathbb{R}^d \). This is no longer true when \( d = 3 \); in fact there is countable collection of pairwise
non-conformally-isomorphic forms $\lambda^n$ in $\mathbb{R}^3$ such that each $\lambda^n$ is conformally isomorphic to $\lambda$ at infinity but not globally. A fundamental result of Eliashberg gives a complete classification of contact forms. According to Eliashberg’s Theorem [El], any contact form in $\mathbb{R}^3$ is conformally isomorphic to one of the following forms

- (i) The standard form $\bar{\lambda}$.
- (ii) The form $\hat{\lambda} = \frac{\sin r}{r}(x dy - y dx) + \cos r \, dz$, where $r^2 = x^2 + y^2$.
- (iii) A countable collection of pairwise non-conformally-isomorphic forms $\{\lambda^n : n \in \mathbb{Z}\}$, where each $\lambda^n$ is conformally-isomorphic to $\bar{\lambda}$ outside the ball $B_1$ but not globally in $\mathbb{R}^3$.

The above classification is related to the important notion of overtwisted contact forms. In fact $\hat{\lambda}$ is globally overtwisted whereas $\lambda^n$s are overtwisted only in a neighborhood of the origin. (We refer to [El] or [Ge] for the definition of overtwisted forms).

**Example 2.3** Let us assume that $d = 3$, and $u$ is an axisymmetric steady solution to Euler Equation. Using the notation of Example 2.1, $u$ is a contact solution if and only if

$$u \cdot \xi = r^{-1}(b(r)c'(r) - b'(r)c(r)) \neq 0.$$  

For example, if $b(r) = 1, c(r) = r$, then we get

$$\alpha = r^2 \, d\theta + dz = x dy - y dx + dz,$$

which is isomorphic to $\bar{\lambda}$. On the other hand, choosing $c(r) = r \sin r, b(r) = \cos r$ would yield exactly $\hat{\lambda}$.  

**Remark 2.2** In view of Eliashberg’s classification in dimension three, Clebsch variables would exist only if we are searching for a solution that is conformally isomorphic to $\lambda$. However, if we search for a solution that is conformally isomorphic to $\bar{\lambda}$ for example, then we need to find scalar-valued functions $R, \Theta, Z, f : \mathbb{R}^3 \times [0, T] \to \mathbb{R}$, such that

$$fu = R(\sin R)\nabla\Theta + (\cos R)\nabla Z.$$
3 Symplectic Diffusions

We study stochastic flows associated with diffusions. More precisely consider SDE

\[ dx(t) = V_0(x(t), t)dt + \sum_{i=1}^{k} V_i(x(t), t) \circ dW^i(t), \]

where \((W^i : i = 1, \ldots, k)\) are standard one dimensional Brownian motions on some filtered probability space \((\Omega, \{F_t\}, \mathbb{P})\), and \(V_0, \ldots, V_k\) are \(C^r\) vector fields in \(\mathbb{R}^n\). Here we are using Stratonovich stochastic differentials for the second term on the right-hand of (3.1) and a solution to the SDE (3.1) is a diffusion with the infinitesimal generator

\[ L = V_0 \cdot \nabla + \frac{1}{2} \sum_{i=1}^{k} (V_i \cdot \nabla)^2, \]

or in short \(L = V_0 + \frac{1}{2} \sum_{i=1}^{k} V_i^2\), where we have simply written \(V\) for the \(V\)-directional derivative operator \(V \cdot \nabla\). We assume that the random flow \(\phi_{s,t}\) of (3.1) is well defined almost surely. More precisely for \(\mathbb{P}\)–almost all realization of \(\omega\), we have a flow \(\{\phi_{s,t} : 0 \leq s \leq t\}\) where \(\phi_{s,t}(\cdot, \omega) : \mathbb{R}^n \to \mathbb{R}^n\) is a \(C^{r-1}\) diffeomorphism and \(\phi_{s,t}(a, \omega) =: x(t)\) is a solution of (3.1) subject to the initial condition \(x(s) = a\). (We also write \(\phi_t\) for \(\phi_{0,t}\).) For example a uniform bound on the \(C^r\)-norm of the coefficients \(V_0, \ldots, V_k\) would guarantee the existence of such a stochastic flow provided that \(r \geq 2\). We also remark that we can formally differentiate (3.1) with respect to the initial condition and derive a SDE for \(\Lambda_{s,t}(x) = \Lambda_t(x) := D_x \phi_{s,t}(x)\):

\[ d\Lambda_t(x) = D_x V_0(\phi_{s,t}(x), t) \Lambda_t(x)dt + \sum_{i=1}^{k} D_x V_i(\phi_{s,t}(x), t) \Lambda_t(x) \circ dW^i(t). \]

Given a differential \(\ell\)-form \(\alpha(x; v_1, \ldots, v_\ell)\), we define

\[ (\phi_{s,t}^* \alpha)(x; v_1, \ldots, v_\ell) = \alpha(\phi_{s,t}(x); \Lambda_{s,t}(x)v_1, \ldots, \Lambda_{s,t}(x)v_\ell). \]

Given a vector field \(V\), we write \(\mathcal{L}_V\) for the Lie derivative in the direction \(V\). More precisely, for every differential form \(\alpha\),

\[ \mathcal{L}_V \alpha = (\hat{d} \circ i_V + i_V \circ \hat{d}) \alpha, \]

where \(\hat{d}\) and \(i_V\) denote the exterior derivative and \(V\)–contraction operator respectively. (To avoid a confusion between the stochastic differential and exterior derivative, we are using a hat for the latter.) We are now ready to state a formula that is the stochastic analog of Cartan’s formula and it is a rather straight forward consequence of (3.2). We refer to Kunita [K2] for a proof.
Proposition 3.1 Set \( V = (V_0, V_1, \ldots, V_m) \) and
\[
A_V = \mathcal{L}_{V_0} + \frac{1}{2} \sum_{i=1}^k \mathcal{L}_{V_i}^2.
\]
We also \( \eta^t \) for \( \phi_{s,t} \eta \) for any form \( \eta \). We have

\[
d\alpha^t = (\mathcal{L}_{V_0} \alpha)^t \ dt + \sum_{i=1}^k (\mathcal{L}_{V_i} \alpha)^t \circ dW^i(t)
\]

\[
= (A_V \alpha)^t \ dt + \sum_{i=1}^k (\mathcal{L}_{V_i} \alpha)^t \ dW^i(t).
\]

Example 3.1

- (i) If \( \alpha = f \) is a 0-form, then \( A_V f = Lf \) is simply the infinitesimal generator of the underlying diffusion.

- (ii) If \( \alpha = \rho \ dx_1 \wedge \cdots \wedge dx_n \), is a volume form, then
  \[
  \mathcal{L}_V \alpha = (\nabla \cdot (\rho \alpha)) \ dx_1 \wedge \cdots \wedge dx_n,
  \]
  and
  \[
  A_V \alpha = (L^* \rho) \ dx_1 \wedge \cdots \wedge dx_n,
  \]
  where \( L^* \) is the adjoint of the operator \( L \).

- (iii) If \( \alpha = \rho \ dx_1 \wedge \cdots \wedge dx_n \), is a volume form, then \( \alpha^t = \phi_{s,t} \alpha = \alpha \) if and only if \( \nabla \cdot (\rho V_i) = 0 \) for \( i = 0, 1, \ldots, k \). For example, if \( W = (W^1, \ldots, W^n) \) is a \( n \)-dimensional standard Brownian motion and
  \[
  dx = V_0(x, t) dt + dW,
  \]
  then the flow of this diffusion preserves the standard volume form \( \bar{\alpha} = dx_1 \wedge \cdots \wedge dx_n \), if and only if \( \nabla \cdot V_0 = 0 \).

- (iv) If \( \alpha = \rho \ dx_1 \wedge \cdots \wedge dx_n \), is a volume form, and \( L^* \rho = 0 \), then \( \alpha \), regarded as a measure, is an invariant measure for the diffusion (3.1). However, \( \alpha^t = \phi_{s,t} \alpha \) is a volume-form-valued martingale.

We now make two definitions:

Definition 3.1 Let \( \alpha \) be a symplectic form.

- (i) We say that the diffusion (3.1) is (strongly) \( \alpha \)-symplectic if its flow is symplectic with respect \( \alpha \), almost surely. That is \( \phi_{s,t} \alpha = \alpha \), a.s.
• (ii) We say that the diffusion (3.1) is weakly symplectic if $\alpha^t := \phi_t^* \alpha$, is a martingale.

Using Proposition 3.1 it is not hard to deduce

**Proposition 3.2**

• (i) The diffusion (3.1) is (strongly) $\alpha$-symplectic if and only if the vector fields $V_0, V_1, \ldots, V_k$ are $\alpha$-Hamiltonian, i.e. $\mathcal{L}_{V_0} \alpha = \mathcal{L}_{V_1} \alpha = \cdots = \mathcal{L}_{V_k} \alpha = 0$.

• (ii) The diffusion (3.1) is weakly $\alpha$-symplectic if and only if $A_V \alpha = 0$.

We discuss two systematic ways of producing weakly symplectic diffusions.

**Recipe (i)**

Given a symplectic form $\alpha$, we write $X^H = X^\alpha_H$ for the Hamiltonian vector field associated with the Hamiltonian function $H$. Note that by non-degeneracy of $\alpha$, there exists a unique vector field $X = X^\alpha(\nu)$ such that $i_X \alpha = \nu$ for every 1-form $\nu$ and $X^H = -\mathcal{X}^\alpha(dH)$. In the following proposition, we show that given $V_1, V_2, \ldots, V_k$, we can always find a unique $\hat{V}_0$ such that the diffusion associated with $V = (X^H + \hat{V}_0, V_1, \ldots, V_k)$ is weakly $\alpha$-symplectic.

**Proposition 3.3** The diffusion (3.1) is weakly $\alpha$-symplectic if and only if there exists a Hamiltonian function $H$, such that

\[
V_0 = X^H - \frac{1}{2} \sum_{j=1}^k \mathcal{X}^\alpha \left( i_{V_j} \hat{d} i_{V_j} \alpha \right).
\]

**Proof.** By definition,

\[
A_V \alpha = \hat{d} \left[ i_{V_0} \alpha + \frac{1}{2} \sum_{j=1}^k \left( i_{V_j} \hat{d} i_{V_j} \alpha \right) \right].
\]

Hence $A_V \alpha = 0$ means that for some function $H$,

\[
i_{V_0} \alpha + \frac{1}{2} \sum_{j=1}^k \left( i_{V_j} \hat{d} i_{V_j} \alpha \right) = -\hat{d}H.
\]

From this we can readily deduce (3.5). \qed

**Recipe (ii)** We now give a useful recipe for constructing $\bar{\omega}$-diffusions where $\bar{\omega}$ is the standard symplectic form and $n = 2d$.

**Proposition 3.4** Given a Hamiltonian function $H$, consider a diffusion $x(t) = (q(t), p(t))$ that solves

\[
dx(t) = J \nabla H(x(t), t) dt + \sum_{i=1}^k V_i(x(t), t) dW^i(t),
\]

16
with \( V_j = \begin{bmatrix} A_j \\ B_j \end{bmatrix} \), for \( j = 1, \ldots, k \), where \( A_j = (A_j^1, \ldots, A_j^d) \), and \( B_j = (B_j^1, \ldots, B_j^d) \). Then
\[
\mathcal{A}_V \tilde{\omega} = \frac{1}{2} \hat{d} \gamma, \quad \text{where} \quad \gamma = Z_1 \cdot dq + Z_2 \cdot dp, \quad \text{with}
\]

\[
(3.6) \quad Z_1^i = \sum_{r,j} \left( \frac{\partial A_j^r}{\partial q_i} B_j^r - \frac{\partial B_j^r}{\partial q_i} A_j^r \right), \\
Z_2^i = \sum_{r,j} \left( \frac{\partial A_j^r}{\partial p_i} B_j^r - \frac{\partial B_j^r}{\partial p_i} A_j^r \right).
\]

**Proof.** The Stratonovich differential is related to Itô differential by
\[
a \circ dW = a dW + \frac{1}{2} [da, dW].
\]
As a result, the diffusion \( x(t) \) satisfies (3.1) for \( V_0 = J \nabla H - \frac{1}{2} \dot{V}_0 \) with \( \dot{V}_0 = \begin{bmatrix} A_0 \\ B_0 \end{bmatrix} \), where

\[
(3.7) \quad A_0^i = \sum_{r,j} \left( \frac{\partial A_j^r}{\partial q_i} A_j^r + \frac{\partial A_j^r}{\partial p_i} B_j^r \right), \\
B_0^i = \sum_{r,j} \left( \frac{\partial B_j^r}{\partial q_i} A_j^r + \frac{\partial B_j^r}{\partial p_i} B_j^r \right).
\]

By definition,
\[
(3.8) \quad \mathcal{A}_V \tilde{\omega} = \frac{1}{2} \hat{d} \gamma := \frac{1}{2} \hat{d} \left( \eta - i_{\dot{V}_0} \tilde{\omega} \right),
\]
where
\[
\eta = \sum_{j=1}^k i_{V_j} \hat{d} i_{V_j} \tilde{\omega}.
\]
To calculate \( \gamma \) and \( \eta \), let us write \( \beta(F) \) for the 1-form \( F \cdot dx \) and observe
\[
i_{V} \tilde{\omega} = \beta(JV), \quad \hat{d} \beta(F)(v, w) = C(F)v \cdot w,
\]
where \( C(F) = DF - (DF)^* \) with \( DF \) denoting the matrix of the partial derivatives of \( F \) with respect to \( x \). From this we deduce

\[
(3.9) \quad \eta = \sum_{j=1}^k \beta \left( C(JV_j)V_j \right).
\]
A straightforward calculation yields

\[ C(JV_j) = \begin{bmatrix} X_{11}^j & X_{12}^j \\ X_{21}^j & X_{22}^j \end{bmatrix} \]

where

\[ X_{11}^j = \left[ \frac{\partial B_j^i}{\partial q_r} - \frac{\partial B_j^i}{\partial q_i} \right]_{i,r=1}^n, \quad X_{12}^j = \left[ \frac{\partial B_j^i}{\partial p_r} + \frac{\partial A_j^i}{\partial q_i} \right]_{i,r=1}^n, \]
\[ X_{21}^j = \left[ -\frac{\partial A_j^i}{\partial q_r} - \frac{\partial B_j^i}{\partial p_i} \right]_{i,r=1}^n, \quad X_{22}^j = \left[ -\frac{\partial A_j^i}{\partial p_r} + \frac{\partial A_j^i}{\partial p_i} \right]_{i,r=1}^n. \]

From this we deduce

\[ C(JV_j)V_j = \begin{bmatrix} Y_1^j \\ Y_2^j \end{bmatrix}, \]

where

\[ Y_1^j = \sum_r \left( \frac{\partial B_j^i}{\partial q_r} - \frac{\partial B_j^i}{\partial q_i} \right) A_r^j + \sum_r \left( \frac{\partial B_j^i}{\partial p_r} + \frac{\partial A_j^i}{\partial q_i} \right) B_r^j \bigg|_{i=1}^n, \]
\[ Y_2^j = \sum_r \left( \frac{\partial A_j^i}{\partial p_r} - \frac{\partial A_j^i}{\partial p_i} \right) B_r^j - \sum_r \left( \frac{\partial A_j^i}{\partial q_r} + \frac{\partial B_j^i}{\partial p_i} \right) A_r^j \bigg|_{i=1}^n. \]

Summing these expressions over \( j \) yields

\[ \sum_j C(JV_j)V_j = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} B_0 \\ -A_0 \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} + J\bar{V}_0, \]

where \( Z_1 \) and \( Z_2 \) are defined by (3.7). This (3.8) and (3.9) complete the proof. □

An immediate consequence of Proposition 3.4 is Corollary 3.1.

**Corollary 3.1** Let \( x(t) = (q(t), p(t)) \) be a diffusion satisfying

\[
\begin{align*}
\text{dq} &= H_p(q, p) \, dt + \sqrt{2\nu} \, dW, \\
\text{dp} &= -H_q(q, p) \, dt + \sqrt{2\nu} \Gamma(q, t) \, dW.
\end{align*}
\]

where \( \Gamma \) is a continuously differentiable \( d \times d \)-matrix valued function and \( W = (W^1, \ldots, W^d) \) is a standard Brownian motion in \( \mathbb{R}^d \). Then the process \( x(t) \) is weakly \( \bar{\omega} \)-symplectic.

**Proof.** Observe that \( x(t) \) satisfies (3.6) for \( A = \sqrt{2\nu}I_d \) and \( B \) that is independent of \( p \). From this we deduce that \( Z_2 = 0 \) and \( Z_1 = -\sqrt{2\nu} \nabla_q(tr\Gamma) \). We are done because \( \gamma = -\sqrt{2\nu} d(tr\Gamma) \), and \( A\bar{\omega} = \frac{d}{\nu} \gamma = 0 \). □
4 Martingale Circulation

Proof of Theorem 1.1. Step 1. As in Section 1, we write $D$ and $\nabla$ for $q$-differentiation. For $x$-differentiation however, we write $D_x$ and $\nabla_x$ instead. Let us write $x'(t) = (q'(t), p'(t))$ for a diffusion that satisfies

$$dq'(t) = p'(t) \, dt + \sqrt{2\nu} \, d\bar{W}$$

(4.1)

$$dp'(t) = -\nabla P(q'(t), t) \, dt + \sqrt{2\nu} \, Dw(q'(t), t) \, d\bar{W}.$$  

for a time dependent $C^1$ vector field $w$ in $\mathbb{R}^d$ and a standard Brownian motion $\bar{W}$. The flow of this diffusion is denoted by $\phi_t$. We then apply Corollary 3.1 for $H(q, p, t) = \frac{1}{2} |p|^2 + P(q, t)$ and $\Gamma = Dw$, to assert that the diffusion $x'$ is weakly $\omega$-symplectic. Let us now assume that $w$ satisfies the backward Navier-Stokes equation

$$w_t + (Dw)w + \nabla P + \nu \Delta w = 0, \quad \nabla \cdot w = 0.$$ 

(4.2)

We observe that if the process $q'(t)$ is a diffusion satisfying

$$dq'(t) = w(q'(t), t) \, dt + \sqrt{2\nu} \, d\bar{W}.$$ 

(4.3)

and $p'(t) = w(q'(t), t)$, then by Ito’s formula,

$$dp'(t) = [w_t + (Dw)w + \nu \Delta w](q'(t), t) \, dt + \sqrt{2\nu} Dw(q'(t), t) \, d\bar{W}$$

(4.4)

This means that if $\check{Q}_t$ denotes the flow of the SDE (4.3), then

$$\phi_t(q, w(q, 0)) = (\check{Q}_t(q), w(\check{Q}_t(q), t)), \quad D_x \phi_t(q, w(q, 0)) \left[ \begin{array}{c} a \\ Dw(q, 0)a \end{array} \right] = \left[ \begin{array}{c} (D\check{Q}_t(q))a \\ (Dw(\check{Q}_t(q), t))(D\check{Q}_t(q))a \end{array} \right].$$

(4.5)

By the conclusion of Corollary 3.1, the process

$$\check{M}_t(x; v_1, v_2) = [J(D\phi_t(x))v_1] \cdot [(D\phi_t(x))v_2]$$

is a 2-form-valued martingale. By choosing

$$x = (q, w(q, 0)), \quad v_1 = \left[ \begin{array}{c} a_1 \\ Dw(q, 0)a_1 \end{array} \right], \quad v_2 = \left[ \begin{array}{c} a_2 \\ Dw(q, 0)a_2 \end{array} \right],$$

(4.6)

for the arguments of $\check{M}_t$, we learn that $\check{M}_t(q; a_1, a_2) := \check{M}_t(x; v_1, v_2)$ is a 2-form-valued martingale in $\mathbb{R}^d$. Using (4.5) we have

$$\check{M}_t(q; a_1, a_2) = J \left[ \begin{array}{c} (D\check{Q}_t(q))a_1 \\ (Dw(\check{Q}_t(q), t))(D\check{Q}_t(q))a_1 \end{array} \right] \cdot \left[ \begin{array}{c} (D\check{Q}_t(q))a_2 \\ (Dw(\check{Q}_t(q), t))(D\check{Q}_t(q))a_2 \end{array} \right]$$

$$= \left[ (Dw - (Dw)^*)(\check{Q}_t(q), t) \right] (D\check{Q}_t(q))a_1 \cdot (D\check{Q}_t(q))a_2$$

$$= \check{Q}_t d\check{\alpha}_t(q; a_1, a_2),$$

(4.7)
where \( \bar{\alpha}_t = w(q, t) \cdot dq \). In summary, \( \bar{M}_t = \bar{Q}_t^* \bar{d}\bar{\alpha}_t \) is a martingale (proving our first claim in Remark 1.1(ii)).

When \( d = 3 \),

\[
\bar{Q}_t^* \bar{d}\bar{\alpha}_t(q; a_1, a_2) = \left( (\eta^t \circ \bar{Q}_t(q)) \times (D\bar{Q}_t(q)) a_1 \right) \cdot (D\bar{Q}_t(q)) a_2
\]

where \( \eta^t(\cdot) = \nabla \times w(\cdot, t) \) and \([a, b, c]\) is the determinant of a matrix with column vectors \( a, b, c \). Since \( w \) is divergence-free, the flow \( \bar{Q}_t \) is volume preserving (see Example 3.1(iii)). Hence

\[
\bar{M}_t(q; a_1, a_2) = [(D\bar{A}_t \circ \bar{Q}_t(q)) \eta^t \circ \bar{Q}_t(q), a_1, a_2],
\]

where \( \bar{A}_t = \bar{Q}_t^{-1} \). Since \( \bar{M}_t \) is a martingale, we deduce that the process

\[
\bar{M}_t(q) = (D\bar{A}^t \circ \bar{Q}_t(q))(\eta^t \circ \bar{Q}_t(q)),
\]

is a martingale (proving our second claim in Remark 1.1(ii)).

**Step 2.** Suppose that now \( u \) is a solution to the forward Navier-Stokes equation (1.15) and recall that when \( d = 3 \), we write \( \xi = \nabla \times u \). We set \( w(q, t) = -u(q, T - t) \) for \( t \in [0, T] \). Then \( w \) satisfies (3.2) in the interval \( t \in [0, T] \). Recall that \( q(t) \) is the solution of SDE (1.16) with the flow \( Q_t \). We choose \( \tilde{W}(t) = W(T - t) - W(T) \) in the equation (4.3). According to a theorem of Kunita (see [K1] or Theorem 13.15 in page 139 of [RW]), the flows \( Q \) and \( \bar{Q} \) are related by the formula

\[
\bar{Q}_t = Q_{T-t} \circ Q_T^{-1} = B_t.
\]

Observe that \( \bar{\alpha}_t = -\alpha_{T-t} \) and

\[
\bar{M}_t = \bar{Q}_t^* \bar{d}\bar{\alpha}_t = -B_t^* \bar{d}\alpha_{T-t} = -\beta_t.
\]

Hence \( (\beta_t : t \in [0, T]) \) is a martingale because \( \bar{M}_t \) is a martingale by Step 1. Also, when \( d = 3 \),

\[
\bar{M}_t = ((D\bar{A}_t)\eta^t) \circ \bar{Q}_t(q) = -((DB_t^{-1}) \xi^{T-t}) \circ B_t.
\]

This completes the proof of Part (i).

**Step 3.** The process \( x'(t) \) is a diffusion of the form (3.1) with \( k = d \) and

\[
V_i(x', t) = V_i(q, t) = \sqrt{2\nu} \left[ \begin{array}{c} e_i \\ w_{q, i} \end{array} \right], \quad i = 1, \ldots, d;
\]

where \( e_i = [\delta^i_j]_{j=1}^d \) is the unit vector in the \( i \)-th direction. A straightforward calculation yields that for the standard symplectic form \( \bar{\omega} = \sum_j dp_j \wedge dq_j \),

\[
\mathcal{L}_{V_i(\cdot, t)} \bar{\omega} = \sqrt{2\nu} \left( w_{q, i}(\cdot, t) \cdot dq_i - dp_i \right) = \sqrt{2\nu} (\gamma_i^t - dp_i),
\]

\[
\mathcal{L}_{V_i(\cdot, t)} \bar{\omega} = \sqrt{2\nu} \bar{d}\gamma_i^t = \sqrt{2\nu} \sum_{j,k} w^k_{q, ij}(\cdot, t) dq_j \wedge dq_k,
\]

20
where \( w = (w^1, \ldots, w^d) \). From this and (3.4) we deduce

\[
\dot{M}_t = \hat{\omega} + \sqrt{2\nu} \sum_{i=1}^{d} \int_{0}^{t} \phi_s^* \left( \dot{d}\gamma_s^i \right) \, dW^i(s),
\]

because by Step 1, we know that \( \mathcal{A}V\omega = 0 \). As in the calculation (4.7), we may choose as in (4.6) for the arguments of \( \dot{d}\gamma_s^i \) to deduce

\[
d\dot{M}_t = \sqrt{2\nu} \sum_{i=1}^{d} \dot{Q}_s^i \zeta_s^i \, dW^i(t),
\]

where \( \zeta_s^i \) is really \( \dot{d}\gamma_s^i \) but now regarded as a 2-form in \( \mathbb{R}^d \). Note

\[
\zeta_s^i(a_1, a_2) = C(w_q^i(\cdot, t))a_1 \cdot a_2.
\]

Let us write \( Z_t \) and \( Y^i_t \) for the antisymmetric matrices associated with the forms \( \dot{Q}_s^i(d\alpha_s) \) and \( \dot{Q}_s^i\zeta_s^i \), respectively. We then have

\[
dZ_t = \sqrt{2\nu} \sum_{i=1}^{d} Y^i_t \, dW^i(t),
\]

From this, we readily deduce that the quadratic variation of the process \( Z_t \) is given by

\[
2\nu \int_{0}^{t} \sum_{i=1}^{d} \| Y^i_s \|^2 \, ds.
\]

We now reverse time as in Step 2 to complete the proof of Part (ii). Part (iii) is an immediate consequence of the identity

\[
\mathbb{E}\| Z_T \|^2 = \mathbb{E}\| Z_0 \|^2 + 2\nu \mathbb{E} \int_{0}^{T} \sum_{i=1}^{d} \| Y^i_s \|^2 \, ds.
\]

\( \square \)

**Example 4.1** In this example, we examine the consequences of Theorem 1.1(i) for axisymmetric solutions of Navier-Stokes Equation when \( d = 3 \). To ease the notation, we assume that \( u \) is backward solution of Navier-Stokes Equation (4.2). We follow the notation of Example 2.1 and write \( u = ae(\theta) + cf(\theta) + be_3 \), where \( a, b, \) and \( c \) are functions of \((r, z, t)\). In cylindrical coordinates, the SDE (4.1) has infinitesimal generator

\[
\mathcal{A} = a \partial_r + b \partial_z + c \partial_\theta + \nu \left( \partial_r^2 + r^{-1} \partial_r + \partial_z^2 + r^{-2} \partial_\theta^2 \right) =: \hat{\mathcal{A}} + \hat{c} \partial_\theta + \nu r^{-2} \partial_\theta^2,
\]

21
and can be written as

\[
\begin{aligned}
dr &= \left[a(r, z, t) + \nu r^{-1}\right] \, dt + \nu' dB^1(t), \\
\frac{dz}{dt} &= b(r, z, t) \, dt + \nu' dB^2(t), \\
\frac{d\theta}{dt} &= \tilde{c}(r, z, t) \, dt + \nu' r^{-1} dB^3(t),
\end{aligned}
\]

where \( B^1, B^2 \) and \( B^3 \) are 3 independent standard Brownian motions, and \( \nu' = \sqrt{2\nu} \). If we write \( \hat{Q}_t(r, z, \theta) \) for the flow of SDE (4.1) in the cylindrical coordinates, then

\[
\hat{Q}_t(r, z, \theta) = \left( \psi_t(r, z), \theta + \int_0^t \tilde{c}(\psi_s(r, z), s) \, ds + \int_0^t R_s^{-1}(r, z) \, dB^3(s) \right),
\]

where \( \psi_t(r, z) = (R_t(r, z), Z_t(z, r)) \) is the flow of the first two equations of (4.8). As in Example 2.1, the divergence free condition \( \nabla \cdot u = 0 \) is equivalent to \( \psi_t^* \gamma = \gamma \) for the area form \( \gamma = rdr \wedge dz \). This can be shown with the aid of Proposition 3.1 as in example 3.1(iii); this time we show that \( \mathcal{L}_v \gamma = 0 \), where \( v = (a + \nu r^{-1}, b) \). As in Example 2.1, we write \( \alpha_t = \hat{\alpha}_t + \tilde{c}(\cdot, t) \, d\theta \). We have

\[
Q^*_t \alpha_t = \psi_t^* \hat{\alpha}_t + (\tilde{c}_t \circ \psi_t) \, d\theta + (\tilde{c}_t \circ \psi_t) \int_0^t \hat{d}(\hat{c}_s \circ \psi_s) \, ds + \int_0^t \hat{d}R_{s}^{-1} \, dB^3(s).
\]

Now, according to Theorem 1.1(i), the process \( \hat{d}(Q^*_t \alpha_t) \), is a martingale. This is equivalent to asserting that the following processes are martingale:

1. \( m_t := \tilde{c}(\psi_t(r, z), t) \),
2. \( m'_t := \hat{d} \left[ \psi_t^* \hat{\alpha}_t + (\tilde{c}_t \circ \psi_t) \int_0^t \hat{d}(\hat{c}_s \circ \psi_s) \, ds + (\tilde{c}_t \circ \psi_t) \int_0^t \hat{d}R_{s}^{-1} \, dB^3(s) \right] \).

To simplify \( m'_t \), first observe that as in Example 2.1,

\[
\hat{d}(\psi_t^* \hat{\alpha}_t) = \psi_t^*(d\hat{\alpha}_t) = \psi_t^*(\hat{\eta}_t \gamma) = (\hat{\eta}_t \circ \psi_t) \gamma,
\]

because \( \psi_t^* \gamma = \gamma \). On the other hand if \( f_t = \int_0^t \hat{d}(\hat{c}_s \circ \psi_s) \, ds \), then \( f_t \) is a process of a bounded variation. Since \( m_t \) is a martingale, we learn that \( d(m_t f_t) = f_t \, dm_t + m_t \, df_t \), so the process

\[
\bar{m}_t := m_t f_t - \int_0^t m_s \, ds = (\tilde{c}_t \circ \psi_t) \int_0^t \hat{d}(\hat{c}_s \circ \psi_s) \, ds - \int_0^t (\tilde{c}_s \circ \psi_s) \hat{d}(\hat{c}_s \circ \psi_s) \, ds,
\]

is a martingale. Moreover since \( m_t \) and \( \int_0^t \hat{d}R_{s}^{-1} \, dB^3(s) \) are two orthogonal martingale, we learn that the process

\[
(\tilde{c}_t \circ \psi_t) \int_0^t \hat{d}R_{s}^{-1} \, dB^3(s),
\]

is a martingale.
is a martingale. From this and the fact that $m_t$ is a martingale, we deduce that

$$m''_t = (\hat{\eta}_t \circ \psi_t)\gamma - \int_0^t (\hat{\epsilon}_s \circ \psi_s) d(\hat{\epsilon}_s \circ \psi_s) \, ds,$$

is a martingale. We are now in a position to repeat our calculation in Example 2.1 to learn that the process

$$N_t = \hat{\eta}_t \circ \psi + 2 \int_0^t (r^{-2}bb_z) \circ \psi_s ds,$$

is a martingale. In summary, Theorem 1.1(i) for an axisymmetric backward solution means that the processes $m_t$ and $N_t$ are martingales. This can be expressed analytically by the following equations:

$$\bar{c}_t + \hat{\epsilon}c = 0, \quad \hat{\epsilon}_t + \hat{\epsilon}\hat{\eta}_t + 2r^{-2}bb_z = 0.$$

Remark 4.1 If we drop the noise from the third equation in (4.8) by replacing $B^3$ with 0, the conclusion of Example 4.1 will not be affected. Basically the $\theta$ process is driven by $r$ and $z$ components and adding noise to the dynamics of $\theta$ would be rather artificial.

5 Contact Diffusions

Recall that contact forms are certain 1-forms that are non-degenerate in some rather strong sense. To explain this, recall that when $\alpha$ is a contact form in dimension $n = 2d + 1$, then the set $l_x = \{v : d\alpha_x(v, w) = 0 \text{ for all } w \in T_xM\}$ is a line. Also, if we define the kernel of $\alpha$ by

$$\eta_x^\alpha = \eta_x = \{v : \alpha(x; v) = 0\},$$

then the contact condition really means that $l_x$ and $\eta_x$ give a decomposition of $\mathbb{R}^n$ that depends solely on $\alpha$:

$$\mathbb{R}^n = \eta_x \oplus l_x.$$

We also define the Reeb vector field $R(x) = R^\alpha(x)$ to be the unique vector such that

$$R(x) \in l_x, \quad \alpha(x; R(x)) = 1.$$

The role of Hamiltonian vector fields in the contact geometry are played by contact vector fields.

Definition 5.1 A vector field $X$ is called an $\alpha$-contact vector field if $\mathcal{L}_X \alpha = f \alpha$ for some scalar-valued positive continuous function $f$. It is known that for a given a “Hamiltonian” $H : M \to \mathbb{R}$, there exists a unique contact $\alpha$-vector field $X_H = X_H^\alpha$ such that $i_{X_H} \alpha = \alpha(X_H) = H$. The function $f$ can be expressed in
terms of $H$ with the aid of the Reeb’s vector field $R = R^\alpha$; indeed, $f = dH(R^\alpha)$, and as a result,

$$\mathcal{L}_{X_H} \alpha = dH(R^\alpha) \alpha.$$ 

In our Euclidean setting, we consider a form $\alpha = u \cdot dx$ for a vector field $u$ and

$$\beta(v_1, v_2) := d\alpha(v_1, v_2) = C(u)v_1 \cdot v_2,$$

where $C(u) = Du - (Du)^*$. (Recall that we are writing $A^*$ for the transpose of $A$.) Since $C^* = -C$, we have that $\det C = (-1)^n \det C$. This implies that $C$ cannot be invertible if the dimension is odd. Hence the null space $l_x$ of $C(u)(x)$ is never trivial and our assumption $\dim l_x = 1$ really means that this null space has the smallest possible dimension. Now (5.1) simply means that $u(x) \cdot R(x) \neq 0$. Of course $R$ is chosen so that $u(x) \cdot R(x) \equiv 1$. Writing $u^\perp$ and $R^\perp$ for the space of vectors perpendicular to $u$ and $R$ respectively, then $\eta = u^\perp$, and we may define a matrix $C'(u)$ which is not exactly the inverse of $C(u)$ (because $C(u)$ is not invertible), but it is specified uniquely by two requirements:

- (i) $C'(u)$ restricted to $R^\perp$ is the inverse of $C(u) : u^\perp \rightarrow R^\perp$.
- (ii) $C'(u)R = 0$.

The contact vector field associated with $H$ is given by

$$X_H = -C'(u)\nabla H + HR.$$

In particular, when $n = 3$, the form $\alpha = u \cdot dx$ is contact if and only if $u \cdot \xi$ is never 0, where $\xi = \nabla \times u$ is the curl (vorticity) of $u$. In this case the Reeb vector field is given by $R = \xi/(u \cdot \xi)$, and

$$\mathcal{L}_\xi u = \nabla(u \cdot Z) + \xi \times Z,$$

We also write $\bar{u} = u/(u \cdot \xi)$. The contact vector field associated with $H$ is given by

$$X_H = \bar{u} \times \nabla H + HR.$$

Let $x(t)$ be a diffusion satisfying (3.1) and assume that this diffusion has a random flow $\phi_t$. Given a contact form $\alpha$, set $\alpha^t = \phi_t^* \alpha$ as before.

**Definition 5.2.**

- (i) We say that the diffusion (3.1) is strongly $\alpha$-contact, if for some scaler-valued semimartingale $Z_t$ of the form,

$$dZ_t = g_0(x(t), t) \, dt + \sum_{i=1}^{k} g_i(x(t), t) \circ dW^i(t),$$

for $g_0, \ldots, g_k > 0$, we have

$$d\alpha^t = \alpha^t \, dZ_t.$$
(ii) We say that the diffusion (3.1) is weakly \( \alpha \)-contact, if there exists a continuous scalar-valued function positive \( f(x,t) \) such that

\[
M_t = \alpha^t - \int_0^t f(x(s), s) \alpha^s \, ds,
\]

is a martingale.

We end this section with two propositions.

**Proposition 5.1** The following statements are equivalent:

• (i) The diffusion (3.1) is strongly \( \alpha \)-contact.

• (ii) There exists a scaler-valued process \( A_t \) of the form

\[
dA_t = h_0(x(t), t) \, dt + \sum_{i=1}^k h_i(x(t), t) \circ dW_i(t).
\]

with \( h_1, \ldots, h_k > 0 \) and \( h_0 + \sum_{i=1}^k h_i^2 > 0 \), such that \( \alpha^t = e^{A_t} \alpha \).

• (iii) The vector fields \( V_0, \ldots, V_k \) are \( \alpha \)-contact.

**Proposition 5.2** The following statements are equivalent:

• (i) The diffusion (3.1) is weakly \( \alpha \)-contact.

• (ii) For some continuous scalar-valued positive function \( f(x,t) \), we have \( \mathcal{A} \alpha = f \alpha \).

The proof of Proposition 5.2 is omitted because it is an immediate consequence of (3.4) and the definition.

**Proof of Proposition 5.1.** Suppose that the vector fields \( V_0, \ldots, V_k \) are \( \alpha \)-contact. Then there exist scalar-valued functions \( g_0(x,t), \ldots, g_k(x,t) \) such that \( \mathcal{L}_{V_i} \alpha = g_i \alpha \). From this and Proposition 3.1 we learn that \( d\alpha^t = \alpha^t \, dZ_t \) for \( Z_t \) as in (5.2). Hence (iii) implies (i).

Now assume (i) and set

\[
Y_t = \exp\left( -Z_t + \frac{1}{2} [Z]_t \right),
\]

where \([Z]\) denotes the quadratic variation of \( Z \). We have

\[
dY_t = Y_t (-dZ_t + d[Z]_t),
\]

\[
d (Y_t \alpha^t) = \alpha^t Y_t (-dZ_t + d[Z]_t) + Y_t \, d\alpha^t + d[Y, \alpha]_t
\]

\[
= \alpha_t Y_t \, d[Z]_t + d[Y, \alpha]_t = \alpha_t Y_t \, d[Z]_t - \alpha_t Y_t \, d[\alpha]_t = 0,
\]

25
with $[Y, \alpha]_t$ denoting the quadratic co-variation process of $Y_t$ and $\alpha_t$. Hence $Y_t \alpha_t = \alpha_t$ and we have (ii) for $A_t = Z_t - \frac{1}{2}[Z]_t$.

We now assume (ii). We certainly have

$$d\alpha_t = \alpha e^{A_t} \left( dA_t + \frac{1}{2} d[A]_t \right) = \alpha_t \left( dA_t + \frac{1}{2} d[A]_t \right)$$

$$= \alpha_t \left( g_0(x(t), t) \, dt + \sum_{i=1}^{k} g_i(x(t), t) \circ dW^i(t) \right),$$

for $g_0 = h_0 + \frac{1}{2} \left( \sum_i h_i^2 \right)$ and $g_i = h_i$ for $i = 1, \ldots, k$. Comparing this to (3.4) yields $\mathcal{L}_V \alpha = g_0 \alpha$ for $i = 0, \ldots, k$. Hence (iii) is true and this completes the proof. \hfill \Box

**Acknowledgments.** The author wishes to thank Yakov Eliashberg, Helmut Hofer and Michael Hutchings for many fruitful discussions on Contact and Symplectic Geometry. After completing this manuscript, the author learned from Gregory Eyink that his student Theo Drivas has been preparing a paper that is closely related to this work. Special thanks to Gregory Eyink for a historical comment regarding formulas (1.13) and (2.4).

**References**


