Equilibrium Fluctuations for Coagulating-Fragmenting Brownian Particles

Fraydoun Rezakhanlou

Department of Mathematics
UC Berkeley

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National Taiwan University
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Outline

1. The Model
2. Scaling Limit
3. Fluctuations
4. Equilibrium
5. Idea of Proof

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- (Configuraion) $x_i \in \mathbb{R}^d, \quad m_i \in \mathbb{N}, \quad r_i \in (0, \infty), \quad i \in I$ are positions (centers) , masses and radii of particles (bubbles).
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  - x_i and x_j coagulate with rate $\alpha(m_i, m_j) V_\epsilon(x_i - x_j; m_i, m_j)$. The new particle of mass $m = m_i + m_j$ is at x_i with probability m_i/m.

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  - x_i fragments into two particles of masses m and $m_i - m$ with rate $\beta(m, m_i - m)V^\epsilon(x_i - y; m_i - m, m)$. The new particles are at x_i and y.

(Details)

- We assume $d = 2$, $V \geq 0$ of compact support and total integral 1.

The central object to study is the cluster density of a given size; Empirical measures

$$g_n^\varepsilon(dx, t) = K_\varepsilon^{-1} \sum_i \delta_{x_i(t)}(dx) 1\!\!1(m_i(t) = n),$$

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- Set $K_\varepsilon = |\log \varepsilon|$. Think of K_ε as the number of particles per unit area.

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Theorem (FR and Hammond when there is no fragmentation)

$g_n^\varepsilon(dx,t)$ converges to $f_n(x,t)dx$ where f_n is a solution to the Smoluchowski's equation.

Smoluchowski's equation (solution is unique)

$$\frac{\partial f_n}{\partial t}(x,t) = d(n)\Delta_x f_n(x,t) + Q_n^{+,c}(\mathbf{f}) - Q_n^{-,c}(\mathbf{f}) + Q_n^{+,f}(\mathbf{f}) - Q_n^{-,f}(\mathbf{f}),$$

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The function $\eta(m, n)$ is calculated in terms of the microscopic details of the model. In the case $d = 2$, η is independent of the function V and the parameter χ, and is simply given by

$$\eta(m, n) = \frac{2\pi(d(m) + d(n))}{2\pi(d(m) + d(n)) + \alpha(m, n)}$$

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Fluctuation fields $\xi_n^\varepsilon(dx,t)$

$$\sqrt{K_\varepsilon}\left(K_\varepsilon^{-1}\sum_i \delta_{x_i(t)}(dx)\ 1\!\!1(m_i(t)=n) - f_n(x,t)dx\right)$$

Heuristics: Roughly,

$$g_n^\varepsilon = f_n + (K_\varepsilon)^{-1/2}\xi_n + o((K_\varepsilon)^{-1/2})$$

with $\mathbf{g}^\varepsilon$ satisfying

$$\begin{aligned}\frac{\partial g_n^\varepsilon}{\partial t} =& d(n)\Delta_x g_n^\varepsilon + Q_n^{+,c}(\mathbf{g}^\varepsilon) - Q_n^{-,c}(\mathbf{g}^\varepsilon) + Q_n^{+,f}(\mathbf{g}^\varepsilon) - Q_n^{-,f}(\mathbf{g}^\varepsilon)\\ &+ (K_\varepsilon)^{-1/2}\,\gamma_n + o((K_\varepsilon)^{-1/2}).\end{aligned}$$

Conjecture

As $\varepsilon \to 0$, the process ξ_n^ε converges to ξ_n where ξ_n is the unique solution to the Uhlenbeck–Ornstein equation

$$\frac{\partial \xi_n}{\partial t} = d(n)\Delta_x \xi_n + \mathcal{L}_n^c \boldsymbol{\xi} + \mathcal{L}_n^f \boldsymbol{\xi} + \gamma_n$$

Here $\boldsymbol{\xi} = (\xi_n : n \in \mathbb{N})$, and

$$\mathcal{L}_n^c = \mathcal{L}_n^{+,c} - \mathcal{L}_n^{-,c}, \quad \mathcal{L}_n^f = \mathcal{L}_n^{+,f} - \mathcal{L}_n^{-,f}$$

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γ_n is a space-time white noise with variance

$$\mathbb{E}\left(\sum_n \iint J_n \gamma_n dxdt\right)^2$$

given by the sum of

- $2\iint \sum_n d(n) f_n |\nabla J_n|^2 dxdt$

for any smooth test function $J = (J_n : n \in \mathbb{N})$ of compact support in $\mathbb{R}^d \times (0, \infty)$.

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Take a collection of positive numbers $\boldsymbol{\lambda} = (\lambda_n : n \in \mathbb{N})$ such that

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- ***Remark:*** For such a collection, $f_n(x,t) \equiv \lambda_n$ solves the Smoluchowski's equation because $\hat{\alpha}(m,n)\lambda_n\lambda_m = \hat{\beta}(m,n)\lambda_{n+m}$, so

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- Given such $\boldsymbol{\lambda}$, we construct a reversible invariant measure: Let $\mathbf{x}^n$ to be a Poisson point process with intensity $K_\varepsilon \lambda_n$. Assume that $(\mathbf{x}^n, n \in \mathbb{N})$ are independent. Set $\omega = (\mathbf{x}, \mathbf{m})$ with $\mathbf{x} = \bigcup_{n=1}^{\infty} \mathbf{x}^n$ and $\mathbf{m}(a) = n$ for $a \in \mathbf{x}^n$.

Theorem (FR and Ranjbar)

Assume the system is in equilibrium as above. As $\varepsilon \to 0$, the process ξ_n^ε converges to ξ_n where ξ_n is the unique solution to the Uhlenbeck–Ornstein equation

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Comments: (Relevant Parameters:)
$\hat{\alpha} = \eta\alpha$, $\tilde{\alpha} = \eta^2\alpha$ and similarly define $\hat{\beta}$, $\tilde{\beta}$.
Recall

$$\eta(m,n) = \frac{2\pi(d(m)+d(n))}{2\pi(d(m)+d(n))+\alpha(m,n)} < 1.$$

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Auxiliary function u^ε

$$(d(m)+d(n))\Delta u^\varepsilon(x) = \alpha(m,n)\left[V_\varepsilon(x;m,n)u^\varepsilon(x) + V^\varepsilon(x;m,n)\right].$$

Here $u^\varepsilon(x) = u^\varepsilon(x;m,n)$.

Dynamics of $x_i - x_j$ has an infinitesimal generator of a killed Brownian motion:

$$\Gamma^\varepsilon = (d(m)+d(n))\Delta - \alpha(m,n)V_\varepsilon(\cdot;m,n),$$

with $m = m_i$ and $n = m_j$. Now the function $u^\varepsilon = \Gamma_\varepsilon^{-1}V^\varepsilon$ is smoother than V^ε and this allows us to perturb its argument by a small vector z. Γ_ε^{-1} is relevant because of the time average.

Auxiliary function u^ε

Main Step

We replace $V^\varepsilon(a)$ with $W^\varepsilon(a+z)$ for any z satisfying $|z| \le |\log\log\varepsilon|^{-\theta}$ with $0 < \theta < 1/2$. Here $a = x_i - x_j$ and

$$W^\varepsilon(a; m, n) = V^\varepsilon(a; m, n)(1 + K_\varepsilon^{-1} u^\varepsilon(a; m, n)).$$

Remark

- First we can afford $|z| \le |\log\varepsilon|^{\theta'}$ with $\theta' < 1/2$. Good enough for going from α to $\hat{\alpha}$. No CLT used. Less singular kernel. Not good enough.

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- We can now go to z satisfying $|z| \le |\log\log\varepsilon|^{-\theta}$
- We have

$$\lim_{\varepsilon\to 0} \sup_{|x|\le k} \left| u^\varepsilon(\varepsilon x)|\log\varepsilon|^{-1} + 1 - \eta \right| = 0$$