1. (a) \( \Delta x = (b-a)/n = (4-0)/2 = 2 \)

\[
L_2 = \sum_{i=1}^{2} f\left( x_{i-1} \right) \Delta x = f\left( x_0 \right) \cdot 2 + f\left( x_1 \right) \cdot 2 = 2\left[ f(0) + f(2) \right] = 2(0.5 + 2.5) = 6
\]

\[
R_2 = \sum_{i=1}^{2} f\left( x_i \right) \Delta x = f\left( x_1 \right) \cdot 2 + f\left( x_2 \right) \cdot 2 = 2\left[ f(2) + f(4) \right] = 2(2.5 + 3.5) = 12
\]

\[
M_2 = \sum_{i=1}^{2} f\left( \bar{x}_i \right) \Delta x = f\left( \bar{x}_1 \right) \cdot 2 + f\left( \bar{x}_2 \right) \cdot 2 = 2\left[ f(1) + f(3) \right] \approx 2(1.6 + 3.2) = 9.6
\]

\[
(b)
\]

\( L_2 \) is an underestimate, since the area under the small rectangles is less than the area under the curve, and \( R_2 \) is an overestimate, since the area under the large rectangles is greater than the area under the curve. It appears that \( M_2 \) is an overestimate, though it is fairly close to \( I \). See the solution to Exercise 45 for a proof of the fact that if \( f \) is concave down on \([a,b]\), then the Midpoint Rule is an overestimate of \( \int_a^b f(x) dx \).

(c) \( T_2 = \left( \frac{1}{2} \Delta x \right) \left[ f\left( x_0 \right) + 2f\left( x_1 \right) + f\left( x_2 \right) \right] = \frac{2}{2} \left[ f(0) + 2f(2) + f(4) \right] = 0.5 + 2(2.5) + 3.5 = 9 \).

This approximation is an underestimate, since the graph is concave down. Thus, \( T_2 = 9 < I \). See the solution to Exercise 45 for a general proof of this conclusion.

(d) For any \( n \), we will have \( L_n < T_n < I < M_n < R_n \).

7. \( f(x) = \sqrt{1+x^2} \), \( \Delta x = \frac{2-0}{8} = \frac{1}{4} \)

(a) \( T_8 = \frac{1}{4 \cdot 2} \left[ f(0) + 2f\left( \frac{1}{4} \right) + 2f\left( \frac{1}{2} \right) + \cdots + 2f\left( \frac{3}{2} \right) + 2f\left( \frac{7}{4} \right) + f(2) \right] \approx 2.413790 
\]

(b) \( M_8 = \frac{1}{4} \left[ f\left( \frac{1}{8} \right) + f\left( \frac{3}{8} \right) + \cdots + f\left( \frac{13}{8} \right) + f\left( \frac{15}{8} \right) \right] \approx 2.411453 
\]

(c) \( S_8 = \frac{1}{4 \cdot 3} \left[ f(0) + 4f\left( \frac{1}{4} \right) + 2f\left( \frac{1}{2} \right) + 4f\left( \frac{3}{4} \right) + 2f(1) + 4f\left( \frac{5}{4} \right) + 2f\left( \frac{3}{2} \right) + 4f\left( \frac{7}{4} \right) + f(2) \right] \approx 2.412216 \)

13. \( f(x) = e^{1/x} \), \( \Delta x = \frac{2-1}{4} = \frac{1}{4} \)

(a) \( T_4 = \frac{1}{4 \cdot 2} \left[ f(1) + 2f(1.25) + 2f(1.5) + 2f(1.75) + f(2) \right] \approx 2.031893 \)
(b) \( M_4 = \frac{1}{4} [f(1.125)+f(1.375)+f(1.625)+f(1.875)] \approx 2.014207 \)

(c) \( S_4 = \frac{1}{4} \frac{1}{3} [f(1)+4f(1.25)+2f(1.5)+4f(1.75)+f(2)] \approx 2.020651 \)

18. \( f(x) = \frac{e^x}{x} \), \( \Delta x = \frac{4-2}{10} = \frac{1}{5} \)

(a) \( T_{10} = \frac{1}{5} \frac{1}{2} \left[ f(2)+2[f(2.2)+f(2.4)+f(2.6)+\cdots+f(3.8)]+f(4) \right] \approx 14.704592 \)

(b) \( M_{10} = \frac{1}{5} \left[ f(2.1)+f(2.3)+f(2.5)+f(2.7)+\cdots+f(3.7)+f(3.9) \right] \approx 14.662669 \)

(c) \( S_{10} = \frac{1}{5} \frac{1}{3} \left[ f(2)+4f(2.2)+2f(2.4)+4f(2.6)+\cdots+2f(3.6)+4f(3.8)+f(4) \right] \approx 14.676696 \)

23. (a) Using a CAS, we differentiate \( f(x) = e^{\cos x} \) twice, and find that \( f''(x) = e^{\cos x} \left( \sin^2 x - \cos x \right) \).

From the graph, we see that the maximum value of \( |f''(x)| \) occurs at the endpoints of the interval \([0, 2\pi]\). Since \( f''(0) = -e \), we can use \( K = e \) or \( K = 2.8 \).

(b) A CAS gives \( M_{10} \approx 7.954926518 \). (In Maple, use student[middlesum].)

(c) Using Theorem 3 for the Midpoint Rule, with \( K = e \), we get \( |E_M| \leq \frac{e(2\pi-0)^3}{24 \cdot 10^2} \approx 0.280945995 \).

With \( K = 2.8 \), we get \( |E_M| \leq \frac{2.8(2\pi-0)^3}{24 \cdot 10^2} = 0.289391916 \).

(d) A CAS gives \( I \approx 7.954926521 \).

(e) The actual error is only about \( 3 \times 10^{-9} \), much less than the estimate in part (c).

(f) We use the CAS to differentiate twice more, and then graph \( f^{(4)}(x) = e^{\cos x} \left( \sin^4 x - 6\sin^2 x \cos x + 3 - 7\sin^2 x \cos x \right) \).

From the graph, we see that the maximum value of \( |f^{(4)}(x)| \) occurs at the endpoints of the interval \([0, 2\pi]\). Since \( f^{(4)}(0) = 4e \), we can use
27. \( \int_{-8}^{14} x^3 \, dx = \left[ \frac{2}{3} x^{3/2} \right]_{1}^{14} = \frac{2}{3} (8-1) = \frac{14}{3} \approx 4.666667 \)

\( n = 6 \):

\[ \Delta x = \frac{(4-1)}{6} = \frac{1}{2} \]

\[ T_6 = \frac{1}{2} \cdot \frac{1}{2} \left[ \sqrt{1 + 2\sqrt{1.5 + 2\sqrt{2} + 2\sqrt{2.5 + 2\sqrt{3} + 2\sqrt{3.5 + \sqrt{4}}} \right] \approx 4.661488 \]

\[ M_6 = \frac{1}{2} \left[ \sqrt{1.25 + \sqrt{1.75 + \sqrt{2.25 + \sqrt{2.75 + \sqrt{3.25 + \sqrt{3.75}}} \right] \approx 4.669245 \]

\[ S_6 = \frac{1}{2} \cdot \frac{1}{3} \left[ \sqrt{1 + 4\sqrt{1.5 + 2\sqrt{2} + 4\sqrt{2.5 + 2\sqrt{3} + 4\sqrt{3.5 + \sqrt{4}}}} \right] \approx 4.666563 \]

\[ E_T \approx \frac{14}{3} - 4.661488 \approx 0.005178 \] , \( E_M \approx \frac{14}{3} - 4.669245 \approx -0.002578 \), \( E_S \approx \frac{14}{3} - 4.666563 \approx 0.000104 \).

\( K = 4e \) or \( K = 10.9 \).

\[(g) \text{ A CAS gives } S_{10} \approx 7.953789422. (\text{In Maple, use student[simpson].}) \]

\[(h) \text{ Using Theorem 4 with } K = 4e, \text{ we get } |E_S| \leq \frac{4e(2\pi-0)^5}{180 \cdot 10^4} \approx 0.059153618. \text{ With } K = 10.9, \text{ we get} \]

\[ |E_S| \leq \frac{10.9(2\pi-0)^5}{180 \cdot 10^4} \approx 0.059299814. \]

\[(i) \text{ The actual error is about } 7.954926521 - 7.953789422 \approx 0.00114. \text{ This is quite a bit smaller than the estimate in part (h), though the difference is not nearly as great as it was in the case of the Midpoint Rule.} \]

\[(j) \text{ To ensure that } |E_S| \leq 0.0001, \text{ we use Theorem 4: } |E_S| \leq \frac{4e(2\pi)^5}{180 \cdot n^4} \leq 0.0001 \Rightarrow \frac{4e(2\pi)^5}{180 \cdot 0.0001} \leq n^4 \]

\[ \Rightarrow n \geq 5, 915, 362 \Leftrightarrow n \geq 49.3. \text{ So we must take } n \geq 50 \text{ to ensure that } \left| I - S_n \right| \leq 0.0001. (K = 10.9 \text{ leads to the same value of } n.) \]
n = 12:
\[ \Delta x = (4 - 1)/12 = \frac{1}{4} \]
\[ T_{12} = \frac{1}{4 \cdot 2} \left( f(1) + 2[f(1.25) + f(1.5) + \cdots + f(3.5) + f(3.75)] + f(4) \right) \approx 4.665367 \]
\[ M_{12} = \frac{1}{4} \left[ f(1.125) + f(1.375) + f(1.625) + \cdots + f(3.875) \right] \approx 4.667316 \]
\[ S_{12} = \frac{1}{4 \cdot 3} \approx 4.666659 \]
\[ E_T \approx \frac{14}{3} - 4.665367 \approx 0.001300, \ E_M \approx \frac{14}{3} - 4.667316 \approx -0.000649, \]
\[ E_S \approx \frac{14}{3} - 4.666659 \approx 0.000007. \]

Note: These errors were computed more precisely and then rounded to six places. That is, they were not computed by comparing the rounded values of \( T_n, M_n, \) and \( S_n \) with the rounded value of the actual integral.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( T_n )</th>
<th>( M_n )</th>
<th>( S_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>4.661488</td>
<td>4.669245</td>
<td>4.666563</td>
</tr>
<tr>
<td>12</td>
<td>4.665367</td>
<td>4.667316</td>
<td>4.666659</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( n )</th>
<th>( E_T )</th>
<th>( E_M )</th>
<th>( E_S )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>0.005178</td>
<td>-0.002578</td>
<td>0.000104</td>
</tr>
<tr>
<td>12</td>
<td>0.001300</td>
<td>-0.000649</td>
<td>0.000007</td>
</tr>
</tbody>
</table>

Observations:
(a) \( E_T \) and \( E_M \) are opposite in sign and decrease by a factor of about 4 as \( n \) is doubled.
(b) The Simpson’s approximation is much more accurate than the Midpoint and Trapezoidal approximations, and seems to decrease by a factor of about 16 as \( n \) is doubled.

29. \( \Delta x = (4 - 0)/4 = 1 \)
(a) \[ T_4 = \frac{1}{2} \left[ f(0) + 2f(1) + 2f(2) + 2f(3) + f(4) \right] \approx \frac{1}{2} \left[ 0 + 2(3) + 2(5) + 2(3) + 1 \right] = 11.5 \]
(b) \[ M_4 = 1 \cdot \left[ f(0.5) + f(1.5) + f(2.5) + f(3.5) \right] \approx 1 + 4.5 + 4.5 + 2 = 12 \]
(c) \[ S_4 = \frac{1}{3} \left[ f(0) + 4f(1) + 2f(2) + 4f(3) + f(4) \right] \approx \frac{1}{3} \left[ 0 + 4(3) + 2(5) + 4(3) + 1 \right] = 11.6 \]