1. The auxiliary equation is \( r^2 - 2r - 15 = 0 \Rightarrow (r-5)(r+3) = 0 \Rightarrow r=5, r=-3 \). Then the general solution is 
\[ y = c_1 e^{5x} + c_2 e^{-3x}. \]

2. The auxiliary equation is \( r^2 + 4r + 13 = 0 \Rightarrow r = -2 \pm 3i \), so 
\[ y = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x). \]

3. The auxiliary equation is \( r^2 + 3 = 0 \Rightarrow r = \pm \sqrt{3}i \). Then the general solution is 
\[ y = c_1 \cos \left( \sqrt{3}x \right) + c_2 \sin \left( \sqrt{3}x \right). \]

4. The auxiliary equation is \( 4r^2 + 4r + 1 = 0 \Rightarrow (2r+1)^2 = 0 \Rightarrow r = -\frac{1}{2} \), so the general solution is 
\[ y = c_1 e^{-x/2} + c_2 xe^{-x/2}. \]

5. \( r^2 - 4r + 5 = 0 \Rightarrow r = 2 \pm i \), so 
\[ y_c(x) = e^{2x}(c_1 \cos x + c_2 \sin x). \] Try \( y_p(x) = Ae^{2x} \Rightarrow y_p' = 2Ae^{2x} \) and \( y_p'' = 4Ae^{2x} \) . Substitution into the differential equation gives 
\[ 4Ae^{2x} - 8Ae^{2x} + 5Ae^{2x} = e^{2x} \Rightarrow A = 1 \] and the general solution is 
\[ y(x) = e^{2x}(c_1 \cos x + c_2 \sin x) + e^{2x}. \]

6. \( r^2 + r - 2 = 0 \Rightarrow r = 1, r = -2 \) and 
\[ y_c(x) = c_1 e^x + c_2 e^{-2x}. \] Try \( y_p(x) = Ax^2 + Bx + C \Rightarrow y_p' = 2Ax + B \) and \( y_p'' = 2A \). Substitution gives 
\[ 2A + 2Ax + B - 2Ax - 2Bx - 2C = x^2 \Rightarrow A = B = -\frac{1}{2}, \quad C = -\frac{3}{4} \] so the general solution is 
\[ y(x) = c_1 e^x + c_2 e^{-2x} - \frac{1}{2} x - \frac{1}{2} x - \frac{3}{4}. \]

7. \( r^2 - 2r + 1 = 0 \Rightarrow r = 1 \) and 
\[ y_c(x) = c_1 e^x + c_2 xe^x. \] Try \( y_p(x) = (Ax + B) \cos x + (Cx + D) \sin x \Rightarrow 
\[ y_p' = (C - Ax - B) \sin x + (A + Cx + D) \cos x \] and 
\[ y_p'' = (2C - A - 2x) \cos x + (-2A - D - Cx) \sin x. \] Substitution gives 
\[ (-2Cx + 2C - 2A - 2D) \cos x + (2Ax - 2A + 2B - 2C) \sin x = x \cos x \Rightarrow A = 0, \quad B = C = D = -\frac{1}{2}. \] The general solution is 
\[ y(x) = c_1 e^x + c_2 xe^x - \frac{1}{2} \cos x - \frac{1}{2} (x + 1) \sin x. \]

8. \( r^2 + 4 = 0 \Rightarrow r = \pm 2i \) and 
\[ y_c(x) = c_1 \cos 2x + c_2 \sin 2x. \] Try 
\[ y_p(x) = Ax \cos 2x + B \sin 2x \] so that no term of \( y_p \) is a solution of the complementary equation. Then 
\[ y_p' = (A + 2Bx) \cos 2x + (B - 2Ax) \sin 2x \] and
The auxiliary equation is given by

\[ y'' = (4B - 4Ax)\cos 2x + (-4A - 4Bx)\sin 2x. \]

Substitution gives \( 4B\cos 2x - 4A\sin 2x = \sin 2x \Rightarrow A = -\frac{1}{4} \) and \( B = 0 \). The general solution is \( y(x) = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4} \) \( x \cos 2x \).\]

9. \( r^2 - r - 6 = 0 \Rightarrow r = -2, r = 3 \) and \( y_c(x) = c_1 e^{-2x} + c_2 e^{3x} \). For \( y' = -y \) \( -6y = 1 \), try \( y_p(x) = B \). Then \( y'(x) = y''(x) = 0 \) and substitution into the differential equation gives \( A = -\frac{1}{6} \). For \( y' = (B - 2Bx)e^{-2x} \) and \( y'' = (4B - 4B)e^{-2x} \), and substitution gives \( -5Be^{-2x} = -e^{-2x} \Rightarrow B = -\frac{1}{5} \). The general solution then is

\[ y(x) = c_1 e^{-2x} + c_2 e^{3x} + y_p(x) = c_1 e^{-2x} + c_2 e^{3x} - \frac{1}{6} - \frac{1}{5} x e^{-2x}. \]

10. Using variation of parameters, \( y_c(x) = c_1 \cos x + c_2 \sin x \), \( u_1'(x) = -\csc x \sin x = -1 \Rightarrow u_1(x) = -x \), and \( u_2(x) = \frac{\csc x \cos x}{x} = \cot x \Rightarrow u_2(x) = \ln |\sin x| \Rightarrow y_p = -x \cos x + \sin x \ln |\sin x| \). The solution is

\[ y(x) = (c_1 - x) \cos x + (c_2 + \ln |\sin x|) \sin x. \]

11. The auxiliary equation is \( r^2 + 6r = 0 \) and the general solution is \( y(x) = c_1 + c_2 e^{-6x} = k_1 + k_2 e^{6(x-1)} \). But \( 3 = y(1) = k_1 + k_2 \) and \( 12 = y'(1) = -6k_2 \). Thus \( k_2 = -2, k_1 = 5 \) and the solution is \( y(x) = 5 - 2e^{-6(x-1)} \).

12. The auxiliary equation is \( r^2 - 6r + 25 = 0 \) and the general solution is \( y(x) = e^{3x} (c_1 \cos 4x + c_2 \sin 4x) \). But \( 2 = y(0) = c_1 \) and \( 1 = y'(0) = 3c_1 + 4c_2 \). Thus the solution is \( y(x) = e^{3x} \left( 2 \cos 4x - \frac{5}{4} \sin 4x \right) \).

13. The auxiliary equation is \( r^2 - 5r + 4 = 0 \) and the general solution is \( y(x) = c_1 e^x + c_2 e^{4x} \). But \( 0 = y(0) = c_1 + c_2 \) and \( 1 = y'(0) = c_1 + 4c_2 \), so the solution is \( y(x) = \frac{1}{3} (e^{4x} - e^x) \).

14. \( y(x) = c_1 \cos (x/3) + c_2 \sin (x/3) \). For \( 9y' = y = 3x \), try \( y_p(x) = Ax + B \). Then \( y_p(x) = 3x \). For \( 9y' = y = e^{-x} \), try

\[ y = c_1 \cos (x/3) + c_2 \sin (x/3). \]
y(x) = Ae^{-x}. Then y = e^{-x} + 9e^{-x} = 10e^{-x}. Thus the general solution is

\[ y(x) = c_1 \cos\left(\frac{x}{3}\right) + c_2 \sin\left(\frac{x}{3}\right) + 3x + \frac{1}{10} e^{-x}. \]

But 1 = y(0) = c_1 + \frac{1}{10} and 2 = y'(0) = \frac{1}{3} c_2 + \frac{3}{10} = \frac{9}{10}

and \( c_2 = \frac{27}{10}\). Hence the solution is

\[ y(x) = \frac{1}{10} \left[ 9 \cos\left(\frac{x}{3}\right) - 27 \sin\left(\frac{x}{3}\right) \right] + 3x + \frac{1}{10} e^{-x}. \]

15. Let \( y(x) = \sum_{n=0}^{\infty} c_n x^n \). Then \( y' = \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n \) and the differential equation becomes \( \sum_{n=0}^{\infty} \left[ (n+2)(n+1)c_{n+2} + (n+1)c_n \right] x^n = 0 \). Thus the recursion relation is

\[ c_{n+2} = -\frac{1}{n(n+2)} c_n \] for \( n = 0, 1, 2, \ldots \). But \( c_0 = y(0) = 0 \), so \( c_{2n} = 0 \) for \( n = 0, 1, 2, \ldots \). Also \( c_1 = y'(0) = 1 \), so

\[ c_{2n+1} = \frac{(-1)^n 2^n n!}{(2n+1)!} \] for \( n = 0, 1, 2, \ldots \). Thus the solution to the initial–value problem is

\[ y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n n!}{(2n+1)!} x^{2n+1}. \]

16. Let \( y(x) = \sum_{n=0}^{\infty} c_n x^n \). Then \( y' = \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n \) and the differential equation becomes \( \sum_{n=0}^{\infty} \left[ (n+2)(n+1)c_{n+2} - (n+2)c_n \right] x^n = 0 \). Thus the recursion relation is

\[ c_{n+2} = \frac{c_n}{n+1} \] for \( n = 0, 1, 2, \ldots \). Given \( c_0 \) and \( c_1 \), we have \( c_2 = \frac{c_0}{2} \), \( c_4 = \frac{c_2}{3} = \frac{c_0}{3} \),

\[ c_6 = \frac{c_4}{5} = \frac{c_0}{1 \cdot 3 \cdot 5}, \ldots, c_{2n} = \frac{c_0}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)} \]

\[ = \frac{c_0}{2^n 1! (2n-1)!} \] . Similarly \( c_3 = \frac{c_1}{2}, c_5 = \frac{c_3}{4} = \frac{c_1}{2 \cdot 4} \),

\[ c_7 = \frac{c_5}{6} = \frac{c_1}{2 \cdot 4 \cdot 6}, \ldots, c_{2n+1} = \frac{c_1}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2n} = \frac{c_1}{2^n n!} \] . Thus the general solution is

\[ y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 \sum_{n=1}^{\infty} \frac{2^{n-1} (n-1)! x^n}{(2n-1)!} + x \sum_{n=0}^{\infty} \frac{2^{n+1} x^{2n+1}}{2^n n!} \]

\[ = x \sum_{n=0}^{\infty} \frac{(1 \cdot 2 x)^n}{n!} = x e^{x^2} \]

\[ , \text{ so } y(x) = c_1 x e^{x^2/2} + c_0 + c_1 \sum_{n=1}^{\infty} \frac{2^{n-1} (n-1)! x^{2n}}{(2n-1)!}. \]